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DC Power Grids With Constant-Power Loads—Part II: Nonnegative Power Demands, Conditions for Feasibility, and High-Voltage Solutions

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***Abstract*—In this two-part article, we develop a unifying framework for the analysis of the feasibility of the power flow equations for dc power grids with constant-power loads. Part II of this article, explores further implications of the results in Part I. We present a necessary and sufficient linear matrix inequality (LMI) condition for the feasibility of a vector of power demands (under small perturbation), which extends a necessary condition in the literature. The alternatives of these LMI conditions are also included. In addition, we refine these LMI conditions to obtain a necessary and sufficient condition for the feasibility of nonnegative power demands, which allows for an alternative approach to determine power flow feasibility. Moreover, we prove two novel sufficient conditions, which generalize known sufficient conditions for power flow feasibility in the literature. Finally, we prove that the unique long-term voltage semistable operating point associated to a feasible vector of power demands is a strict high-voltage solution. A parametrization of such operating points, which is dual to the parametrization in Part I, is obtained, as well as a parametrization of the boundary of the set of feasible power demands.**

***Index Terms*—Power flow analysis, dc power grids, constant-power loads, voltage stability.**

IV. Introduction of Part II

THE feasibility of the power flow equations is of crucial importance for the long-term safe operation of a power grid. Classical papers such as [1]–[3] have studied this problem for ac power grids, and over the past decade, the research for ac power grids has been reinvigorated by articles such as [4]–[8]. Unfortunately, a complete understanding of this problem is still lacking.

Similar to the ac case, the somewhat simpler case concerning dc power grids is also not well-understood. A notable advancement is [9], which presents an algorithm to decide on the feasibility of the dc power flow equations with constant-power loads. However, a full characterization of the feasibility of the dc power flow equations is not found in the literature. For a more detailed introduction, we refer to Part I of this article.

The aim of this twin article is to provide an in-depth analysis of the power flow equations of dc power grids with constant-power loads, and develop a framework which unifies and extends known results in the literature. In Part I, we presented a complete geometric characterization of the feasibility of the associated power flow equations. More importantly, we obtained necessary and sufficient conditions for their feasibility, and presented a method to compute the corresponding long-term voltage semistable operating point, which was shown to be unique. These advances fill an important gap in the literature, and provide a deep insight in the nature of power flow feasibility and voltage stability of power grids with constant-power loads. In Part II of this article continues this approach by studying nonnegative power demands, sufficient conditions for feasibility, and high-voltage solutions. We refer to Part I for a list of the main results M1–M11 of this twin article.

A. Organization of Part II

Section V presents a necessary and sufficient LMI condition for the feasibility of a vector of power demands, and a similar condition for feasibility under small perturbation (M5). In addition, the LMI alternatives of these results are presented.

Section VI focuses on nonnegative power demands, and studies when such power demands are feasible. First, we give an alternative parametrization of $\mathcal{D}$ and discuss its relation to the parametrization of $\mathcal{D}$ in Part I (M6). By means of this parametrization, we study the boundary of $\mathcal{F}$ (M7), and derive a parametrization for the boundary of feasible power demands in the nonnegative orthant (M8a). This allows us to refine the necessary and sufficient condition M5 for nonnegative power demands (M8b).

Section VII recovers and generalizes several sufficient conditions in the literature in the context of dc power grids. More

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specifically, we prove two sufficient conditions (M10), which generalize the sufficient conditions in [7] and [8]. In addition, we show that any power demand, which is element-wise dominated by a feasible power demand is feasible as well (M9).

Section VIII focuses on the long-term voltage semistable operating points. We show that any such operating point is a strict high-voltage solution. As a consequence, the notions of long-term voltage stable operation points, dissipation-minimizing operation points, and (strict) high-voltage solutions coincide (M11). Section IX concludes this article.

B. Notation and matrix definitions

For a vector $x = \begin{pmatrix} x_1 & \cdots & x_k \end{pmatrix}^\top$, we denote

$$[x] := \operatorname{diag}(x_1, \dots, x_k).$$

We let $\mathbb{1}$ and $\mathbb{0}$ denote the all-ones and all-zeros vector, respectively, and let I denote the identity matrix. We let their dimensions follow from their context. All vector and matrix inequalities are taken to be element-wise. We write $x \lneq y$ if $x \leq y$ and $x \neq y$. We let $\|x\|_p$ denote the p-norm of $x \in \mathbb{R}^k$.

We define $\boldsymbol{n} := \{1, \dots, n\}$. All matrices are square $n \times n$ matrices, unless stated otherwise. The submatrix of a matrix A with rows and columns indexed by $\alpha, \beta \subseteq \boldsymbol{n}$, respectively, is denoted by $A_{[\alpha,\beta]}$. The same notation $v_{[\alpha]}$ is used for subvectors of a vector v. We let α^{c} denote the set-theoretic complement of α with respect to $\boldsymbol{n}$. For a set S, the notation $\operatorname{int}(S)$, $\operatorname{cl}(S)$, ∂S, and $\operatorname{conv}(S)$ is used for the interior, closure, boundary and convex hull of S, respectively.

We list some classical definitions from matrix theory.

Definition 4.1 (see[10], Ch. 5): A matrix A is a *Z-matrix* if $A_{ij} \leq 0$ for all $i \neq j$.

Definition 4.2 (see[10], Th. 5.3): A Z-matrix is an *M-matrix* if all its eigenvalues have nonnegative real part.

Definition 4.3 (see[10], pp. 71): A matrix A is *irreducible* if for every nonempty set $\alpha \subsetneqq \boldsymbol{n}$ we have $A_{[\alpha,\alpha^{\mathrm{c}}]} \neq 0$.

Definition 4.4: The Schur complement of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the principal submatrix D is denoted by

$$M/D := A - BD^{-1}C.$$

V. Necessary and Sufficient Conditions for Feasibility

We continue Part I of this article by restating the geometric characterization of $\mathcal{F}$ in Theorem 3.18 in terms of an LMI condition. In the context of Problem 2.6, Barabanov et al. [8] presented a necessary LMI condition for the feasibility of power demands, and states that the LMI condition is also sufficient when the set of feasible power demands is closed and convex, as is the case here. The following theorem recovers this result and extends the result for power demands which are feasible under small perturbation.

Theorem 5.1 (M5a): A vector $\widetilde{P}_c$ of power demands is feasible (i.e., $\widetilde{P}_c \in \mathcal{F}$) if and only if there does not exist a positive vector $\nu \in \mathbb{R}^n$ such that the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} [\nu]Y_{LL} + Y_{LL}[\nu] & [\nu]\mathcal{I}_L^* \\ ([\nu]\mathcal{I}_L^*)^\top & 2\nu^\top \widetilde{P}_c \end{pmatrix} = 2 \begin{pmatrix} h(\nu) & \frac{1}{2}[\nu]\mathcal{I}_L^* \\ \frac{1}{2}([\nu]\mathcal{I}_L^*)^\top & \nu^\top \widetilde{P}_c \end{pmatrix} \tag{63}$$

is positive definite. Similarly, $\widetilde{P}_c$ is feasible under small perturbation (i.e., $\widetilde{P}_c \in \operatorname{int}(\mathcal{F})$) if and only if there does not exist a positive vector $\nu \in \mathbb{R}^n$ such that (63) is positive semidefinite.

Proof: We will prove the logical transposition.

($\Leftarrow$): Without loss of generality, we assume that $\|\nu\|_1 = 1$. If (63) is positive semidefinite, then $h(\nu)$ is positive semidefinite. It follows from Lemmas B.8 and B.9 that $h(\nu)$ is an irreducible M-matrix. Let $v > \mathbb{0}$ be a Perron vector of $h(\nu)$. Suppose that $h(\nu)$ is singular, then $h(\nu)v = \mathbb{0}$ by Proposition A.2. However, note that for $t \in \mathbb{R}$ we have

$$\begin{pmatrix} tv \\ 1 \end{pmatrix}^\top \begin{pmatrix} h(\nu) & \frac{1}{2}[\nu]\mathcal{I}_L^* \\ \frac{1}{2}([\nu]\mathcal{I}_L^*)^\top & \nu^\top \widetilde{P}_c \end{pmatrix} \begin{pmatrix} tv \\ 1 \end{pmatrix} = tv^\top [\nu]\mathcal{I}_L^* + \nu^\top \widetilde{P}_c$$

which is a nonconstant line in t since $v^\top [\nu]\mathcal{I}_L^* > 0$, and is not bounded from below. This contradicts the assumption that (63) is positive semidefinite. Hence, $h(\nu)$ must be positive definite and $\nu \in \Lambda_1$. Alternatively, if (63) is positive definite, then $h(\nu)$ is positive definite. If $h(\nu)$ is positive definite, then by the Haynsworth inertia additivity formula (see [11, Sec. 0.10]) (63) is positive definite (semidefinite) if and only if

$$\nu^\top \widetilde{P}_c - \tfrac{1}{4}([\nu]\mathcal{I}_L^*)^\top h(\nu)^{-1}[\nu]\mathcal{I}_L^* > (\geq)\, 0. \tag{64}$$

Using (35) and (38), we note that (64) is equivalent to

$$\nu^\top \widetilde{P}_c > (\geq)\, \tfrac{1}{4}([\nu]\mathcal{I}_L^*)^\top h(\nu)^{-1}[\nu]\mathcal{I}_L^* = \|\varphi(\nu)\|^2_{h(\nu)}. \tag{65}$$

Theorem 3.18 implies that $\widetilde{P}_c$ is not feasible if and only if there exists $\lambda \in \Lambda$ such that $\widetilde{P}_c \notin H_\lambda$, or equivalently, $\lambda^\top \widetilde{P}_c > \|\varphi(\lambda)\|^2_\lambda$. Thus, if (63) is positive definite, then the strict inequality in (65) holds and $\widetilde{P}_c$ is not feasible. Moreover, if equality in (65) holds then

$$\nu^\top \widetilde{P}_c = \|\varphi(\nu)\|^2_{h(\nu)} = \nu^\top P_c(\varphi(\nu)).$$

Lemma 3.9 implies that $\widetilde{P}_c = P_c(\varphi(\nu))$, and thus $\widetilde{P}_c \in \partial\mathcal{F}$ by Theorem 3.14. Thus, if (63) is positive semidefinite, then $\widetilde{P}_c \notin \mathcal{F}$ or $\widetilde{P}_c \in \partial\mathcal{F}$, and therefore, $\widetilde{P}_c \notin \operatorname{int}(\mathcal{F})$. □

($\Rightarrow$): The converse is obtained by reversing the steps.

Theorem 5.1 presents a necessary and sufficient LMI conditions for the feasibility (under small perturbation) of a dc power grid with constant-power loads. A more common formulation of Theorem 5.1 as an LMI condition can be obtained by replacing $[\nu]$ by a positive definite diagonal matrix D, and replacing $\nu^\top \widetilde{P}_c$ by $\mathbb{1}^\top D\widetilde{P}_c$ (*cf.* [8]).

Note that Theorem 5.1 shows that power flow feasibility and the positive definiteness of (63) are mutually exclusive. By considering the LMI alternative of the latter condition, we may obtain an equivalence of power flow feasibility instead.

Theorem 5.2 (M5b): A vector $\widetilde{P}_c$ of power demands is feasible (i.e., $\widetilde{P}_c \in \mathcal{F}$) if and only if there exists a nonzero symmetric positive semidefinite matrix $Z = Z^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$\operatorname{trace}\left(Z \begin{pmatrix} h(e_i) & \frac{1}{2}[e_i]\mathcal{I}_L^* \\ \frac{1}{2}([e_i]\mathcal{I}_L^*)^\top & e_i^\top \widetilde{P}_c \end{pmatrix} \right) = 0 \tag{66}$$

for all $i = 1,\ldots,n$. Similarly, $\widetilde{P}_c$ is feasible under small perturbation (i.e., $\widetilde{P}_c \in \operatorname{int}(\mathcal{F})$) if and only if there exists a symmetric positive definite matrix $Z = Z^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ such that (66) holds for all i.

Proof: For $i = 1,\ldots,n$, we define

$$A_i := \begin{pmatrix} h(e_i) & \frac{1}{2}[e_i]\mathcal{I}_L^* \\ \frac{1}{2}([e_i]\mathcal{I}_L^*)^\top & e_i^\top \widetilde{P}_c \end{pmatrix}.$$

By Theorem 5.1, a vector $\widetilde{P}_c$ of power demands is feasible (i.e., $\widetilde{P}_c \in \mathcal{F}$) if and only if (63), which is equivalent to $2\sum_i \lambda_i A_i$, is not positive definite for all positive vectors $\lambda \in \mathbb{R}^n$. Note that $h(\lambda)$ [and hence (63)] is never positive definite if $\lambda \not> \mathbb{0}$ since Y_{LL} has positive diagonal elements. Hence, a vector $\widetilde{P}_c$ of power demands is feasible (i.e., $\widetilde{P}_c \in \mathcal{F}$) if and only if (63) is not positive definite for all $\lambda \in \mathbb{R}^n$. It follows from [12, Th. 1] that (63) is not positive definite for all $\lambda \in \mathbb{R}^n$ if and only if there exists a nonzero symmetric positive semidefinite $Z = Z^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ such that $\operatorname{trace}(ZA_i) = 0$ for all i. Analogously, by [12, Th. 2], it follows that (63) is not positive semidefinite for all $\lambda \in \mathbb{R}^n$ if and only if there exists a symmetric positive definite $Z = Z^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ such that $\operatorname{trace}(ZA_i) = 0$ for all i. □

Theorems 5.1 and 5.2 give a description of the power flow feasibility in terms of LMI problems and semidefinite programming problems. Since computational tools for such problems are widely available, these results may have a promising application for the assessment of power flow feasibility in a practical setting. As an example, the existence of equilibria to the dynamical dc power grid (15) of Section II-C can be determined by Theorems 5.1 or 5.2. This study of these results and their performance for benchmark power grids are an interesting topic for further research.

VI. Nonnegative Feasible Power Demands

In this section, we study the feasibility of nonnegative power demands (i.e., power demands $\widetilde{P}_c$ such that $\widetilde{P}_c \geq \mathbb{0}$). Recall that in Part I, we consider constant-power loads which could both drain power and inject power. However, practical applications of dc power grids often deal with constant-power loads that do not inject power into the network, in which case the power demands are nonnegative. The goal of this section is to refine the result of Part I for such power demands. In particular, we show that the necessary and sufficient LMI condition for the feasibility of a vector of power demands $\widetilde{P}_c \in \mathbb{R}^n$ (see Theorem 5.1) can be refined, leading to a condition that provides an alternative method to determine if the power flow is feasible.

This section is structured as follows. We first identify the operating points corresponding to a nonnegative power demand (see Lemma 6.1). In addition, we present a refinement for the geometric characterization of Theorem 3.18 (see Lemma 6.3), which motivates us to study the boundary of $\mathcal{F}$ in more detail. To study this boundary, we deduce an alternative parametrization of $\mathcal{D}$ (see Theorem 6.6), which is in a sense dual to the parametrization in Theorem 3.7. We subsequently give a parametrization of the boundary of $\mathcal{F}$ (see Theorem 6.8). This parametrization gives rise to a parametrization of the boundary of $\mathcal{F}$ in the nonnegative orthant (see Theorem 6.12). We then reformulate the geometric characterization (see Corollary 6.14), and refine the necessary and sufficient LMI condition of Theorem 5.1 for nonnegative power demands (see Theorem 6.15).

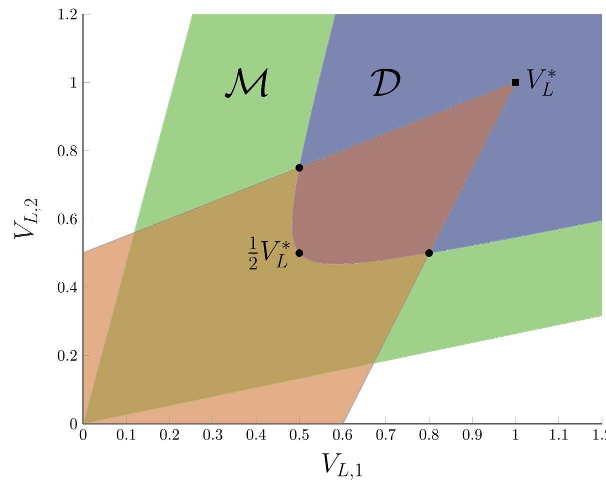

Fig. 1. Plot of the voltage domain for a power grid with two load nodes. The blue area corresponds to the set $\mathcal{D}$ of long-term voltage stable operating points. The brown area indicates the operating points corresponding to a nonnegative power demand. The green area corresponds to the vectors in $\mathcal{M}$, which contains the set $\operatorname{cl}(\mathcal{D})$. The black operating points correspond to the black power demands in Fig. 2.

A. Operating Points and a Geometric Characterization for Nonnegative Power Demands

We are interested in the nonnegative feasible power demands, which are described by the set $\mathcal{F} \cap \mathcal{N}$, where

$$\mathcal{N} := \{\, \nu \in \mathbb{R}^n \mid \nu \geq \mathbb{0} \,\}$$

denote the nonnegative vectors. The following lemma characterizes the operating points, which correspond to a nonnegative power demand.

Lemma 6.1: A feasible power demand $\widetilde{P}_c$ is nonnegative (i.e., $\widetilde{P}_c \in \mathcal{F} \cap \mathcal{N}$) if and only if the operating points $\widetilde{V}_L$ associated to $\widetilde{P}_c$ satisfy $Y_{LL}\widetilde{V}_L \leq Y_{LL}V_L^* = \mathcal{I}_L^*$.

Proof: Since operating points are assumed to be positive, we have $\widetilde{V}_L > \mathbb{0}$. Hence

$$\widetilde{P}_c = P_c(\widetilde{V}_L) = [\widetilde{V}_L]Y_{LL}(V_L^* - \widetilde{V}_L) \geq \mathbb{0}$$

if and only if $Y_{LL}(V_L^* - \widetilde{V}_L) = \mathcal{I}_L^* - Y_{LL}\widetilde{V}_L \geq \mathbb{0}$, where we used (6). □

Fig. 1 illustrates the location of these operating points in the voltage domain.

Lemma 6.1 shows that all operating points corresponding to a positive power demand lie in the polyhedral set

$$\left\{\, \widetilde{V}_L \in \mathbb{R}^n \;\middle|\; \widetilde{V}_L > \mathbb{0},\, Y_{LL}\widetilde{V}_L \leq \mathcal{I}_L^* \,\right\}. \tag{67}$$

Note that equality holds in $Y_{LL}\widetilde{V}_L \leq \mathcal{I}_L^*$ if and only if $\widetilde{V}_L = V_L^*$, which corresponds to the power demand $\widetilde{P}_c = \mathbb{0}$. The next result shows that the vector of open-circuit voltages V_L^* element-wise strictly dominates all operating points corresponding to a nonzero nonnegative power demand.

Corollary 6.2: Let $\widetilde{P}_c \neq \mathbb{0}$ be a nonnegative feasible power demand, then any operating point $\widetilde{V}_L$ associated to $\widetilde{P}_c$ satisfies $\widetilde{V}_L < V_L^*$. Hence, (67) is bounded.

Proof: The matrix Y_{LL} is an irreducible M-matrix, and hence, its inverse is positive by [10, Th. 5.12]. By Lemma 6.1, we have $Y_{LL}(V_L^* - \widetilde{V}_L) \geq \mathbb{0}$. Since $P_c(V_L^*) = \mathbb{0}$ and $\widetilde{P}_c \neq \mathbb{0}$ it follows that $\widetilde{V}_L \neq V_L^*$ and, therefore, $Y_{LL}(V_L^* - \widetilde{V}_L) \gneqq \mathbb{0}$. Multiplying this inequality by the positive matrix ${Y_{LL}}^{-1}$ implies that $V_L^* - \widetilde{V}_L > \mathbb{0}$. □

Using Theorem 3.18, we present a geometric characterization of $\mathcal{F} \cap \mathcal{N}$.

Lemma 6.3: The set $\mathcal{F} \cap \mathcal{N}$ is closed, convex, bounded, and is the intersection over all $\lambda \in \Lambda_1$ of the half-spaces H_λ for which $P(\varphi(\lambda))$ is nonnegative, i.e.,

$$\mathcal{F} \cap \mathcal{N} = \mathcal{N} \cap \bigcap_{\lambda \in \Lambda_1 :\, P_c(\varphi(\lambda)) \geq \mathbb{0}} H_\lambda.$$

Proof: The set $\mathcal{F} \cap \mathcal{N}$ is the intersection of closed convex sets, and is, therefore, closed and convex. Note that $h(\mathbb{1}) = Y_{LL}$, $\varphi(\mathbb{1}) = \frac{1}{2}V_L^*$, $P_c(\varphi(\mathbb{1})) = P_{\text{max}}$, and that (22) is equivalent to the inclusion

$$\mathcal{F} \subseteq H_{\mathbb{1}} = \left\{ y \,\middle|\, \mathbb{1}^\top y \leq \mathbb{1}^\top P_{\text{max}} \right\}.$$

It follows that $\mathcal{F} \cap \mathcal{N} \subseteq H_{\mathbb{1}} \cap \mathcal{N}$. The set $\mathcal{F} \cap \mathcal{N}$ is bounded since $H_{\mathbb{1}} \cap \mathcal{N}$ is bounded. It follows from Theorem 3.18 that

$$\mathcal{F} \cap \mathcal{N} = \mathcal{N} \cap \bigcap_{\lambda \in \Lambda_1} H_\lambda.$$

Since $\mathcal{F} \cap \mathcal{N}$ is closed and convex, it coincides with the intersection of its supporting half-spaces (see Section III-C). Theorem 3.12 identifies all supporting half-spaces of $\mathcal{F}$, and in particular shows that $P_c(\varphi(\lambda))$ is the unique point of support associated to the half-space H_λ. By definition, H_λ is also a supporting half-space for $\mathcal{F} \cap \mathcal{N}$ if and only if $P_c(\varphi(\lambda)) \in \mathcal{F} \cap \mathcal{N}$, which is equivalent to $P_c(\varphi(\lambda)) \geq \mathbb{0}$. □

The power demands $P_c(\varphi(\lambda))$ for $\lambda \in \Lambda_1$ describe the boundary of $\mathcal{F}$ (see Corollary 3.20 and Theorem 3.7). Lemma 6.3 characterizes all nonnegative feasible power demands in terms of the boundary in the nonnegative orthant (i.e., $\partial F \cap \mathcal{N}$). In its current form, this requires the identification of all λ such that $\lambda \in \lambda$ and $P_c(\varphi(\lambda)) \geq \mathbb{0}$, which is a nontrivial computational problem. In the remainder of this section, we deduce an alternative parametrization of the boundary of $\mathcal{F}$ in the nonnegative orthant. This parametrization leads to a more constructive description of all such λ.

B. Alternative Parametrization of $\mathcal{D}$

In order to parametrize the boundary of $\mathcal{F}$ in the nonnegative orthant, we study the set $\mathcal{D}$ of long-term voltage stable operating points in more detail. In Part I of this article, we have parametrized the set $\mathcal{D}$ and its boundary by means of the set λ_1 (see Theorem 3.7). In the following, we present an alternative parametrization of $\mathcal{D}$, which is dual to the parametrization in Theorem 3.7, in the sense that we parametrize $\mathcal{D}$ by the (right) Perron vector of $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$ instead of its transpose $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)^\top$.

We introduce the following definitions. For a vector $\mu \in \mathbb{R}^n$, we introduce the notation

$$g(\mu) := [\mu]Y_{LL} + [Y_{LL}\mu]. \tag{68}$$

Note that $g(\mu)$ is linear in μ, and that for any vector v we have $g(\mu)v = g(v)\mu$. By using (11) and (6), we observe that

$$\begin{aligned}\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L) &= [Y_{LL}V_L^*] - [\widetilde{V}_L]Y_{LL} - [Y_{LL}\widetilde{V}_L] \\ &= [\mathcal{I}_L^*] - g(\widetilde{V}_L).\end{aligned} \tag{69}$$

Analogous to λ, we define the set

$$\mathcal{M} := \{\, \mu \mid g(\mu) \text{ is a nonsingular M-matrix} \,\}.$$

Appendix E lists several properties of the set $\mathcal{M}$. In particular, Lemma E.1 shows that $\mathcal{M}$ is an open cone, which lies in the positive orthant, and that $\mathcal{M}$ is simply connected.

Recall that Z-matrices, M-matrices, and irreducible matrices were defined in Definitions 4.1–4.3. Appendix A lists multiple properties of such matrices. Recall the following proposition from Part I of this article.

Proposition 6.4 (Proposition 3.1 of Part I): Let A be an irreducible Z-matrix. There is a unique eigenvalue r of A with smallest (i.e., "most negative") real part. The eigenvalue r, known as the *Perron root*, is real and simple. A corresponding eigenvector v, known as a *Perron vector*, is unique up to scaling, and can be chosen such that $v > \mathbb{0}$.

The next lemma relates the Perron root and Perron vector of the Jacobian of P_c to the set $\mathcal{M}$.

Lemma 6.5: Let $r \in \mathbb{R}$ and $\mu \in \mathbb{R}^n$ such that $r \geq \mathbb{0}$ and $\mu > \mathbb{0}$. The Jacobian $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$ is an irreducible M-matrix with Perron root r and Perron vector μ if and only if $g(\mu)$ is an M-matrix (i.e., $\mu \in \mathcal{M}$) and $\widetilde{V}_L$ satisfies

$$\widetilde{V}_L = g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1}). \tag{70}$$

Proof: (⇒): The matrix Y_{LL} is an irreducible Z-matrix and $\mu > \mathbb{0}$, and so $g(\mu)$ is an irreducible Z-matrix by Propositions A.3 and A.4 of Part I. We let s and $v > \mathbb{0}$ denote, respectively, the Perron root and Perron vector of $g(\mu)$. The matrix $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$ is an M-matrix, and therefore, a Z-matrix. Lemma 3.2 states that $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$ is a Z-matrix if and only if $V_L^* > \mathbb{0}$, and so $\widetilde{V}_L > \mathbb{0}$. Using the fact that (r, μ) is an eigenpair to $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$ and substituting (69), we observe that

$$r\mu = -\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)\mu = g(\widetilde{V}_L)\mu - [\mathcal{I}_L^*]\mu = g(\mu)\widetilde{V}_L - [\mu]\mathcal{I}_L^*.$$

By rearranging terms it follows that

$$[\mu]\mathcal{I}_L^* + r\mu = g(\mu)\widetilde{V}_L. \tag{71}$$

Multiplying (71) by $v^\top$ results in

$$v^\top([\mu]\mathcal{I}_L^* + r\mu) = v^\top g(\mu)\widetilde{V}_L = sv^\top\widetilde{V}_L. \tag{72}$$

Note that $\widetilde{V}_L > \mathbb{0}$, $v > \mathbb{0}$, $\mu > \mathbb{0}$, $r \geq 0$, and $\mathcal{I}_L^* \gneq \mathbb{0}$. It follows that the left-hand side of (72) is positive. Since $v^\top \widetilde{V}_L$ is also positive, we deduce that the Perron root s is positive. This means that $g(\mu)$ is a nonsingular M-matrix (i.e., $\mu \in \mathcal{M}$), and that (70) follows from (71).

($\Leftarrow$): If $\mu \in \mathcal{M}$, then $\mu > \mathbb{0}$ by Lemma E.1. The rest of the proof follows by reversing the steps of the "$\Rightarrow$"-part. □

Note that (70) is invariant under scaling of μ, and since $\mathcal{M}$ is a cone we may normalize μ. For this purpose, we define

$$\mathcal{M}_1 := \mathcal{M} \cap \{ \mu \mid \|\mu\|_1 = 1 \} = \mathcal{M} \cap \{ \mu \mid \mathbb{1}^\top \mu = 1 \}.$$

Lemma 6.5 and Corollary 3.4 give rise to an alternative parametrization of $\mathcal{D}$.

Theorem 6.6 (M6): The set $\mathcal{D}$ of long-term voltage stable operating points, its closure $\mathrm{cl}(\mathcal{D})$, and its boundary $\partial\mathcal{D}$ are parametrized by

$$\mathcal{D} = \{ g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1}) \mid \mu \in \mathcal{M}_1, r > 0 \}$$
$$\mathrm{cl}(\mathcal{D}) = \{ g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1}) \mid \mu \in \mathcal{M}_1, r \geq 0 \}$$
$$\partial\mathcal{D} = \{ g(\mu)^{-1}[\mu]\mathcal{I}_L^* \mid \mu \in \mathcal{M}_1 \}.$$

Furthermore, the map

$$(\mu, r) \mapsto g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1})$$

from $\mathcal{M}_1 \times \mathbb{R}_{\geq 0}$ to $\mathrm{cl}(\mathcal{D})$ is a bicontinuous map.

The proof of Theorem 6.6 is analogous to the proof of Theorem 3.7, and is therefore omitted.

To simplify notation, we define for $\mu \in \mathcal{M}$ the map

$$\psi(\mu) := g(\mu)^{-1}[\mu]\mathcal{I}_L^*. \tag{73}$$

Note that Theorem 6.6 implies that $\psi(\mathcal{M}_1) = \partial\mathcal{D}$, which is a parametrization of the boundary of $\mathcal{D}$.

Fig. 1 illustrates that $\mathrm{cl}(\mathcal{D})$ is in fact a subset of $\mathcal{M}_1$, which is shown is Lemma 1.2.

Theorems 3.7 and 6.6 present two different parametrizations of $\partial\mathcal{D}$. The next lemma relates these two parametrizations, and will be instrumental for identifying which $\lambda \in \Lambda$ satisfy $P_c(\varphi(\lambda)) \geq \mathbb{0}$ in Lemma 6.3.

Lemma 6.7: Let $\widetilde{V}_L \in \partial\mathcal{D}$, then there exist

1) a unique vector $\lambda \in \Lambda_1$ such that $\widetilde{V}_L = \varphi(\lambda)$;
2) a unique vector $\mu \in \mathcal{M}_1$ such that $\widetilde{V}_L = \psi(\mu)$;
3) a positive scalar c such that

$$[\lambda]\widetilde{V}_L = c\mu. \tag{74}$$

Consequently, μ may be expressed in terms of λ, and vice versa, by

$$\mu = (\lambda^\top \varphi(\lambda))^{-1}[\lambda]\varphi(\lambda) \in \mathcal{M}_1 \tag{75}$$
$$\lambda = (\mathbb{1}^\top [\psi(\mu)]^{-1}\mu)^{-1}[\psi(\mu)]^{-1}\mu \in \Lambda_1. \tag{76}$$

Proof: The existence and uniqueness of λ and μ follows, respectively, from Theorems 3.7 and 6.6. Since Y_{LL} is symmetric we have

$$-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)[\widetilde{V}_L] = [\widetilde{V}_L]Y_{LL}[\widetilde{V}_L] + [\widetilde{V}_L][Y_{LL}(\widetilde{V}_L - V_L^*)]$$
$$= -[\widetilde{V}_L]\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)^\top. \tag{77}$$

Note that $-\frac{\partial P_c}{\partial V_L}$ and its transpose are singular M-matrices by Corollary 3.4, and are irreducible by Lemma 3.2 since $\widetilde{V}_L > \mathbb{0}$. Proposition A.2 states that the kernels of $-\frac{\partial P_c}{\partial V_L}$ and its transpose are spanned by any of their respective Perron vectors. Hence, if $\lambda \in \Lambda_1$ is such that $\widetilde{V}_L = \varphi(\lambda)$, then λ in a Perron vector of $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)^\top$ by Lemma 3.6. We deduce from (77) that

$$\mathbb{0} = -[\widetilde{V}_L]\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)^\top \lambda = -\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)[\widetilde{V}_L]\lambda.$$

It follows that $[\widetilde{V}_L]\lambda$ spans in the kernel of $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$. Lemma 6.5 implies that (74) holds for some scalar c. Since $\widetilde{V}_L > \mathbb{0}$, $\lambda > \mathbb{0}$, and $\mu > \mathbb{0}$, we have $c > \mathbb{0}$. Moreover, since $\mu^\top \mathbb{1} = 1$, multiplying (74) by $\mathbb{1}^\top$ yields $c = \lambda^\top \widetilde{V}_L = \lambda^\top \varphi(\lambda)$. By taking c to the other side of (74) we obtain (75). Similarly, since $\lambda^\top \mathbb{1} = 1$, multiplying (74) by $\mathbb{1}^\top [\widetilde{V}_L]^{-1}$ yields $1 = \mathbb{1}^\top [\widetilde{V}_L]^{-1}\mu c = \mathbb{1}^\top [\psi(\mu)]^{-1}\mu c$. By multiplying (74) by $[\psi(\mu)]^{-1}$ we obtain (76). □

Lemma 6.7, and in particular (74), establishes a duality between the two parametrizations of $\partial\mathcal{D}$. Note that (75) and (76) describe their correspondence.

C. Two Parametrizations of the Boundary of $\mathcal{F}$

We continue by studying parametrizations of the boundary of $\mathcal{F}$. Corollary 3.20 states that $\partial\mathcal{D}$ is in one-to-one correspondence with $\partial\mathcal{F}$. Since $\partial\mathcal{D}$ is parametrized both by $\varphi(\lambda)$ for $\lambda \in \Lambda_1$ (see Theorem 3.7) and by $\psi(\mu)$ for $\mu \in \mathcal{M}_1$ (see Theorem 6.6), it follows that $\partial\mathcal{F}$ can be parametrized as

$$\partial\mathcal{F} = \{ P_c(\varphi(\lambda)) \mid \lambda \in \Lambda_1 \} = \{ P_c(\psi(\mu)) \mid \mu \in \mathcal{M}_1 \}.$$

The following theorem gives an alternative formulation for both of these parametrizations.

Theorem 6.8 (M7): Let $\widetilde{P}_c \in \partial F$, then there exist unique vectors $\widetilde{V}_L \in \partial\mathcal{D}$ and $\mu \in \mathcal{M}_1$ such that $\widetilde{P}_c = P_c(\widetilde{V}_L)$ and $\widetilde{V}_L = \psi(\mu)$. These vectors satisfy

$$\widetilde{P}_c = [\widetilde{V}_L]^2[\mu]^{-1}Y_{LL}\mu. \tag{78}$$

This implies that the boundary of $\mathcal{F}$ is parametrized by

$$\partial\mathcal{F} = \{ [\psi(\mu)]^2[\mu]^{-1}Y_{LL}\mu \mid \mu \in \mathcal{M}_1 \}. \tag{79}$$

Proof: The existence and uniqueness of $\widetilde{V}_L$ and μ follows, respectively, from Corollary 3.20 and Theorem 6.6. By (73), (68), and (6) we have

$$\psi(\mu) = g(\mu)^{-1}[\mu]\mathcal{I}_L^*$$
$$= ([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[\mu]Y_{LL}V_L^*$$
$$= V_L^* - ([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[Y_{LL}\mu]V_L^*. \tag{80}$$

We deduce that

$$[\mu]Y_{LL}(V_L^* - \psi(\mu))$$
$$= [\mu]Y_{LL}([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[Y_{LL}\mu]V_L^*. \tag{81}$$

Observe that for any two square matrices A, B such that $A+B$ is nonsingular we have the identity[1]

$$A(A+B)^{-1}B = B(A+B)^{-1}A. \tag{82}$$

Using (82) with $A=[\mu]Y_{LL}$ and $B=[Y_{LL}\mu]$ in (81) yields

$$\begin{aligned}&[\mu]Y_{LL}(V_L^*-\psi(\mu))\\&=[Y_{LL}\mu]([\mu]Y_{LL}+[Y_{LL}\mu])^{-1}[\mu]Y_{LL}V_L^*\\&=[Y_{LL}\mu]\psi(\mu)=[\psi(\mu)]Y_{LL}\mu\end{aligned} \tag{83}$$

where we substituted (80). By (83) it follows that

$$\begin{aligned}P_c(\psi(\mu)) &= [\psi(\mu)]Y_{LL}(V_L^*-\psi(\mu))\\ &= [\psi(\mu)][\mu]^{-1}[\psi(\mu)]Y_{LL}\mu = [\psi(\mu)]^2[\mu]^{-1}Y_{LL}\mu\end{aligned}$$

which proves (79). Since $\widetilde{P}_c = P_c(\psi(\mu))$ and $\widetilde{V}_L=\psi(\mu)$ we have $\widetilde{P}_c=[\widetilde{V}_L]^2[\mu]^{-1}Y_{LL}\mu$, which proves (78). □

The duality of Lemma 6.7 implies the following corollary.

Corollary 6.9 (M7): Let $\widetilde{P}_c\in\partial F$, then there exists unique vectors $\widetilde{V}_L\in\partial D$ and $\lambda\in\Lambda_1$ such that $\widetilde{P}_c=P_c(\widetilde{V}_L)$ and $\widetilde{V}_L=\varphi(\lambda)$. These vectors satisfy

$$\widetilde{P}_c=[\widetilde{V}_L][\lambda]^{-1}Y_{LL}[\lambda]\widetilde{V}_L. \tag{84}$$

This implies that the boundary of $\mathcal{F}$ is parametrized by

$$\partial\mathcal{F}=\left\{\,[\varphi(\lambda)][\lambda]^{-1}Y_{LL}[\lambda]\varphi(\lambda)\;\middle|\;\lambda\in\Lambda_1\,\right\}. \tag{85}$$

D. Boundary of $\mathcal{F}$ in the Nonnegative Orthant

Theorem 6.8 gives an explicit relation between the boundary of $\mathcal{F}$ and the vectors $\mu\in\mathcal{M}_1$. The following lemma characterizes all $\mu\in\mathcal{M}_1$ for which the corresponding power demand in $\partial\mathcal{F}$ lies in the nonnegative orthant.

Lemma 6.10: Given $\widetilde{P}_c\in\partial\mathcal{F}$, let $\widetilde{V}_L\in\partial\mathcal{D}$ and $\mu\in\mathcal{M}_1$ be the unique vectors so that $\widetilde{P}_c=P_c(\widetilde{V}_L)$ and $\widetilde{V}_L=\psi(\mu)$, then $\widetilde{P}_c\in\mathcal{N}$ if and only if $Y_{LL}\mu\in\mathcal{N}$. Consequently, the boundary of $\mathcal{F}$ in the nonnegative orthant is parametrized by

$$\partial\mathcal{F}\cap\mathcal{N}=\{\,P_c(\psi(\mu))\mid\mu\in\mathcal{M}_1, Y_{LL}\mu\in\mathcal{N}\,\}.$$

Proof: The existence and uniqueness of $\widetilde{V}_L$ and μ follow, respectively, from Corollary 3.20 and Theorem 6.8. Note that $\widetilde{V}_L>\mathbb{0}$, and $\mu>\mathbb{0}$ by Lemma E.1. Hence, it follows from (78) that $\widetilde{P}_c\geq\mathbb{0}$ if and only if $Y_{LL}\mu\geq\mathbb{0}$. The parametrization follows directly from Theorem 6.8. □

Lemma 6.10 shows that any power demand $\widetilde{P}_c$ in $\partial\mathcal{F}\cap\mathcal{N}$ is uniquely associated to the vector $Y_{LL}\mu$ in $\mathcal{N}$. Conversely, we now show that any nonzero vector ν in $\mathcal{N}$ is, up to scaling of ν, is uniquely associated to a power demand in $\partial\mathcal{F}\cap\mathcal{N}$. We require the following lemma.

Lemma 6.11: For each nonzero vector $\nu\in\mathcal{N}$, we have ${Y_{LL}}^{-1}\nu\in\mathcal{M}$.

Proof: It suffices to show that $g({Y_{LL}}^{-1}\nu)$ is a nonsingular M-matrix. Note that

$$g({Y_{LL}}^{-1}\nu)=[{Y_{LL}}^{-1}\nu]Y_{LL}+[\nu].$$

The matrix Y_{LL} is a nonsingular irreducible M-matrix, and its inverse is a positive matrix by [10, Th. 5.12]. Since $\nu\gneqq\mathbb{0}$ it follows that ${Y_{LL}}^{-1}\nu>\mathbb{0}$. Hence, $[{Y_{LL}}^{-1}\nu]Y_{LL}$ is a nonsingular M-matrix by Proposition A.3:5. Since $\nu\gneqq\mathbb{0}$, Proposition A.3:6 implies that $[{Y_{LL}}^{-1}\nu]Y_{LL}+[\nu]$ is a nonsingular M-matrix. □

We normalize the nonzero vectors in $\mathcal{N}$ by

$$\begin{aligned}\mathcal{N}_1 &:= \mathcal{N}\cap\{\,\nu\mid\|\nu\|_1=1\,\}\\ &= \left\{\,\nu\in\mathbb{R}^n\;\middle|\;\nu\geq\mathbb{0},\ \mathbb{1}^\top\nu=1\,\right\}.\end{aligned} \tag{86}$$

We remark that $\mathcal{N}_1$ is known as the standard $n-1$-simplex.

Lemmas 6.10 and 6.11 suggest that each $\nu\in\mathcal{N}_1$ is uniquely associated to a vector $\mu\in\mathcal{M}_1$ for which the associated power demand $P_c(\psi(\mu))$ is nonnegative. Since there is a one-to-one correspondence between $\mathcal{M}_1$ and Λ_1 by Lemma 6.7, this would mean that there is a one-to-one correspondence between $\mathcal{N}_1$, and the vectors $\lambda\in\Lambda_1$ for which the associated power demand $P_c(\varphi(\lambda))$ is nonnegative. To this end, we define for nonzero $\nu\in\mathcal{N}$ the map

$$\begin{aligned}\chi(\nu) &:= \left[\psi({Y_{LL}}^{-1}\nu)\right]^{-1}{Y_{LL}}^{-1}\nu\\ &= \left[[{Y_{LL}}^{-1}\nu]^{-1}g\left({Y_{LL}}^{-1}\nu\right)^{-1}[{Y_{LL}}^{-1}\nu]\mathcal{I}_L^*\right]^{-1}\mathbb{1}.\end{aligned} \tag{87}$$

Since Y_{LL} is symmetric we have for all $\mu>\mathbb{0}$ that

$$[\mu]^{-1}g(\mu)[\mu]=(Y_{LL}+[\mu]^{-1}[Y_{LL}\mu])[\mu]=g(\mu)^\top \tag{88}$$

by using (68), and hence $\chi(\nu)$ can also be written as

$$\chi(\nu)=\left[g\left({Y_{LL}}^{-1}\nu\right)^{-\top}\mathcal{I}_L^*\right]^{-1}\mathbb{1}. \tag{89}$$

The following theorem establishes a one-to-one correspondence between the set $\mathcal{N}_1$ and the sets $\partial\mathcal{F}$, $\partial\mathcal{D}$, $\mathcal{M}_1$, and Λ_1 for which their associated power demands are nonnegative. In addition, we present a parametrization of the boundary of $\mathcal{F}$ restricted to the nonnegative orthant, in terms of $\mathcal{N}_1$.

Theorem 6.12 (M8a): There is a one-to-one correspondence between the following sets:

i) the nonnegative feasible power demands $\widetilde{P}_c$ on the boundary of $\mathcal{F}$ (i.e., $\widetilde{P}_c\in\partial\mathcal{F}\cap\mathcal{N}$);
ii) the operating points $\widetilde{V}_L$ on the boundary of $\mathcal{D}$ such that $Y_{LL}\widetilde{V}_L\leq\mathcal{I}_L^*$;
iii) the vectors $\mu\in\mathcal{M}_1$ such that $Y_{LL}\mu\in\mathcal{N}$;
iv) the vectors $\lambda\in\Lambda_1$ such that $Y_{LL}[\lambda]\varphi(\lambda)\in\mathcal{N}$;
v) the vectors $\nu\in\mathcal{N}_1$.

These correspondences satisfy the equations

$$\begin{aligned}&\widetilde{P}_c=P_c(\widetilde{V}_L); && \mu\propto[\lambda]\varphi(\lambda)\propto{Y_{LL}}^{-1}\nu\\ &\widetilde{V}_L=\psi(\mu)=\psi({Y_{LL}}^{-1}\nu) && \lambda\propto[\psi(\mu)]^{-1}\mu\propto\chi(\nu)\\ &\quad=\varphi(\lambda)=\varphi(\chi(\nu)); && \nu\propto Y_{LL}[\lambda]\varphi(\lambda)\propto Y_{LL}\mu\end{aligned}$$

where by $\propto$ we mean that equality holds up to a positive scaling factor. In particular, χ is a one-to-one correspondence between $\mathcal{N}_1$ and the set iv), up to scaling. Moreover, the boundary of $\mathcal{F}$ in the nonnegative orthant is parametrized by

$$\partial\mathcal{F}\cap\mathcal{N}=\left\{\,P_c(\psi({Y_{LL}}^{-1}\nu))\;\middle|\;\nu\in\mathcal{N}_1\,\right\}$$

and the corresponding operating points are parametrized by

$$\left\{\,\widetilde{V}_L\in\partial\mathcal{D}\;\middle|\;P_c(\widetilde{V}_L)\geq\mathbb{0}\,\right\}=\left\{\,\psi({Y_{LL}}^{-1}\nu))\;\middle|\;\nu\in\mathcal{N}_1\,\right\}.$$

[1] This identity may be verified by adding $A(A+B)^{-1}A$ to both sides of the equation and simplifying.

Proof: (i $\leftrightarrow$ ii): The map P_c from $\partial\mathcal{D}$ to $\partial\mathcal{F}$ is a one-to-one by Corollary 3.20. Lemma 6.1, therefore, implies that the map P_c from i) and ii) is one-to-one.

(i $\leftrightarrow$ iii): The map ψ from $\mathcal{M}_1$ to $\partial\mathcal{D}$ is one-to-one by Theorem 6.6, and hence, $P_c \circ \psi$ from $\mathcal{M}_1$ to $\partial\mathcal{F}$ is one-to-one. Lemma 6.10, therefore, implies that the map $P_c \circ \psi$ from i) to iii) is one-to-one.

(iii $\leftrightarrow$ iv): Lemma 6.7 establishes that $\mathcal{M}_1$ and λ_1 are in one-to-one correspondence, and that $\widetilde{V}_L = \psi(\mu) = \varphi(\lambda)$. Note that (75) and (76) imply that $\mu \propto [\lambda]\varphi(\lambda)$ and $\lambda \propto [\psi(\mu)]^{-1}\mu$. Substituting (74) in iii) results in iv) and are therefore equivalent.

(v $\leftrightarrow$ iii): Lemma 6.11 shows that the map $v \mapsto (\mathbb{1}^\top {Y_{LL}}^{-1}\nu)^{-1}{Y_{LL}}^{-1}\nu$ is a map $\mathcal{N}_1$ to $\mathcal{M}_1$. This map is injective since ${Y_{LL}}^{-1}$ is nonsingular, and is therefore one-to-one on its image, which is exactly the set iii). This shows that $\mu \propto {Y_{LL}}^{-1}\nu$ and $\nu \propto Y_{LL}\mu$.

Since $\mu \propto [\lambda]\varphi(\lambda)$ and $\nu \propto Y_{LL}\mu$, it follows that $\nu \propto Y_{LL}[\lambda]\varphi(\lambda)$. Due to (73) and (88), we have

$$[\mu]^{-1}\psi(\mu) = [\mu]^{-1}g(\mu)^{-1}[\mu]\mathcal{I}_L^* = g(\mu)^{-\top}\mathcal{I}_L^*.$$

Since $\lambda \propto [\psi(\mu)]^{-1}\mu$, it follows that $\lambda \propto [g(\mu)^{-\top}\mathcal{I}_L^*]^{-1}\mathbb{1}$. Because $\mu \propto {Y_{LL}}^{-1}\nu$, we deduce that $\lambda \propto \chi(\nu)$ by (89). Thus, the map χ from $\mathcal{N}_1$ to Λ is one-to-one, up to scaling.

Finally, the parametrizations follow directly from (i $\leftrightarrow$ iii) and (v $\leftrightarrow$ iii). □

Remark 6.13: From a computation standpoint, the parametrization of $\partial\mathcal{F} \cap \mathcal{N}$ in Theorem 6.12 is cheaper to compute than the parametrizations of $\partial\mathcal{F}$ in Theorem 6.8 or Corollary 6.9. Indeed, to compute the set $\partial\mathcal{F}$, we require to identify either $\mathcal{M}_1$ or Λ_1 by Theorem 6.8 or Corollary 6.9, respectively, which both are sets that are (in essence) described in terms of the eigenvalues of $n \times n$ matrices. In contrast, the parametrization of $\partial\mathcal{F} \cap \mathcal{N}$ in Theorem 6.12 is in terms of the set $\mathcal{N}_1$, which is merely an $n-1$-simplex and requires no additional computation.

E. Refined Results for Nonnegative Power Demands

We conclude this section by presenting a refinement of Theorems 3.18 and 5.1 for nonnegative power demands. This is obtained by applying Theorem 6.12 to Lemma 6.3.

Theorem 6.12 states that the map χ is a one-to-one correspondence between the set $\nu \in \mathcal{N}_1$ and vectors $\lambda \in \Lambda_1$ for which the associated power demand $P_c(\varphi(\lambda))$ is nonnegative. More specifically, we have

$$\begin{aligned}&\{\lambda \in \Lambda_1 |\, P_c(\varphi(\lambda)) \geq \mathbb{0}\ \} \\ &\quad = \left\{\ (\mathbb{1}^\top \chi(\nu))^{-1}\chi(\nu)\ \middle|\ \nu \in \mathcal{N}_1\ \right\} \subseteq \Lambda_1. \qquad (90)\end{aligned}$$

By substituting this result in Lemma 6.3, we obtain a geometric characterization of $\mathcal{F}$ in terms of $\mathcal{N}_1$.

Corollary 6.14: The set $\mathcal{F} \cap \mathcal{N}$ is the intersection over all $\nu \in \mathcal{N}_1$ of the half-spaces $H_{\chi(\nu)}$, i.e.,

$$\mathcal{F} \cap \mathcal{N} = \mathcal{N} \cap \bigcap_{\nu \in \mathcal{N}_1} H_{\chi(\nu)}.$$

Proof: The statement follows from substituting (90) in Lemma 6.3, and by noting the half-spaces H_λ are invariant under scaling of λ. □

We may now present a necessary and sufficient condition for a vector of nonnegative power demands to be feasible. This condition can be regarded as a refinement of Theorem 5.1 for nonnegative power demands, and is obtained from Corollary 6.14 by rewriting the half-spaces $H_{\chi(\nu)}$.

Theorem 6.15 (M8b): Let $\widetilde{P}_c$ be a nonnegative power demand (i.e., $\widetilde{P}_c \in \mathcal{N}$). The following are equivalent:

1) $\widetilde{P}_c$ is feasible (i.e., $\widetilde{P}_c \in \mathcal{F} \cap \mathcal{N}$);
2) the inequality
$$\chi(\nu)^\top \widetilde{P}_c \leq \tfrac{1}{2}\nu^\top V_L^* \qquad (91)$$
holds for all $\nu \in \mathcal{N}_1$;
3) the inequality
$$\|\widetilde{P}_c\|_\bullet := \max_{\nu \in \mathcal{N}_1} \left\{ \frac{\chi(\nu)^\top \widetilde{P}_c}{\frac{1}{2}\nu^\top V_L^*} \right\} \leq 1 \qquad (92)$$
holds

where $\chi(\nu)$ was defined in (87), where V_L^* are the open-circuit voltages (6), and where $\mathcal{N}_1$ is the standard $n-1$-simplex (86). More explicitly, (91) is equivalent to

$$\mathbb{1}^\top \left[\left([{Y_{LL}}^{-1}\nu] + {Y_{LL}}^{-1}[\nu] \right)^{-1} V_L^* \right]^{-1} \widetilde{P}_c \leq \tfrac{1}{2}\nu^\top V_L^*.$$

Similarly, $\widetilde{P}_c$ is feasible under small perturbation (i.e., $\widetilde{P}_c \in \operatorname{int}(\mathcal{F}) \cap \mathcal{N}$) if and only if the inequality in (91) holds strictly for all $\nu \in \mathcal{N}_1$, if and only if inequality in (92) holds strictly. □

Proof: (1 $\Leftrightarrow$ 2): Corollary 6.14 implies that $\widetilde{P}_c \in \mathcal{F} \cap \mathcal{N}$ if and only if $\widetilde{P}_c \in \mathcal{N}$ and $\widetilde{P}_c \in H_{\chi(\nu)}$ for all $\nu \in \mathcal{N}_1$. By definition of H_λ, the latter is equivalent to

$$\chi(\nu)^\top \widetilde{P}_c \leq \|\varphi(\chi(\nu))\|^2_{h(\chi(\nu))} \qquad (93)$$

for all $\nu \in \mathcal{N}_1$. We continue by rewriting the right-hand side of (93). Note that

$$\begin{aligned}\|\varphi(\chi(\nu))\|^2_{h(\chi(\nu))} &= \varphi(\chi(\nu))^\top h(\chi(\nu))\varphi(\chi(\nu)) \\ &= \tfrac{1}{2}\varphi(\chi(\nu))^\top [\chi(\nu)]\mathcal{I}_L^* \qquad (94)\end{aligned}$$

where we substituted (38) and (35). By substituting (87) in (94) it follows that the right-hand side of (93) equals

$$\tfrac{1}{2}\varphi(\chi(\nu))^\top [\psi({Y_{LL}}^{-1}\nu)]^{-1}[{Y_{LL}}^{-1}\nu]\mathcal{I}_L^*. \qquad (95)$$

Theorem 6.12 states that $\varphi(\chi(\nu)) = \psi({Y_{LL}}^{-1}\nu)$, and hence, from (95), we deduce that the right-hand side of (93) equals

$$\tfrac{1}{2}\mathbb{1}^\top [{Y_{LL}}^{-1}\nu]\mathcal{I}_L^* = \tfrac{1}{2}\nu^\top {Y_{LL}}^{-1}\mathcal{I}_L^* = \tfrac{1}{2}\nu^\top V_L^*$$

where we used (6). The left-hand side of (91) can be rewritten by observing in (89) that

$$\begin{aligned}g\left({Y_{LL}}^{-1}\nu\right)^{-\top}\mathcal{I}_L^* &= (Y_{LL}[{Y_{LL}}^{-1}\nu] + [\nu])^{-1}\mathcal{I}_L^* \\ &= ([{Y_{LL}}^{-1}\nu] + {Y_{LL}}^{-1}[\nu])^{-1}{Y_{LL}}^{-1}\mathcal{I}_L^* \\ &= ([{Y_{LL}}^{-1}\nu] + {Y_{LL}}^{-1}[\nu])^{-1}V_L^*\end{aligned}$$

where we used (68) and (6) of the Part I of this article.

(2 $\Leftrightarrow$ 3): Note that the right-hand side of (91) is positive since $\nu \gneq \mathbb{0}$ since $\nu \in \mathcal{N}_1$, and $V_L^* > \mathbb{0}$ due to Lemma 2.3. The equivalence of 2 and 3 follows immediately. The maximum is

well-defined since we maximize a continuous function over the compact set $\mathcal{N}_1$.

Lemma 3.9 states that we have equality in (93) if and only if $\widetilde{P}_c = P_c(\varphi(\chi(\nu)))$. Theorem 6.12 implies that $\widetilde{P}_c \in \partial\mathcal{F} \cap \mathcal{N}$ if and only if there exists $\nu \in \mathcal{N}_1$ such that $\widetilde{P}_c = P_c(\varphi(\chi(\nu)))$. Hence, $\widetilde{P}_c \notin \partial\mathcal{F} \cap \mathcal{N}$ if and only if equality in (91) does not hold for all $\nu \in \mathcal{N}_1$. Thus, $\widetilde{P}_c \in \mathrm{int}(\mathcal{F}) \cap \mathcal{N}$ if and only if the inequality in (91) holds strictly for all $\nu \in \mathcal{N}_1$, if and only if inequality in (92) holds strictly. □

The scalar $\|\widetilde{P}_c\|_\bullet$ defined in (92) provides an exact measure for the feasibility of the power flow in the power grid, generalizing the feasibility measures in [6] and [7]. Indeed, Theorem 6.15 tells us that the power flow is feasible if and only if $\|\widetilde{P}_c\|_\bullet \leq 1$. The relation between Theorem 6.15 and the sufficient conditions from [7] and [6] is discussed in Section VII. Moreover, if $\widetilde{P}_c$ is feasible then the scalar $1 - \|\widetilde{P}_c\|_\bullet$ provides a measure for how close the power flow is to unfeasibility.

We note that the necessary and sufficient condition presented in Theorem 6.15 does not require the definiteness of any matrices, in contrast to the necessary and sufficient conditions in Theorems 5.1 and 5.2. Instead, Theorem 6.15, and in particular (92), seeks to maximize a continuous scalar-valued function over a compact domain. The prospect of this observation is that may also use non-LMI-based computational techniques, such as gradient descent algorithms to determine if the power flow is feasible. However it is noted that an effective application of this approach and a comparison with the LMI approach of Theorems 5.1 and 5.2 requires a more in-depth study of the maximization problem in (92), and in particular of the possible concavity of the map $\nu \mapsto \frac{\chi(\nu)^\top \widetilde{P}_c}{\nu^\top V_L^*}$. These topics lie beyond the scope of this article.

Remark 6.16: Note that the numerator and denominator in (92) are invariant under scaling of ν. Hence, we may similarly define $\|\cdot\|_\bullet$ by taking the maximization over all $\nu \in \mathcal{N}$ so that $\frac{1}{2}\nu^\top V_L^* = 1$, in which case the denominator in (92) equals 1.

Remark 6.17: Similar results for positive power demand are obtained by taking $\mathcal{N} = \{\nu \in \mathbb{R}^n | \nu > \mathbb{0}\}$ throughout this section. In particular, analogous to Theorem 6.15, it can be shown that a vector of positive power demands $\widetilde{P}_c > \mathbb{0}$ is feasible if and only if (91) holds for all $\nu > \mathbb{0}$, and similar for feasibility under small perturbation.

VII. Sufficient Conditions for Power Flow Feasibility

In the remainder of this article, we return to the case where power demands are not restricted to the nonnegative orthant. In this section, we prove two novel sufficient conditions for the feasibility of a vector of power demands, which generalize the sufficient conditions found in [6] and [7]. In addition, we show how the conditions in [6] and [7] are recovered from the conditions proposed in this section.

The benefit of these sufficient conditions for power flow feasibility over a necessary and sufficient condition such as Theorem 5.1 is that they are cheaper to compute, and may, therefore, be more suitable for practical applications. However, since these sufficient condition are not necessary, they cannot guarantee that a power demand is not feasible.

This section is structured as follows. First, we show that each feasible vector of power demands gives rise to a sufficient condition for power flow feasibility (see Lemma 7.1), and derive a sufficient condition from Theorem 6.15 (Corollary 7.3). Next, we propose a weaker sufficient condition (see Theorem 7.5), which generalizes the condition in [7] and identifies for which vectors the latter condition is tight. Finally, we show that Theorem 7.5 generalizes the sufficient condition in [6], and argue why the latter condition is not tight in general (see Lemma 7.9).

A. Sufficient Conditions by Element-Wise Domination

Lemma 7.1 (M9): Let $\widetilde{P}_c$ be a feasible power demand (i.e., $\widetilde{P}_c \in \mathcal{F}$). If a power demand $\widehat{P}_c$ satisfies $\widehat{P}_c \lneq \widetilde{P}_c$, then $\widehat{P}_c$ is feasible under small perturbation (i.e., $\widehat{P}_c \in \mathrm{int}(\mathcal{F})$).

Proof: Since $\widetilde{P}_c \in \mathcal{F}$ we have by Theorem 3.18 that

$$\lambda^\top \widetilde{P}_c \leq \|\varphi(\lambda)\|^2_{h(\lambda)}$$

for all $\lambda \in \lambda$, where we used (42). Note that $\lambda > \mathbb{0}$ for $\lambda \in \Lambda$ by Lemma B.6. Since $\widehat{P}_c \lneq \widetilde{P}_c$, we have

$$\lambda^\top \widehat{P}_c < \lambda^\top \widetilde{P}_c \leq \|\varphi(\lambda)\|^2_{h(\lambda)} \tag{96}$$

for all $\lambda \in \Lambda$. Hence, $\widehat{P}_c \in \mathcal{F}$ by Theorem 3.18 and (42). Since the inequality in (96) is strict, Lemma 40 implies that $\widehat{P}_c \neq P_c(\varphi(\lambda))$ for all $\lambda \in \Lambda$, and therefore, $\widehat{P}_c \notin \partial\mathcal{F}$ by Corollary 3.20 and Theorem 3.7. □

Lemma 7.1 shows that any feasible power demand gives rise to a sufficient condition for power flow feasibility. In particular, note that the power demand $\mathbb{0} = P_c(V_L^*)$ is feasible under small perturbation. Lemma 7.1, therefore, implies the following corollary.

Corollary 7.2: Any nonpositive power demand is feasible under small perturbation.

We remark that a vector of nonpositive power demands corresponds to a case in which none of the power loads drain power from the grid and, therefore, behave as sources. Intuitively it is clear that such a vector of power demands is feasible. Consequently, some of the sources may act as loads and drain the power that is not dissipated in the lines.

Recall that Theorem 6.15 gives a necessary and sufficient condition for the feasibility of a nonnegative power demand. Lemma 7.1 allows us to extend Theorem 6.15 to a sufficient condition for vectors of power demands which have negative entries. We define $\mathbf{max}(a, b) \in \mathbb{R}^n$ as the vector obtained by taking the element-wise maximum of $a, b \in \mathbb{R}^n$, i.e.,

$$\mathbf{max}(a, b)_i := \max(a_i, b_i).$$

Note for $\widetilde{P}_c \in \mathbb{R}^n$ that $\mathbf{max}(\widetilde{P}_c, \mathbb{0})$ is nonnegative, and that $\widetilde{P}_c \leq \mathbf{max}(\widetilde{P}_c, \mathbb{0})$. Hence, Theorem 6.15 and Lemma 7.1 directly imply the following sufficient condition for the feasibility of a vector of power demands.

Corollary 7.3 (M10): A vector of power demands $\widetilde{P}_c \in \mathbb{R}^n$ is feasible (i.e., $\widetilde{P}_c \in \mathcal{F}$) if

$$\chi(\nu)^\top \mathbf{max}(\widetilde{P}_c, \mathbb{0}) \leq \tfrac{1}{2}\nu^\top V_L^*$$

for all $\nu \in \mathcal{N}_1$, or equivalently, if

$$\|\mathbf{max}(\widetilde{P}_c, \mathbb{0})\|_\bullet \leq 1$$

where $\|\cdot\|_\bullet$ was defined in (92).

Note that Corollary 7.3 is necessary and sufficient for nonnegative power demands by Theorem 6.15.

B. Generalization of the Sufficient Condition of Simpson-Porco et al. (2016)

We proceed by studying known sufficient conditions in the literature and comparing them to Corollary 7.3. The paper [7] studies the decoupled reactive power flow equations for lossless ac power grids with constant power loads. The analysis and results in [7] translate naturally to dc power grids. In [7], a sufficient condition for the feasibility of a vector of constant power demands is proposed, which we state in the context of dc power grids.

Proposition 7.4 (see[7, Supplementary Theorem 1]): Let $\widetilde{P}_c$ be a nonnegative vector of power demands (i.e., $\widetilde{P}_c \in \mathcal{N}$), then $\widetilde{P}_c$ is feasible under small perturbation (i.e., $\widetilde{P}_c \in \operatorname{int}(\mathcal{F})$) if

$$\|(\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\widetilde{P}_c\|_\infty < 1. \tag{97}$$

This sufficient condition for feasibility is tight since we have

$$\|(\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}P_{\max}\|_\infty = 1$$

where $P_{\max} \in \partial\mathcal{F}$ is the maximizing power demand defined in Lemma 2.18, and lies on the boundary of $\mathcal{F}$.

Proposition 7.4 applies only to nonnegative power demands and is not necessary in general. It is, therefore, weaker than Theorem 6.15 and Corollary 7.3. The proof of Proposition 7.4 in [7] relies on a fixed point argument. The following result generalizes Proposition 7.4, and identifies all power demands for which condition (97) is tight (i.e., the power demands on the boundary of $\mathcal{F}$ so that equality in (97) holds).

Theorem 7.5 (M10): A vector of power demands $\widetilde{P}_c$ is feasible (i.e., $\widetilde{P}_c \in \mathcal{F}$) if

$$(\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\mathbf{max}(\widetilde{P}_c, \mathbb{0}) \le \mathbb{1} \tag{98}$$

and feasible under small perturbation (i.e., $\widetilde{P}_c \notin \operatorname{int}(\mathcal{F})$) if $\widetilde{P}_c$ is not of the form

$$\begin{aligned}(\widetilde{P}_c)_{[\alpha]} &= \tfrac{1}{4}[(V_L^*)_{[\alpha]}]((Y_{LL}/(Y_{LL})_{[\alpha^c,\alpha^c]})(V_L^*)_{[\alpha]}\\ (\widetilde{P}_c)_{[\alpha^c]} &= \mathbb{0}\end{aligned} \tag{99}$$

for all nonempty $\alpha \subseteq \boldsymbol{n}$.

The proof of Theorem 7.5 is found in Appendix F. Note that Theorem 7.5 is weaker than Corollary 7.3, but is cheaper to compute. Proposition 7.4 is recovered from Theorem 7.5 as follows.

Proof of Proposition 7.4: Let $\widetilde{P}_c \in \mathcal{N}$ which implies that $\widetilde{P}_c = \mathbf{max}(\widetilde{P}_c, \mathbb{0})$. Let $\widetilde{P}_c$ satisfy (97), which is therefore equivalent to

$$-\mathbb{1} < (\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\mathbf{max}(\widetilde{P}_c, \mathbb{0}) < \mathbb{1}. \tag{100}$$

It follows from Theorem 7.5 that $\widetilde{P}_c \in \mathcal{F}$. The latter inequality in (100) is strict, and therefore, $\widetilde{P}_c$ lies in the interior of the set described by (98), and hence, $\widetilde{P}_c \in \operatorname{int}(\mathcal{F})$. ☐

Theorem 7.5 states that the power demands described by (99) are the only power demands which satisfy (98) and lie on the boundary of $\mathcal{F}$. The condition in (97) is therefore tight for such power demands.

Remark 7.6: Note that if $\alpha = \boldsymbol{n}$ in (99), we obtain the maximizing power, since

$$\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*]\mathbb{1} = \tfrac{1}{4}[V_L^*]\mathcal{I}_L^* = P_{\max}$$

by (6) and (23). The proof of Theorem 7.5 shows that the power demands described by (99) correspond to the maximizing power demands of all power grids obtained by Kron-reduction (see, e.g., [13]). The power flow of such power grids is equivalent to power flow of the full power grid, with the additional restriction that the currents at the loads indexed by α vanish (i.e., $(\mathcal{I}_L)_{[\alpha]} = \mathbb{0}$).

C. On the Sufficient Condition of Bolognani and Zampieri (2015)

We conclude this section by showing that Theorem 7.5 also generalizes the sufficient condition in [6]. The paper [6] studies the power flow equation of an ac power grid with constant power loads and a single source node. The analysis and results in [6] translate naturally to dc power grids with a single source node. The next lemma show that the results in [6] apply to dc power grids with multiple sources as well, which allows us to compare Theorem 7.5 and [6].

Lemma 7.7: Let $\mathcal{P}$ denote a dc power grid with constant power loads with n loads and m sources. Let $\widehat{\mathcal{P}}$ denote the dc power grid with n loads and a single source, of which the Kirchhoff matrix satisfies

$$\widehat{Y} = \begin{pmatrix}\widehat{Y}_{LL} & \widehat{Y}_{LS}\\ \widehat{Y}_{SL} & \widehat{Y}_{SS}\end{pmatrix} = \begin{pmatrix}[V_L^*]Y_{LL}[V_L^*] & -[V_L^*]\mathcal{I}_L^*\\ -(\mathcal{I}_L^*)^\top[V_L^*] & (V_L^*)^\top\mathcal{I}_L^*\end{pmatrix}. \tag{101}$$

and where the source voltage equals $\widehat{V}_S = 1$. The feasibility of the power flow equations of $\mathcal{P}$ and $\widehat{\mathcal{P}}$ is equivalent.

Proof: We first verify that $\widehat{Y}$ is indeed a Kirchhoff matrix. Note that $[V_L^*]Y_{LL}[V_L^*]\mathbb{1} = [V_L^*]\mathcal{I}_L^*$ by (6), and so $\widehat{Y}\mathbb{1} = \mathbb{0}$. Also, since Y_{LL} is an irreducible Z-matrix, and $V_L^* > \mathbb{0}$ and $\mathcal{I}_L^* \gneqq \mathbb{0}$, $\widehat{Y}$ is also an irreducible Z-matrix, and therefore, a Kirchhoff matrix. The powers injected at the loads in power grid $\widehat{\mathcal{P}}$ satisfy

$$\begin{aligned}\widehat{P}_L(\widehat{V}_L) &= [\widehat{V}_L](\widehat{Y}_{LL}\widehat{V}_L + \widehat{Y}_{LS}\widehat{V}_S)\\ &= [\widehat{V}_L]([V_L^*]Y_{LL}[V_L^*]\widehat{V}_L - [V_L^*]\mathcal{I}_L^*)\\ &= [\widehat{V}_L][V_L^*](Y_{LL}[V_L^*]\widehat{V}_L - Y_{LL}V_L^*) = P_L([V_L^*]\widehat{V}_L).\end{aligned}$$

where we used (6) and (4). We, therefore, have $\widehat{P}_L(\widehat{V}_L) = P_L(V_L)$ by taking $V_L = [V_L^*]\widehat{V}_L$. Hence, given $P_c \in \mathbb{R}^n$, we have that $\widehat{P}_L(\widehat{V}_L) = P_c$ is feasible for some $\widehat{V}_L > \mathbb{0}$ if and only if $P_L(V_L) = P_c$ is feasible for some $V_L > \mathbb{0}$. ☐

We continue by formulating the sufficient condition in [6]. We follow [6] and define for $p \in [1, \infty]$ the matrix norm

$$\|A\|_p^\star := \max_j\{\|A_{[j,\boldsymbol{n}]}\|_p\} \tag{102}$$

where $A_{[j,\boldsymbol{n}]}$ denotes the jth row of A. The sufficient condition for power flow feasibility of [6] in the context of dc power grids is given as follows.

Proposition 7.8 (see[6, Th. 1]): Let $\widetilde{P}_c \in \mathbb{R}^n$ be a vector of power demands, then $\widetilde{P}_c$ is feasible under small perturbation (i.e., $\widetilde{P}_c \in \operatorname{int}(\mathcal{F})$) if for some $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\|(\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\|_q^\star \, \|\widetilde{P}_c\|_p < 1. \tag{103}$$

The proof of Proposition 7.8 in [6] also relies on a fixed point argument. Proposition 7.8 is recovered from Lemma 7.1 and Proposition 7.4 as follows.

Proof: Let $\widetilde{P}_c$ satisfy (103). Let $\widehat{P}_c \in \mathcal{N}$ be such that $(\widehat{P}_c)_i = |(\widetilde{P}_c)_i|$. It follows that $\|\widehat{P}_c\|_p = \|\widetilde{P}_c\|_p$, and hence $\widehat{P}_c$ satisfies (103). The matrix Y_{LL} is an M-matrix, and hence, $[V_L^*]Y_{LL}[V_L^*]$ is a M-matrix by Proposition A.3.5. Its inverse is a positive matrix by [10, Th. 5.12]. Let $v^j \in \mathbb{R}^n$ be such that $(v^j)^\top$ is the jth row of $(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}$. We have $v^j > \mathbb{0}$. By (102) it follows from (103) that for all j

$$\|v^j\|_q \|\widehat{P}_c\|_p < 1.$$

By Hölder's inequality (see, e.g., [14, pp. 303]) we have

$$\|[v^j]\widehat{P}_c\|_1 \le \|v^j\|_q \|\widehat{P}_c\|_p < 1.$$

Since $[v^j]P_c \ge \mathbb{0}$ we know that $\|[v^j]\widehat{P}_c\|_1 = (v^j)^\top \widehat{P}_c$. This implies that $(v^j)^\top \widehat{P}_c < 1$ for all i, and hence

$$(\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\widehat{P}_c < \mathbb{1}.$$

Hence, (97) holds for $\widehat{P}_c$, and $\widehat{P}_c \in \operatorname{int}(\mathcal{F})$ by Proposition 7.4. Since $\widetilde{P}_c \le \widehat{P}_c$, Lemma 7.1 implies that $\widetilde{P}_c \in \operatorname{int}(\mathcal{F})$. □

Our proof of Proposition 7.8 shows that the sufficient condition in [6] for nonnegative power demands is more conservative in comparison to the sufficient condition in [7]. This also shows that Theorem 7.5 generalizes both results. The following lemma gives a more intuitive interpretation of condition (103), by showing that (103) describes the largest open p-ball such that (97) holds for nonnegative power demands.

Lemma 7.9: Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The sufficient condition for power flow feasibility (103) in Proposition 7.8 describes the open ball centered at $\mathbb{0}$

$$\mathcal{B} := \{\, y \in \mathbb{R}^n \mid \|y\|_p < r \,\}$$

where the radius $r = (\|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\|_q^\star)^{-1} > 0$ is the largest scalar such that (97) holds for all $\widetilde{P}_c \in \mathcal{B} \cap \mathcal{N}$.

Proof: We show that there exists a nonnegative vector of power demands on the boundary of $\mathcal{B}$ such that such that equality in (97) holds, which therefore defines the radius r. We continue our proof of Proposition 7.8. Let j be such that $\|v^j\|_q = \|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\|_q^\star$. If $p \ne 1$, then equality in Hölder's inequality $\|[v^j]\widetilde{P}_c\|_1 \le \|v^j\|_q \|\widetilde{P}_c\|_p$ holds if $(\widetilde{P}_c)_i = c((v^j)_i)^{q-1}$ for all i and for any $c \in \mathbb{R}^n$. Consider the positive vector of power demands $\widehat{P}_c$ given by $(\widehat{P}_c)_i = c((v^j)_i)^{q-1}$ and where $c^{-1} = \|v^j\|_q^q$. For this vector, we have $\|[v^j]\widehat{P}_c\|_1 = \|v^j\|_q \|\widehat{P}_c\|_p = 1$. Hence, by following proof of Proposition 7.8, $\widehat{P}_c$ satisfies equality in both (97) and (103). Thus, $\widehat{P}_c \in \partial\mathcal{B} \cap \mathcal{N}$, and $\|\widehat{P}_c\|_p = (\|v^j\|_q)^{-1} = r$. If $p = 1$, the same holds when we take $\widehat{P}_c = e_i \|v^j\|_\infty^{-1}$, where i is a single index such that $(v^j)_i = \|v^j\|_\infty$, and $(\widehat{P}_c)_i = 0$ otherwise. □

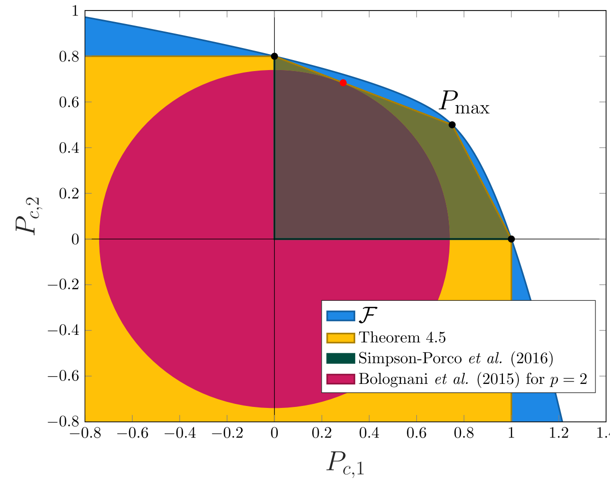

Fig. 2. Depiction of the set $\mathcal{F}$ of feasible power demands for a power grid with two loads. The yellow area corresponds to the sufficient condition in Theorem 7.5. The green shaded area corresponds to the set described by the sufficient condition in [7] (see Proposition 7.4), and does not include the yellow boundary. The black points are the power demands for which the condition in [7] is tight, and correspond to the black operating points in Fig. 1. The red area corresponds to the sufficient condition in [6] (see Proposition 7.8). The red point indicates a point of intersection of the boundary of the condition in [6] with either the boundary of the condition in [7], or the boundary of the condition in Theorem 7.5.

Note that $\widehat{P}_c$ constructed in the proof of Lemma 7.9 is not necessarily of the form (99). Since (97) is tight only for such points, this suggests that condition (103) is not tight in general. This is can be observed for $p = q = 2$ in Fig. 2 by the red dot, which does not lie on the boundary of $\mathcal{F}$.

VIII. Desirable Operating Points

We conclude this article by showing that for each feasible vector of power demands the different definitions of desirable operating points in Part I (Definitions 2.10–2.13) identify the same unique operating point. It was shown in [9] that for each feasible power demand there exists a unique operating point, which is a high-voltage solution, and that this operating point is "almost surely" long-term voltage stable. In addition, Matveev et al. [9] state that this operating point is the unique long-term voltage stable operating point if all power demands have the same sign. The following theorem sharpens these results by showing that the long-term voltage stable operating point associated to a feasible vector of power demands is a strict high-voltage solution.

Theorem 8.1 (M11): Let $\widetilde{P}_c$ be a feasible vector of power demands (i.e., $\widetilde{P}_c \in \mathcal{F}$). Let $\widetilde{V}_L \in \operatorname{cl}(\mathcal{D})$ be such that $\widetilde{V}_L$ is an operating point associated to $\widetilde{P}_c$ (i.e., $\widetilde{P}_c = P_c(\widetilde{V}_L)$), which exists and is unique by Theorem 3.17. Suppose there exists a vector $\widetilde{V}_L' \in \mathbb{R}^n$ such that $\widetilde{V}_L' \ne \widetilde{V}_L$ and $\widetilde{P}_c = P_c(\widetilde{V}_L')$, then $\widetilde{V}_L' < \widetilde{V}_L$. Hence, $\widetilde{V}_L$ is a strict high-voltage solution. Moreover, $\frac{1}{2}(\widetilde{V}_L' + \widetilde{V}_L)$ lies on the boundary of $\mathcal{D}$.

Proof: If $\widetilde{P}_c \in \partial\mathcal{F}$, then by Corollary 3.20, the operating point $\widetilde{V}_L \in \partial\mathcal{D}$ is the unique operating point associated to $\widetilde{P}_c$. Hence, a second operating point $\widetilde{V}_L'$ does not exist. The uniqueness of $\widetilde{V}_L$ implies that $\widetilde{V}_L$ is a high-voltage solution and is dissipation-minimizing.

If $\widetilde{P}_c \in \mathrm{int}(\mathcal{F})$, then by Corollary 3.21, we have $\widetilde{V}_L \in \mathcal{D}$. We define the vectors $v := \frac{1}{2}(\widetilde{V}_L + \widetilde{V}_L')$ and $\mu := \frac{1}{2}(\widetilde{V}_L - \widetilde{V}_L')$, and the line $\gamma(\theta) := v + \theta\mu$. Note that $\gamma(1) = \widetilde{V}_L$ and $\gamma(-1) = \widetilde{V}_L'$. Since $\widetilde{V}_L \in \mathcal{D}$ and $\widetilde{V}_L' \notin \mathrm{cl}(D)$ we have $\widetilde{V}_L \neq \widetilde{V}_L'$, and so $\mu \neq \mathbb{0}$. Lemma 3.9 implies that

$$P_c(\gamma(\theta)) = P_c(v + \theta\mu) = P_c(v) + \theta \frac{\partial P_c}{\partial V_L}(v)\mu - \theta^2 [\mu] Y_{LL}\mu. \quad (104)$$

Since $\widetilde{P}_c = P_c(\gamma(1)) = P_c(\gamma(-1))$, it follows from (104) that

$$\frac{\partial P_c}{\partial V_L}(v)\mu = \mathbb{0}. \quad (105)$$

We, therefore, have

$$P_c(\gamma(\theta)) = P_c(v) - \theta^2[\mu] Y_{LL}\mu \quad (106)$$

which describes a half-line contained in $\mathcal{F}$. Note also that $P_c(\gamma(\theta)) = P_c(\gamma(-\theta))$ and $\gamma(\theta) \neq \gamma(-\theta)$ if $\theta \neq 0$, which shows that the map $P_c(V_L)$ gives rise to a two-to-one correspondence between the line $\gamma(\theta)$ and the half-line (106) for $\theta \neq 0$. The line $\gamma(\theta)$ crosses the boundary of $\mathcal{D}$ since $\gamma(1) \in \mathcal{D}$ and $\gamma(-1) \notin \mathcal{D}$. Let $\widehat{\theta}$ be such that $\gamma(\widehat{\theta}) \in \partial\mathcal{D}$. Corollary 3.20 implies that there does not exists $\widehat{V}_L \neq \gamma(\widehat{\theta})$ such that $P_c(\gamma(\widehat{\theta})) = P_c(\widehat{V}_L)$. Hence, due to the two-to-one correspondence between $\gamma(\theta)$ and (106) for $\theta \neq 0$, we conclude that $\widehat{\theta} = 0$ and $\gamma(0) = v \in \partial\mathcal{D}$. Corollary 3.4 implies that $-\frac{\partial P_c}{\partial V_L}(v)$ is a singular M-matrix. Note that μ lies in the kernel of $-\frac{\partial P_c}{\partial V_L}(v)$ due to (105), and it follows from Lemma E.2 that $\pm\mu > \mathbb{0}$ and that μ spans the kernel of $-\frac{\partial P_c}{\partial V_L}(v)$. Since $\gamma(\theta)$ intersects $\partial\mathcal{D}$ only when $\theta = 0$, and since $\gamma(1) \in \mathcal{D}$ and $\gamma(-1) \notin \mathrm{cl}(\mathcal{D})$, it follows that $\gamma(\theta) \in \mathcal{D}$ if and only if $\theta > 0$. However, if $\mu < \mathbb{0}$ then $\gamma(\theta) = v + \theta\mu$ is a negative vector for sufficiently large θ, which contradicts that all vectors in $\mathrm{cl}(\mathcal{D})$ are positive. We conclude that $\mu > \mathbb{0}$, which by definition of μ implies that $\widetilde{V}_L > \widetilde{V}_L'$. The operating point $\widetilde{V}_L$ is a strict high-voltage solution by Definition 2.13. □

We conclude by proving that the different types of desirable operating points defined in Section II-B describe one and the same operating point.

Theorem 8.2 (M11): Let $\widetilde{P}_c$ be a feasible vector of power demands, and let $\widetilde{V}_L$ be an associated operating point (i.e., $\widetilde{P}_c = P_c(\widetilde{V}_L)$). The following statements are equivalent:

i) $\widetilde{V}_L$ is long-term voltage semistable (i.e., $\widetilde{V}_L \in \mathrm{cl}(\mathcal{D})$);
ii) $\widetilde{V}_L$ is the unique long-term voltage semistable operating point associated to $\widetilde{P}_c$;
iii) $\widetilde{V}_L$ is dissipation-minimizing;
iv) $\widetilde{V}_L$ is the unique dissipation-minimizing operating point associated to $\widetilde{P}_c$;
v) $\widetilde{V}_L$ is a high-voltage solution;
vi) $\widetilde{V}_L$ is a strict high-voltage solution.

Proof: Theorem 3.17 guarantees the existence and uniqueness of a long-term voltage semistable operating point $\widehat{V}_L$ associated to $\widetilde{P}_c$. It therefore suffices to show that $\widehat{V}_L$ is the unique operating point which satisfies statements iii)–vi) individually. Theorem 8.1 implies that $\widehat{V}_L$ is a (strict) high-voltage solution. Note that there exists at most one high-voltage solution, since $\widetilde{V}_L \leq \widetilde{V}_L'$ and $\widetilde{V}_L' \leq \widetilde{V}_L$ imply that $\widetilde{V}_L = \widetilde{V}_L'$. Corollary 2.14 implies that $\widehat{V}_L$ is the unique dissipation-minimizing operating point. □

Theorem 8.2 shows that the desirable operating points defined in Section II-B coincide, and that we may speak of a single desired operating point. Moreover, in the context of the dynamical power grid (15) of Section II-C, Theorem 8.2 states that there exists a stable equilibrium that always minimizes the total dissipation at steady state among all equilibria, and elementwise strictly dominates the voltage potentials of all other equilibria.

Remark 8.3: Note that none of the equivalent statements in Theorem 8.2 depends on Y_{SS}, which is the matrix that describes the interconnection of lines between the sources. Indeed, recall that Y_{SS} does not appear in (9), and recall from the proof of Proposition 2.12 that Y_{SS} is not relevant for finding a dissipation-minimizing operating point. However, from (14), we recall that the matrix Y_{SS} does affect the total dissipated power in the grid. Consequently, Y_{SS} does affect the total power that is dissipated when the operating point of Theorem 8.2 is chosen. Put differently, the minimal total power that is dissipated in the lines for a given vector P_c of constant power demands is not independent of the lines between the sources, despite the fact that the operating point which achieves this minimum is independent of these lines.

Remark 8.4: In [9], it was shown that for a feasible vector of power demands the algorithm proposed in [9], converges to a high-voltage solution. By Theorem 8.2, this means that this algorithm converges to the unique long-term voltage semistable operating point associated to these power demands.

IX. Conclusion

In this article, we constructed a framework for the analysis of the feasibility of the power flow equations for dc power grids. Within this framework, we unified and generalized the results in the literature concerning this feasibility problem, and gave a complete characterization of feasibility.

In Part II of this article, we showed that the feasibility (under small perturbation) of a power demand can be decided by an necessary and sufficient LMI condition. In addition, we gave a necessary and sufficient condition for the feasibility (under small perturbation) for nonnegative power demands, which provides an alternative method to determine power flow feasibility. We have presented two novel sufficient conditions for the feasibility of a power demand, which were shown to generalize known sufficient conditions in the literature. In addition, we proved that any power demand dominated by a feasible power demand is also feasible. Finally, we showed that the operating points corresponding to a power demand which are long-term voltage semistable, dissipation-minimizing, or a (strict) high-voltage solution, are one and the same.

Further directions of research may concern the question if and how the approach and/or results in this article generalize to general ac power grids. Other interesting directions of research concern, the feasibility of the power flow equations with uncertain parameters, conditions for long-term voltage (semi)stability of an operating point, and the (non)convexity of the set of such operating points. Furthermore, control schemes which implement the proposed conditions for power flow feasibility are of particular interest. Finally, it would be interesting to see how the approach of this twin article may be applied to problems outside the topic of power systems.

Appendix

E. Properties of $\mathcal{M}$

Lemma E.1: If $\mu \in \mathcal{M}$, then $\mu > \mathbb{0}$. Moreover, $\mathcal{M}$ is an open cone and is simply connected.

Proof: If $\mu \in \mathcal{M}$, then $g(\mu) = [\mu]Y_{LL} + [Y_{LL}\mu]$ is a nonsingular M-matrix, and therefore, a Z-matrix. Recall that Y_{LL} is an irreducible Z-matrix, which implies that $(Y_{LL})_{[i,i^c]} \lneqq 0$ for all i. If $\mu_i < 0$, then $g(\mu)_{[i,i^c]} = \mu_i (Y_{LL})_{[i,i^c]} \gneqq 0$, which contradicts the fact that $g(\mu)$ is a Z-matrix. Hence, $\mu \geq \mathbb{0}$. We will show that a vector μ, which contains zeros does not yield an M-matrix. Suppose $\mu \in \mathcal{M}$ such that $\mu_{[\alpha]} = \mathbb{0}$ and $\mu_{[\alpha^c]} > \mathbb{0}$ for some nonempty set $\alpha \subseteq \boldsymbol{n}$. Since $\mu_{[\alpha]} = \mathbb{0}$, the following submatrices of $[\mu]Y_{LL} + [Y_{LL}\mu]$ satisfy

$$([\mu]Y_{LL} + [Y_{LL}\mu])_{[\alpha,\alpha^c]} = [\mu_{[\alpha]}](Y_{LL})_{[\alpha,\alpha^c]} = 0 \tag{107}$$

and

$$\begin{aligned}
&([\mu]Y_{LL} + [Y_{LL}\mu])_{[\alpha,\alpha]} \\
&= [\mu_{[\alpha]}](Y_{LL})_{[\alpha,\alpha]} + [(Y_{LL}\mu)_{[\alpha]}] \\
&= 0 + \left[(Y_{LL})_{[\alpha,\alpha]}\mu_{[\alpha]} + (Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]}\right] \\
&= [(Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]}].
\end{aligned} \tag{108}$$

It follows from (107) that $[\mu]Y_{LL} + [Y_{LL}\mu]$ is block-triangular. Hence, the eigenvalues of each diagonal block are also eigenvalues of $[\mu]Y_{LL} + [Y_{LL}\mu]$. The principal submatrix given in (108) is such a diagonal block. Note from (108) that this block is diagonal, and so its eigenvalues are the elements of the vector $(Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]}$. Since Y_{LL} is an irreducible Z-matrix we have $(Y_{LL})_{[\alpha,\alpha^c]} \lneqq 0$. Recall that $\mu_{[\alpha^c]} > \mathbb{0}$, which implies that

$$(Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]} \lneqq \mathbb{0}$$

and so $[\mu]Y_{LL} + [Y_{LL}\mu]$ has nonpositive eigenvalues. However, since $[\mu]Y_{LL} + [Y_{LL}\mu]$ is an M-matrix, its Perron root is positive and is a lower bound for all other eigenvalues, which is a contradiction. We conclude that $\mu > \mathbb{0}$.

The matrix $g(\mu)$ is linear in μ. Hence, scaling of μ gives rise to a scaling of the eigenvalues of $g(\mu)$, and in particular of the Perron root of $g(\mu)$. Hence, $\mathcal{M}$ is a cone. The set of nonsingular M-matrices is open, and so $\mathcal{M}$ is an open set.

The set $\partial\mathcal{D}$ is simply connected by Theorem 3.7. Theorem 6.6 shows that there exists a bicontinuous map between $\partial\mathcal{D}$ and $\mathcal{M}_1$. Topological properties are preserved by bicontinuous maps, and hence, $\mathcal{M}_1$ is also simply connected. Its conic hull $\mathcal{M}$ is, therefore, also simply connected. □

Lemma E.2: The set of long-term voltage semistable operating points is contained in $\mathcal{M}$ (i.e., $\mathrm{cl}(\mathcal{D}) \subseteq \mathcal{M}$).

Proof: Recall from Corollary 3.4 that if $\widetilde{V}_L \in \mathrm{cl}(\mathcal{D})$, then $-\frac{\partial P_c}{\partial V_L}(\widetilde{V}_L)$ is an M-matrix. This means that $g(\widetilde{V}_L) - [\mathcal{I}_L^*]$ is an M-matrix by (69). By adding $[\mathcal{I}_L^*]$ to $g(\widetilde{V}_L) - [\mathcal{I}_L^*]$, Proposition A.3:6 implies that $g(\widetilde{V}_L)$ is an M-matrix since $\mathcal{I}_L^* \gneqq \mathbb{0}$. □

F. Proof of Theorem 7.5

For the sake of notation we follow Lemma 7.7 and define $\widehat{Y}_{LL} := [V_L^*]Y_{LL}[V_L^*]$, which is an irreducible nonsingular M-matrix. It follows from [10, Th. 5.12] that the inverse of $\widehat{Y}_{LL}$ is positive. Let S be the set of $\widetilde{P}_c$ defined by

$$\widetilde{P}_c \geq \mathbb{0}; \quad \widehat{Y}_{LL}^{-1}\widetilde{P}_c \leq \tfrac{1}{4}\mathbb{1} \tag{109}$$

which corresponds to all $\widetilde{P}_c \in \mathcal{N}$ so that (98) holds. The set S is convex and (109) describes the intersection of $2n$ closed half-spaces. The normals to these half-spaces are given by the canonical basis vectors $e_1, \ldots, e_n$ and the rows of $\widehat{Y}_{LL}^{-1}$. The set S is bounded since $\widehat{Y}_{LL}^{-1}$ is positive. Weyl's Theorem [15, pp. 88] states that S is the convex hull of the points which lie on the boundary of n half-spaces in (109) so that their corresponding normals span $\mathbb{R}^n$. To this end, we define $P_c^{\emptyset} := \mathbb{0}$, which lies on the boundary of the n half-spaces described by $\widetilde{P}_c \geq \mathbb{0}$. Similarly, we let $\alpha \subseteq \boldsymbol{n}$ be nonempty and let $P_c^{\alpha} \in S$ be a point described by Weyl's Theorem for which $(\widehat{Y}_{LL}^{-1}\widetilde{P}_c)_{[\alpha]} = \frac{1}{4}\mathbb{1}$ and $(\widehat{Y}_{LL}^{-1}\widetilde{P}_c)_{[\alpha^c]} < \frac{1}{4}\mathbb{1}$. The corresponding normals are given by the rows of $\widehat{Y}_{LL}^{-1}$ indexed by α. Since $\widehat{Y}_{LL}$ is positive definite we know that $(\widehat{Y}_{LL})_{[\alpha,\alpha]}$ is positive definite and, therefore, nonsingular. The only choice of normals of the half-spaces, which complete the span of $\mathbb{R}^n$ are e_i for $i \in \alpha^c$, which implies $(P_c^{\alpha})_{[\alpha^c]} = \mathbb{0}$. Since $(P_c^{\alpha})_{[\alpha^c]} = \mathbb{0}$, we have

$$\tfrac{1}{4}\mathbb{1} = (\widehat{Y}_{LL}^{-1}P_c^{\alpha})_{[\alpha]} = (\widehat{Y}_{LL}^{-1})_{[\alpha,\alpha]}(P_c^{\alpha})_{[\alpha]}$$

and therefore, $(P_c^{\alpha})_{[\alpha]} = \frac{1}{4}(\widehat{Y}_{LL}^{-1})_{[\alpha,\alpha]}^{-1}\mathbb{1}$. By the block matrix inverse formula [11, Eq. (0.8.1)], we observe that

$$(P_c^{\alpha})_{[\alpha]} = \tfrac{1}{4}(\widehat{Y}_{LL}/(\widehat{Y}_{LL})_{[\alpha^c,\alpha^c]})\mathbb{1}. \tag{110}$$

The abovementioned process exhaustively describes all points specified by Weyl's Theorem, and hence, we have

$$S = \mathrm{conv}(\{\, P_c^{\alpha} \mid \alpha \subseteq \boldsymbol{n} \,\}).$$

Recall that $P_c^{\emptyset} = \mathbb{0} = P_c(V_L^*) \in \mathrm{int}(\mathcal{F})$. The points P_c^{α} for nonempty $\alpha \subseteq \boldsymbol{n}$ correspond to the power demands described in (99) through substitution of $\widehat{Y}_{LL} = [V_L^*]Y_{LL}[V_L^*]$. We show that these points lie on the boundary of $\mathcal{F}$. Note that for $\alpha = \boldsymbol{n}$ we have by (110), (6), and (23) that

$$P_c^{\boldsymbol{n}} = \tfrac{1}{4}\widehat{Y}_{LL}\mathbb{1} = \tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*]\mathbb{1} = \tfrac{1}{4}[V_L^*]\mathcal{I}_L^* = P_{\max}$$

which lies on the boundary of $\mathcal{F}$. Consider any feasible power demand $\widetilde{P}_c \in \mathcal{F}$ such that $(\widetilde{P}_c)_{[\alpha]} = \mathbb{0}$ with $\alpha \neq \emptyset, \boldsymbol{n}$. Let $\widetilde{V}_L > \mathbb{0}$ be so that $\widetilde{P}_c = P_c(\widetilde{V}_L)$. By (10), we have

$$\mathbb{0} = P_c(\widetilde{V}_L)_{[\alpha^c]} = [(\widetilde{V}_L)_{[\alpha^c]}](Y_{LL}(V_L^* - \widetilde{V}_L))_{[\alpha^c]} \tag{111}$$

where we used (6). Since $(\widetilde{V}_L^{\alpha})_{[\alpha^c]} > \mathbb{0}$ it follows from (111) that $(Y_{LL}(V_L^* - \widetilde{V}_L))_{[\alpha^c]} = \mathbb{0}$. Since $(Y_{LL})_{[\alpha^c,\alpha^c]}$ is nonsingular, we may solve for $(\widetilde{V}_L)_{[\alpha^c]}$. Similar to [11, Eq. (0.7.4)], substitution of $(\widetilde{V}_L)_{[\alpha^c]}$ in $P_c(\widetilde{V}_L)_{[\alpha^c]}$ yields

$$P_c(\widetilde{V}_L)_{[\alpha]} = [(\widetilde{V}_L)_{[\alpha]}] Y_{LL}/(Y_{LL})_{[\alpha^c,\alpha^c]}((V_L^*)_{[\alpha]} - (\widetilde{V}_L)_{[\alpha]})$$

which corresponds to the power flow equations of a Kron-reduced power grid (see, e.g., [13], [16]). Analogous to Lemma 2.18, the maximizing feasible power demand for the Kron-reduced power grid is obtained by taking $(\widetilde{V}_L)_{[\alpha]} = \frac{1}{2}(V_L^*)_{[\alpha]}$, which corresponds in the power demand P_c^α. Hence, P_c^α lies on the boundary of $\mathcal{F}$. Since $\mathcal{F}$ is convex by Theorem 3.18, and $P_c^\alpha \in \mathcal{F}$ for all $\alpha \subseteq \boldsymbol{n}$, we have that $S \subseteq \mathcal{F} \cap \mathcal{N}$. Each supporting half-space of $\mathcal{F}$ has a unique point of support (see Theorem 3.12), and so the boundary of $\mathcal{F}$ does not contain a line piece. Consequently, the all points in S other than the points P_c^α for $\alpha \neq \emptyset$ lie in the interior of $\mathcal{F}$. Lemma 7.1 implies (98) from (109). □

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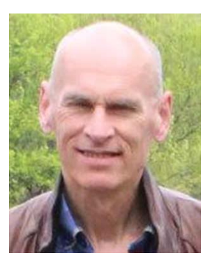

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