Kybernetika

Xiaowei Wei; Fu Gui Shi

L-fuzzy ideal degrees in effect algebras

Kybernetika, Vol. 58 (2022), No. 6, 996–1015

Persistent URL: `http://dml.cz/dmlcz/151540`

KYBERNETIKA — VOLUME *58* (2022), NUMBER 6, PAGES 996–1015

L-FUZZY IDEAL DEGREES IN EFFECT ALGEBRAS

Xiaowei Wei and Fu-Gui Shi

In this paper, considering L being a completely distributive lattice, we first introduce the concept of L-fuzzy ideal degrees in an effect algebra E, in symbol $\mathfrak{D}_{ei}$. Further, we characterize L-fuzzy ideal degrees by cut sets. Then it is shown that an L-fuzzy subset A in E is an L-fuzzy ideal if and only if $\mathfrak{D}_{ei}(A) = \top$, which can be seen as a generalization of fuzzy ideals. Later, we discuss the relations between L-fuzzy ideals and cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convexity. The morphism between two effect algebras is an (L, L)-fuzzy convexity-preserving mapping.

Keywords: effect algebra, L-fuzzy ideal degree, cut set, (L, L)-fuzzy convexity

Classification: 03B52, 03G27, 52A01

1. INTRODUCTION

In 1994, Foulis and Bennett [5] introduced effect algebras to model unsharp quantum logics. We know that the ideals of effect algebras (pseudo-effect algebras) have attracted a lot of attention [13, 39, 42]. Since Zadeh introduced the concept of fuzzy sets, many branches of mathematics were discussed in fuzzy cases [17, 23, 34, 35]. In particular, Liu and Wang [11] proposed the concept of fuzzy ideals for effect algebras in the unit interval $[0, 1]$. Later, Liu [10] introduced and investigated fuzzy ideals and fuzzy filters in pseudo-effect algebras. In order to better study fuzzy sets, cut sets were introduced, which can be seen as a bridge between fuzzy sets and classic sets. The reader is referred to [9, 37] for more information of cut sets.

Many branches of mathematics have the concept of convexities [31], such as vector spaces, metric spaces, lattices, graphs, matroids and so on. At present, for the convex theory, the research has formed a system, as follows: Rosa [24] first proposed the concept of fuzzy convexities, which are called L-convex structures nowadays [2, 22, 25, 30, 36, 40, 45]. Afterwards, Shi and Xiu [28] gave a new approach to fuzzification of convexity and proposed the concept of M-fuzzifying convex structures [19, 20, 33]. Later, Shi and Xiu [29] further introduced the definition of (L, M)-fuzzy convex structures, which provided a more general framework of fuzzy convex structures [21, 43, 44].

Groups, rings and fields are important parts of algebra. Williams, Latha and Chandrasekeran discussed the fuzzification of bi-Γ-ideals in Γ-semigroups and studied some properties [38]. Öztürk, Jun and Yazarli introduced a kind of fuzzy Γ-ring and discussed

DOI: 10.14736/kyb-2022-6-0996

some properties [18]. Malik and Mordeson [14] introduced the concepts of the fuzzy weak direct sum and the fuzzy complete direct sum of fuzzy subrings of commutative rings. Later, Mehmood, Shi and Hayat [16] introduced a new approach to the fuzzification of rings. Further, Mehmood and Shi [15] discussed the M-hazy vector spaces over M-hazy field. Recently, Shi and Xin [27] gave the concept of L-fuzzy subgroup degrees and L-fuzzy normal subgroup degrees, which generalized the notion of degrees to which a fuzzy subset is a fuzzy subgroup to L-fuzzy setting. Later, Li and Shi [9], Wen et al. [37] characterized L-fuzzy convex structures by L-convex fuzzy sublattice degrees and L-convex degrees on vector spaces, respectively. The L-fuzzy convexity is also called the (L, L)-fuzzy convex structure.

In this paper, considering L being a completely distributive lattice, we first introduce the definition of L-fuzzy ideal degrees and will characterize (L, L)-fuzzy convexities by L-fuzzy ideal degrees on an effect algebra. If the L-fuzzy ideal degree of an L-fuzzy subset equals to the maximum element in a lattice, then the L-fuzzy subset is an L-fuzzy ideal, which can be seen as a generalization of the fuzzy ideal on effect algebras. We further characterize L-fuzzy ideal degrees by four types of cut sets. We also discuss the relations between L-fuzzy ideals and their cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convex structure. The morphism between two effect algebras is an (L, L)-fuzzy convexity-preserving mapping.

2. PRELIMINARIES

2.1. Effect algebras

Definition 2.1. (Foulis and Bennett [5]) An effect algebra is a partial algebra $\big(E, +, 0, 1\big)$, where $0, 1$ are two different constants and $+$ is a partial binary operation satisfying the following:

(E1) If $x + y$ is defined, then $y + x$ is also defined, and $x + y = y + x$;

(E2) $x + y$ and $\big(x + y\big) + z$ are defined if and only if $y + z$ and $x + \big(y + z\big)$ are defined, and $\big(x + y\big) + z = x + \big(y + z\big)$;

(E3) For any $x \in E$, there exists a unique $y \in E$ such that $x+y$ is defined and $x+y = 1$;

(E4) If $x + 1$ is defined, then $x = 0$.

We often denote the effect algebra $\big(E, +, 0, 1\big)$ briefly by E. For any $x \in E$, we denote the unique y in condition (**E3**) by x'. The operation $+$ of an effect algebra $\big(E, +, 0, 1\big)$ can induce a partial order $\leq$ as follows: $x \leq y$ if and only if there exists $z \in E$ such that $x + z$ is defined and $x + z = y$. If $x + y$ is defined, then it is denoted by $x \perp y$.

In order to better understand effect algebras, we give the following examples, which are the most important effect algebras.

Example 2.2. (1) Let $\mathcal{H}$ be a complete Hilbert space and $B\big(\mathcal{H}\big)$ the set of all bounded linear operators on $\mathcal{H}$, $E\big(\mathcal{H}\big) = \big\{\, A \mid A \in B\big(\mathcal{H}\big),\, 0 \leq A \leq I \,\big\}$. For $A, B \in E\big(\mathcal{H}\big)$, if we define

$$A \perp B \Longleftrightarrow A + B \leq I,$$

then $\big(E(\mathcal{H}), +, 0, I\big)$ is an effect algebra.

(2) Let $E=[0,1]$. For any $x,y\in[0,1]$, $x\perp y$ if and only if $x+y\leq 1$, then $\big(E,+,0,1\big)$ is an effect algebra.

Definition 2.3. (Dvurečenskij and Pulmannová [3]) Let E and F be two effect algebras. A mapping $f:E\longrightarrow F$ is called a morphism provided that

(M1) $f(1_E)=1_F$;

(M2) If $x,y\in E$ and $x\perp y$, then $f(x)\perp f(y)$ and $f(x)+f(y)=f(x+y)$.

Lemma 2.4. (Dvurečenskij and Pulmannová [3]) Let $f:E\longrightarrow F$ be a morphism between two effect algebras. Then

(1) f is order-preserving, i. e., $x\leq y$ implies $f(x)\leq f(y)$ for all $x,y\in E$;

(2) $f(x')=f(x)'$ for all $x\in E$.

Definition 2.5. (Dvurečenskij and Pulmannová [3]) Let E and F be two effect algebras. A morphism $f:E\longrightarrow F$ is called a monomorphism provided that $f(x)\leq f(y)$ implies $x\leq y$ for all $x,y\in E$.

Definition 2.6. (Dvurečenskij and Pulmannová [3]) Let E be an effect algebra. A nonempty subset I of E is said to be an ideal provided that

(I1) If $x\in I$ and $y\in E$ with $y\leq x$, then $y\in I$;

(I2) If $x,y\in I$ and $x\perp y$, then $x+y\in I$.

2.2. Cut sets and (L,L)-fuzzy convexities

A partially ordered set $(L,\leq)$ [1] is said to be a lattice if any two elements λ and μ in L have a smallest upper bound, denoted by $\lambda\vee\mu$, as well as a greatest lower bound, denoted by $\lambda\wedge\mu$. Let $(L,\leq)$ be a partially ordered set and $\Lambda\subseteq L$ be a nonempty subset. If for any $\lambda,\mu\in\Lambda$, there always exists $\theta\in\Lambda$ such that $\lambda\leq\theta$ and $\mu\leq\theta$, then Λ is called upward directed.

Let L be a lattice. If for any $B\subseteq L$, $\bigvee B$ and $\bigwedge B$ exist, then L is called a complete lattice. An element λ in a complete lattice L is said to be a prime element if $\mu\wedge\theta\leq\lambda$ implies $\mu\leq\lambda$ or $\theta\leq\lambda$. An element λ is said to be co-prime if $\lambda\leq\mu\vee\theta$ implies $\lambda\leq\mu$ or $\lambda\leq\theta$ [6]. Every complete lattice is always a bounded lattice such that the unit is the top element and the zero is the bottom element. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in a complete lattice L is defined as follows: for $\lambda,\mu\in L$, $\lambda\prec\mu$ if and only if for any subset $A\subseteq L$, such that $\mu\leq\bigvee A$ implies $\lambda\leq\theta$ for some $\theta\in A$ [4]. The set $\big\{\,\lambda\,|\,\lambda\prec\mu\,\big\}$ is said to be the greatest minimal family of μ, denoted by $\beta(\mu)$ [32]. Moreover, for any $\mu\in L$, we define $\alpha(\mu)=\big\{\,\lambda\in L\,|\,\lambda\prec^{op}\mu\,\big\}$. A complete lattice L is a completely distributive lattice if and only if $\mu=\vee\beta(\mu)=\wedge\alpha(\mu)$ for all $\mu\in L$ [32]. In a completely distributive lattice L, α is an $\wedge$-$\cup$ mapping and β is a union-preserving mapping. There also exists an implication operator $\rightarrow: L \times L \longrightarrow L$ as the right adjoint for the meet operator $\wedge$, which is defined by

$$\lambda \rightarrow \mu = \bigvee \Big\{ \theta \in L \,|\, \lambda \wedge \theta \leq \mu \Big\},$$

for all $\lambda, \mu \in L$.

In this paper, if not otherwise specified, we always assume that L is a completely distributive lattice, the smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively.

Lemma 2.7. (Höhle and Šostak [8]) Let L be a completely distributive lattice and the operation $\rightarrow$ be the implication operator corresponding to $\wedge$. For any $\lambda, \mu, \theta \in L$ and $\{\lambda_i\}_{i\in I} \subseteq L$, then the following statements hold:

(1) $\top \rightarrow \lambda = \lambda$;

(2) $\lambda \leq \theta \rightarrow \mu \Longleftrightarrow \lambda \wedge \theta \leq \mu$;

(3) $\lambda \rightarrow \mu = \top \Longleftrightarrow \lambda \leq \mu$;

(4) $\lambda \rightarrow \big(\bigwedge\limits_{i\in I} \lambda_i\big) = \bigwedge\limits_{i\in I} \big(\lambda \rightarrow \lambda_i\big)$, hence $\lambda \rightarrow \mu \leq \lambda \rightarrow \theta$ whenever $\mu \leq \theta$;

(5) $\big(\bigvee\limits_{i\in I} \lambda_i\big) \rightarrow \mu = \bigwedge\limits_{i\in I} \big(\lambda_i \rightarrow \mu\big)$, hence $\lambda \rightarrow \mu \leq \theta \rightarrow \mu$ whenever $\theta \leq \lambda$;

(6) $(\lambda \rightarrow \mu) \wedge (\mu \rightarrow \theta) \leq \lambda \rightarrow \theta$.

Lemma 2.8. (Li and Shi [9], Wen et al. [37]) Let L be a completely distributive lattice and $\lambda, \mu \in L$. Then the following statements are equivalent:

(1) $\lambda \leq \mu$;

(2) for any $\delta \in J(L)$, $\delta \leq \lambda$ implies $\delta \leq \mu$;

(3) for any $\delta \in P(L)$, $\lambda \not\leq \delta$ implies $\mu \not\leq \delta$;

(4) for any $\delta \in \beta(\top)$, $\delta \in \beta(\lambda)$ implies $\delta \in \beta(\mu)$;

(5) for any $\delta \in \alpha(\perp)$, $\delta \notin \alpha(\lambda)$ implies $\delta \notin \alpha(\mu)$.

In what follows, we will recall some famous examples of t-norms on interval $[0, 1]$.

Example 2.9. (1) The minimum t-norm $x * y = x \wedge y$. The corresponding implication is defined by

$$x \rightarrow y = \begin{cases} 1, & x \leq y; \\ y, & x > y. \end{cases}$$

(2) The product t-norm $x * y = x \cdot y$. The corresponding implication is defined by

$$x \rightarrow y = \begin{cases} 1\,, & x \leq y; \\ y/x\,, & x > y. \end{cases}$$

(3) The Łukasiewicz t-norm $x * y = \max\{x+y-1, 0\}$. The corresponding implication is defined by $x \to y = \min\{1{,}1 - x + y\}$.

An L-fuzzy subset [7] of a set X is a mapping from X to L, and the family of all L-fuzzy subsets on X will be denoted by L^X, called the L-power set of X. $\top_X$ and $\bot_X$ denote the largest element and the smallest element in L^X, respectively.

Let $f : X \longrightarrow Y$ be a mapping between two nonempty sets. Define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by

$$f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x) \quad \text{and} \quad f_L^{\leftarrow}(B)(x) = B\big(f(x)\big),$$

for all $A \in L^X, B \in L^Y$, $x \in X$ and $y \in Y$. Then the L-fuzzy subset $f_L^{\rightarrow}(A)$ is called the image of A under f, and $f_L^{\leftarrow}(B)$ the preimage of B.

If L is a completely distributive lattice, then we can define

$$A_{[\lambda]} = \big\{\, x \in X \mid A(x) \geq \lambda \,\big\},\ A^{(\lambda)} = \big\{\, x \in X \mid A(x) \nleq \lambda \,\big\},$$

$$A_{(\lambda)} = \big\{x \in X \mid \lambda \in \beta\big(A(x)\big)\,\big\},\ A^{[\lambda]} = \big\{x \in X \mid \lambda \notin \alpha\big(A(x)\big)\,\big\},$$

for all $A \in L^X$ and $\lambda \in L$.

In [29], Shi and Xiu introduced the notion of (L, M)-fuzzy convexities. When $L = M$, we called it (L, L)-fuzzy convex structure. In what follows, we will recall it.

Definition 2.10. A mapping $\mathfrak{C} : L^X \longrightarrow L$ is said to be an (L, L)-fuzzy convex structure on X if it satisfies the following three conditions:

(C1) $\mathfrak{C}(\top_X) = \mathfrak{C}(\bot_X) = \top$;

(C2) If $\{A_i\}_{i\in I} \subseteq L^X$, then $\mathfrak{C}\big(\bigwedge_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{C}(A_i)$;

(C3) If $\{A_i\}_{i\in I} \subseteq L^X$ is nonempty and upward directed, then $\mathfrak{C}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{C}(A_i)$.

If $\mathfrak{C}$ is an (L, L)-fuzzy convex structure on X, then $(X, \mathfrak{C})$ is said to be an (L, L)-fuzzy convexity space. Every (L, L)-fuzzy convex structure on X is also called an (L, L)-fuzzy convexity on X.

Definition 2.11. Let $(X, \mathfrak{C}_x)$ and $(Y, \mathfrak{C}_y)$ be two (L, L)-fuzzy convexity spaces. Then a mapping $f : X \longrightarrow Y$ is called

(1) an (L, L)-fuzzy convexity-preserving mapping if $\mathfrak{C}_x(f_L^{\leftarrow}(B)) \geq \mathfrak{C}_y(B)$ for all $B \in L^Y$;

(2) an (L, L)-fuzzy convex-to-convex mapping if $\mathfrak{C}_y(f_L^{\rightarrow}(A)) \geq \mathfrak{C}_x(A)$ for all $A \in L^X$.

3. L-FUZZY IDEAL DEGREES

In this section, we will introduce the concept of L-fuzzy ideal degrees and investigate it by cut sets, further discuss some properties of L-fuzzy ideal degrees from the perspective of convexity. If not otherwise specified, E denotes an effect algebra and L is a completely distributive lattice with $\to$ (implication operator) corresponding to $\wedge$ (lattice infimum).

3.1. L-fuzzy ideal degrees

Definition 3.1. Let E be an effect algebra and A be an L-fuzzy subset in E. Then the L-fuzzy ideal degree $\mathfrak{D}_{ei}(A)$ of A is defined by

$$\mathfrak{D}_{ei}(A) = \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big).$$

It is obvious that the $\mathfrak{D}_{ei}$ is a mapping from L^E to L. In [11], authors proposed the concept of L-fuzzy ideals with $L = [0,1]$, which are said to be fuzzy ideals.

A mapping $A : E \longrightarrow [0,1]$ is called a fuzzy ideal of an effect algebra E provided that

(II1) $A(y) \leq A(x)$ if $x \leq y$;

(II2) $A(x) \wedge A(y) \leq A(x+y)$ if $x \perp y$,

for all $x, y \in E$.

In what follows, we will generalize the concept of fuzzy ideals from [0,1] to a lattice.

Definition 3.2. Let E be an effect algebra and $\mathfrak{D}_{ei}(A)$ an L-fuzzy ideal degree of an L-fuzzy subset A in E. If $\mathfrak{D}_{ei}(A) = \top$, then the A is called an L-fuzzy ideal.

Remark 3.3. Let E be an effect algebra and $\mathfrak{D}_{ei}(A)$ an L-fuzzy ideal degree of an L-fuzzy subset A in E. If $\mathfrak{D}_{ei}(A) = \top$, then

$$\big(A(y) \to A(x)\big) \wedge \big(A(z) \wedge A(w) \to A(z+w)\big) = \top,$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. It follows that

$$A(y) \leq A(x) \text{ and } A(z) \wedge A(w) \leq A(z+w),$$

for $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Hence, we could obtain that L-fuzzy ideals can be seen as generalizations of fuzzy ideals from [0,1] to a lattice L.

In the sequel, we will give some examples of L-fuzzy ideal degrees.

Example 3.4. Let $E = \{0, x, x', 1\}$ with $0 \leq x \leq x' \leq 1$, $x + x' = 1$ be an effect algebra.

(1) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$. If A is a constant value mapping on E, then

$$\mathfrak{D}_{ei}(A) = \top.$$

In this case, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, the L-fuzzy subset A satisfies

$$A(y) \leq A(x) \text{ and } A(z) \wedge A(w) \leq A(z+w).$$

Hence, the L-fuzzy subset A is an L-fuzzy ideal.

(2) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{0.1}{0} + \frac{0.3}{x} + \frac{0.7}{x'} + \frac{1}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = 0.1$ and $A(z) \wedge A(w) \leq A(z+w)$ if $z \perp w$. Since $x \leq x'$, but we have

$$A(x') = 0.7 \nleq 0.3 = A(x).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

(3) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{1}{0} + \frac{0.7}{x} + \frac{0.3}{x'} + \frac{0.1}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = 0.1$ with $A(y) \leq A(z)$ if $z \leq y$. Since $x + x' = 1$, but we have

$$A(x) \wedge A(x') = 0.7 \wedge 0.3 \nleq 0.1 = A(1).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

(4) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{0}{0} + \frac{0.7}{x} + \frac{1}{x'} + \frac{0.3}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = \perp$. In this case, we have $0 \leq x$ and $x + x' = 1$. But

$$A(x) = 0.7 \nleq 0 = A(0) \text{ and } A(x) \wedge A(x') = 0.7 \wedge 1 \nleq 0.3 = A(1).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

Lemma 3.5. Let E be an effect algebra and A be an L-fuzzy subset in E.

(1) If $A(x) = \top$ for all $x \in E$, then $\mathfrak{D}_{ei}(A) = \top$;

(2) If $A(x) = \perp$ for all $x \in E$, then $\mathfrak{D}_{ei}(A) = \top$.

Proof. It is easy and omitted. □

Lemma 3.6. Let E be an effect algebra and A an L-fuzzy subset in E. For any $\lambda \in L$, $\lambda \leq \mathfrak{D}_{ei}(A)$ if and only if for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, then

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Proof. Necessity: Take any $\lambda \in L$. If $\lambda \leq \mathfrak{D}_{ei}(A)$, then

$$\lambda \leq \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big).$$

Hence, it follows that

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

which means

$$\lambda \leq A(y) \to A(x) \text{ and } \lambda \leq A(z) \wedge A(w) \to A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Hence, we obtain

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$.

Sufficiency: Take any $\lambda \in L$. For any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, by the assumption, we have

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it follows that

$$\lambda \leq A(y) \to A(x) \text{ and } \lambda \leq A(z) \wedge A(w) \to A(z+w),$$

which means

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. That is to say, we obtain

$$\lambda \leq \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

as desired. □

Theorem 3.7. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y,\ z \perp w\Big\}.$$

P r o o f. Take any $t \in L$. Then it follows that

$$\begin{aligned} t \leq \mathfrak{D}_{ei}(A) &\iff t \wedge A(y) \leq A(x) \text{ and } t \wedge A(z) \wedge A(w) \leq A(z+w), \text{ for any } x, y, z, w \in E \\ &\qquad\qquad \text{with } x \leq y \text{ and } z \perp w \text{ (by Lemma 3.6)} \\ &\Longrightarrow t \leq \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \forall\, x \leq y, z \perp w\Big\}. \end{aligned}$$

On the other hand, assume that

$$t \leq \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \forall\, x \leq y, z \perp w\Big\}.$$

For any $\alpha \prec t$, it follows from the definition of binary relation $\prec$ that $\alpha \leq \lambda$ for some $\lambda \in L$ satisfying

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x \leq y$ and $z \perp w$. Then we know

$$\alpha \leq \lambda \leq A(y) \to A(x) \text{ and } \alpha \leq \lambda \leq \big(A(z) \wedge A(w)\big) \to A(z+w),$$

for all $\alpha \in L$ with $\alpha \prec t$. By $t = \bigvee \{\, \alpha \in L \mid \alpha \prec t \,\}$, we know

$$t \leq A(y) \to A(x) \text{ and } t \leq \big(A(z) \wedge A(w)\big) \to A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. That is to say,

$$t \wedge A(y) \leq A(x) \text{ and } t \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Then it follows from Lemma 3.6 that $t \leq \mathfrak{D}_{ei}(A)$. Hence, we obtain

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any}\, x \leq y,\ z \perp w \Big\},$$

as desired. $\square$

In the following, cut sets of L-fuzzy subset A may be empty. If cut sets of A are empty, then we still consider the empty set as a special ideal of E, when we discuss that cut sets of L-fuzzy subset A are ideals. That is to say, the empty set is a special ideal of E.

Theorem 3.8. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L \mid \forall \mu \leq \lambda,\ A_{[\mu]} \text{ is an ideal of E} \Big\}.$$

Proof. Assume that $A_{[\mu]}$ is an ideal of E for $\lambda \in L$ with $\mu \leq \lambda$. Take any $x, y \in E$ with $x \leq y$. Let $\theta = \lambda \wedge A(y)$. Then we have $\theta \leq \lambda$ and $\theta \leq A(y)$, which imply $y \in A_{[\theta]}$. By the assumption, we know that $A_{[\theta]}$ is an ideal of E. Then it shows that

$$x \in A_{[\theta]},$$

which means $\theta \leq A(x)$. It follows that

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Hence, it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda\in L \mid \lambda\wedge A(y)\leq A(x),\ \lambda\wedge A(z)\wedge A(w)\leq A(z+w), \text{ for any } x\leq y, z\perp w\Big\}\\ &\geq \bigvee\Big\{\ \lambda\in L\mid \forall\,\mu\leq\lambda,\ A_{[\mu]} \text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $\lambda\wedge A(y)\leq A(x)$ and $\lambda\wedge A(z)\wedge A(w)\leq A(z+w)$ for all $x,y,z,w\in E$ with $z\perp w$ and $x\leq y$. For any $\mu\leq\lambda$, we need to prove $A_{[\mu]}$ is an ideal of E.

(**I1**) If $y\in A_{[\mu]}$ with $x\leq y$, then $\mu\leq A(y)$. It implies that

$$\mu\leq\lambda\wedge A(y)\leq A(x).$$

Then it follows that $x\in A_{[\mu]}$.

(**I2**) If $z,w\in A_{[\mu]}$ and $z\perp w$, then

$$\mu\leq\lambda\wedge A(z)\wedge A(w)\leq A(z+w).$$

Hence, it follows that

$$z+w\in A_{[\mu]}.$$

That is to say, $A_{[\mu]}$ is an ideal of E. Then it implies that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda\in L \mid \lambda\wedge A(y)\leq A(x),\ \lambda\wedge A(z)\wedge A(w)\leq A(z+w), \text{for any } x\leq y,\ z\perp w\Big\}\\ &\leq \bigvee\Big\{\ \lambda\in L\mid \forall\,\mu\leq\lambda,\ A_{[\mu]} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Theorem 3.9. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A)=\bigvee\Big\{\ \lambda\in L\mid \mu\notin\alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\}.$$

P r o o f. Assume that $\lambda\in L$ with $\lambda\wedge A(y)\leq A(x)$ and $\lambda\wedge A(z)\wedge A(w)\leq A(z+w)$ for all $x,y,z,w\in E$ with $x\leq y$ and $z\perp w$. For $\mu\notin\alpha(\lambda)$, we need to prove that $A^{[\mu]}$ is an ideal of E.

(**I1**) If $x\leq y$ and $y\in A^{[\mu]}$, then $\mu\notin\alpha(A(y))$. It follows from

$$\lambda\wedge A(y)\leq A(x)$$

that

$$\alpha(A(x))\subseteq\alpha(\lambda\wedge A(y))=\alpha(\lambda)\cup\alpha(A(y)),$$

which means $\mu\notin\alpha(A(x))$. Hence, we obtain $x\in A^{[\mu]}$.

(**I2**) If $z,w\in A^{[\mu]}$ and $z\perp w$, then

$$\mu\notin\alpha(A(z))\cup\alpha(A(w))\cup\alpha(\lambda)=\alpha(\lambda\wedge A(z)\wedge A(w)).$$

It follows from

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$$

that

$$\alpha(A(z+w)) \subseteq \alpha(\lambda \wedge A(z) \wedge A(w)).$$

Hence, we obtain

$$\mu \notin \alpha(A(z+w)).$$

Then it follows that $z+w \in A^{[\mu]}$, which means

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\}\\ &\leq \bigvee \Big\{\ \lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $A^{[\mu]}$ is an ideal of E for $\lambda \in L$ with $\mu \notin \alpha(\lambda)$. In the sequel, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, we need to prove

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Suppose that $\mu \notin \alpha(\lambda \wedge A(y))$. It follows from

$$\alpha(\lambda \wedge A(y)) = \alpha(\lambda) \cup \alpha(A(y))$$

that

$$\mu \notin \alpha(\lambda) \text{ and } \mu \notin \alpha(A(y)).$$

It implies that $y \in A^{[\mu]}$. By the assumption, we know that $A^{[\mu]}$ is an ideal of E, which means $x \in A^{[\mu]}$. Then it follows that

$$\mu \notin \alpha(A(x)).$$

Hence, we obtain

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it shows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\ \lambda \in L \mid \lambda \wedge A(y) \leq A(x),\ \ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\}\\ &\geq \bigvee \Big\{\ \lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. $\square$

Theorem 3.10. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E} \Big\}.$$

Proof. Assume that $\lambda \in L$ with $\lambda \wedge A(y) \leq A(x)$ and $\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$ for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. If $\mu \in P(L)$ and $\lambda \nleq \mu$, then we need to prove that $A^{(\mu)}$ is an ideal of E.

Assume that $y \in A^{(\mu)}$. If $x \notin A^{(\mu)}$, then $A(x) \leq \mu$. It follows from

$$\lambda \wedge A(y) \leq A(x)$$

that

$$\lambda \wedge A(y) \leq \mu.$$

By $\mu \in P(L)$ and $y \in A^{(\mu)}$, i.e., $A(y) \nleq \mu$, we have $\lambda \leq \mu$. This is a contradiction. Hence, it follows that

$$x \in A^{(\mu)}.$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$z, w \in A^{(\mu)} \text{ implies } z + w \in A^{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y) \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\}\\ &\leq \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E } \Big\}.\end{aligned}$$

Conversely, assume that $A^{(\mu)}$ is an ideal of E for $\lambda \in L$ and $\mu \in P(L)$ with $\lambda \nleq \mu$. In what follows, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, we need to prove

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

For any $x, y \in E$ with $x \leq y$, let $\mu \in P(L)$ and $\lambda \wedge A(y) \nleq \mu$. Then we have

$$\lambda \nleq \mu \text{ and } A(y) \nleq \mu.$$

It follows that $y \in A^{(\mu)}$. By the assumption, we know $A^{(\mu)}$ is an ideal of E, then $x \in A^{(\mu)}$. Further, it implies that

$$A(x) \nleq \mu.$$

Hence, we have

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{ for any } x \leq y, z \perp w\Big\}\\ &\geq \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E } \Big\},\end{aligned}$$

as desired. □

Theorem 3.11. Let E be an effect algebra and A an L-fuzzy subset in E. If $\beta(\lambda\wedge\mu) = \beta(\lambda)\cap\beta(\mu)$ for all $\lambda,\mu\in L$, then

$$\mathfrak{D}_{ei}(A) = \bigvee\Big\{\lambda\in L \mid \forall\,\mu\in\beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\}.$$

Proof. Assume that $\lambda\in L$ such that $\lambda\wedge A(y)\le A(x)$ and $\lambda\wedge A(z)\wedge A(w)\le A(z+w)$ for all $x,y,z,w\in E$ with $x\le y$ and $z\perp w$. For any $\mu\in\beta(\lambda)$, we need to prove that $A_{(\mu)}$ is an ideal of E.

(**I1**) If $y\in A_{(\mu)}$ and $x\le y$, then

$$\mu\in\beta(A(y))\cap\beta(\lambda)=\beta(A(y)\wedge\lambda)\subseteq\beta(A(x)).$$

It follows that $x\in A_{(\mu)}$.

(**I2**) If $z,w\in A_{(\mu)}$ and $z\perp w$, then

$$\mu\in\beta(A(z))\cap\beta(A(w))\cap\beta(\lambda)=\beta(A(z)\wedge A(w)\wedge\lambda)\subseteq\beta(A(z+w)).$$

Hence, we obtain

$$z+w\in A_{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda\in L \mid \lambda\wedge A(y)\le A(x),\ \lambda\wedge A(z)\wedge A(w)\le A(z+w),\ \text{for any}\ x\le y,\ z\perp w\Big\}\\ &\le \bigvee\Big\{\lambda\in L \mid \forall\,\mu\in\beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $A_{(\mu)}$ is an ideal of E for $\lambda\in L$ with $\mu\in\beta(\lambda)$. For any $x,y,z,w\in E$ with $x\le y$ and $z\perp w$, we need to prove

$$\lambda\wedge A(y)\le A(x) \text{ and } \lambda\wedge A(z)\wedge A(w)\le A(z+w).$$

(i) Assume that $x,y\in E$ with $x\le y$. Let $\mu\in\beta(\lambda\wedge A(y))$. Then it follows from

$$\beta(\lambda\wedge A(y))=\beta(A(y))\cap\beta(\lambda)$$

that

$$\mu\in\beta(\lambda) \text{ and } \mu\in\beta(A(y)).$$

It implies $y\in A_{(\mu)}$. By the assumption, we know that $A_{(\mu)}$ is an ideal of E. Then it shows $x\in A_{(\mu)}$. Hence, we have

$$\mu\in\beta(A(x)).$$

It follows that $\lambda\wedge A(y)\le A(x)$.

(ii) Assume that $z,w\in E$ and $z\perp w$. Let $\mu\in\beta(\lambda\wedge A(z)\wedge A(w))$. It follows from

$$\beta(\lambda\wedge A(z)\wedge A(w))=\beta(\lambda)\cap\beta(A(z))\cap\beta(A(w))$$

that

$$\mu\in\beta(\lambda),\mu\in\beta(A(z)) \text{ and } \mu\in\beta(A(w)).$$

It implies that

$$z, w \in A_{(\mu)}.$$

By the assumption, we know that $A_{(\mu)}$ is an ideal of E and $z \perp w$. Then it shows that

$$z + w \in A_{(\mu)},$$

which means $\mu \in \beta(A(z + w))$. It follows that

$$\lambda \wedge A(z) \wedge A(w) \leq A(z + w).$$

Hence, we have

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z + w), \text{for any } x \leq y, z \perp w\Big\} \\ &\geq \bigvee \Big\{\lambda \in L | \forall\, \mu \in \beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Zhang [41] discussed the relations between fuzzy ideals and fuzzy filters in dual effect algebras. Liu [10], Liu and Wang [11] studied the connections between a fuzzy filter $\mathcal{F}$ and its cut sets $\mathcal{F}_{[\lambda]}$ for all $\lambda \in [0, 1]$ in effect algebras and pseudo-effect algebras, respectively. In the sequel, on one hand, we investigate L-fuzzy ideals by cut sets $A_{[\lambda]}$ for all $\lambda \in L$, which generalizes the unit interval $[0, 1]$ to a lattice L. On the other hand, we characterize L-fuzzy ideals by another three kinds of cut sets. In particular, we think that the empty set is a special ideal of an effect algebra E. By [9, 37], we obtain the following corollaries immediately.

Corollary 3.12. Let E be an effect algebra and A an L-fuzzy subset in E. Then the following statements are equivalent:

(1) A is an L-fuzzy ideal of E;

(2) for every $\lambda \in L$, $A_{[\lambda]}$ is an ideal;

(3) for every $\lambda \in J(L)$, $A_{[\lambda]}$ is an ideal;

(4) for every $\lambda \in L$, $A^{[\lambda]}$ is an ideal;

(5) for every $\lambda \in P(L)$, $A^{[\lambda]}$ is an ideal;

(6) for every $\lambda \in P(L)$, $A^{(\lambda)}$ is an ideal.

Corollary 3.13. Let E be an effect algebra and A an L-fuzzy subset in E. Then the following statements are equivalent when $\beta(\lambda \wedge \mu) = \beta(\lambda) \cap \beta(\mu)$ for all $\lambda, \mu \in L$.

(1) A is an L-fuzzy ideal of E;

(2) for every $\lambda \in J(L)$, $A_{(\lambda)}$ is an ideal;

(3) for every $\lambda \in L$, $A_{(\lambda)}$ is an ideal.

In what follows, we will characterize L-fuzzy ideals by L_β-nested sets and L_α-nested sets. We also can refer to [12, 26] for more information on nested sets. By [26], we can immediately obtain the following Theorems 3.14 and 3.15.

Theorem 3.14. Let E be an effect algebra and $\big\{A(\lambda) \mid \lambda \in L\big\}$ be an L_β-nest of ideals of E. Then there exists an L-fuzzy ideal A such that

(1) $A_{(\lambda)} \subseteq A(\lambda) \subseteq A_{[\lambda]}$ for all $\lambda \in L$;

(2) $A_{(\lambda)} = \bigcup\limits_{\lambda\in\beta(\nu)} A(\nu)$ for all $\lambda \in L$;

(3) $A_{[\nu]} = \bigcap\limits_{\lambda\in\beta(\nu)} A(\lambda)$ for all $\nu \in L$.

Theorem 3.15. Let E be an effect algebra and $\big\{A(\lambda) \mid \lambda \in L\big\}$ be an L_α-nest of ideals of E. Then there exists an L-fuzzy ideal A such that

(1) $A^{(\lambda)} \subseteq A(\lambda) \subseteq A^{[\lambda]}$ for all $\lambda \in L$;

(2) $A^{(\lambda)} = \bigcup\limits_{\nu\in\alpha(\lambda)} A(\nu)$ for all $\lambda \in P(L)$;

(3) $A^{[\lambda]} = \bigcap\limits_{\lambda\in\alpha(\nu)} A(\nu)$ for all $\lambda \in P(L)$.

3.2. (L, L)-fuzzy convexities are induced by L-fuzzy ideal degrees

In this subsection, we will characterize convex properties of L-fuzzy ideal degrees. By morphisms between effect algebras, we obtain one kind of mappings between convexity spaces. Firstly, we will investigate the structural properties of convexity spaces by L-fuzzy ideal degrees.

Theorem 3.16. Let E be an effect algebra and $\mathfrak{D}_{ei}$ an L-fuzzy ideal degree. Then $\mathfrak{D}_{ei}$ is an (L, L)-fuzzy convexity on E.

Proof. By Lemma 3.5, we only need to prove (**C2**) and (**C3**).

(**C2**) Let $\big\{A_i\big\}_{i\in I}$ be a family of L-fuzzy subsets in E. Then it follows that

$$\begin{aligned}
\mathfrak{D}_{ei}\big(\bigwedge_{i\in I} A_i\big) &= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(\bigwedge_{i\in I} A_i(y) \to \bigwedge_{i\in I} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(z) \wedge \bigwedge_{i\in I} A_i(w) \to \bigwedge_{i\in I} A_i(z+w)\Big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(z+w)\Big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(z+w)\Big)\\
&\geq \bigwedge_{i\in I} \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(A_i(y) \to A_i(x)\Big) \wedge \Big(A_i(z) \wedge A_i(w) \to A_i(z+w)\Big)\\
&= \bigwedge_{i\in I} \mathfrak{D}_{ei}\big(A_i\big).
\end{aligned}$$

(**C3**) Let $\{A_i\}_{i\in I}$ be a upward directed family of L-fuzzy subsets in E. Then we need to prove

$$\mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{D}_{ei}\big(A_i\big).$$

Take any $\lambda \in L$ with $\lambda \leq \bigwedge\limits_{i\in I} \mathfrak{D}_{ei}\big(A_i\big)$. Then it follows that $\lambda \leq \mathfrak{D}_{ei}\big(A_i\big)$ for all $i \in I$. By Lemma 3.6, we know

$$\lambda \wedge A_i(y) \leq A_i(x) \text{ and } \lambda \wedge A_i(z) \wedge A_i(w) \leq A_i(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$, $z \perp w$ and $i \in I$. In what follows, we need to prove

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(y)\big) \leq \bigvee_{i\in I} A_i(x) \text{ and } \lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$.

For any $\eta \prec \lambda \wedge \big(\bigvee\limits_{i\in I} A_i(z)\big) \wedge \big(\bigvee\limits_{i\in I} A_i(w)\big)$, there exist $i \in I$ and $j \in I$ such that

$$\eta \leq A_i(z),\ \eta \leq A_j(w) \text{ and } \eta \leq \lambda.$$

Since $\{A_i\}_{i\in I}$ is upward directed, there exists $k \in I$ such that $A_i \leq A_k$ and $A_j \leq A_k$. Then it follows that

$$A_i(z) \leq A_k(z) \text{ and } A_j(w) \leq A_k(w),$$

which means that

$$\eta \leq \lambda \wedge A_k(z) \wedge A_k(w) \leq A_k(z+w) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $z, w \in E$ with $z \perp w$. Hence, we obtain

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $z, w \in E$ with $z \perp w$. Similarly, we obtain

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(y)\big) \leq \bigvee_{i\in I} A_i(x),$$

for all $x, y \in E$ with $x \leq y$. Then it follows from Lemma 3.6 that

$$\lambda \leq \mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big),$$

which implies $\mathfrak{D}_{ei}\big(\bigvee\limits_{i\in I} A_i\big) \geq \bigwedge\limits_{i\in I} \mathfrak{D}_{ei}\big(A_i\big)$. Hence, we obtain that $\mathfrak{D}_{ei}$ is an (L, L)-fuzzy convexity, as desired. $\square$

Theorem 3.17. Let E and F be two effect algebras and $f : E \longrightarrow F$ be an effect algebra morphism. Then $f : (E, \mathfrak{D}_{ei}) \longrightarrow (F, \mathfrak{D}_{fi})$ is an (L, L)-fuzzy convexity-preserving mapping.

Proof. Take any L-fuzzy subset A in F. Then

$$\begin{aligned}
&\mathfrak{D}_{ei}\big(f_L^{\leftarrow}(A)\big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \big(f_L^{\leftarrow}(A)(y) \to f_L^{\leftarrow}(A)(x)\big) \wedge \big(f_L^{\leftarrow}(A)(z) \wedge f_L^{\leftarrow}(A)(w) \to f_L^{\leftarrow}(A)(z+w)\big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(A(f(y)) \to A(f(x))\Big) \wedge \Big(A(f(z)) \wedge A(f(w)) \to A(f(z)+f(w))\Big)\\
&\geq \bigwedge_{\substack{x_1,y_1,z_1,w_1\in F\\ z_1\perp w_1, x_1\leq y_1}} \big(A(y_1) \to A(x_1)\big) \wedge \big(A(z_1) \wedge A(w_1) \to A(z_1+w_1)\big)\\
&= \mathfrak{D}_{fi}(A).
\end{aligned}$$

Hence, we obtain that f is an (L, L)-fuzzy convexity-preserving mapping, as desired. □

In the sequel, we will discuss the relations between L-fuzzy ideals and their inverse images by L-fuzzy ideal degrees.

Theorem 3.18. Let E and F be two effect algebras and $f : E \longrightarrow F$ a monomorphism. If B is an L-fuzzy ideal of F, then $f_L^{\leftarrow}(B)$ is an L-fuzzy ideal of E.

Proof. It can be obtained from Theorem 3.17. □

Remark 3.19. In this paper, we first introduce the concept of L-fuzzy ideal degrees and further investigate it. In order to highlight the idea of fuzzy mathematics, we discuss L-fuzzy ideal degrees, which emphasize the ideal of many-valued logics. The concept can reveal essential characterizations of different mathematical structures. There are some papers for different mathematical structures on degrees of mathematical structures, such as [9, 27, 37, 44] and (Y.-Y. Dong, F.-G. Shi, L-fuzzy Sub-Effect Algebras).

4. CONCLUSIONS

In this paper, considering L being a completely distributive lattice, we first introduce the concept of L-fuzzy ideal degrees. Then, we characterize L-fuzzy ideal degrees by four types of cut sets. By L-fuzzy ideal degrees, we could give the concept of L-fuzzy ideals, which can be seen as generalizations of fuzzy ideals. We also discuss the relations between L-fuzzy ideals and cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convexity. These morphisms between effect algebras are (L, L)-fuzzy convexity-preserving mappings.

ACKNOWLEDGEMENT

This work is supported by the Natural Science Foundation of China (No. 11871097, 12071033, 11971448) and Beijing Institute of Technology Science and Technology Innovation Plan Cultivation Project (No. 2021CX01030). We are grateful to the anonymous reviewers and editors for their valuable comments, which helps to improve the original version of our manuscript greatly.

(Received June 26, 2021)

REFERENCES

[1] B. A. Davey and H. A. Priestley: Introduction to Lattice and Order. Cambridge University Press, Cambridge 2002.

[2] Y. Y. Dong and J. Li: Fuzzy convex structures and prime fuzzy ideal space on residuated lattices. J. Nonlinear Convex Analysis *21* (2020), 12, 2725–2735.

[3] A. Dvurečenskij and S. Pulmannová: New trends in Quantum Structures. Springer-Science Business Media, B.V. 2000.

[4] P. Dwinger: Characterizations of the complete homomorphic images of a completely distributive lattice I. Mathematics *85* (1982), 4, 403–414. DOI:10.1016/1385-7258(82)90034-8

[5] D. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logice. Foundat. Physics *24* (1994), 1331–1352. DOI:10.1007/BF02283036

[6] G. Gierz et al.: Continuous lattices and domains, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge 2003.

[7] J. A. Goguan: *L*-fuzzy sets. J. Math. Analysis Appl. *18* (1967), 145–174. DOI:10.1016/0022-247X(67)90189-8

[8] U. Höhle and A. P. Šostak: Axiomatic foundations of fixed basis fuzzy topology. Math. Fuzzy Sets *3* (1999), 123–272. DOI:10.1007/978-1-4615-5079-2_5

[9] J. Li and F.-G. Shi: *L*-fuzzy convexity induced by *L*-convex fuzzy sublattice degree. Iranian J. Fuzzy Systems *14* (2017), 5, 83–102.

[10] D. L. Liu: Fuzzy filters and fuzzy ideals in pseudo-effect algebras. Computer Engrg. Appl. (2011), 50–52. DOI:10.1111/j.1740-9713.2011.00484.x

[11] D. L. Liu and G. J. Wang: Fuzzy filters in effect algebras. Fuzzy Systems Math. *23* (2009), 6–17.

[12] C. Z. Luo: Fuzzy sets and nested sets. Fuzzy Math. *4* (1983), 113–126.

[13] Z. H. Ma: Note on ideals of effect algebras. Inform. Sci. *179* (2009), 505–507. DOI:10.1016/j.ins.2008.07.018

[14] D. S. Malik and J. N. Mordeson: Fuzzy direct sums of fuzzy rings. Fuzzy Sets Systems *45* (1992), 83–91. DOI:10.1016/0165-0114(92)90094-K

[15] F. Mehmood and F.-G. Shi: M-hazy vector spaces over M-hazy field. Mathematics *9* (2021), 1118. DOI:10.3390/math9101118

[16] F. Mehmood, F.-G. Shi, and K. Hayat: A new approach to the fuzzification of rings. J. Nonlinear Convex Analysis *21* (12) (2020), 2637–2646.

[17] J. Močkoř: α-cuts and models of fuzzy logic. Int. J. General Systems *42* (2013), 67–78. DOI:10.1080/03081079.2012.710438

[18] M. A. Öztürk, Y. B. Jun, and H. Yazarli: A new view of fuzzy gamma rings. Hacettepe J. Math. Statist. *39* (2010), 3, 365–378.

[19] B. Pang: Convergence structures in M-fuzzifying convex spaces. Quaest. Math. *43* (2020), 11, 1541–1561. DOI:10.2989/16073606.2019.1637379

[20] B. Pang: L-fuzzifying convex structures as L-convex structures. J. Nonlinear Convex Anal. *21* (2020), 12, 2831–2841.

[21] B. Pang: Bases and subbases in (L, M)-fuzzy convex spaces. Computat. Appl. Math. (2020), 39–41. DOI:10.1007/s40314-020-1065-4

[22] B. Pang and F.-G. Shi: Strong inclusion orders between L-subsets and its applications in L-convex spaces. Quaest. Math. *41* (2018), 8, 1021–1043. DOI:10.2989/16073606.2018.1436613

[23] D. Pei and T. Fan: On generalized fuzzy rough sets. Int. J. General Systems *38* (2009), 3, 255–271. DOI:10.1080/03081070701553678

[24] M. V. Rosa: On fuzzy topology, fuzzy convexity spaces and fuzzy local convexity. Fuzzy Sets Systems *62* (1994), 97–100. DOI:10.1016/0165-0114(94)90076-0

[25] C. Shen and F.-G. Shi: L-convex systems and the categorical isomorphism to Scott-hull operators. Iranian J. Fuzzy Systems *15* (2018), 2, 23–40.

[26] F.-G. Shi: Theory of L_β-nested set and L_α-nested sets and applications. Fuzzy Systems Math. (1995), 65–72.

[27] F.-G. Shi and X. Xin: L-fuzzy subgroup degrees and L-fuzzy normal subgroup degrees. J. Advanced Res. Pure Math. *3* (2011), 4, 92–108. DOI:10.5373/jarpm.762.020211

[28] F.-G. Shi and Z. Y. Xiu: A new approach to the fuzzification of convex structures. J. Appl. Math. *3* (2014), 1–12. DOI:10.1155/2014/249183

[29] F.-G. Shi and Z. Y. Xiu: (L, M)-fuzzy convex structures. J. Nonlinear Sci. Appl. *10* (2017), 3655–3669. DOI:10.22436/jnsa.010.07.25

[30] Y. Shi and H. L. Huang: A characterization of strong Q-concave spaces. J. Nonlinear Convex Anal. *21* (2020), 12, 2771–2781.

[31] M. L. J. van de Vel: Theory of Vonvex Structures. North Holland, N. Y. 1993.

[32] G. J. Wang: Theory of topological molecular lattices. Fuzzy Sets Systems *47* (1992), 351–376. DOI:10.1016/0165-0114(92)90301-J

[33] K. Wang and F.-G. Shi: M-fuzzifying topological convex spaces. Iranian J. Fuzzy Systems *15*, (2018), 6, 159–174.

[34] X. W. Wei, B. Pang, and J. S. Mi: Axiomatic characterizations of L-valued rough sets using a single axiom. Inform, Sci. *580* (2021), 283–310. DOI:10.1016/j.ins.2021.08.078

[35] X. W. Wei, B. Pang, and J. S. Mi: Axiomatic characterizations of L-fuzzy rough sets by L-fuzzy unions and L-fuzzy intersections. Int. J. General Systems *51* (2022), 3, 277–312. DOI:10.1080/03081079.2021.2024178

[36] X. W. Wei and B. Wang: Fuzzy (restricted) hull operators and fuzzy convex structures on L-sets. J. Nonlinear Convex Anal. *21* (2020), 12, 2805–2815.

[37] Y. F. Wen, Y. Zhong, and F.-G. Shi: L-fuzzy convexity induced by L-convex degree on vector spaces. J. Intell. Fuzzy Systems *33* (2017), 4031–4041. DOI:10.3233/JIFS-17912

[38] D. R. P. Williams, K. B. Latha, and E. Chandrasekeran: Fuzzy bi-Γ-ideals in Γ-semigroups. Hacettepe J. Math. Statist. *38* (2009), 1, 1–15.

[39] J. Wu: Ideals, filters and supports in pseudo-effect algebras. Int. J. Theoret. Physics *43* (2004), 349–358. DOI:10.1023/B:IJTP.0000028868.66790.0c

[40] H. Yang and E. Q. Li: A new approach to interval operators in L-convex spaces. J. Nonlinear Convex Anal. *21* (2020), 12, 2705–2714.

[41] Q. W. Zhang: Ideals and filters in dual effect algebras. Hennan Sci. *34* (2016), 8, 1211–1214. DOI:10.1038/nbt1216-1214

[42] Q. W. Zhang: Note on ideals in effect algebras. Hennan Sci. *35* (2017), 10, 1567–1569. DOI:10.1016/j.ajem.2017.04.002

[43] F. F. Zhao and H. L. Huang: The relationships among L-ordered hull operators, restricted L-hull operators and strong L-fuzzy convex structures. J. Nonlinear Convex Anal. *21* (2020), 12, 2817–2829.

[44] Y. Zhong and F.-G. Shi: Characterizations of (L, M)-fuzzy topological degrees. Iranian J. Fuzzy Syst. *15* (2018), 4, 129–149.

[45] X. W. Zhou and F.-G. Shi: Some new results on six types mappings between L-convex spaces. Filomat. *34* (2020), 4767–4781. DOI:10.2298/FIL2014767Z

Xiaowei Wei, Beijing Key Laboratory on MCAACI, School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 102488. P. R. China.
e-mail: weixiaowei2016@126.com

Fu-Gui Shi, Corresponding Author. School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 102488- P. R. China.
e-mail: fuguishi@bit.edu.cn