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Petr Dostál

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ALMOST LOG-OPTIMAL TRADING STRATEGIES FOR SMALL TRANSACTION COSTS IN MODEL WITH STOCHASTIC COEFFICIENTS

Petr Dostál

We consider a non-consuming agent investing in a stock and a money market interested in the portfolio market price far in the future. We derive a strategy which is almost log-optimal in the long run in the presence of small proportional transaction costs for the case when the rate of return and the volatility of the stock market price are bounded Itô processes with bounded coefficients and when the volatility is bounded away from zero.

Keywords: small transaction costs, logarithmic utility function, non-constant coefficients

Classification: 60H30, 60G44, 91G80

1. INTRODUCTION

The purpose of this paper is to provide certain technical tools for solving the investment problem in the long run with small proportional transaction costs. Primarily, we focus on the most aggressive investor with HARA[1] utility function unbounded from below whose aim is to maximize the long run growth rate of the wealth process $(\mathcal{W}_t)_{t\geq 0}$ up to a certain admissible error. Such an error should be small (of the highest possible order) when the level of transaction fees, described by a parameter $\lambda > 0$, is small. Our task can be viewed as follows

$$\text{(almost)}\ \max\ \liminf_{t\to\infty}\ \tfrac{1}{t}\,\mathbb{E}\ln(\mathcal{W}_t).$$

Since the usual investor is not the most aggressive one, we also consider a modification of the necessary technical results so that then they can be used in order to derive a certain strategy, for a more risk averse investor, which is also

1. time consistent,
2. robust with respect to time change in the model,
3. independent on the form of our model in the future.

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[1]The formal definitionof of a HARA utility function (unbounded from below) is given in (1.1) and in the paragraph below (1.1).

We do not believe that there exists a strategy satisfying properties (1,2,3) that also maximizes the long run growth rate of the certainty equivalent of the wealth process if the investor faces the following power utility function

$$\mathcal{U}_\gamma(y) \stackrel{\text{def}}{=\!=} \tfrac{1}{y}\, y^\gamma, \quad \gamma < 0, \tag{1.1}$$

unbounded from below, in general. Note that the certainty equivalent is a deterministic value giving the same expected utility as the prescribed random variable. Also note that the abbreviation HARA means that the functions from (1.1), and also the logarithmic one $\mathcal{U}_0(y) \stackrel{\text{def}}{=\!=} \ln y$, have constant *hyperbolic absolute risk aversion* described by the value (here not depending on y) defined as

$$p = -y\,\mathcal{U}_\gamma''(y)/\mathcal{U}_\gamma'(y) = 1 - \gamma. \tag{1.2}$$

See [14] for information on how the derived technical tools can be used in order to obtain certain results for the investor with $\gamma < 0$. Further, see Theorems 5.9 and 5.13 in this paper in order to agree that we are able to find a strategy (dependent also on the level of transaction fees λ) with the wealth process, denoted here as $(\hat{\mathcal{W}}_t)_{t\geq 0}$, such that for $\lambda > 0$ small enough

$$\limsup_{t\to\infty} \tfrac{1}{t}\, \mathbb{E}[\ln \mathcal{W}_t - \ln \hat{\mathcal{W}}_t] \leq K_{5.13}\, \lambda^{6/7}. \tag{1.3}$$

Here, $K_{5.13}$ is a fixed finite positive number and $(\mathcal{W}_t)_{t\geq 0}$ is the wealth process of an admissible strategy. Note that the symbol "$\mathbb{E}$" stands for the expected value. If it is omitted in (1.3), then the corresponding inequality holds up to a null set by Theorem 5.13. Finally, note that the value $6/7$ is obtained when minimizing the rate of the error of our estimate and that this value can be replaced by 1 under certain additional technical assumptions, see Theorem 5.13 and $q \in \{6/7, 1\}$ in Definition 5.3.

It should be mentioned that it is crucially important for us to have a strategy that is independent of the form of the model in the future, especially in the case when our model is obtained from another one with the help of filtering techniques. If this property is not satisfied, then the applicability of the strategy (when we face models with random coefficients) is significantly reduced.

The results of this paper can be compared with the ones obtained in [27], where the authors focus on the long term expected growth rate in a Markovian setting. In the above-mentioned paper, the authors rule out shortselling and allow the drift and diffusion coefficients of the returns process to be unbounded, and they always obtain an error bound with $q = 1$ under their assumptions. Here, it should be mentioned that our strategy is designed in order to be applicable also in the case when the coefficient of the model are random. In that case, we would obtain from the methods of robust filtering a model which is typically non-Markovian.

For non-Markovian dynamics, rigorous results for log optimal portfolios with small transaction costs are obtained in Section 3.3.1 of the PhD thesis [1]. The market model considered there is very similar to the one considered here, with general Itô dynamics assumed for the returns process and the frictionless optimal portfolio weight. If all coefficient processes are uniformly bounded as in this paper, then the corresponding results seem to apply, leading to an almost optimal portfolio even for a finite time horizon, but the error term in the main result of [1], Theorem 3.2.6, is only guaranteed to be of order $o(\lambda^{2/3})$. The same rate of error is achieved in [23]. Note that the strategy proposed in [23] is (under circumstances considered in this paper) very close to our (almost optimal) strategy introduced in Remark 5.11 in the case when we consider $q = 1$. More precisely, their strategy should be rather compared with the strategy mentioned above the remark. In case when $q = 6/7$, there are slight but substantial differences between these two strategies, and this is perhaps the reason why their results regarding the reached rate of error are not so strong.

For further reading, we recommend the overview of the literature given in [23], but briefly, the reader interested in the classical Merton approach to the (corresponding) consumption-investment problem is advised to look at [2, 3, 12, 20, 21, 26, 28, 29, 33], while the reader interested in the properties of the log-optimal investment is referred to [4, 5, 6, 8, 9, 19, 25, 31, 32, 34, 35]. For the martingale approach to the investment problem, see [13, 14, 15, 16].

The reader of this paper may be interested also in papers related to optimal strategies calculated within finite time horizons, see Bichuch and Sircar [7]; using the linear programming approach, see Cai, Rosenbaum, and Tankov [10], or recently also the machine learning approach, see Mulvey, Sun, Wang, and Ye [30].

To the log-optimal investment, it can be written that the logarithmic utility function is one of the most desirable ones, and it dates back to Daniel Bernoulli in the eighteenth century. Using logarithmic utility is known as the Kelly criterion, see Kelly [25], and the objective is to maximize the exponential growth rate rather than to use a utility function. Breiman [8] and Algoet and Cover [4] showed that maximizing the logarithmic utility leads to an asymptotically maximal growth rate and asymptotically minimal expected time to reach a presigned goal. Bell and Cover [5, 6] showed that the expected log-optimal portfolio is also game theoretically optimal in a single play or in multiple plays of the stock market for a wide variety of pay-off functions. Browne and Whitt [9] used the Bayesian approach to derive optimal gambling and investment policies for cases in which the underlying stochastic process has parameter values that are unobserved random variables.

Note that the stochastic control approach to the investment problem was established by Merton [28], further known as the investment-consumption problem. He found a closed-form solution for the case of no transaction cost where the stock market price is a geometric Brownian motion, further known also as the Merton model. Magill and Constantinides [26] formulated the problem in the presence of transaction costs and conjectured that the proportion of the total wealth invested in the stock should be kept within a certain interval. This problem was solved under restrictive conditions by Davis and Norman [12] and analysed by Shreve and Soner [33]. Constantinides [11] numerically computed the effect of transaction costs on the value function for the problem and the width of the no-transaction region. His conjecture has been made precise by formal power series expansions in a variety of models. A justification for the leading term in the expansion is given by Janeček and Shreve [20]. Morton and Pliska [29] studied optimal portfolio management policies for an investor who must pay a transaction cost equal to a fixed fraction of his/her portfolio value each time he/she trades.

Akian, Sulem and Taksar [3] showed that the ergodic singular stochastic control problem corresponds to the limit of a discounted control problem for vanishing discount factor. Note that the dynamic programming approach to the investment-consumption problem with proportional transaction costs leads to the variational inequality and the corresponding viscosity solution. The reader who is not familiar with viscosity solutions, but wishes to be, is referred to Akian, Menaldi and Sulem [2]. On the other hand, the reader who is not interested in viscosity solutions may prefer the martingale approach to the investment problem.

The paper is organized as follows. The next (short) section is devoted to basic notation. In Section 3, we introduce the model of proportional transaction costs and specify what is meant by an admissible strategy. The wealth process and the position process are introduced there. Their dynamics are described in Subsection B.1. In Section 4, the model for the stock market price is specified. In Section 5, we first introduce the policies associated with a specific strategy which is in the second part of the section shown to be almost log-optimal. The main results of this paper are stated there, see Theorems 5.12 and 5.13. Further sections are complementary. Section A is an additional part of the paper, and it is (together with the supplement to this paper) a reaction to the requirement of one of the reviewers who wanted some ,,numerical confirmation" of the derived theoretical results. This and further sections are placed behind the references, and they are denoted as appendices (both on demand of another reviewer) although only the last section plays the role of a true appendix.

In Section B, the necessary dynamics of important processes is described and, in Subsection B.2, the properties of the policies are treated, and there is introduced a function which helps us apply the martingale approach to obtain the desired results. Its properties are described in Section C. Section D is devoted to proofs of results from the previous sections, and within this section, the proofs are written in the correct order. The last section is the (true) appendix.

Note that the proved technical results are a little bit broader than it is necessary to obtain the main result of this paper, Theorem 5.13. It involves only considering one additional parameter γ (or p), and it helps make the results from [14] correct.

2. BASIC NOTATION

Notation 2.1. Let $(\Omega, \mathscr{A}, \mathbb{P}, \mathscr{F})$ be a filtered probability space. We denote by $\mathbb{CA}(\mathscr{F})$ the set of all continuous $\mathscr{F}$-adapted real valued processes and put

$$\mathbb{CA}_b(\mathscr{F}) \stackrel{\text{def}}{=\!=} \{X \in \mathbb{CA}(\mathscr{F});\ X \text{ is bounded}\}.$$

Similarly, we denote by $\overset{\frown}{\mathbb{C}}\mathbb{A}(\mathscr{F})$ the set of all $\mathscr{F}$-adapted rcll processes (right-continuous with finite left-hand limits). Further, $\mathbb{CM}(\mathscr{F})$ stands for the set of all continuous $\mathscr{F}$-martingales and $\mathbb{CM}_{loc}(\mathscr{F})$ for all continuous local $\mathscr{F}$-martingales,

$$\mathbb{CI}(\mathscr{F}) \stackrel{\text{def}}{=\!=} \{X \in \mathbb{CA}(\mathscr{F});\ X \text{ is non-decreasing}\}, \quad \mathbb{CFV}(\mathscr{F}) \stackrel{\text{def}}{=\!=} \{X - Y; X, Y \in \mathbb{CI}(\mathscr{F})\},$$

and similarly, we consider $\overset{\frown}{\mathbb{C}}\mathbb{I}(\mathscr{F})$, $\overset{\frown}{\mathbb{C}}\mathbb{FV}(\mathscr{F})$ with $\mathbb{C}$ replaced by $\overset{\frown}{\mathbb{C}}$ in the previous line. We also denote by $\mathbb{CS}(\mathscr{F})$ the set of all continuous $\mathscr{F}$-semimartingales and

$$\overset{\frown}{\mathbb{C}}\mathbb{S}(\mathscr{F}) \stackrel{\text{def}}{=\!=} \{X + M; X \in \overset{\frown}{\mathbb{C}}\mathbb{FV}(\mathscr{F}), M \in \mathbb{CM}_{loc}(\mathscr{F})\} \supseteq \mathbb{CS}(\mathscr{F}).$$

Further, we write $\int X\,\mathrm{d}Y \overset{\text{def}}{=\!=} (\int_0^t X\,\mathrm{d}Y)_{t\geq 0}$ if X, Y are processes with the index set $[0,\infty)$ such that the corresponding integrals are well defined.

Remark 2.2. In this paper, we do not work with discontinuous (local) martingales since every considered discontinuity is caused by trading and since this discontinuity affects only the "trend" part of the processes.

Definition 2.3. A continuous real valued process X is said to be *Lipschitz* if there exists a constant $L \in [0,\infty)$ such that $|X_t - X_s| \overset{\text{as}}{\leq} L|t-s|$ whenever $s,t \in [0,\infty)$. In this case, we also say that the process X is *L-Lipschitz*.

Notation 2.4. By $\mathbb{CS}_l(\mathscr{F})$ we denote the set of all continuous $\mathscr{F}$-semimartingales with Lipschitz quadratic variation, i. e., we put

$$\begin{aligned} \mathbb{CS}_l(\mathscr{F}) &\overset{\text{def}}{=\!=} \{X \in \mathbb{CS}(\mathscr{F}); \langle X\rangle \text{ is Lipschitz}\}, \\ \mathbb{CM}_l(\mathscr{F}) &\overset{\text{def}}{=\!=} \{X - X_0; X \in \mathbb{CS}_l(\mathscr{F}) \cap \mathbb{CM}_{loc}(\mathscr{F})\} \subseteq \mathbb{CM}(\mathscr{F}), \end{aligned} \tag{2.1}$$

The last relation can be easily obtained from [22, Corollary 15.9]. Note that if τ is an integrable $\mathscr{F}$-stopping time and $X \in \mathbb{CM}_l(\mathscr{F})$, then X_τ is a centered random variable and $\lim_{t\to\infty} \frac{1}{t} X_t \overset{\text{as}}{=} 0$. By $\mathbb{PM}(\mathscr{F})$ we denote the set of all $\mathscr{F}$-progressive processes, and we put $\mathbb{PM}_b(\mathscr{F}) \overset{\text{def}}{=\!=} \{X \in \mathbb{PM}(\mathscr{F}); \exists n \in \mathbb{N}\ |X| \leq n\}$. Keep in mind that any Lipschitz $\mathscr{F}$-adapted process starting from zero is almost surely equal to $\int X_s\,\mathrm{d}s$ for some $X \in \mathbb{PM}_b(\mathscr{F})$. Then we have that

$$\mathbb{CS}_l(\mathscr{F}) = \{X \in \mathbb{CS}(\mathscr{F}); \exists Y \in \mathbb{PM}_b(\mathscr{F})\ \langle X\rangle \overset{\text{as}}{=} \int Y_s\,\mathrm{d}s\}.$$

We consider the set of all *bounded Itô processes with bounded coefficients* defined as

$$\mathbb{BI}_b(\mathscr{F}) \overset{\text{def}}{=\!=} \{X \in \mathbb{CA}_b(\mathscr{F}) \cap \mathbb{CS}_l(\mathscr{F}); \exists Y \in \mathbb{PM}_b(\mathscr{F})\ X - \int Y_s\,\mathrm{d}s \in \mathbb{CM}(\mathscr{F})\}.$$

For the justification of this notion, see Lemma 4.6. Note that if $X \in \mathbb{PM}_b(\mathscr{F})$ and if W is a standard $\mathscr{F}$-Brownian motion, then $\int X\,\mathrm{d}W \in \mathbb{CM}_l(\mathscr{F}) \subseteq \mathbb{CM}(\mathscr{F})$. We omit the reference to the considered filtration if it is clear from the context which filtration is considered. Further, whenever $(X_t)_{t\geq 0}$ is an rcll-process, we denote its jump at time $t \in (0,\infty)$ from the left as $\Delta X_t \overset{\text{def}}{=\!=} X_t - X_{t-}$.

3. STOCK TRADING WITH PROPORTIONAL TRANSACTION COSTS

In this section, we introduce the model of proportional transaction costs (together with the wealth and the position process) and specify what is meant by an admissible strategy in this paper. Further specific assumptions required in our main results (Theorems 5.12 and 5.13 presented in Section 5) are introduced in the next section.

In this paper, the agent may invest in one risky asset called stock with the market price $0 < S = (S_t)_{t\geq 0}$ and in money market with interest rate $r = (r_t)_{t\geq 0}$, which is assumed in this paper to be zero without any loss of generality. The market price S is modelled as a positive continuous semimartingale defined on a complete probability space $(\Omega, \mathscr{A}, P)$ endowed with a complete right-continuous filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ so that there exists $F \in \mathbb{CS}$ starting from zero such that

$$\mathrm{d}S = S\,\mathrm{d}F, \qquad \text{i.e.,} \qquad S \overset{\text{as}}{=} S_0 \exp\{F - \tfrac{1}{2}\langle F\rangle\}. \tag{3.1}$$

Further specification of the process F will be given in the next section.

Definition 3.1. By a *(trading) strategy* we mean a pair (φ, ψ) where $\varphi, \psi \in \overset{\bullet\!\frown\!\circ}{\mathbb{C}}\mathbb{FV}$ stand for the *number of shares* held in the stock and in the bank account, respectively. The corresponding *wealth process* is defined as follows

$$\mathcal{W} \overset{\text{def}}{=\!=} \psi + \varphi S \in \overset{\bullet\!\frown\!\circ}{\mathbb{C}}\mathbb{A}. \tag{3.2}$$

The agent buys the stock for the ask price $S^{\uparrow}$ and gets the bid price $S^{\downarrow}$ for it where

$$S^{\uparrow} \overset{\text{def}}{=\!=} (1+\lambda^{\uparrow})S, \qquad S^{\downarrow} \overset{\text{def}}{=\!=} (1-\lambda^{\downarrow})S. \tag{3.3}$$

Here, we assume that transaction fees $\lambda^{\uparrow} \in (0,\infty), \lambda^{\downarrow} \in (0,1)$ are *symmetric* as follows

$$\lambda^{\uparrow} \overset{\text{def}}{=\!=} e^{\lambda/2} - 1, \qquad \lambda^{\downarrow} \overset{\text{def}}{=\!=} 1 - e^{-\lambda/2}. \tag{3.4}$$

Remark 3.2. Note that if we multiplied S by a positive constant $c \in (1-\lambda^{\downarrow}, 1+\lambda^{\uparrow})$, we would obtain a new model of the stock market price $0 < \tilde{S} = cS \in \mathbb{CS}$ with the same dynamics as S, i. e.,

$$\mathrm{d}\tilde{S} = \tilde{S}\,\mathrm{d}F,$$

cf. (3.1), but the corresponding transaction taxes $\tilde{\lambda}^{\uparrow}, \tilde{\lambda}^{\downarrow}$ would differ. On the other hand, whenever different (non-symmetric) transaction taxes $\tilde{\lambda}^{\uparrow} \in (0,\infty)$ and $\tilde{\lambda}^{\downarrow} \in (0,1)$ are given, then there exist a positive value $\tilde{c} \in (1-\tilde{\lambda}^{\downarrow}, 1+\tilde{\lambda}^{\uparrow})$ such that $S = \tilde{c}\,\tilde{S}$ satisfies our assumptions (3.1,3.3,3.4) with

$$\lambda = \ln\frac{1+\tilde{\lambda}^{\uparrow}}{1-\tilde{\lambda}^{\downarrow}} = \ln\frac{1+\lambda^{\uparrow}}{1-\lambda^{\downarrow}}, \qquad \text{namely} \qquad \tilde{c} \overset{\text{def}}{=\!=} \sqrt{(1+\tilde{\lambda}^{\uparrow})(1-\tilde{\lambda}^{\downarrow})}.$$

Remark 3.3. If (φ, ψ) is a trading strategy, then by assumption $\varphi \in \overset{\bullet\!\frown\!\circ}{\mathbb{C}}\mathbb{FV}$. Hence, there exist $\varphi^{\uparrow}, \varphi^{\downarrow} \in \overset{\bullet\!\frown\!\circ}{\mathbb{C}}\mathbb{I}$ starting from 0 such that

$$\varphi = \varphi_0 + \varphi^{\uparrow} - \varphi^{\downarrow}$$

and that $\varphi^{\uparrow}, \varphi^{\downarrow}$ do not grow at the same time, i. e., that the Lebesgue-Stieltjes measures induced by both processes are mutually singular. A simple argument referring to the uniqueness of Hahn decomposition used on compact intervals gives that the processes $\varphi^{\uparrow}, \varphi^{\downarrow}$ introduced above are uniquely determined by the process φ. Then $\varphi^{\uparrow}, \varphi^{\downarrow}$ are interpreted as the *number of shares bought and sold up to time t*, respectively, and the assumption that they do not grow at the same time is just a natural requirement that the corresponding strategy does not buy and sell the stock simultaneously.

Definition 3.4. Let (φ, ψ) be a trading strategy. Then the corresponding *value of transaction costs* up to $t \geq 0$ is defined as follows

$$C_t^\varphi \stackrel{\text{def}}{=\!=} \int_0^t S_s(\lambda^\uparrow \mathrm{d}\varphi_s^\uparrow + \lambda^\downarrow \mathrm{d}\varphi_s^\downarrow). \tag{3.5}$$

The above integral is understood in the Lebesgue-Stieltjes sense, and as the integrators are right-continuous, the integral from 0 to t is understood as integral over $(0, t]$. Consequently, $C^\varphi = (C_t^\varphi)_{t \geq 0} \in \overset{\bullet\!-\!\circ}{\mathbb{C}}\mathbb{I}$.

Definition 3.5. We say that the strategy (φ, ψ) is λ-*self-financing* if

$$\psi = \psi_0 - \int S^\uparrow \,\mathrm{d}\varphi^\uparrow + \int S^\downarrow \,\mathrm{d}\varphi^\downarrow. \tag{3.6}$$

Here and in the whole paper, $\varphi^\uparrow, \varphi^\downarrow$ are as in Remark 3.3. From (3.6) and the *integration by parts formula* $S_t\varphi_t - S_0\varphi_0 - \int_0^t S \,\mathrm{d}\varphi \stackrel{\text{as}}{=} \int_0^t \varphi \,\mathrm{d}S$ for $S \in \mathbb{CS}$ and $\varphi \in \overset{\bullet\!-\!\circ}{\mathbb{C}}\mathbb{FV}$, we obtain that $\mathcal{W} = \psi + \varphi S \in \overset{\bullet\!-\!\circ}{\mathbb{C}}\mathbb{S}$ and that

$$\mathcal{W} - \mathcal{W}_0 \stackrel{\text{as}}{=} \int \varphi \,\mathrm{d}S - C^\varphi \stackrel{\text{as}}{=} \int S[\varphi \,\mathrm{d}F - \lambda^\uparrow \mathrm{d}\varphi^\uparrow - \lambda^\downarrow \mathrm{d}\varphi^\downarrow]. \tag{3.7}$$

Note that the integrals with respect to $\mathrm{d}S, \mathrm{d}F$ are considered in the classical sense of continuous stochastic integration and that the ones with respect to $\mathrm{d}\varphi^\uparrow, \mathrm{d}\varphi^\downarrow, \mathrm{d}\varphi$ are considered in the Lebesgue-Stieltjes sense.

Definition 3.6. A λ-self-financing strategy (φ, ψ) is called λ-*admissible* if the wealth processes computed from the ask and bid prices $S^\uparrow, S^\downarrow$ are positive, i. e., if

$$\mathcal{W}^\uparrow \stackrel{\text{def}}{=\!=} \psi + \varphi S^\uparrow > 0, \quad \mathcal{W}^\downarrow \stackrel{\text{def}}{=\!=} \psi + \varphi S^\downarrow > 0. \tag{3.8}$$

Briefly, we also say that the strategy is *admissible*. It will be helpful to realize that

$$\mathcal{W}^\uparrow = \mathcal{W} + \lambda^\uparrow \varphi S, \quad \mathcal{W}^\downarrow = \mathcal{W} - \lambda^\downarrow \varphi S.$$

Definition 3.7. Whenever (φ, ψ) is a λ-admissible strategy, we introduce the corresponding *position process* $\pi = (\pi_t)_{t \geq 0}$ as

$$\pi_t \stackrel{\text{def}}{=\!=} \varphi_t S_t / \mathcal{W}_t. \tag{3.9}$$

Further, we denote $\mathcal{A}_\lambda \stackrel{\text{def}}{=\!=} (-1/\lambda^\uparrow, 1/\lambda^\downarrow)$ and we call it the *set of admissible positions*. A λ-admissible strategy is called *strictly* λ-*admissible* if the corresponding position π attains values in a compact subset of $\mathcal{A}_\lambda$.

Remark 3.8. The position process π of a λ-admissible strategy describes the ratio of the investors' wealth invested in the risky asset, and it attains values in $\mathcal{A}_\lambda$. To see the latter statement, first realize that the corresponding wealth process $\mathcal{W}$ is positive as $\mathcal{W} \geq \mathcal{W}^\uparrow \wedge \mathcal{W}^\downarrow > 0$, and then the desired property follows from the following relations

$$0 < \mathcal{W}^\uparrow = \mathcal{W}(1 + \lambda^\uparrow \pi), \quad 0 < \mathcal{W}^\downarrow = \mathcal{W}(1 - \lambda^\downarrow \pi). \tag{3.10}$$

Remark 3.9. For $\tilde{\lambda} \in (0,\lambda)$ and a λ-admissible strategy (φ,ψ) with the wealth process $\mathcal{W}$, there exists a strictly $\tilde{\lambda}$-admissible strategy with the wealth process $\tilde{\mathcal{W}} \geq \mathcal{W}$ starting from the same initial wealth, i. e., $\tilde{\mathcal{W}}_0 = \mathcal{W}_0$. It is sufficient to consider a $\tilde{\lambda}$-self-financing strategy $(\tilde{\varphi},\tilde{\psi})$ with $\tilde{\varphi} = \varphi$ and $\tilde{\psi}_0 = \psi_0$. Then obviously $\tilde{\mathcal{W}}_0 = \mathcal{W}_0$ and $\tilde{\psi} \geq \psi$, which gives that $\tilde{\mathcal{W}} \geq \mathcal{W}$. Consequently, we obtain that the corresponding position $\tilde{\pi} \overset{\text{def}}{=\!=} \tilde{\varphi} S/\tilde{\mathcal{W}}$ attains values in $\mathcal{A}_\lambda$, and the closure of $\mathcal{A}_\lambda$ is a compact subset of $\mathcal{A}_{\tilde{\lambda}}$.

Lemma 3.10. Let (φ,ψ) be a λ-admissible strategy with the wealth and the position $\mathcal{W},\pi$ and with the ask and bid wealth processes $\mathcal{W}^\uparrow,\mathcal{W}^\downarrow$. Then

(i) $\pi, \mathcal{W}, \mathcal{W}^{-1}, \mathcal{W}^\uparrow, \mathcal{W}^\downarrow \in \overset{\circ\!\!\!\circ}{\mathbb{C}}\mathbb{A}$.

(ii) $\pi_{t-} \in \mathcal{A}_\lambda$ and $\mathcal{W}_{t-} \geq \mathcal{W}^\uparrow_{t-} \wedge \mathcal{W}^\downarrow_{t-} > 0$ hold whenever $t \in (0,\infty)$.

Proof. See Subsection D.1 in Section Proofs. □

Remark 3.11. An easy calculation shows that if we followed a λ-self-financing strategy and if we decided to withdraw from the market at time $t \in (0,\infty)$, the remaining wealth would be equal to $\mathcal{W}_{t-} - S_t(\lambda^\uparrow \varphi^-_{t-} + \lambda^\downarrow \varphi^+_{t-}) = \mathcal{W}^\uparrow_{t-} \wedge \mathcal{W}^\downarrow_{t-}$. Hence, any admissible strategy always keeps a safe way to withdraw from the market with positive remaining wealth.

4. MODEL SET-UP

In this section, we will specify the assumptions of our model of the stock market price $S = (S_t)_{t\geq 0}$ generally introduced in (3.1) in terms of the driving process $F = (F_t)_{t\geq 0}$, see Assumptions 4.1 and 4.7. Here, we also introduce the key process of this paper called the displacement, see Definition 4.12. The main results of this paper (Theorems 5.12 and 5.13) are presented in the next section.

Assumption 4.1. There are a standard $\mathcal{F}$-Brownian motion W and $\alpha, \beta \in \mathbb{B}\mathbb{I}_b(\mathcal{F})$ s.t.

$$F \overset{\text{as}}{=\!=} \textstyle\int \alpha_s \,\mathrm{d}s + \int \sigma \,\mathrm{d}W \qquad \text{where} \qquad \sigma \overset{\text{def}}{=\!=} e^\beta. \tag{4.1}$$

If this assumption is satisfied, we simply write that ($\mathbb{A}$4.1) holds.

Remark 4.2. Assumption 4.1 will be assumed from here (even without further remarks) if not stated otherwise. Note there are parts of the paper that are completely independent of it such as Subsection B.1 from beginning to Lemma B.7, the corresponding part in the section Proofs (from the beginning of the section to the end of Subsection D.4), Theorem 5.9 and the Appendix.

Definition 4.3. Let W, F, α, σ be as in Assumption 4.1, but with a less restrictive requirement that $\alpha, \beta \in \mathbb{P}\mathbb{M}$. Then we introduce the process $\theta = (\theta_t)_{t\geq 0}$ where

$$\theta_t \overset{\text{def}}{=\!=} \sigma_t^{-2} \alpha_t = \arg\max_{\varpi \in \mathbb{R}} \ (\alpha_t \varpi - \tfrac{1}{2}\,\sigma_t^2 \varpi^2), \tag{4.2}$$

and we call it *the log-optimal proportion.*

Remark 4.4. Exceptionally, let $\varphi, \psi, \mathcal{W}$ be as in Definition 3.1 but with the restriction that φ, ψ are progressively measurable instead of $\varphi, \psi \in \overset{\circ}{\mathbb{C}}\mathbb{FV}$. Further, assume that $\mathcal{W} > 0$ and that $\pi = (\pi_t)_{t \geq 0}$ is as in (3.9). Here, we will consider the frictionless market, which means that the self-financing condition is now of the form that

$$\mathbb{CS} \ni \mathcal{W} \overset{\text{as}}{=} \mathcal{W}_0 + \int \varphi \, \mathrm{d}S \overset{\text{as}}{=} \mathcal{W}_0 + \int \mathcal{W}_t \pi_t [\alpha_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t].$$

This includes the assumption that $\int_0^t |\varphi_s| S_s (|\alpha_s| + \sigma_s^2 |\varphi_s| S_s) \, \mathrm{d}s \overset{\text{as}}{<} \infty, t \in [0, \infty)$. With the help of the Itô rule, we obtain that in this case

$$\ln \mathcal{W} \overset{\text{as}}{=} \ln \mathcal{W}_0 + \int (\alpha_t \pi_t - \tfrac{1}{2} \sigma_t^2 \pi_t^2) \, \mathrm{d}t,$$

which means that the log-optimal proportion θ from (4.2) corresponds to the position maximizing the drift part of the logarithm of the wealth process among all self-financing strategies in the frictionless market.

Remark 4.5. In Assumption 4.1, instead of assuming that $\alpha, \beta \in \mathbb{BI}_b$, we could equivalently assume that $\theta, \ln \sigma \in \mathbb{BI}_b$. In order to realize this, it is enough to use Lemma E.1 and the equality $\theta = \alpha \sigma^{-2} = \alpha e^{-2 \ln \sigma}$.

Lemma 4.6. Let $\mathscr{F}$ be a right-continuous complete filtration, and let $X \in \mathbb{BI}_b(\mathscr{F})$. Let W be a standard $\mathscr{F}$-Brownian motion, and let $\hat{W}$ be a standard Brownian motion independent of $\mathscr{F}_\infty$. Then there exist a standard $\tilde{\mathscr{F}}$-Brownian motion $\tilde{W}$ independent of W and processes $\mathfrak{a}, \mathfrak{b}, \tilde{\mathfrak{b}} \in \mathbb{PM}_b(\mathscr{F})$ such that

$$X \overset{\text{as}}{=} X_0 + \int \mathfrak{a}_s \, \mathrm{d}s + \int \mathfrak{b} \, \mathrm{d}W + \int \tilde{\mathfrak{b}} \, \mathrm{d}\tilde{W} \tag{4.3}$$

where $\tilde{\mathscr{F}} \overset{\text{def}}{=} (\mathscr{F}_t \vee \sigma(\hat{W}_s; s \leq t))_{t \geq 0}$ is the smallest extension of $\mathscr{F}$ such that $\hat{W} \in \mathbb{CA}(\tilde{\mathscr{F}})$.

Proof. See Subsection D.5 in Section Proofs. □

Assumption 4.7. Let ($\mathbb{A}$4.1) hold, and let $\theta = \sigma^{-2}\alpha$ be the corresponding log-optimal proportion. We say that Assumption 4.7 is satisfied or that ($\mathbb{A}$4.7) holds if there exist a standard $\mathcal{F}$-Brownian motion $\tilde{W}$ independent of W, $\mathfrak{a}^\theta \in \mathbb{PM}_b$, $\mathfrak{b}^\theta, \tilde{\mathfrak{b}}^\theta \in \mathbb{BI}_b$ such that

$$\mathbb{CS} \ni \theta \overset{\text{as}}{=} \theta_0 + \int \mathfrak{a}_t^\theta \mathrm{d}t + \int \mathfrak{b}^\theta \, \mathrm{d}W + \int \tilde{\mathfrak{b}}^\theta \, \mathrm{d}\tilde{W}. \tag{4.4}$$

Remark 4.8. Again, ($\mathbb{A}$4.7) will be assumed from here (even without further remarks). Obviously, the parts that are independent of ($\mathbb{A}$4.1) are also independent of ($\mathbb{A}$4.7).

Remark 4.9. Note that under Assumption 4.7 the processes $\mathfrak{b}^\theta, |\tilde{\mathfrak{b}}^\theta|$ are continuous and that they are almost surely uniquely determined as follows

$$\mathfrak{b}_t^\theta \overset{\text{as}}{=} \tfrac{\mathrm{d}\langle \theta, W \rangle_t}{\mathrm{d}t}, \qquad |\tilde{\mathfrak{b}}_t^\theta| \overset{\text{as}}{=} \sqrt{\tfrac{\mathrm{d}\langle \theta \rangle_t}{\mathrm{d}t} - \left(\tfrac{\mathrm{d}\langle \theta, W \rangle_t}{\mathrm{d}t}\right)^2}, \qquad t \geq 0. \tag{4.5}$$

Definition 4.10. Besides the log-optimal proportion θ from Definition 4.3, we also consider $\Theta \overset{\text{def}}{=} \theta / p$ and call it *the Merton proportion*, see (1.2).

Note that if the stock market price is a geometric Brownian motion, then Θ attains just one deterministic value which is usually called the Merton proportion. Under Assumption 4.7, we have that

$$\mathbb{B}\mathbb{I}_b \ni \Theta \overset{\text{as}}{=} \Theta_0 + \int \mathfrak{a}^\Theta_t \,\mathrm{d}t + \int \mathfrak{b}^\Theta \,\mathrm{d}W + \int \tilde{\mathfrak{b}}^\Theta \,\mathrm{d}\tilde{W} \tag{4.6}$$

where $\mathfrak{a}^\Theta \overset{\text{def}}{=} \mathfrak{a}^\theta/p \in \mathbb{P}\mathbb{M}_b$ and $(\mathfrak{b}^\Theta, \tilde{\mathfrak{b}}^\Theta)^\intercal \overset{\text{def}}{=} (\mathfrak{b}^\theta, \tilde{\mathfrak{b}}^\theta)^\intercal/p \in \mathbb{B}\mathbb{I}^2_b$. See also Remark 4.5, which easily gives that $\Theta, \sigma \in \mathbb{B}\mathbb{I}_b$ holds in this case. Hence, under Assumption 4.7,

$$\mathfrak{C} \overset{\text{def}}{=} (\Theta, \mathfrak{b}^\Theta, \tilde{\mathfrak{b}}^\Theta, \sigma)^\intercal \in \mathbb{B}\mathbb{I}^4_b. \tag{4.7}$$

This process $\mathfrak{C}$ will be called *the complementary process*. It contains additional information that helps construct an almost optimal strategy.

Remark 4.11. If π is the position process of an admissible strategy, then $\pi \in \mathring{\mathbb{C}}\mathbb{S}$. In order to be more specific, $\pi - \int \pi(1-\pi)\,\mathrm{d}F \in \mathring{\mathbb{C}}\mathbb{F}\mathbb{V}$. It follows from Lemma B.5. This information is used only for an explanation accompanying the following definition.

Definition 4.12. Let Assumption 4.7 be satisfied. The key process of our analysis is the process $D \overset{\text{def}}{=} \pi - \Theta$ called the *displacement* and its diffusion coefficient. According to (4.6) and Remark 4.11, we have that $D \in \mathring{\mathbb{C}}\mathbb{S}$ and that

$$D - \left[\int \sigma\pi(1-\pi)\,\mathrm{d}W - \left(\int \mathfrak{b}^\Theta \,\mathrm{d}W + \int \tilde{\mathfrak{b}}^\Theta \,\mathrm{d}\tilde{W}\right)\right] \in \mathring{\mathbb{C}}\mathbb{F}\mathbb{V}.$$

Then the infinitesimal fluctuation of the diffusion part of D is described by the function

$$\mathbb{D}(x,c) \overset{\text{def}}{=} \mathbb{D}_c(x) \overset{\text{def}}{=} [c_4 x(1-x) - c_2]^2 + c_3^2, \qquad G \overset{\text{def}}{=} \binom{\pi}{\mathfrak{C}} = (\pi, \mathfrak{C}^\intercal)^\intercal. \tag{4.8}$$

Note that $\mathbb{D}(G) = [\sigma\pi(1-\pi) - \mathfrak{b}^\Theta]^2 + (\tilde{\mathfrak{b}}^\Theta)^2$.

Notation 4.13. We count the coordinates of G from zero so that its 0th coordinate is π and that its ith coordinate coincides with the ith coordinate of $\mathfrak{C}$ if $i \in \{1,2,3,4\}$. Then we can write that $G \in \mathring{\mathbb{C}}\mathbb{S}^{\mathbf{5}}$ where $\mathbf{5} \overset{\text{def}}{=} \{0,1,2,3,4\}$. If A is a set, we denote by 1_A its indicator function attaining the value 1 on A and zero otherwise. This notation enables us to denote the ith canonical vector as $1_{\{i\}}$ and to denote $\mathfrak{d} \overset{\text{def}}{=} 1_{\{0\}} - 1_{\{1\}} \in \{-1,0,1\}^{\mathbf{5}}$, with a slight abuse of notation: $\mathfrak{d} = (1,-1,0,0,0)^\intercal \in \mathbb{R}^{\mathbf{5}}$. Note that $\mathfrak{d}^\intercal G = D$ is the displacement. Similarly, we use the notation $1_{[\dots]}$ for the indicator of a statement.

5. ASYMPTOTIC (ALMOST) OPTIMALITY

In this section, we introduce an almost optimal strategy. See Theorems 5.12 and 5.13 for the corresponding statements and Definition 5.8 for the definition of a (pure jump) strategy defined according to the prescribed four policies. In the first part of this section, we introduce the corresponding policies, and in the second one, we show how the corresponding strategy is defined and that it is almost optimal for small values of transaction fees in Theorems 5.12 and 5.13.

5.1. Almost optimal policies

From now, we will consider only $\lambda \in (0,1)$.

In this subsection, we introduce four „policies" that enable us to define an almost optimal strategy, with the help of Definition 5.8, in the next subsection. Our almost optimal strategy is considered in Theorem 5.12, for example, where the properties of the long run growth rate of its wealth process are presented. Here, we start with motivation coming from the case of constant coefficients, where the optimal strategy can be found, and it (roughly speaking) just keeps the position process within a certain interval. Then we introduce very useful objects in Definition 5.3 that enable us to define a certain function in Definition 5.5, which helps us define the inner two „policies" in Notation 5.7.

Remark 5.1. In the case of constant coefficients, roughly speaking, we are able to find a function $f \in C^2(\mathcal{A}_\lambda)$ and $\nu \in \mathbb{R}$ such that the process $\mathcal{U}_\gamma(\mathcal{W}_t \exp\{-f(\pi_t) - \nu t\}), t \geq 0$, is a martingale in the optimal case and that it is supermartingale when considering any other "admissible" strategy. See [16] for details.

Remark 5.2. The classical martingale approach, considered in [16], is based on the Itô rule, which leads to a certain PDE. This PDE can be reduced to an ODE that can be solved almost explicitly in the case of constant coefficients. More precisely, the solution is explicit in the terms of $\omega \in (0, |\Theta| \wedge |1-\Theta|)$ such that $\nu = \frac{p}{2}\sigma^2(\Theta^2 - \omega^2)$ in the regular case when $\Theta \in \mathbb{R}\backslash\{0,1\}$. The value ω is closely related to the with of the no-trade region, which is approximately of the form $(\Theta - \omega, \Theta + \omega)$ for small values of the transaction fees corresponding to small values of ω. In the limiting case, the ODE on this interval is approximately of the form

$$f''(x) \sim \kappa[\omega^2 - (x-\Theta)^2] \tag{5.1}$$

for some constant $\kappa \in (0,\infty)$. This approximation stands behind Definitions 5.3,5.5 and Notation B.15.

Definition 5.3. Let Assumption 4.7 be satisfied. Since $\mathfrak{C} \in \mathbb{BI}_b^{\mathbf{4}}$ and $\ln\sigma \in \mathbb{BI}_b$, there exists a compact convex set $\mathbb{K} \subseteq \mathbb{R}^3 \times (0,\infty)$ such that $\mathfrak{C}$ attains values within $\mathbb{K}$. This set is fixed from here. If

$$\forall\, c \in \mathbb{K} \qquad \mathbb{D}_c(c_1) > 0, \tag{5.2}$$

we put $a \stackrel{\text{def}}{=\!=} \infty, q \stackrel{\text{def}}{=\!=} 1$, and $a \stackrel{\text{def}}{=\!=} \frac{2}{7} \in (0,\frac{1}{2}), q \stackrel{\text{def}}{=\!=} \frac{6}{7}$ otherwise. Then we consider

$$\kappa_{\lambda,c} \stackrel{\text{def}}{=\!=} \frac{p}{c_4^{-2}\mathbb{D}_c(c_1) + \lambda^a}, \qquad \omega_{\lambda,c} \stackrel{\text{def}}{=\!=} \sqrt[3]{\tfrac{3\lambda}{4\kappa_{\lambda,c}}}. \tag{5.3}$$

For an interpretation of the parameter q, see the second part of Remark 5.4. Further, in order to abbreviate the notation, we will write $\mathbb{\lambda} \stackrel{\text{def}}{=\!=} (\lambda, c)^{\intercal}$ instead of λ, c.

Remark 5.4. If the condition (5.2) is satisfied and if $\lambda \in (0,1)$, then the "correction" $\lambda^a = 0$, and the function $(0,1) \times \mathbb{K} \ni \mathbb{\lambda} \mapsto \kappa_{\mathbb{\lambda}}$ attains values within a compact subset of $(0,\infty)$. This can be considered as the regular case when the diffusion coefficient of the displacement is bounded away from zero. Here, $\omega_{\mathbb{\lambda}}$ is of the order $O(\lambda^{1/3})$ as $\lambda \to 0^+$.

In the opposite case, the diffusion coefficient of the displacement can be close to zero, and in order to prevent $\kappa_{\mathbb{\lambda}}$ from being close to infinity, we have added the "correction term" λ^a, where the value a has be chosen as in Definition 5.3 in order to minimize some estimate of the order of the reached error, which corresponds to the value of q. The reader interested in the details where the values $a = 2/7$ and $q = 6/7$ come from is referred to the end of the proof Lemma C.7 in Subsection D.15. See also the main result of this paper, Theorem 5.13.

Definition 5.5. Before the definition of $f_{\mathbb{\lambda}}$, we need to introduce $h_{\mathbb{\lambda}}$ as follows. Put

$$h_\lambda(x,c) \stackrel{\text{def}}{=} h_{\mathbb{\lambda}}(x) \stackrel{\text{def}}{=} \kappa_{\mathbb{\lambda}}\big[\tfrac{1}{2}\,\omega_{\mathbb{\lambda}}^2 z^2 - \tfrac{1}{12}\,z^4 + \tfrac{\varepsilon^2}{6} z^6\big] \qquad \text{where} \qquad z \stackrel{\text{def}}{=} x - c_1. \tag{5.4}$$

Here, $\varepsilon > 0$ is an arbitrary but fixed value. This is the reason why it does not have to be emphasized in the notation. We chose $\varepsilon > 0$ (instead of $\varepsilon = 0$) in order to be able to ensure that the function satisfies some boundary conditions that are the subject of the next lemma.

Lemma 5.6. There exist $\lambda_{5.6} \in (0,1)$ and $n \in \mathbb{N}$ such that

$$\begin{array}{l} h'_{\mathbb{\lambda}}(c_1 + \omega_{\mathbb{\lambda}}) > +\,\zeta^{\downarrow}_\lambda(c_1 + \omega_{\mathbb{\lambda}}), \\ h'_{\mathbb{\lambda}}(c_1 - \omega_{\mathbb{\lambda}}) < -\,\zeta^{\uparrow}_\lambda(c_1 - \omega_{\mathbb{\lambda}}), \end{array} \qquad |c_1 \pm \omega_{\mathbb{\lambda}}| \le n, \quad [-n,n] \subseteq \mathcal{A}_\lambda \tag{5.5}$$

holds if $\lambda \in (0, \lambda_{5.6})$ and $c \in \mathbb{K}$ where

$$\zeta^{\uparrow}_\lambda(x) \stackrel{\text{def}}{=} \tfrac{\lambda^{\uparrow}}{1+\lambda^{\uparrow}x}, \qquad \zeta^{\downarrow}_\lambda(x) \stackrel{\text{def}}{=} \tfrac{\lambda^{\downarrow}}{1-\lambda^{\downarrow}x}. \tag{5.6}$$

Proof. See Subsection D.8 in Section Proofs. □

For an interpretation of the functions $\zeta^{\uparrow}_\lambda, \zeta^{\downarrow}_\lambda$, see (B.5), where we can see that their values at π play the role of the multiplicative factors of the infinitesimal decrease of the logarithm of the wealth process corresponding to the infinitesimal change of the position caused by transactions. In the following notation, we introduce functions $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ that help define policies $\underline{\pi}, \overline{\pi}$ in Notation 5.10 and subsequently also an almost optimal strategy introduced in Remark 5.11.

Notation 5.7. Let $\tilde{\mathbb{G}}_\lambda$ be the set of all $c \in \mathbb{R}^3 \times (0,\infty)$ such that $c_1 \pm \omega_{\mathbb{\lambda}} \in \mathcal{A}_\lambda$ and that the inequalities in (5.5) on the left hold. The set $\tilde{\mathbb{G}}_\lambda$ is obviously open as $c \mapsto \omega_{\mathbb{\lambda}}, (x,c) \mapsto h'_{\mathbb{\lambda}}(x)$ and $\zeta^{\uparrow}_\lambda, \zeta^{\downarrow}_\lambda$ are continuous functions and as $\mathcal{A}_\lambda \subseteq \mathbb{R}$ is an open set. Let $\lambda \in (0, \lambda_{5.6})$. As $h'_{\mathbb{\lambda}}(c_1) = 0$ and as $\zeta^{\uparrow}_\lambda, \zeta^{\downarrow}_\lambda$ attain only positive values on $\mathcal{A}_\lambda$, we get from the continuity of $h'_{\mathbb{\lambda}}, \zeta^{\uparrow}_\lambda, \zeta^{\downarrow}_\lambda$ and from the definition of $\tilde{\mathbb{G}}_\lambda$ that for every $c \in \tilde{\mathbb{G}}_\lambda$

$$\underline{\pi}_{\mathbb{\lambda}} \stackrel{\text{def}}{=} \sup\{x \in \mathcal{A}_\lambda, h'_{\mathbb{\lambda}}(x) \le -\zeta^{\uparrow}_\lambda(x)\} \in (c_1 - \omega_{\mathbb{\lambda}}, c_1), \quad h'_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}}) = -\zeta^{\uparrow}_\lambda(\underline{\pi}_{\mathbb{\lambda}}), \tag{5.7}$$

$$\overline{\pi}_{\mathbb{\lambda}} \stackrel{\text{def}}{=} \inf\{x \in \mathcal{A}_\lambda, h'_{\mathbb{\lambda}}(x) \ge +\zeta^{\downarrow}_\lambda(x)\} \in (c_1, c_1 + \omega_{\mathbb{\lambda}}), \quad h'_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}}) = \zeta^{\downarrow}_\lambda(\overline{\pi}_{\mathbb{\lambda}}). \tag{5.8}$$

Note that $\mathbb{K} \subseteq \tilde{\mathbb{G}}_\lambda$ holds if $\lambda \in (0, \lambda_{5.6})$ by Lemma 5.6.

5.2. Almost optimal strategy

In this subsection, we present the main results of this paper, Theorems 5.12 and 5.13. First, we introduce a strategy based on four policies, and we show (in Theorem 5.9) that it exists. Further, we introduce a process (in Notation 5.10) whose long run average is close to the long run growth rate of the wealth process of our almost optimal strategy, see Theorem 5.12.

Definition 5.8. In the whole definition, we assume that (φ, ψ) is a λ-admissible strategy. It is called *a pure jump strategy* if it is of the form

$$(\varphi, \psi) = \textstyle\sum_{k=0}^{\infty} (\Phi_k, \Psi_k)\, 1_{[\tau_k, \tau_{k+1})} \tag{5.9}$$

where $(\tau_k)_{k=0}^{\infty}$ is an increasing sequence of stopping times tending to ∞ as $k \to \infty$ with $\tau_0 = 0$ and where Φ_k, Ψ_k are $\mathcal{F}_{\tau_k}$-measurable real valued random variables whenever $k \in \mathbb{N}_0$. Let $\mathfrak{a}^{\uparrow} < \mathfrak{b}^{\uparrow} < \mathfrak{b}^{\downarrow} < \mathfrak{a}^{\downarrow}$ be continuous adapted processes with values in $\mathcal{A}_\lambda$. The strategy (φ, ψ) from (5.9) with the position π is called $[\,\mathfrak{a}^{\uparrow}(\,\mathfrak{b}^{\uparrow},\mathfrak{b}^{\downarrow})\,\mathfrak{a}^{\downarrow}]$-*strategy* if $\mathfrak{a}^{\uparrow}_t < \pi_t < \mathfrak{a}^{\downarrow}_t$ holds for every $t \in [0, \infty)$ and if for every $t \in (0, \infty)$

$$\cup_{k=1}^{\infty} [t = \tau_k] \subseteq [\pi_{t-} = \mathfrak{a}^{\uparrow}_t] \cup [\pi_{t-} = \mathfrak{a}^{\downarrow}_t], \qquad \begin{array}{l} [\pi_{t-} = \mathfrak{a}^{\uparrow}_t] \subseteq [\pi_t = \mathfrak{b}^{\uparrow}_t], \\ [\pi_{t-} = \mathfrak{a}^{\downarrow}_t] \subseteq [\pi_t = \mathfrak{b}^{\downarrow}_t]. \end{array} \tag{5.10}$$

Theorem 5.9. Let $\mathfrak{a}^{\uparrow} < \mathfrak{b}^{\uparrow} < \mathfrak{b}^{\downarrow} < \mathfrak{a}^{\downarrow}$ be continuous adapted processes with values in $\mathcal{A}_\lambda$. Given $\mathcal{F}_0$-measurable random variables $\mathfrak{w} > 0$ and $\mathfrak{p} \in (\mathfrak{a}^{\uparrow}_0, \mathfrak{a}^{\downarrow}_0)$, there exists a λ-admissible $[\,\mathfrak{a}^{\uparrow}(\,\mathfrak{b}^{\uparrow},\mathfrak{b}^{\downarrow})\,\mathfrak{a}^{\downarrow}]$-*strategy* with the initial wealth $\mathfrak{w}$ and the initial *position* $\mathfrak{p}$.

P r o o f. See Subsection D.22. □

In the following notation, we will introduce processes ν, ϖ that play similar roles as the constant values ν, ω from Remarks 5.1 and 5.2. In particular, if $p = 1$, the value ν_t, roughly speaking, will (more or less) represent the instantaneous rate of the long term exponential growth of the wealth process. The value of the process ϖ can be viewed as the half-width of the "no trade" region of a strategy with the policies $\Theta \pm \varpi$ that could be shown to have similar properties to the one considered in Remark 5.11. See also Remark 5.16.

Notation 5.10. Let ($\mathbb{A}$4.7) hold. We briefly write

$$\nu \overset{\text{def}}{=\!=} \tfrac{1}{2}\, \sigma^2 (\theta^2 - \varpi^2) \qquad \text{where} \qquad \varpi \overset{\text{def}}{=\!=} (\omega_{\lambda, \mathfrak{C}_t})_{t \geq 0}, \tag{5.11}$$

and similarly, we put $\underline{\pi} \overset{\text{def}}{=\!=} (\underline{\pi}_{\lambda, \mathfrak{C}_t})_{t \geq 0}$ and $\bar{\pi} \overset{\text{def}}{=\!=} (\overline{\pi}_{\lambda, \mathfrak{C}_t})_{t \geq 0}$. See (4.7,5.3) for $\mathfrak{C}, \omega_{\lambda, c}$.

Remark 5.11. If $\lambda \in (0, \lambda_{5.6})$, any λ-admissible $[\Theta - \varpi\,(\,\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$-strategy is strictly λ-admissible. It follows from (5.5) in Lemma 5.6 and from Definition 5.8. Further, whenever we write about $[\Theta - \varpi\,(\,\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$-strategy, we implicitly assume that it is λ-admissible without emphasizing the considered λ in the notation. We hope that it will not lead to confusion. To agree that this strategy exists, see the following paragraph.

By (4.7), $\mathfrak{C} \in \mathbb{BI}_b^4 \subseteq \mathbb{CA}^4$ holds under (implicitly assumed) Assumption 4.7. This ensures that $\Theta \pm \varpi \in \mathbb{CA}$, and similarly, we get from Lemma B.14 that $\underline{\pi}, \bar{\pi} \in \mathbb{CA}$ if $\lambda \in (0, \lambda_{B.14})$. For such λ, Theorem 5.9 gives that the strategy $[\Theta - \varpi\,(\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$ exists. Here, we used that $\lambda_{B.14} \in (0, \lambda_{5.6})$, which ensures that $\Theta \pm \varpi$ attain values in $\mathcal{A}_\lambda$ if $\lambda \in (0, \lambda_{B.14})$.

Here, the main results of this paper come. The first one, Theorem 5.12, says (roughly speaking) that the long run growth rate of the wealth process corresponding to our almost optimal strategy is very close (for small transaction fees) to the long run average of the process ν from Notation 5.10. The corresponding error is of order $O(\lambda^q)$. The second result, Theorem 5.13, shows that the wealth process of any admissible strategy does not have significantly higher long run growth rate than the (above mentioned) long run average of the process ν.

Theorem 5.12. Let ($\mathbb{A}$4.7) hold, let $p = 1$, i. e., $\gamma = 0$, and let $q \in \{\frac{6}{7}, 1\}$ be from Definition 5.3. Then there exist $\lambda_{5.12} \in (0, \infty), K_{5.12} \in (K_{B.18}, \infty)$ such that the following holds whenever $\lambda \in (0, \lambda_{5.12})$. Let (φ, ψ) be a λ-admissible $[\Theta - \varpi\,(\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$-strategy with the wealth process $\mathcal{W}$, and let ν be from (5.11). Then

$$|\mathbb{E}[\ln \tfrac{\mathcal{W}_\tau}{\mathcal{W}_0}] - \mathbb{E}[\textstyle\int_0^\tau \nu_s \,\mathrm{d}s]| \le K_{5.12}(1 + \mathbb{E}[\tau])\lambda^q \tag{5.12}$$

holds whenever τ is an integrable stopping time. In particular,

$$\limsup_{t\to\infty} \frac{1}{t} |\mathbb{E}[\ln \tfrac{\mathcal{W}_t}{\mathcal{W}_0}] - \mathbb{E}[\textstyle\int_0^t \nu_s \,\mathrm{d}s]| \le K_{5.12}\, \lambda^q.$$

Proof. See Subsection D.19 in section Proofs. □

Theorem 5.13. Let ($\mathbb{A}$4.7) hold, let $p = 1$, $q \in \{\frac{6}{7}, 1\}$ be from Definition 5.3. Then there exist $K_{5.13}, \lambda_{5.13} \in (0, \infty)$ such that the following holds whenever $\lambda \in (0, \lambda_{5.13})$. Let $\hat{\mathcal{W}}$ be the wealth process of a λ-admissible $[\Theta - \varpi\,(\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$-strategy, and let $\mathcal{W}, \pi$ be the wealth process and the position of a λ-admissible strategy such that $\mathcal{W}_0 = \hat{\mathcal{W}}_0$. Then

$$\limsup_{t\to\infty} \tfrac{1}{t}\, (\ln \mathcal{W}_t - \ln \hat{\mathcal{W}}_t) \overset{\text{as}}{\le} K_{5.13}\, \lambda^q, \tag{5.13}$$

$$\limsup_{t\to\infty} \tfrac{1}{t}\, \mathbb{E}[\ln \mathcal{W}_t - \ln \hat{\mathcal{W}}_t] \le K_{5.13}\, \lambda^q. \tag{5.14}$$

Moreover, if π attains values within a compact set $\mathfrak{X} \subseteq \mathcal{A}_\lambda$, then

$$\mathbb{E}[\ln(\mathcal{W}_\tau/\hat{\mathcal{W}}_\tau)] \le K_{5.13}\, \lambda^q(1 + \mathbb{E}[\tau]) + 2 \sup_{x\in\mathfrak{X}} \max\{|\ln(1 + \lambda^{\uparrow} x)|, |\ln(1 - \lambda^{\downarrow} x)|\} \tag{5.15}$$

whenever τ is an integrable stopping time.

Proof. See Subsection D.21 in section Proofs. □

Remark 5.14. Note that Assumption ($\mathbb{A}$4.7), including ($\mathbb{A}$4.1), is absolutely essential in the statements of Theorems 5.12 and 5.13, regarding their formal proofs in this paper. Further, see Remark A.4 for the comments showing how ($\mathbb{A}$4.7) can be verified in some special cases and how to cope with some technical difficulties that may arise.

Remark 5.15. The statements of Theorems 5.12 and 5.13 remain valid if the policies $(\underline{\pi}, \bar{\pi})$ are replaced by other continuous adapted processes $A < B$ (dependent also on $\lambda > 0$) such that $\Theta - \varpi < A \leq \underline{\pi}$ and $\bar{\pi} \leq B < \Theta + \varpi$ hold for $\lambda > 0$ small enough, say for $\lambda \in (0, \lambda_{B.14})$. The corresponding proofs can be obtained by a minor modification, but this modification has to be applied to the whole tree of proofs including the corresponding auxiliary statements.

Remark 5.16. Similarly as in Remark 5.15, instead of the strategy $[\Theta - \varpi\,(\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$, we could consider a strategy that just keeps the position within the interval $[\Theta - \varpi, \Theta + \varpi]$. The corresponding statements would remain valid, and the proofs of the theorems can be used as a guide that can help the reader prove them, but the rigorous proof of that would significantly increase the length of this paper. On the other hand, for practical purposes (including simulations), it is always easier to apply (approximately) a strategy mentioned in this remark, although its existence (and its almost optimality) is not proved in this paper.

A. APPENDIX

This section was added additionally as a reaction to the requirement of one of the reviewers who wanted some ,,numerical confirmation" of the derived theoretical results. The corresponding numerical illustration is a part of the supplement to this paper, and this section serves as a theoretical background for the corresponding calculations and comparisons.

The following lemma says how the long run growth rate of the wealth process can be approximated based on the values of the position process.

Lemma A.1. Assume ($\mathbb{A}$4.1). Let $\mathcal{W}$ and π be the wealth and the position process of a pure jump strategy. Then

$$\lim_{t\to\infty} \tfrac{1}{t}\big[\ln \tfrac{\mathcal{W}_t}{\mathcal{W}_0} - \textstyle\sum_{0<s\leq t} \Delta \ln \mathcal{W}_s - \int_0^t \tfrac{1}{2}\,\sigma_s^2[\theta_s^2 - (\pi_s - \theta_s)^2]\,\mathrm{d}s\big] \overset{\text{as}}{=} 0, \tag{A.1}$$

and $\Delta \ln \mathcal{W}_s = \ln \min\{1, \frac{1+\lambda^{\uparrow}\pi_{s-}}{1+\lambda^{\uparrow}\pi_s}, \frac{1-\lambda^{\downarrow}\pi_{s-}}{1-\lambda^{\downarrow}\pi_s}\}$, $s \in (0, \infty)$.

Proof. Here, we will use Remark D.10. It says that (a) the continuous parts of $\pi^{\uparrow}, \pi^{\downarrow}$ are equal to zero and (b) that (D.88) holds, which gives the expression of $\Delta \ln \mathcal{W}_s$ from the statement of the lemma since $1_{[\pm\Delta\varphi_t>0]} = 1_{[\pm\Delta\pi_t>0]}$ holds (if (φ, ψ) is the considered strategy) and as $\pi_t, \pi_{t-} \in \mathcal{A}_\lambda$, see Remark 3.8 and Lemma 3.10 (ii). Hence, it remains to show (A.1).

From ($\mathbb{A}$4.1) and Lemma B.7, we have that

$$\ln \tfrac{\mathcal{W}_t}{\mathcal{W}_0} - \textstyle\sum_{0<s\leq t} \Delta \ln \mathcal{W}_s \overset{\text{as}}{=} \int \tfrac{1}{2}\,\sigma_s^2[\theta_s^2 - (\pi_s - \theta_s)^2]\,\mathrm{d}s + \int \sigma\pi\,\mathrm{d}W,$$

and to get (A.1), it remains to show that $\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma\pi\,\mathrm{d}W \overset{\text{as}}{=} 0$, but this follows from the Strong Law of Large Numbers for martingales as both $\sigma \in \mathbb{BI}_b \subseteq \mathbb{PM}_b$ and $\pi \in \ddot{\mathbb{C}}\mathbb{A}$ are bounded progressive processes, see Lemma 3.10 (i). Here, we used again Remark 3.8, which says that π attains values within a (bounded) interval $\mathcal{A}_\lambda$. □

In the following example, we consider three basic cases which are considered in this paper (and in the corresponding supplement) for an illustration.

Example A.2. Here, we will consider $\sigma = 1$. Note that this restriction is (almost) without loss of generality (when it comes to examples) due to the nature of the considered investment problem (and of its suggested solution in this paper).

(a) First, we consider the case when $\theta = \vartheta \in \mathbb{R}\backslash\{0,1\}$.

(b) In the second case, we consider $\theta = \arctan(U)$ where $U \in \mathbb{CA}(\mathcal{F})$ is (an Ornstein-Uhlembeck process) such that $W = U - U_0 + \Xi \int U_s\,\mathrm{d}s$ is a standard $\mathcal{F}$-Brownian motion where $\Xi \in (0,\infty)$ is a real number.

(c) The last case is obtained from case (b) if W is in (b) replaced by a standard $\mathcal{F}$-Brownian motion (say $\tilde{W}$) independent of W.

Remark A.3. Note that the assumption ($\mathbb{A}$4.7) is satisfied in all of the cases (a,b,c) from Example A.2. We have (a) $\mathfrak{C} = (\vartheta, 0, 0, 1)^\intercal$ and $\mathbb{D}_{\mathfrak{C}}(\theta) = \theta^2(1-\theta)^2$,

$$\text{(b) } \mathfrak{C} = (\theta, (1+U^2)^{-1}, 0, 1)^\intercal, \quad \mathbb{D}_{\mathfrak{C}}(\theta) = [\theta(1-\theta) - (1+U^2)^{-1}]^2,$$
$$\text{(c) } \mathfrak{C} = (\theta, 0, (1+U^2)^{-1}, 1)^\intercal, \quad \mathbb{D}_{\mathfrak{C}}(\theta) = \theta^2(1-\theta)^2 + (1+U^2)^{-2}.$$

The process $\mathfrak{C}$ attains values within the compact set (a) $\mathbb{K} = \{(\vartheta, 0, 0, 1)^\intercal\}$,

$$\text{(b) } \mathbb{K} = \{(\vartheta, \cos^2(\vartheta), 0, 1)^\intercal; |\vartheta| \le \tfrac{\pi}{2}\}, \quad \text{(c) } \mathbb{K} = \{(\vartheta, 0, \cos^2(\vartheta), 1)^\intercal; |\vartheta| \le \tfrac{\pi}{2}\}$$

where π stands for the Ludolphine number. For our choices, (5.2) holds, which means that we have $a = \infty$ and $q = 1$ in Definition 5.3. Note that $\mathbb{D}$ is defined in (4.8).

Remark A.4. The purpose of this remark is to show that $\theta \in \mathbb{BI}_b$ hold in case (b) from Example A.2. Case (c) can be treated similarly, and case (a) does not require any treatment to obtain the corresponding result.

(i) We easily obtain that the following functions are continuous and bounded

$$\mathbb{R} \ni x \mapsto g'(x),\ g'(x)x,\ g''(x),\ g''(x)x,\ g'''(x) \quad \text{where} \quad g = \arctan.$$

(ii) From the Itô rule and (i), we obtain that (4.4) holds with

$$\mathfrak{a}^\theta = \tfrac{1}{2}g''(U) - \Xi U g'(U) \in \mathbb{PM}_b, \quad \mathfrak{b}^\theta = g'(U) \in \mathbb{PM}_b, \quad \tilde{\mathfrak{b}}^\theta = 0 \in \mathbb{BI}_b$$

if there exists a standard $\mathcal{F}$-Brownian motion $\tilde{W}$ independent of W, which is easy to ensure in our examples. See point (iv) below.

(iii) Further, we obtain again from the Itô rule and from (i) that $\mathfrak{b}^\theta \in \mathbb{BI}_b$ as follows

$$\mathfrak{b}^\theta = g'(U) \overset{\text{as}}{=} \mathfrak{b}^\theta_0 + \int [\tfrac{1}{2}\,g'''(U_t) - \Xi U_t g''(U_t)]\,\mathrm{d}t + \int g''(U)\,\mathrm{d}W \in \mathbb{BI}_b.$$

(iv) Hence, if there exists a standard $\mathcal{F}$-Brownian motion $\tilde{W}$ independent of W, Assumption 4.7 is satisfied. This condition can be easily ensured with the help of Lemma 4.6 if we are allowed to consider an extended filtration instead of $\mathcal{F}$ and if there exists a standard Brownian motion independent of ,,anything already mentioned". This last condition is easy to satisfy by extension of our probability space (which means that our probability space is then replaced by the extended one).

The following lemma says that the inner two ,,policies" can be obtained as the only roots of certain functions for $\lambda > 0$ small enough, and the subsequent remark (following after the proof) specifies the corresponding condition under which the uniqueness of the roots is ensured.

Lemma A.5. There exists $\lambda_{A.5} \in (0, \lambda_{5.6})$ such that the following points (i,ii) hold whenever $\lambda \in (0, \lambda_{A.5})$.
(i) $\underline{\mathbb{\pi}}_{\mathbb{\lambda}}$ is the only root of the function $h'_{\mathbb{\lambda}} + \zeta^{\uparrow}_{\lambda}$ on the interval $(c_1 - \mathbb{\omega}_{\mathbb{\lambda}}, c_1)$,
(ii) $\overline{\mathbb{\pi}}_{\mathbb{\lambda}}$ is the only root of $h'_{\mathbb{\lambda}} - \zeta^{\downarrow}_{\lambda}$ on $(c_1, c_1 + \mathbb{\omega}_{\mathbb{\lambda}})$, cf. (5.7,5.8).

Remark A.6. Let $g : I \to \mathbb{R}$ be a continuous function where I is a non-degenerate open interval. If $[g = 0] \overset{\text{def}}{=\!=} \{x \in I; g(x) = 0\} \subseteq [g' > 0]$, then $\operatorname{card}[g = 0] \le 1$.

P r o o f. (of Lemma A.5) We show only that (ii) holds for $\lambda > 0$ small enough as (i) can be shown similarly. The unicity of the root will be verified once we use Remark A.6 for $g \overset{\text{def}}{=\!=} h'_{\mathbb{\lambda}}/\zeta^{\downarrow}_{\lambda} - 1$ and $I \overset{\text{def}}{=\!=} (c_1, c_1 + \mathbb{\omega}_{\mathbb{\lambda}})$. Hence, it remains to verify the condition $[g = 0] \subseteq [g' > 0]$. Since $\frac{\mathrm{d}}{\mathrm{d}x}\zeta^{\downarrow}_{\lambda}(x) = \zeta^{\downarrow}_{\lambda}(x)^2$ and $\zeta^{\downarrow}_{\lambda} > 0$, the condition to be verified (for $\lambda > 0$ small enough) is of the form

$$\forall\, x \in I \quad \big[h'_{\mathbb{\lambda}}(x) = \zeta^{\downarrow}_{\lambda}(x) \quad \Rightarrow \quad h''_{\mathbb{\lambda}}(x)/h'_{\mathbb{\lambda}}(x)^2 > 1\big]. \tag{A.2}$$

On I, we have the following inequalities

$$0 < h'_{\mathbb{\lambda}}(x) = \kappa_{\mathbb{\lambda}}\big[\,\mathbb{\omega}^2_{\mathbb{\lambda}}(x - c_1) - \tfrac{1}{3}\,(x - c_1)^3 + \varepsilon^2(x - c_1)^5\big] \le \kappa_{\mathbb{\lambda}}\mathbb{\omega}^3_{\mathbb{\lambda}}[\tfrac{2}{3} + \varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}], \tag{A.3}$$

$$h''_{\mathbb{\lambda}}(x) = \kappa_{\mathbb{\lambda}}\big[\,\mathbb{\omega}^2_{\mathbb{\lambda}} - (x - c_1)^2 + 5\varepsilon^2(x - c_1)^4\big] \ge \kappa_{\mathbb{\lambda}}\mathbb{\omega}^2_{\mathbb{\lambda}} \min\{\tfrac{1}{2}, 5\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}\}. \tag{A.4}$$

In the inequality in (A.4), we have used the following relation (for w, b positive)

$$\inf_{0<y<w}(w - y + by^2) = \begin{cases} bw^2 & \text{if } 1 \ge 2bw \\ w - \frac{1}{4b} > \frac{w}{2} & \text{if } 1 < 2bw \end{cases} \ge w\min\{bw, \tfrac{1}{2}\}.$$

Then we get that

$$\begin{aligned} 0 &< h'_{\mathbb{\lambda}}(x)^2/h''_{\mathbb{\lambda}}(x) \le \big(\kappa_{\mathbb{\lambda}}\mathbb{\omega}^3_{\mathbb{\lambda}}[\tfrac{2}{3} + \varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}]\big)^2/\big(\kappa_{\mathbb{\lambda}}\mathbb{\omega}^2_{\mathbb{\lambda}}\min\{\tfrac{1}{2}, 5\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}\}\big) \\ &= \kappa_{\mathbb{\lambda}}\mathbb{\omega}^4_{\mathbb{\lambda}}[\tfrac{2}{3} + \varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}]^2/\min\{\tfrac{1}{2}, 5\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}\} = \kappa_{\mathbb{\lambda}}\mathbb{\omega}^4_{\mathbb{\lambda}}[\tfrac{2}{3} + \varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}]^2\max\{2, 1/(5\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}})\} \\ &= \tfrac{3}{4}\,\lambda\,\mathbb{\omega}_{\mathbb{\lambda}}[\tfrac{2}{3} + \varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}]^2\max\{2, 1/(5\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}})\} \mathrel{\underset{\cdot}{=}} o_{\mathbb{K}}(\mathbb{\omega}^3_{\mathbb{\lambda}})\max\{1, 1/(10\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}})\} \mathrel{\underset{\cdot}{=}} o_{\mathbb{K}}(\mathbb{\omega}_{\mathbb{\lambda}}) \mathrel{\underset{\cdot}{=}} o_{\mathbb{K}}(1), \end{aligned}$$

see (B.12) in Lemma B.12. This ensures that (A.2) holds for $\lambda > 0$ small enough. □

Remark A.7. It follows from the proof of Lemma A.5 that the unicity in the point (ii) of the statement of the lemma holds whenever $\kappa_{\mathbb{\lambda}}\mathbb{\omega}^4_{\mathbb{\lambda}}[\frac{2}{3} + \varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}]^2 < \min\{\frac{1}{2}, 5\varepsilon^2\mathbb{\omega}^2_{\mathbb{\lambda}}\}$. Note that the same condition ensures also the unicity in the point (i) since to get the missing part of the proof, it is enough to consider $g \overset{\text{def}}{=\!=} h'_{\mathbb{\lambda}}/\zeta^{\uparrow}_{\lambda} + 1$ with $I \overset{\text{def}}{=\!=} (c_1 - \mathbb{\omega}_{\mathbb{\lambda}}, c_1)$ as $\frac{\mathrm{d}}{\mathrm{d}x}\zeta^{\uparrow}_{\lambda}(x) = -\zeta^{\uparrow}_{\lambda}(x)^2$.

A.1. Constant coefficients

In this subsection, the processes $\sigma \in (0,\infty)$ and $\theta \in \mathbb{R}\backslash\{0,1\}$ are assumed to be constant, and we consider here only the case when $p = 1$. Note that the restriction to $\lambda \in (0,1)$ does not apply (exceptionally) to this subsection.

Remark A.8. Consider $p = 1$ and $\theta \in \mathbb{R}\backslash\{0,1\}$. According to Theorem 5.5 in [16], if the strategy of keeping the position just within the interval $[\alpha,\beta]$ is applied, where $\alpha,\beta \in \mathcal{A}_\lambda\backslash\{0,1\}, \alpha < \beta$, the rate of the exponential growth of the portfolio market price is of the form $\frac{1}{2}\sigma^2 u(\alpha,\beta)$ where $u(\alpha,\beta)$ is given by the formulas (39,40) in [16]. In our settings, which includes the assumption that the transaction fees are symmetric, see (3.4), this value can be rewritten as follows

$$u_{\theta,\lambda}(\alpha,\beta) \overset{\text{def}}{=\!=} \begin{cases} \frac{\xi_-(\beta,\lambda)-\xi_+(\alpha,\lambda)}{\ln|1/\alpha-1|-\ln|1/\beta-1|} & \text{if } \ \rho \overset{\text{def}}{=\!=} \theta - \frac{1}{2} = 0, \\ 2\rho\frac{\xi_+(\alpha,\lambda)|1/\alpha-1|^{-2\rho}-\xi_-(\beta,\lambda)|1/\beta-1|^{-2\rho}}{|1/\alpha-1|^{-2\rho}-|1/\beta-1|^{-2\rho}} & \text{if } \ \theta \in \mathcal{A}_\lambda\backslash\{0,\frac{1}{2},1\}, \end{cases} \tag{A.5}$$

where $\xi_\pm(x,\lambda) \overset{\text{def}}{=\!=} 1/[1+(\frac{1}{x}-1)e^{\mp\lambda/2}]$. See also Remark 1 in Section 4 in [16].

Remark A.9. According to Corollary 6.5 combined with Theorem 4.3 (both) from [16], the maximal long run growth rate of the portfolio market price is of the form $\nu = \frac{\sigma^2}{2}(\theta^2-\omega^2)$ where $\omega \in (0,\omega_0)$ is the unique solution of the equation given by the first equality in

$$-\lambda = \mathcal{I}(\omega) \overset{\text{def}}{=\!=} \int_{\theta-\omega}^{\theta+\omega} \frac{\omega^2-(\theta-x)^2}{x(1-x)[2\rho x-(\theta^2-\omega^2)]}\,\mathrm{d}x = \int_{\theta-\omega}^{\theta+\omega}\left[\frac{1}{x(1-x)}+\frac{1}{2\rho x-(\theta^2-\omega^2)}\right]\mathrm{d}x, \tag{A.6}$$

where $\mathcal{I}$ comes from Lemm 6.3 in [16] and $\omega_0 \overset{\text{def}}{=\!=} |\theta| \wedge |1-\theta|$. Equivalently, we can write that

$$\lambda = \begin{cases} \frac{4}{1-2\theta}[\theta \operatorname{arctanh}(\frac{\omega}{\theta}) - (1-\theta)\operatorname{arctanh}(\frac{\omega}{1-\theta})] & \text{if } \ \theta \in \mathcal{A}_\lambda\backslash\{0,\frac{1}{2},1\}, \\ \frac{2\omega}{1/4-\omega^2} + 2\ln\frac{1/2-\omega}{1/2+\omega} & \text{if } \ \theta = \frac{1}{2}. \end{cases} \tag{A.7}$$

Note that the function $\mathcal{I} : (0,\omega_0) \to (-\infty,0)$ is continuous decreasing bijection by Lemm 6.3 in [16], which ensures that $\omega = \mathcal{I}^{-1}(-\lambda) \to 0$ as $\lambda \to 0^+$. Obviously, also $\lambda \to 0$ as $\omega \to 0^+$.

The following lemma provides a certain upper bound for ω.

Lemma A.10. Let $\lambda \in (0,\infty)$ and $\omega \in (0,\omega_0)$ be as in Remark A.9. Put

$$\bar{\omega}_\lambda \overset{\text{def}}{=\!=} \begin{cases} |\theta-1|\tanh\left[\,|\frac{2\theta-1}{\theta-1}|\lambda + \frac{2\theta}{\theta-1}\ln(2\theta-1)\right], & \text{if } \rho > 0, \\ |\theta|\tanh\left[\,|\frac{2\theta-1}{\theta}|\lambda + \frac{2(\theta-1)}{\theta}\ln(1-2\theta)\right], & \text{if } \rho < 0, \end{cases}$$

and $\bar{\omega}_\lambda \overset{\text{def}}{=\!=} \frac{\lambda/2+2/e}{2+\lambda}$ if $\rho = 0$. Then $\omega \in (0,\bar{\omega}_\lambda)$.

Proof. See Subsection A.2. □

Notation A.11. We will use the Laudau notation O (and the asymmetric notation $\fallingdotseq$) formally introduced in Notation B.11 for $\lambda \to 0^+$ in a more general setting. We hope that the reader will accept this notation (in this subsection) even if it is not used only for $\lambda \to 0^+$.

Lemma A.12. Consider $\theta \in \mathbb{R}\backslash\{0,1\}$ and $u_{\theta,\lambda}$ from (A.5). Then as $\lambda \to 0^+$

$$\varpi \overset{\text{def}}{=\!=} \sqrt[3]{\tfrac{3\lambda}{4}\theta^2(1-\theta)^2} \fallingdotseq \omega + \Psi_\theta \omega^3 + O(\omega^5), \tag{A.8}$$

$$u_{\theta,\lambda}(\theta-\varpi,\theta+\varpi) \fallingdotseq \theta^2 - \varpi^2 + 2\Psi_\theta\varpi^4 + O(\varpi^6), \tag{A.9}$$

where ω comes from Remark A.9 and where $\Psi_\theta \overset{\text{def}}{=\!=} \frac{1-2\theta+2\theta^2}{5\theta^2(1-\theta)^2} = \frac{1}{5}[\theta^{-2} + (1-\theta)^{-2}]$.

P r o o f. The relation in (A.8) can be obtained by a straightforward computation, with the help of software on symbolic computing and (A.7). Here, keep in mind that $\lambda \to 0^+$ if and only if $\omega \to 0^+$, roughly speaking. See the end of Remark A.9.

Similarly, we obtain the following relation

$$\mathfrak{x}_\theta(w) \overset{\text{def}}{=\!=} \xi_-(\theta + w, \tfrac{4}{3}\tfrac{w^3}{\theta^2(1-\theta^2)}) \fallingdotseq \Xi_\theta(w) + C_\theta w^6 + O(w^7), \quad w \to 0, \tag{A.10}$$

$$\Xi_\theta(w) \overset{\text{def}}{=\!=} \theta + w - \tfrac{2w^3}{3\theta(1-\theta)} + \tfrac{2(2\theta-1)w^4}{3\theta^2(1-\theta)^2} + \tfrac{2w^5}{3\theta^2(1-\theta)^2}, \qquad C_\theta \in \mathbb{R}. \tag{A.11}$$

From Remark A.8, we obtain that $u_{\theta,\lambda}(\theta-\varpi,\theta+\varpi) = \mathfrak{u}_\theta(\varpi)$ where

$$\mathfrak{u}_\theta(w) \overset{\text{def}}{=\!=} \mathfrak{x}_\theta(w)\mathfrak{B}_\theta(w) + \mathfrak{x}_\theta(-w)\mathfrak{B}_\theta(-w), \qquad \mathfrak{x}_\theta(-w) = \xi_+(\theta - w, \tfrac{4}{3}\tfrac{w^3}{\theta^2(1-\theta^2)}),$$

$$\mathfrak{B}_\theta(w) \overset{\text{def}}{=\!=} \begin{cases} -\frac{1}{\ln|1/(\theta+w)-1| - \ln|1/(\theta-w)-1|} & \text{if } \theta = \frac{1}{2}, \\ 2\rho\frac{|(\theta+w)^{-1}-1|^{-2\rho}}{|(\theta+w)^{-1}-1|^{-2\rho} - |(\theta-w)^{-1}-1|^{-2\rho}} & \text{otherwise}, \end{cases} \qquad |w| \in (0, |\theta| \wedge |1-\theta|),$$

$$\fallingdotseq \tfrac{(1-\theta)\theta}{2w} + \theta - \tfrac{1}{2} - \tfrac{w}{6} - \tfrac{2(11\theta^2-11\theta+3)}{45(\theta-1)^2\theta^2}w^3 + D_\theta w^5 + O(w^6), \qquad D_\theta \in \mathbb{R}, \tag{A.12}$$

as $w \to 0$. Note that the relation in (A.12) was obtained with the help of symbolic computing. Since $\mathfrak{B}_\theta(w) \fallingdotseq O(w^{-1})$ as $w \to 0$ and since $\mathfrak{B}_\theta(-w) = -\mathfrak{B}_\theta(w)$, we get that

$$\mathfrak{u}_\theta(w) \fallingdotseq \Xi_\theta(w)\mathfrak{B}_\theta(w) + \Xi_\theta(-w)\mathfrak{B}_\theta(-w) + O(w^6) \tag{A.13}$$

$$\fallingdotseq \theta^2 - w^2 + 2\Psi_\theta w^4 + O(w^6) \tag{A.14}$$

Note that the relation in (A.14) simply follows from (A.11,A.12). Finally, it is enough to realize that $\varpi \to 0$ as $\lambda \to 0^+$, see (A.8), to get the relation in (A.9). □

Corollary A.13. In the context of Lemma A.12, we have that

$$u_{\theta,\lambda}(\theta-\varpi,\theta+\varpi) \fallingdotseq \theta^2 - \omega^2 + O(\lambda^2), \qquad \lambda \to 0^+. \tag{A.15}$$

P r o o f. Since $\omega \to 0$ as $\lambda \to 0^+$, see the end of Remark A.9, we get from (A.8) that

$$\varpi^2 \fallingdotseq \omega^2 + 2\Psi_\theta\omega^4 + O(\omega^6), \qquad \varpi^4 \fallingdotseq \omega^4 + O(\omega^6), \qquad \tfrac{\varpi}{\omega} \to 1, \qquad \text{as } \lambda \to 0^+. \tag{A.16}$$

From (A.8,A.16), we obtain that $O(\omega^6) \fallingdotseq O(\varpi^6) \fallingdotseq O(\lambda^2)$, which enables to obtain (A.15) from (A.9,A.16). □

Remark A.14. In the case of constant coefficients considered in this subsection, we have the maximal growth rate of the portfolio market price in the form $\frac{\sigma^2}{2}(\theta^2 - \omega^2)$, see Remark A.9. From Remark A.8, we obtain that if $\varpi \in (0, |\theta| \wedge |1-\theta|)$, the strategy of keeping the position just within the interval $[\theta - \varpi, \theta + \varpi]$ leads to the long run growth rate of the portfolio market price in the form $\frac{\sigma^2}{2} u_{\theta,\lambda}(\theta - \varpi, \theta + \varpi)$. Corollary A.13 says that the difference between these two growth rates is of the form $O(\lambda^2)$, which is far less (for $\lambda > 0$ small enough) than the error from the statement of Theorem 5.13.

For example, if $\theta = \frac{1}{2}$, the relation (A.15) from Corollary A.13 can be improved into the form

$$u_{\frac{1}{2},\lambda}(\tfrac{1}{2} - \varpi, \tfrac{1}{2} + \varpi) \risingdotseq \tfrac{1}{4} - \omega^2 + \tfrac{\lambda^2}{400} + O(\lambda^{8/3}), \quad \lambda \to 0^+, \tag{A.17}$$

which means that the error (between the optimal and the achieved long run growth rate of the portfolio market price) from Remark A.14 is in this case of the form $\frac{\sigma^2}{2}(\frac{\lambda^2}{400} + O(\lambda^{8/3}))$.

Proof. (of (A.17)) With the help of symbols introduced in the proof of Lemma A.12 and with the help of arguments (and comments) from its proofs and from the proof of the subsequent corollary, we obtain that

$$\mathbb{\beta}_{\frac{1}{2}}(w) \risingdotseq \tfrac{1}{8w} - \tfrac{w}{6} - \tfrac{8w^3}{45} - \tfrac{352w^5}{945} + Ew^7 + O(w^8), \quad E \in \mathbb{R}, \tag{A.18}$$

$$\mathfrak{x}_{\frac{1}{2}}(w) \risingdotseq \Xi_{\frac{1}{2}}(w) - \tfrac{256}{9}w^7 + O(w^9), \quad w \to 0, \tag{A.19}$$

$$\varpi \risingdotseq \omega + \tfrac{8}{5}\omega^3 + \tfrac{752}{175}\omega^5 + O(\omega^7), \quad \lambda \to 0^+, \tag{A.20}$$

which gives that $\varpi^6 \risingdotseq \omega^6 + O(\omega^8)$ and that

$$\varpi^2 \risingdotseq \omega^2 + \tfrac{16}{5}\omega^4 + \tfrac{1952}{175}\omega^6 + O(\omega^8), \quad \tfrac{16}{5}\varpi^4 \risingdotseq \tfrac{16}{5}\omega^4 + \tfrac{512}{25}\omega^6 + O(\omega^8) \tag{A.21}$$

as $\lambda \to 0^+$. Altogether, we obtain from (A.18,A.19) and (A.11) that

$$\mathbb{u}_{\frac{1}{2}}(w) = \mathfrak{x}_{\frac{1}{2}}(w)\mathbb{\beta}_{\frac{1}{2}}(w) + \mathfrak{x}_{\frac{1}{2}}(-w)\mathbb{\beta}_{\frac{1}{2}}(-w) \risingdotseq \tfrac{1}{4} - w^2 + \tfrac{16}{5}w^4 - 2\tfrac{1648}{315}w^6 + O(w^8) \tag{A.22}$$

as $w \to 0$, and then we get from the part from (A.20) to (A.21) combined with (A.22) that

$$\mathbb{u}_{\theta}(\varpi) \risingdotseq \tfrac{1}{4} - \omega^2 + (\tfrac{512}{25} - \tfrac{1952}{175} - 2\tfrac{1648}{315})\omega^6 + O(\omega^8) \risingdotseq \tfrac{1}{4} - \omega^2 - \tfrac{256}{225}\omega^6 + O(\omega^8) \tag{A.23}$$

as $\lambda \to 0^+$. In order to get (A.17) from (A.23), it is enough to realize that

$$\omega^6 + O(\omega^8) \risingdotseq \varpi^6 + O(\omega^8) \risingdotseq \varpi^6 + O(\varpi^8) \risingdotseq (\tfrac{3\lambda}{64})^2 + O(\lambda^{8/3})$$

holds in this case, see (A.8). □

A.2. Proof of Lemma A.10

Proof. (a) Let $\rho \neq 0$. Then

$$\lambda = \tfrac{1}{\rho}[\mathfrak{f}(1-\theta, \omega) - \mathfrak{f}(\theta, \omega)] \quad \text{where} \quad \mathfrak{f}(x,y) \stackrel{\text{def}}{=} x \ln \tfrac{x+y}{x-y}$$

is a function increasing in the variable $y, |y| < |x|$. Further,

$$\lambda > \underline{\lambda}(y) \overset{\text{def}}{=\!=} \begin{cases} \frac{1}{\rho}[\mathfrak{f}(1-\theta,y) - \mathfrak{f}(\theta,\omega_0)] & \text{if} \quad \rho > 0, \\ \frac{1}{\rho}[\mathfrak{f}(1-\theta,\omega_0) - \mathfrak{f}(\theta,y)] & \text{if} \quad \rho < 0, \end{cases}$$

and this function is increasing on $(0, \omega_0)$. Note that $\underline{\lambda}(\bar{\omega}_\lambda) = \lambda$. Since $\underline{\lambda}$ is increasing, we obtain that $\omega < \bar{\omega}_\lambda$ since the opposite inequality $\omega \geq \bar{\omega}_\lambda$ leads to the following contradiction $\lambda = \lambda(\omega) > \underline{\lambda}(\omega) \geq \underline{\lambda}(\bar{\omega}_\lambda) = \lambda$.

(b) Let $\rho = 0$, i.e., $\theta = \frac{1}{2}$. This case can be treated similarly. Here, we only show what is different. Put $\underline{\lambda}(y) \overset{\text{def}}{=\!=} 2\frac{y-1/e}{1/2-y}$. Then

$$\underline{\lambda}(\omega) \leq 2\frac{\omega+(1/2-\omega)\ln(1/2-\omega)}{1/2-\omega} = \frac{2\omega}{1/2-\omega} + 2\ln(\tfrac{1}{2}-\omega) < \frac{2\omega}{1/4-\omega^2} + 2\ln\frac{1/2-\omega}{1/2+\omega} = \lambda(\omega).$$

We use that $\frac{1}{2} + y \in (\frac{1}{2}, 1)$ holds for any $y \in (0, \omega_0) = (0, \frac{1}{2})$ in this case. □

B. BALANCING FUNCTION AND DYNAMICS

In this section, the necessary dynamics of important processes is described. In subsection B.2, the properties of the policies are treated, and there is introduced a function which helps us apply the martingale approach to obtain the desired results.

B.1. Basic dynamics

The following lemma provides a decomposition of the wealth process of any admissible strategy into a product of two factors. One of them is a continuous semimartingale (a stochastic exponential of $\int \pi \,\mathrm{d}F$), and the second one is a non-increasing pure jump process.

Lemma B.1. Let $\mathcal{W}, \pi$ be the wealth process and the position of an admissible strategy (φ, ψ). Then $\mathcal{W} \in \overset{\bullet\circ}{\mathbb{C}}\mathbb{S}$ and

$$\mathcal{W} \overset{\text{as}}{=} \mathcal{E}^\pi \cdot [\mathcal{W}_0 - \textstyle\int (\mathcal{E}^\pi)^{-1} \,\mathrm{d}\, C^\varphi] \quad \text{where} \quad \mathcal{E}^\pi \overset{\text{def}}{=\!=} \exp\{\textstyle\int \pi \,\mathrm{d}F - \frac{1}{2}\int \pi^2 \,\mathrm{d}\langle F\rangle\}. \tag{B.1}$$

P r o o f. See Subsection D.2 in Section Proofs. □

Remark B.2. The paper is based on continuous stochastic integration described in Chapter 3 in [24]. The jumps of the processes are treated separately. This is a reason why we introduce the so-called continuous part of a process.

Definition B.3. Let $(X_t)_{t\geq 0}$ be a $\mathbb{R}$-valued rcll-process such that $\sum_{s\in(0,t]} |\Delta X_s| < \infty$ holds for every $t \in (0, \infty)$. By its *continuous part* we mean the process

$$X^{(c)} = (X^{(c)}_t)_{t\geq 0} \quad \text{where} \quad X^{(c)}_t \overset{\text{def}}{=\!=} X_t - \textstyle\sum_{s\in(0,t]} \Delta X_s.$$

Obviously, $X^{(c)}_0 = X_0$ and $X^{(c)}$ is a continuous process as it is rcll, and $\Delta X^{(c)}_t = 0$ holds whenever $t \in (0, \infty)$. Finally, note that if X is an adapted process, so is $X^{(c)}$.

The following lemma says how the jumps of the position process look, and it shows that this process has a continuous part with the dynamics described in Notation B.6.

Lemma B.4. Let (φ, ψ) be an admissible strategy with the wealth process $\mathcal{W}$ and the position π. Then

$$\Delta\pi_t = \tfrac{S_t}{\mathcal{W}_t}[(1+\lambda^{\uparrow}\pi_{t-})(\Delta\varphi_t)^+ - (1-\lambda^{\downarrow}\pi_{t-})(\Delta\varphi_t)^-], \qquad t \in (0,\infty).$$

In particular, $\sum_{s\in(0,t]} |\Delta\pi_s| < \infty$ holds if $t \in [0,\infty)$, i. e., π has a continuous part.

P r o o f. See Subsection D.3 in Section Proofs. □

The following lemma describes the dynamics of the position process of an admissible strategy.

Lemma B.5. Let (φ, ψ) be an admissible strategy with the wealth process $\mathcal{W}$ and the position π. Then

$$\mathbb{CS} \ni \pi^{(c)} \overset{\text{as}}{=} \pi_0 + \int \pi(1-\pi)[\,\mathrm{d}F - \pi\,\mathrm{d}\langle F\rangle] + \int \tfrac{S}{\mathcal{W}}[(1+\lambda^{\uparrow}\pi)\,\mathrm{d}\varphi^{\uparrow(c)} - (1-\lambda^{\downarrow}\pi)\,\mathrm{d}\varphi^{\downarrow(c)}].$$

P r o o f. See Subsection D.4 in Section Proofs. □

Processes $\pi^{\uparrow}_t, \pi^{\downarrow}_t$ introduced in the following notation represent the increment and the decrement of the position process caused by the purchase and by the sale of the stock, respectively. This notation helps express the dynamics of the logarithm of the wealth process in terms of the position with the help of functions $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$, see Lemma B.7.

Notation B.6. Let $\mathcal{W}, \pi$ be the wealth and the position processes of an admissible strategy (φ, ψ). Put

$$\pi^{\uparrow}_t \overset{\text{def}}{=\!=} \textstyle\sum_{s\in(0,t]}(\Delta\pi_s)^+ + \int_0^t \tfrac{S}{\mathcal{W}}\,(1+\lambda^{\uparrow}\pi)\,\mathrm{d}\varphi^{\uparrow(c)}, \tag{B.2}$$

$$\pi^{\downarrow}_t \overset{\text{def}}{=\!=} \textstyle\sum_{s\in(0,t]}(\Delta\pi_s)^- + \int_0^t \tfrac{S}{\mathcal{W}}\,(1-\lambda^{\downarrow}\pi)\,\mathrm{d}\varphi^{\downarrow(c)}. \tag{B.3}$$

Then $\pi^{\uparrow} \overset{\text{def}}{=\!=} (\pi^{\uparrow}_t)_{t\geq 0}, \pi^{\downarrow} \overset{\text{def}}{=\!=} (\pi^{\downarrow}_t)_{t\geq 0}$ are in $\overset{\bullet\!-\!\circ}{\mathbb{C}}\mathbb{I}$, and we have by Lemma B.5 that

$$\overset{\bullet\!-\!\circ}{\mathbb{C}}\mathbb{S} \ni \pi \overset{\text{as}}{=} \pi_0 + \int \pi(1-\pi)[\,\mathrm{d}F - \pi\,\mathrm{d}\langle F\rangle] + \pi^{\uparrow} - \pi^{\downarrow}. \tag{B.4}$$

The following lemma describes the dynamics of the wealth process of an admissible strategy with the help of the continuous part of its logarithm.

Lemma B.7. Let $\mathcal{W}, \pi$ be the wealth and the position processes of an admissible strategy (φ, ψ). Then

$$(\ln\mathcal{W})^{(c)} \overset{\text{as}}{=} \ln\mathcal{W}_0 + \int[\pi\,\mathrm{d}F - \tfrac{1}{2}\pi^2\,\mathrm{d}\langle F\rangle - \zeta^{\uparrow}_{\lambda}(\pi)\,\mathrm{d}\pi^{\uparrow(c)} - \zeta^{\downarrow}_{\lambda}(\pi)\,\mathrm{d}\pi^{\downarrow(c)}]. \tag{B.5}$$

P r o o f. See Subsection D.4 in Section Proofs. □

Remark B.8. If (4.1) holds, then (B.4) obtained in Lemma B.5 reads as follows

$$\pi \overset{\text{as}}{=} \pi_0 + \int \mathbb{B}(\pi_t, \theta_t, \sigma_t)\,\mathrm{d}t + \int \mathbb{S}(\pi, \sigma)\,\mathrm{d}W + \pi^{\uparrow} - \mathrm{d}\pi^{\downarrow} \tag{B.6}$$

where

$$\mathbb{B}(x, \vartheta, s) \overset{\text{def}}{=\!=} s^2 x(1-x)(\vartheta - x), \quad \mathbb{S}(x, s) \overset{\text{def}}{=\!=} sx(1-x). \tag{B.7}$$

These functions $\mathbb{B}, \mathbb{S}$ help describe the drift and the diffusion coefficient of the continuous part of the position process.

Remark B.9. Let Assumption 4.7 be satisfied. By (4.6,B.6), $D = \pi - \Theta \in \mathfrak{C}\mathbb{S}$ and

$$D \overset{\text{as}}{=} D_0 + \int \left[\mathbb{B}(\pi_t, \theta_t, \sigma_t) - \mathfrak{a}^{\Theta}_t\right]\mathrm{d}t + \int \left[\mathbb{S}(\pi, \sigma) - \mathfrak{b}^{\Theta}\right]\mathrm{d}W - \int \tilde{\mathfrak{b}}^{\Theta}\,\mathrm{d}\tilde{W} + \pi^{\uparrow} - \pi^{\downarrow}.$$

In particular, $\langle D \rangle = \int \mathbb{D}(G_t)\,\mathrm{d}t$, see Definition 4.12.

The following lemma describes the joint dynamics of the complementary process and of the position process of an admissible strategy.

Lemma B.10. Let Assumption 4.7 be satisfied. There exist $(k_n)_{n=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ and $K_{B.10} \in (0, \infty)$ such that the following holds. If $G = \binom{\pi}{\mathfrak{C}}$ and if π is the position of a λ-admissible strategy, then there exist $\mathfrak{a}^G \in \mathbb{PM}^{\mathbf{5}}_b$, $\mathfrak{n}^G \in \mathbb{PM}^{\mathbf{5}\times\mathbf{5}}_b$ and $\mathfrak{m}^G \in \mathbb{CM}^{\mathbf{5}}_l$ such that

$$\mathfrak{C}\mathbb{S}^{\mathbf{5}} \ni G \overset{\text{as}}{=} G_0 + \int \mathfrak{a}^G_s\,\mathrm{d}s + \mathfrak{m}^G + 1_{\{0\}}(\pi^{\uparrow} - \pi^{\downarrow}), \tag{B.8}$$

$$\langle\!\langle \mathfrak{m}^G \rangle\!\rangle \overset{\text{as}}{=} \int \mathfrak{n}^G_s\,\mathrm{d}s, \qquad \mathbb{D}(G) = \mathfrak{d}^{\intercal}\mathfrak{n}^G\mathfrak{d}, \tag{B.9}$$

$$1^{\intercal}_{\{0\}}\mathfrak{a}^G = \mathbb{B}(\pi, \theta, \sigma), \quad 1^{\intercal}_{\{0\}}\mathfrak{n}^G 1_{\{0\}} = \mathbb{S}^2(\pi, \sigma), \tag{B.10}$$

where $\pi^{\uparrow}, \pi^{\downarrow}$ are as in Notation B.6, see (4.8), and that $1_{[|\pi| \le n]}\mathfrak{a}^G, 1_{[|\pi| \le n]}\mathfrak{n}^G$ attain values within $[-k_n, k_n]^{\mathbf{5}}$ and $[-k_n, k_n]^{\mathbf{5}\times\mathbf{5}}$, respectively, and that $1^{\intercal}_{\{i\}}\mathfrak{a}^G, 1^{\intercal}_{\{i\}}\mathfrak{n}^G 1_{\{j\}}$ are in the absolute value bounded by k_n if $i, j \in \{1, 2, 3, 4\}$.

P r o o f. See Subsection D.6 in Section Proofs. □

B.2. Balancing function and asymptotics

In this subsection, the properties of the policies are treated, and there is introduced a function which helps us apply the martingale approach to obtain the desired results.

Notation B.11. In this paper, we are interested in small values of $\lambda > 0$. For this reason, we introduce the following notation. If $\delta_\lambda, \rho_\lambda$ are functions of $\lambda \in (0, \epsilon)$ for some $\epsilon \in (0, \infty)$ and of another variable, say $x \in \mathbb{R}^A$, where A is a finite set. We write

$$\delta_\lambda \risingdotseq O_B(\rho_\lambda) \tag{B.11}$$

if there exists a constant $C \in (0, \infty)$ such that $|\delta_\lambda| \le C|\rho_\lambda|$ holds on $B \subseteq \mathbb{R}^A$ for $\lambda > 0$ small enough. If this constant $C \in (0, \infty)$ can be chosen arbitrarily small, we write o_B instead of O_B in (B.11). More precisely, we write $\delta_\lambda \risingdotseq o_B(\rho_\lambda)$ if for each

$C \in (0,\infty)$, there exists $\lambda_0 \in (0,\infty)$ such that $|\delta_\lambda| \le C|\rho_\lambda|$ holds for every $\lambda \in (0,\lambda_0)$. If $A=\emptyset, B=\{\emptyset\}$, i. e., if there is no variable x, we omit the index B in (B.11) in both the previous cases. We use asymmetric notation in (B.11) just in order to be able to write briefly $O_B(\delta_\lambda) \mathrel{\underset{\cdot}{=}} O_B(\rho_\lambda)$ instead of saying that every function of the order $O_B(\delta_\lambda)$ is also of the order $O_B(\rho_\lambda)$. Further, $\delta_\lambda \mathrel{\underset{\cdot}{\leq}} O_B(\rho_\lambda)$ means that there exists $\varrho_\lambda \mathrel{\underset{\cdot}{=}} O_B(\rho_\lambda)$ such that $\delta_\lambda \le \varrho_\lambda$.

Lemma B.12. As $\lambda \to 0^+$, we have that

$$\omega_{\mathbb{\lambda}} \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda^{1/3}), \quad \lambda \mathrel{\underset{\cdot}{=}} o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}^2), \quad \kappa_{\mathbb{\lambda}} \mathrel{\underset{\cdot}{=}} o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}^{-1}). \tag{B.12}$$

P r o o f. See Subsection D.7 in Section Proofs. □

Next, we mention the asymptotics of the policies. We will introduce the desired function $f_{\mathbb{\lambda}}$ in Notation B.15. Its further properties are described in Section C.

Lemma B.13. Let us consider $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ from (5.7) and (5.8). Then

$$\overline{\pi}_{\mathbb{\lambda}} \mathrel{\underset{\cdot}{=}} c_1 + \omega_{\mathbb{\lambda}}(1-\varepsilon\omega_{\mathbb{\lambda}}) + o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}^2),$$
$$\underline{\pi}_{\mathbb{\lambda}} \mathrel{\underset{\cdot}{=}} c_1 - \omega_{\mathbb{\lambda}}(1-\varepsilon\omega_{\mathbb{\lambda}}) + o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}^2).$$

P r o o f. See Subsection D.9. □

Lemma B.14. There exists $\lambda_{B.14} \in (0, \lambda_{5.6})$ such that the following holds whenever $\lambda \in (0, \lambda_{B.14})$. There exists an open convex superset $\mathbb{G}_\lambda \supseteq \mathbb{K}$ such that $\mathbb{G}_\lambda \subseteq \tilde{\mathbb{G}}_\lambda$ and that $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}} \in C^2(\mathbb{G}_\lambda)$. Moreover, the derivatives of $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ w.r.t. c_2, c_3, c_4 are of the order $o_{\mathbb{K}}(1)$ and w.r.t. c_1 of the form $1 + o_{\mathbb{K}}(1)$.

P r o o f. See Subsection D.10. □

In the following notation, we introduce the main technical object of this paper which we call here *a balancing function*. This function is designed to play a similar role as (similarly denoted) function f from Remarks 5.1 and 5.2 related to the case of constant coefficients, where the corresponding problem can be solved almost explicitly without admitting any error.

Notation B.15. If $\lambda \in (0, \lambda_{B.14}), c \in \mathbb{G}_\lambda \supseteq \mathbb{K}$, we put $f_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} h_{\mathbb{\lambda}}(x)$ if $x \in [\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}]$,

$$f_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} g^{\downarrow}_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} h_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}}) + \int_{\overline{\pi}_{\mathbb{\lambda}}}^{x} \zeta^{\downarrow}_{\lambda}(y)\,\mathrm{d}y \quad \text{if} \quad x \in (\overline{\pi}_{\mathbb{\lambda}}, 1/\lambda^{\downarrow}), \tag{B.13}$$
$$f_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} g^{\uparrow}_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} h_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}}) + \int_{x}^{\underline{\pi}_{\mathbb{\lambda}}} \zeta^{\uparrow}_{\lambda}(y)\,\mathrm{d}y \quad \text{if} \quad x \in (-1/\lambda^{\uparrow}, \underline{\pi}_{\mathbb{\lambda}}). \tag{B.14}$$

We also write $f_\lambda(x,c) \overset{\text{def}}{=\!=} f_{\mathbb{\lambda}}(x) = f_{\lambda,c}(x)$. Then $f_\lambda \in C(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$ and

$$f_\lambda \in C^2\{\left(\begin{smallmatrix} x \\ c \end{smallmatrix}\right) \in \mathcal{A}_\lambda \times \mathbb{G}_\lambda;\, \underline{\pi}_{\mathbb{\lambda}} \ne x \ne \overline{\pi}_{\mathbb{\lambda}}\}. \tag{B.15}$$

For C^1-property of f_λ, see Theorem B.16. Further, we write f'_λ instead of $\frac{\partial}{\partial x} f_\lambda$.

B.3. Advanced dynamics

In this part, we assume that $\lambda \in (0, \lambda_{B.14})$.

Theorem B.16. Let f_λ be as in Notation B.15 and $\lambda \in (0, \lambda_{B.14})$. Then $f_\lambda \in C^1(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$. If ($\mathbb{A}$4.7) holds and π is a position process of a λ-admissible strategy, then

$$f_\lambda(G)^{(c)} \stackrel{\text{as}}{=} f_\lambda(G_0) + \int \nabla f_\lambda(G)\,\mathrm{d}G^{(c)} + \tfrac{1}{2}\int \operatorname{tr}\{\tilde{\nabla}^2 f_\lambda(G)\,\mathrm{d}\langle\langle G^{(c)}\rangle\rangle\} \tag{B.16}$$

where $\tilde{\nabla}^2 f_\lambda(x,c) \stackrel{\text{def}}{=} \nabla^2 f_{\lambda,c}(x_+)$. See (4.8) for G. Moreover, $\tilde{\nabla}^2 f_\lambda$ has coordinates that are Borel measurable and locally bounded on $\mathcal{A}_\lambda \times \mathbb{G}_\lambda$.

Proof. See Subsection D.11 in section Proofs. □

The following lemma plays a supporting role for the subsequent theorem. It provides a useful decomposition of the logarithm of the wealth process of a strictly admissible strategy.

Lemma B.17. In the context of Lemma B.10, let $\mathcal{W}$ be the wealth process of a strictly λ-admissible strategy (φ, ψ). Let f_λ be as in Notation B.15 and $\lambda \in (0, \lambda_{5.6})$. Then

$$\overset{\circ}{\mathbb{C}}\mathbb{S} \ni V \stackrel{\text{def}}{=} \ln \mathcal{W} - f_\lambda(G) \stackrel{\text{as}}{=} V_0 + \int \mathfrak{a}^V_s\,\mathrm{d}s + \mathfrak{m}^V + \mathfrak{D}^V + (\textstyle\sum_{s\le t}\Delta V_s)_{t\ge 0} \tag{B.17}$$

where $\Delta V_0 \stackrel{\text{def}}{=} 0$ and

$$\mathfrak{a}^V \stackrel{\text{def}}{=} \sigma^2(\theta\pi - \tfrac{1}{2}\pi^2) - \nabla f_\lambda(G)^{\intercal}\mathfrak{a}^G - \tfrac{1}{2}\operatorname{tr}\{\tilde{\nabla}^2 f_\lambda(G)\,\mathfrak{n}^G\} \in \mathbb{PM}_b, \tag{B.18}$$

$$\mathfrak{m}^V \stackrel{\text{def}}{=} \int \sigma\pi\,\mathrm{d}W - \int \nabla f_\lambda(G)^{\intercal}\,\mathrm{d}\mathfrak{m}^G \in \mathbb{CM}_l, \tag{B.19}$$

$$\mathfrak{D}^V \stackrel{\text{def}}{=} -\int [f'_\lambda(G) + \zeta^\uparrow_\lambda(\pi)]\,\mathrm{d}\pi^{\uparrow(c)} + \int [f'_\lambda(G) - \zeta^\downarrow_\lambda(\pi)]\,\mathrm{d}\pi^{\downarrow(c)} \in \mathbb{CFV}. \tag{B.20}$$

Proof. See Subsection D.16 in section Proofs. □

The following theorem says that the long run average of the process ν is very close to the long run growth rate of the wealth process of our almost optimal strategy. The corresponding error is of the order $O(\lambda^q)$.

Theorem B.18. Let Assumption 4.7 be satisfied, and consider $p = 1$, i. e., $\gamma = 0$. Then there exist $K_{B.18} \in (0,\infty)$ and $\lambda_{B.18} \in (0, \lambda_{B.14})$ such that the following holds if $\lambda \in (0, \lambda_{B.18})$. Let (φ, ψ) be a $[\Theta - \varpi\,(\underline{\pi}, \bar{\pi})\,\Theta + \varpi]$-strategy with the wealth $\mathcal{W}$ and the position π and let $V, \mathfrak{a}^V, \mathfrak{m}^V$ be as in (B.17,B.18,B.19), ν as in (5.11) and $G = \binom{\pi}{\mathfrak{c}}$. Then

$$V = \ln \mathcal{W} - f_\lambda(G) \stackrel{\text{as}}{=} V_0 + \int \mathfrak{a}^V_s\,\mathrm{d}s + \mathfrak{m}^V, \qquad \mathfrak{m}^V \in \mathbb{CM}_l, \tag{B.21}$$

and $|\nu - \mathfrak{a}^V| \le K_{B.18}\lambda^q$, where $q \in \{\frac{6}{7}, 1\}$ comes from Definition 5.3. In particular,

$$\limsup_{t\to\infty} \tfrac{1}{t}\,|\ln \mathcal{W}_t - \textstyle\int_0^t \nu_s\,\mathrm{d}s| \stackrel{\text{as}}{\le} K_{B.18}\,\lambda^q. \tag{B.22}$$

Proof. See Subsection D.18 in section Proofs. □

Further, we show that it is not possible to achieve a better asymptotic result of the logarithm of the wealth process than $\int_0^t \nu_s \, \mathrm{d}s + t \cdot O_{\Omega\times\mathbb{R}^+}(\lambda^q)$.

Theorem B.19. Let ($\mathbb{A}$4.7) hold, and let $p = 1$. Let ν be as in (5.11) and $q \in \{\frac{6}{7}, 1\}$ as in Definition 5.3. There exist $\lambda_{B.19}, K_{B.19} \in (0, \infty)$ such that the following holds. If (φ, ψ) is a strictly λ-admissible strategy with the wealth process $\mathcal{W}$ and if $\lambda \in (0, \lambda_{B.19})$, then

$$\ln \mathcal{W} - f_\lambda(G) - \int [\nu_s + K_{B.19}\lambda^q] \, \mathrm{d}s \in \mathbb{CM}_l \ominus \mathbb{C}\mathbb{I} \tag{B.23}$$

where $A \ominus C \stackrel{\text{def}}{=\!=} \{a - c; a \in A, c \in C\}$ if A, C are subsets of a linear space.

Proof. See Subsection D.20 in section Proofs. □

C. PROPERTIES OF THE BALANCING FUNCTION

Lemma C.1. Let f_λ be as in Notation B.15. (i) Then $f_\lambda \mathrel{\overset{\cdot}{=}} O_{[-n,n]\times\mathbb{K}}(\lambda), n \in \mathbb{N}$.
(ii) If $\lambda \in (0, \lambda_{B.14}), x \in \mathcal{A}_\lambda$ and $c \in \mathbb{K}$, then

$$f'_{\mathbb{\lambda}}(x) = f'_{\lambda,c}(x) \in [-\zeta^{\uparrow}_\lambda(x), \zeta^{\downarrow}_\lambda(x)]. \tag{C.1}$$

In particular, $|f'_{\mathbb{\lambda}}(x)| \le |\zeta^{\uparrow}_\lambda(x)| \vee |\zeta^{\downarrow}_\lambda(x)| \mathrel{\overset{\cdot}{=}} O_{[-n,n]\times\mathbb{K}}(\lambda), n \in \mathbb{N}$. See (5.6).

Proof. See Subsection D.12 in section Proofs. □

Lemma C.2. Let $f_{\mathbb{\lambda}}$ be as in Notation B.15. If $i, j \in \{1, 2, 3, 4\}$, then

$$\begin{aligned} \tfrac{\partial}{\partial c_i} f_\lambda(x, c) &\mathrel{\overset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda), \\ \tfrac{\partial^2}{\partial c_i \partial c_j} f_\lambda(x_+, c) &\mathrel{\overset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda) + 1_{[\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}})}(x) \cdot [O_{\mathbb{R}\times\mathbb{K}}(\lambda\kappa_{\mathbb{\lambda}}^{1/2}) + h''_{\mathbb{\lambda}}(x)\mathfrak{d}\mathfrak{d}^{\intercal}]. \end{aligned} \tag{C.2}$$

If (i, j) or (j, i) is in $\{0\}\times\{1, 2, 3, 4\}$, (C.2) holds with the first term on the right omitted.

Proof. It follows from Lemmas D.7 and D.8. See Subsection D.13. □

In the following notation, we introduce functions, depending on the balancing function, that help approximate the difference $\nu - \mathfrak{a}^V$ in Theorem B.18, for example. The first one $\mathbb{Y}_{\lambda,c}$ consists only of the essential terms, and it is motivated by the task (5.1), cf. Lemma C.4.

Notation C.3. For $x \in \mathcal{A}_\lambda$ and $c \in \mathbb{K}$ put

$$\mathbb{Y}_{\lambda,c}(x) \stackrel{\text{def}}{=\!=} \mathbb{Y}_{\mathbb{\lambda}}(x) \stackrel{\text{def}}{=\!=} f''_{\mathbb{\lambda}}(x_+)\,\mathbb{D}_c(x) + \mathbb{Z}_{\mathbb{\lambda}}(x), \qquad \mathbb{Z}_{\mathbb{\lambda}}(x) \stackrel{\text{def}}{=\!=} pc_4^2[(x - c_1)^2 - \omega_{\mathbb{\lambda}}^2], \tag{C.3}$$

cf. (5.1). See (4.8) for $\mathbb{D}_c(x)$.

Lemma C.4. Put $\aleph \stackrel{\text{def}}{=\!=} 1$ if $a = \infty$ and $\aleph \stackrel{\text{def}}{=\!=} \frac{2}{3}\,(a + 1) \le 1$ if $a \in (0, \frac{1}{2})$. Then

$$1_{[|x-c_1|\le\omega_{\mathbb{\lambda}}]}\mathbb{Y}_{\mathbb{\lambda}}(x) \mathrel{\overset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda^{\aleph}). \tag{C.4}$$

Proof. See Subsection D.14 in Section Proofs. □

In the following notation, we introduce a function $\mathbb{T}_{\lambda}$ analogous to the half of $\mathbb{Y}_{\lambda}$, but this new function also contains some additional negligible terms. These new terms enable us to use the Itô rule and the information on the dynamics of the process G described in Lemma B.10. The definition of the set $\mathbb{V}_{\lambda}$ reflects the properties (B.9,B.10) summing up the essential information on the coefficients $\mathfrak{a}^G, \mathfrak{n}^G$ describing the dynamics of the process G.

Notation C.5. For $x \in \mathcal{A}_\lambda, c \in \mathbb{K}, B \in \mathbb{R}^{\mathbf{5}}, \mathcal{D} \in \mathbb{R}^{\mathbf{5}\times\mathbf{5}}$, we set

$$\mathbb{T}_{\lambda}(x, B, \mathcal{D}) \overset{\text{def}}{=\!=} \tfrac{\text{tr}}{2}\{[\nabla^2 f_{\lambda}(x_+) - (1-p)\nabla f_{\lambda}(x)\nabla f_{\lambda}(x)^{\intercal}]\mathcal{D}\} + \nabla f_{\lambda}(x)^{\intercal} B + \tfrac{1}{2}\,\mathbb{Z}_{\lambda}(x),$$

where $\nabla f_{\lambda} \overset{\text{def}}{=\!=} (\frac{\partial}{\partial c_i} f_{\lambda})_{i\in\mathbf{5}}$ and $\nabla^2 f_{\lambda} \overset{\text{def}}{=\!=} (\frac{\partial^2}{\partial c_i \partial c_j} f_{\lambda})_{i,j\in\mathbf{5}}$, and

$$\begin{aligned}\mathbb{V}_{\lambda} \overset{\text{def}}{=\!=} \{(x, B, \mathcal{D}) \in \mathcal{A}_\lambda \times \mathbb{R}^{\mathbf{5}} \times \mathbb{R}^{\mathbf{5}\times\mathbf{5}}; \mathbb{D}_c(x) = \mathfrak{d}^{\intercal}\mathcal{D}\,\mathfrak{d},\\ \mathcal{D}_{0,0} = \mathbb{S}^2(x, c_4), B_0 = p\,\mathbb{B}(x, c_1, c_4)\}.\end{aligned} \tag{C.5}$$

Remark C.6. If $p = 1$, the term containing $1 - p = 0$ vanishes from $\mathbb{T}_{\lambda}$, and then

$$\mathbb{T}_{\lambda}(x, B, \mathcal{D}) = \tfrac{1}{2}\,\mathbb{Y}_{\lambda}(x) + \tfrac{\text{tr}}{2}\{[\nabla^2 f_{\lambda}(x_+) - f''_{\lambda}(x_+)\mathfrak{d}\mathfrak{d}^{\intercal}]\mathcal{D}\} + \nabla f_{\lambda}(x)^{\intercal} B \quad \text{if} \quad \mathbb{D}_c(x) = \mathfrak{d}^{\intercal}\mathcal{D}\,\mathfrak{d},$$

Note that ∇f_{λ} is negligible according to Lemma C.2.

Lemma C.7. Let $q \in \{\frac{6}{7}, 1\}, a \in \{\frac{2}{7}, \infty\}$ be from Definition 5.3. Whenever $n \in \mathbb{N}$,

$$\begin{aligned} &1_{[|x-c_1|\le\omega_{\lambda}]}(1_{\mathbb{V}_{\lambda}}\mathbb{T}_{\lambda})(x, B, \mathcal{D}) \mathrel{\overset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}\times[-n,n]^{\mathbf{5}}\times[-n,n]^{\mathbf{5}\times\mathbf{5}}}(\lambda^q),\\ &1_{[|x-c_1|>\omega_{\lambda}]}(1_{\mathbb{V}_{\lambda}}\mathbb{T}_{\lambda})(x, B, \mathcal{D}) \mathrel{\dot{\geq}} O_{\mathbb{R}\times\mathbb{K}\times R_n\times R_n^2}(\lambda), \qquad R_n \overset{\text{def}}{=\!=} \mathbb{R}\times[-n, n]^4. \end{aligned} \tag{C.6}$$

Proof. See Subsection D.15. □

Lemma C.8. Let $p = 1$. Then there exist $K_{C.8} \in (0, \infty)$ and $\lambda_{C.8} \in (0, \lambda_{5.6})$ such that whenever (φ, ψ) is a λ-admissible strategy, we have that

$$1_{[|\pi-\Theta|\le\varpi]}|\mathbb{T}_{\lambda}(G, \mathfrak{a}^G, \mathfrak{n}^G)| \le K_{C.8}\lambda^q, \quad \nu - \mathfrak{a}^V = \mathbb{T}_{\lambda}(G, \mathfrak{a}^G, \mathfrak{n}^G) \ge -K_{C.8}\lambda^q \tag{C.7}$$

holds if $\lambda \in (0, \lambda_{C.8})$, where $G, \mathfrak{a}^G, \mathfrak{n}^G$ comes from Lemma B.10 and $\mathfrak{a}^V$ from Lemma B.17.

Proof. See Subsection D.17 in section Proofs. □

D. PROOFS

D.1. Proof of Lemma 3.10

Proof. As φ, ψ, S are rcll-processes, we get from (3.2) that also $\mathcal{W}$ is an rcll-process.

(ii) Let $t \in (0, \infty)$. From (3.7), we have that $-\Delta\mathcal{W}_t = \Delta C^{\varphi}_t = S_t[\lambda^{\uparrow}\Delta\varphi^{\uparrow}_t + \lambda^{\downarrow}\Delta\varphi^{\downarrow}_t]$, and from Remark 3.3, we obtain that $\Delta\varphi^{\uparrow}_t = (\Delta\varphi_t)^+, \Delta\varphi^{\downarrow}_t = (\Delta\varphi_t)^-$. Hence, we have that $\Delta\mathcal{W}^{\uparrow}_t = \Delta\mathcal{W}_t + \lambda^{\uparrow}S_t\Delta\varphi_t \le 0, \Delta\mathcal{W}^{\downarrow}_t = \Delta\mathcal{W}_t - \lambda^{\downarrow}S_t\Delta\varphi_t \le 0$. In particular,

$$\mathcal{W}_{t-} \ge \mathcal{W}^{\uparrow}_{t-} \wedge \mathcal{W}^{\downarrow}_{t-} \ge \mathcal{W}^{\uparrow}_t \wedge \mathcal{W}^{\downarrow}_t > 0, \quad t \in (0, \infty), \tag{D.1}$$

and then from (3.10) we obtain that also $\pi_{t-} \in \mathcal{A}_\lambda$.

(i) By assumption, $\mathcal{W}$ is a positive process, and as it is rcll, we get that also $\mathcal{W}^{-1}$ is a right-continuous process and that it has also the left-hand limits in $(0, \infty]$. From (D.1), we obtain that the limits are finite, i.e., the process $\mathcal{W}^{-1}$ is rcll. From (3.9), we get that $\pi = \varphi S \mathcal{W}^{-1}$ is a product of rcll processes and hence, it is also an rcll process, and similarly, we obtain from (3.10) that $\mathcal{W}^{\uparrow}, \mathcal{W}^{\downarrow}$ have the same property. Finally, note that all processes considered in (i) are obviously adapted as φ, ψ and S are adapted processes, see (3.2,3.8,3.9). □

D.2. Proof of Lemma B.1

Cf. the proof of Lemma 2.18 in [17].

Proof. For the property $\mathcal{W} \in \mathbb{S}$, see Definition 3.5. As $\pi \in \mathbb{A}$ by Lemma 3.10, we have that $\int \pi \, \mathrm{d}F$ is a well defined continuous semimartingale. As $Y \overset{\text{def}}{=} \mathcal{W} + C^{\varphi} \in \mathbb{CS}$ is such that $\mathrm{d}Y = \pi \mathcal{W} \, \mathrm{d}F$ and as $1/\mathcal{E}^{\pi} \in \mathbb{CS}$ satisfies $\mathrm{d}\,(\mathcal{E}^{\pi})^{-1} = (\mathcal{E}^{\pi})^{-1}[-\pi \, \mathrm{d}F + \pi^2 \, \mathrm{d}\langle F \rangle]$, we obtain, with the help of calculus of continuous stochastic integration, that

$$Y/\mathcal{E}^{\pi} \overset{\text{as}}{=} Y_0 + \int C^{\varphi} \, \mathrm{d}\,(\mathcal{E}^{\pi})^{-1}. \tag{D.2}$$

As C^{φ} is a non-decreasing adapted rcll-process and as $1/\mathcal{E}^{\pi}$ is a continuous semimartingale, we obtain, with the help of the integration by parts formula, that

$$C^{\varphi}/\mathcal{E}^{\pi} \overset{\text{as}}{=} C_0^{\varphi} + \int C^{\varphi} \, \mathrm{d}(\mathcal{E}^{\pi})^{-1} + \int (\mathcal{E}^{\pi})^{-1} \, \mathrm{d}C^{\varphi}. \tag{D.3}$$

Then we obtain (B.1) if we subtract (D.3) from (D.2). □

D.3. Proof of Lemma B.4

Proof. Note that $S \in \mathbb{CA}$ and $\varphi \in \mathbb{FV}$ hold by assumption and that $\mathcal{W}^{-1}, \pi \in \mathbb{A}$ by Lemma 3.10. Let $t \in (0, \infty)$. Since $\Delta \mathcal{W}_t^{-1} = -\mathcal{W}_t^{-1} \mathcal{W}_{t-}^{-1} \Delta \mathcal{W}_t$, we get that

$$\Delta(\varphi_t \mathcal{W}_t^{-1}) = \varphi_{t-} \Delta \mathcal{W}_t^{-1} + \mathcal{W}_t^{-1} \Delta \varphi_t = \mathcal{W}_t^{-1}[\Delta \varphi_t - \varphi_{t-} \mathcal{W}_{t-}^{-1} \Delta \mathcal{W}_t].$$

From (3.7), we have that $-\Delta \mathcal{W}_t = \Delta C_t^{\varphi} = S_t[\lambda^{\uparrow} \Delta \varphi_t^{\uparrow} + \lambda^{\downarrow} \Delta \varphi_t^{\downarrow}]$, and from Remark 3.3, we obtain that $\Delta \varphi_t^{\uparrow} = (\Delta \varphi_t)^+, \Delta \varphi_t^{\downarrow} = (\Delta \varphi_t)^-$. Hence, as $S_t \varphi_t \mathcal{W}_t^{-1} = \pi_t$, we get that

$$\begin{aligned} \Delta \pi_t = \Delta(S_t \varphi_t / \mathcal{W}_t) &= S_t \cdot \Delta(\varphi_t \mathcal{W}_t^{-1}) = S_t \mathcal{W}_t^{-1}[\Delta \varphi_t - \varphi_{t-} \mathcal{W}_{t-}^{-1} \Delta \mathcal{W}_t] \\ &= S_t \mathcal{W}_t^{-1}[(1 + \lambda^{\uparrow} \pi_{t-})(\Delta \varphi_t)^+ - (1 - \lambda^{\downarrow} \pi_{t-})(\Delta \varphi_t)^-]. \end{aligned}$$

As $S, \mathcal{W}^{-1}, \pi \in \mathbb{A}$, they have locally bounded trajectories. In particular, there exists a non-decreasing real-valued process $(N_t)_{t \geq 0}$ such that $S_t \mathcal{W}_t^{-1}[1 + (\lambda^{\uparrow} + \lambda^{\downarrow})|\pi_{t-}|] \leq N_t$ holds whenever $t \in (0, \infty)$. Then we get that

$$\textstyle\sum_{s \in (0,t]} |\Delta \pi_s| \leq N_t \sum_{s \in (0,t]} |\Delta \varphi_s| \leq N_t v_t(\varphi) < \infty$$

where $v_t(\varphi)$ is a variation of φ on $[0, t]$, which is finite by assumption. □

D.4. Proof of Lemmas B.5 and B.7

Cf. the proof of Lemma 2.20 in [17].

P r o o f. (of Lemma B.5) By Lemma B.4, π has a continuous part $\pi^{(c)}$, and from the point (i) of Lemma 3.10, we easily get that $\pi^{(c)} \in \mathbb{CA}$. Hence, once we show the desired equality, we will also have that $\pi^{(c)} \in \mathbb{CS}$. By assumption, $S, F \in \mathbb{CS}$ and $S \overset{\text{as}}{=} S_0 e^{F-\langle F\rangle/2}$. Then we get from the Itô Lemma, with $0 < \mathcal{E}^\pi$ from (B.1), that

$$Z \overset{\text{def}}{=} S/\mathcal{E}^\pi \overset{\text{as}}{=} Z_0 + \exp\{\textstyle\int(1-\pi)\,\mathrm{d}F - \frac{1}{2}\int(1-\pi^2)\,\mathrm{d}\langle F\rangle\} \in \mathbb{CS},$$
$$Z \overset{\text{as}}{=} Z_0 + \textstyle\int Z(1-\pi)[\mathrm{d}F - \pi\,\mathrm{d}\langle F\rangle]. \tag{D.4}$$

As the considered strategy (φ,ψ) is admissible, we get from Lemma 3.10 that $\mathcal{W}_{t-} > 0$, $t \in (0,\infty)$, and as $\mathcal{W}$ is a positive rcll process, we have that $\inf\{\mathcal{W}_s; s \le t\} > 0$. As $\mathcal{E}^\pi$ is a positive continuous process, we also have that $\inf\{L_s; s \le t\} > 0$ where

$$L \overset{\text{def}}{=} \mathcal{W}/\mathcal{E}^\pi \overset{\text{as}}{=} \mathcal{W}_0 - \textstyle\int(\mathcal{E}^\pi)^{-1}\,\mathrm{d}C^\varphi \tag{D.5}$$

holds by in Lemma B.1. As the filtration $\mathcal{F}$ is complete by assumption, there exists a set $A \in \mathcal{F}_0$ with $\mathbb{P}(A) = 1$ such that we have equality in (D.5) on A. Then $\tilde{L} \overset{\text{def}}{=} L1_A + 1_{\Omega\setminus A} \in \overset{\bullet}{\mathfrak{C}}\mathbb{FV}$ also satisfies $\inf\{\tilde{L}_s; s \le t\} > 0$, and we get from Corollary E.12 that

$$(\tilde{L}^{-1})^{(c)} \overset{\text{as}}{=} \tilde{L}_0^{-1} + \textstyle\int\tilde{L}^{-2}(\mathcal{E}^\pi)^{-1}\,\mathrm{d}C^{\varphi(c)} \overset{\text{as}}{=} \tilde{L}_0^{-1} + \int\mathcal{W}^{-1}\tilde{L}^{-1}\,\mathrm{d}C^{\varphi(c)},$$
$$(\varphi\tilde{L}^{-1})^{(c)} \overset{\text{as}}{=} \varphi_0\tilde{L}_0^{-1} + \textstyle\int\tilde{L}^{-1}[\mathrm{d}\varphi^{(c)} + \pi(\lambda^\uparrow\mathrm{d}\varphi^{(c)} + \lambda^\downarrow\mathrm{d}\varphi^{(c)})] \tag{D.6}$$
$$= \varphi_0\tilde{L}_0^{-1} + \textstyle\int\tilde{L}^{-1}[(1+\lambda^\uparrow\pi)\,\mathrm{d}\varphi^{(c)} + (1-\lambda^\downarrow\pi)\,\mathrm{d}\varphi^{(c)}], \tag{D.7}$$

see (3.5) and (3.9). Since $ZL^{-1} = S\mathcal{W}^{-1}$, we get that $\pi = \varphi S\mathcal{W}^{-1} = \varphi ZL^{-1} \overset{\text{as}}{=} \varphi Z\tilde{L}^{-1}$, and by integration by parts formula and (D.4), we have that

$$\pi \overset{\text{as}}{=} \pi_0 + \textstyle\int\varphi\tilde{L}^{-1}\,\mathrm{d}Z + \int Z\,\mathrm{d}(\varphi\tilde{L}^{-1}) \tag{D.8}$$
$$\overset{\text{as}}{=} \pi_0 + \textstyle\int\pi(1-\pi)[\mathrm{d}F - \pi\,\mathrm{d}\langle F\rangle] + \int Z\,\mathrm{d}(\varphi\tilde{L}^{-1}). \tag{D.9}$$

Note that the continuous part of $\int Z\,\mathrm{d}(\varphi\tilde{L}^{-1})$ is, according to (D.6,D.7), of the form

$$\textstyle\int Z\,\mathrm{d}(\varphi\tilde{L}^{-1})^{(c)} \overset{\text{as}}{=} \int\frac{S}{\mathcal{W}}[(1+\lambda^\uparrow\pi)\,\mathrm{d}\varphi^{(c)} + (1-\lambda^\downarrow\pi)\,\mathrm{d}\varphi^{(c)}]. \tag{D.10}$$

Then we get the statement of the lemma from (D.8,D.9) and (D.10). □

P r o o f. (of Lemma B.7) From the definition of $C^\varphi, \pi^\uparrow, \pi^\downarrow$ in (3.5,B.2,B.3), we get that

$$C^{\varphi(c)} = \textstyle\int S(\lambda^\uparrow\,\mathrm{d}\varphi^{\uparrow(c)} + \lambda^\downarrow\,\mathrm{d}\varphi^{\downarrow(c)}) = \int\mathcal{W}[\zeta_\lambda^\uparrow(\pi)\,\mathrm{d}\pi^{\uparrow(c)} + \zeta_\lambda^\downarrow(\pi)\,\mathrm{d}\pi^{\downarrow(c)}], \tag{D.11}$$

see (5.6). Consider $\tilde{L}, \mathcal{E}^\pi$ as in the previous proof. From Lemma B.1, we obtain that

$$\ln\mathcal{W} \overset{\text{as}}{=} \textstyle\int(\pi\,\mathrm{d}F - \frac{1}{2}\pi^2\,\mathrm{d}\langle F\rangle) + \ln\tilde{L} \quad\text{and}\quad \tilde{L} \overset{\text{as}}{=} L \overset{\text{as}}{=} \mathcal{W}/\mathcal{E}^\pi. \tag{D.12}$$

Further, as $\tilde{L}^{(c)} \overset{\text{as}}{=} \mathcal{W}_0 - \int(\mathcal{E}^\pi)^{-1}\,\mathrm{d}C^{\varphi(c)}$, we get from Corollary E.12 and (D.11,D.12) that

$$(\ln\tilde{L})^{(c)} - \ln\tilde{L}_0 \overset{\text{as}}{=} \textstyle\int\tilde{L}^{-1}\,\mathrm{d}\tilde{L}^{(c)} \overset{\text{as}}{=} -\int\mathcal{W}^{-1}\,\mathrm{d}C^{\varphi(c)} = -\int[\zeta_\lambda^\uparrow(\pi)\,\mathrm{d}\pi^{\uparrow(c)} + \zeta_\lambda^\downarrow(\pi)\,\mathrm{d}\pi^{\downarrow(c)}],$$

and then (B.5) follows immediately from these equalities and (D.12) as $\tilde{L}_0 \overset{\text{as}}{=} \mathcal{W}_0$. □

D.5. Proof of Lemma 4.6

Proof. As $X \in \mathbb{BI}_b(\mathscr{F})$, there exist $\mathfrak{a}, \mathfrak{c} \in \mathbb{PM}_b(\mathscr{F})$ and $M \in \mathbb{CM}(\mathscr{F})$ such that

$$X = X_0 + \int \mathfrak{a}_s \,\mathrm{d}s + M, \quad \langle X \rangle \overset{\text{as}}{=} \langle M \rangle \overset{\text{as}}{=} \int \mathfrak{c}_s \,\mathrm{d}s. \tag{D.13}$$

As W is a standard $\mathscr{F}$-Brownian motion, we get from (D.13) that $\langle X, W \rangle$ have locally absolutely continuous trajectories with $\mathrm{d}\langle X, W\rangle_t / \mathrm{d}t \leq \sqrt{\mathfrak{c}_t}$ up to a null set. Hence, there exists $\mathfrak{b} \in \mathbb{PM}_b(\mathscr{F})$ such that

$$\langle X, W \rangle \overset{\text{as}}{=} \langle M, W \rangle \overset{\text{as}}{=} \int \mathfrak{b}_s \,\mathrm{d}s. \tag{D.14}$$

As $M_0 = X_0 - X_0 = 0$, we have, according to (2.1), that

$$N \overset{\text{def}}{=} M - \int \mathfrak{b} \,\mathrm{d}W \in \mathbb{CM}_l(\mathscr{F}) \subseteq \mathbb{CM}(\mathscr{F}). \tag{D.15}$$

Further, from (D.13,D.14,D.15) we obtain that

$$\langle N, W \rangle \overset{\text{as}}{=} 0, \quad \langle N \rangle \overset{\text{as}}{=} \int (\mathfrak{c}_s - \mathfrak{b}_s^2) \,\mathrm{d}s,$$

and obviously there exists $\tilde{\mathfrak{b}} \in \mathbb{PM}_b(\mathscr{F})$ such that $\int_0^\infty |\mathfrak{b}_t^2 + \tilde{\mathfrak{b}}_t^2 - \mathfrak{c}_t| \,\mathrm{d}t \overset{\text{as}}{=} 0$. Then

$$\tilde{W} \overset{\text{def}}{=} \int 1_{[\tilde{\mathfrak{b}} \neq 0]} \tilde{\mathfrak{b}}^{-1} \,\mathrm{d}N + \int 1_{[\tilde{\mathfrak{b}} = 0]} \,\mathrm{d}\hat{W} \tag{D.16}$$

is a continuous local $\tilde{\mathscr{F}}$-martingale starting from 0 with $\langle \tilde{W} \rangle_t \overset{\text{as}}{=} t, t \geq 0$, uncovariated with W, and therefore, $\tilde{W}$ is by the Lévy Theorem a standard $\tilde{\mathscr{F}}$-Brownian motion independent with W, see [22, Theorem 16.3]. Further, as $\langle N \rangle \overset{\text{as}}{=} \int \tilde{\mathfrak{b}}_s^2 \,\mathrm{d}s$, we get that the first equality in

$$N \overset{\text{as}}{=} \int 1_{[\tilde{\mathfrak{b}} \neq 0]} \,\mathrm{d}N \overset{\text{as}}{=} \int \tilde{\mathfrak{b}} \,\mathrm{d}\tilde{W}, \tag{D.17}$$

while the latter equality is based on (D.16). Then (4.3) follows from (D.13,D.15,D.17). □

D.6. Proof of Lemma B.10

Proof. By (4.7), $\mathfrak{C}_t \in \mathbb{BI}_b^4$. Hence, there exist $\mathfrak{a}^{\mathfrak{C}} \in \mathbb{PM}_b^4$, $\mathfrak{n}^{\mathfrak{C}} \in \mathbb{PM}_b^{4\times 4}$ and $\mathfrak{m}^{\mathfrak{C}} \in \mathbb{CM}_l^4$ such that

$$\mathfrak{C} = \mathfrak{C}_0 + \int \mathfrak{a}_t^{\mathfrak{C}} \,\mathrm{d}t + \mathfrak{m}^{\mathfrak{C}}, \qquad \langle\!\langle \mathfrak{m}^{\mathfrak{C}} \rangle\!\rangle = \int \mathfrak{n}_t^{\mathfrak{C}} \,\mathrm{d}t. \tag{D.18}$$

Then obviously there exists $K \in [0, \infty)$ such that

$$\mathfrak{a}_t^{\mathfrak{C}} \in [-K, K]^4, \quad \mathfrak{n}_t^{\mathfrak{C}} \in [-K, K]^{4\times 4}, \quad \mathfrak{C}_t \in \mathbb{K}, \qquad t \in [0, \infty). \tag{D.19}$$

Here, $\mathbb{K}$ comes from Definition 5.3. As $\mathbb{B}, \mathbb{S}$ from (B.7) are continuous functions and as $\mathbb{K}$ is a compact set, there exists a sequence $(k_n)_{n=1}^\infty \in \mathbb{N}^{\mathbb{N}}$ such that

$$\forall\, n \in \mathbb{N} \quad \forall\, x \in [-n, n] \quad \forall\, c \in \mathbb{K} \qquad |\mathbb{B}(x, p\, c_1, c_4)| \vee \mathbb{S}^2(x, c_4) \vee K \leq k_n. \tag{D.20}$$

Now, let (φ, ψ) be a λ-admissible strategy with the position process π. By (B.6) in Remark B.8, we have that

$$\pi \overset{\text{as}}{=} \pi_0 + \int \mathfrak{a}^\pi_s \,\mathrm{d}s + \int \mathfrak{b}^\pi \,\mathrm{d}W + \pi^\uparrow - \pi^\downarrow \tag{D.21}$$

where $\mathfrak{a}^\pi \overset{\text{def}}{=} \mathbb{B}(\pi, \theta, \sigma) \in \mathbb{PM}_b$, $\mathfrak{b}^\pi \overset{\text{def}}{=} \mathbb{S}(\pi, \sigma) \in \mathbb{PM}_b$. Here, θ stands for the log-optimal proportion introduced in (4.3). From (D.18,D.21), we have that $G = \binom{\pi}{\mathfrak{C}}$ satisfies (B.8) with

$$\mathfrak{a}^G \overset{\text{def}}{=} \binom{\mathfrak{a}^\pi}{\mathfrak{a}^{\mathfrak{C}}} \in \mathbb{PM}^{\mathbf{5}}_b, \quad \mathfrak{m}^G \overset{\text{def}}{=} \binom{\int \mathfrak{b}^\pi \,\mathrm{d}W}{\mathfrak{m}^{\mathfrak{C}}} \in \mathbb{CM}^{\mathbf{5}}_l. \tag{D.22}$$

From (D.19,D.20,D.22), we have that $1_{[|\pi|\le n]}\mathfrak{a}^G$ attains values in $[-k_n, k_n]^{\mathbf{5}}$, and similarly, we get that $\int 1_{[|\pi|\le n]} \,\mathrm{d}\langle\!\langle \mathfrak{m}^G \rangle\!\rangle$ has k_n-Lipschitz coordinates. Hence, what remains to show is that

$$\langle \mathfrak{d}^\intercal \mathfrak{m}^G \rangle \overset{\text{as}}{=} \int \mathbb{D}(G_t) \,\mathrm{d}t, \qquad \langle 1^\intercal_{\{0\}} \mathfrak{m}^G \rangle \overset{\text{as}}{=} \int \mathbb{S}^2(\pi, \sigma) \,\mathrm{d}t \tag{D.23}$$

since then it is immediate that there exists a process $\mathfrak{n}^G \in \mathbb{PM}^{\mathbf{5}\times\mathbf{5}}_b$ satisfying (B.9,B.10) and such that the coordinates of $1_{[|\pi|\le n]}\mathfrak{n}^G$ have values in $[-k_n, k_n]$. By (D.22),(4.6) and the uniqueness of the decomposition of a continuous semimartingale, we have that

$$(1_{\{0\}}, 1_{\{1\}})^\intercal \mathfrak{m}^G \overset{\text{as}}{=} \int (\mathfrak{b}^\pi, \mathfrak{b}^\Theta)^\intercal \mathrm{d}W + \int (0, \tilde{\mathfrak{b}}^\Theta)^\intercal \mathrm{d}\tilde{W}. \tag{D.24}$$

Then we obtain (D.23) from (D.24) immediately since

$$\mathbb{D}(G_t) = [(\mathfrak{b}^\pi_t - \mathfrak{b}^\Theta_t)^2 + (\tilde{\mathfrak{b}}^\Theta_t)^2], \quad \mathbb{S}(\pi_t, \sigma_t) = \mathfrak{b}^\pi_t, \quad t \in [0, \infty),$$

see (4.8,B.7), and since the processes $W, \tilde{W}$ are assumed to be independent. □

D.7. Proof of Lemma B.12

Remark D.1. Consider the case $a = 2/7$. As $\mathbb{D}$ attains only non-negative values, we obtain from (5.3) that $0 < \lambda = \frac{4}{3}\kappa_{\lambda\!\!\lambda}\,\omega^3_{\lambda\!\!\lambda} \le \frac{4}{3}p\lambda^{-a}\omega^3_{\lambda\!\!\lambda}$, hence $0 < \lambda \le (\frac{4}{3}p\,\omega^3_{\lambda\!\!\lambda})^{\frac{1}{1+a}} \mathrel{=\!\!:} o_{\mathbb{K}}(\omega^2_{\lambda\!\!\lambda})$ as $a = 2/7 \in (0, 1/2)$ holds in this case.

P r o o f. (of Lemma B.12) As $(x, c) \mapsto \mathbb{D}_c(x)$ is a continuous function attaining non-negative values and as $\mathbb{K}$ is a compact subset of $\mathbb{R}^3 \times (0, \infty)$, we get that

$$0 \le L_p \overset{\text{def}}{=} \sup\{c_4^{-2}\mathbb{D}_c(c_1); c \in \mathbb{K}\} < \infty.$$

Let $\lambda \in (0, 1)$. As $a \in (0, \infty]$, we have that $\lambda^a \le 1$. Then, by the definition of $\kappa_{\lambda\!\!\lambda}$ and $\omega_{\lambda\!\!\lambda}$ in (5.3), we get that $\kappa_{\lambda\!\!\lambda} \ge p/(L_p + 1)$ and that

$$0 < \omega_{\lambda\!\!\lambda} = \sqrt[3]{\tfrac{3\lambda}{4\kappa_{\lambda\!\!\lambda}}} \le \sqrt[3]{\tfrac{3\lambda(L_p+1)}{4p}} \mathrel{=\!\!:} O(\lambda^{1/3}),$$

which gives that $\omega_{\lambda\!\!\lambda} \mathrel{=\!\!:} O_{\mathbb{K}}(\lambda^{1/3}) \mathrel{=\!\!:} o_{\mathbb{K}}(1)$, i.e., the first relation in (B.12) is verified. Note that the third relation follows from the second one, which remains to be proven.

If $a = \infty$, then $\kappa_{\lambda\!\!\lambda}$ does not depend on $\lambda \in (0, 1)$, and it is bounded on $\mathbb{K}$. In this case, we get that $\lambda \mathrel{=\!\!:} O_{\mathbb{K}}(\omega^3_{\lambda\!\!\lambda}) \mathrel{=\!\!:} o_{\mathbb{K}}(\omega^2_{\lambda\!\!\lambda})$. For the case $a = 2/7$, see Remark D.1. □

D.8. Proof of Lemma 5.6

Proof. We consider both cases together with a new variable $s \in \{-1, 1\}$. By the definition of $h_{\mathbb{\lambda}}$ in (5.4) and $\omega_{\mathbb{\lambda}}$ in (5.3), we have that

$$h'_{\mathbb{\lambda}}(c_1 + s\,\omega_{\mathbb{\lambda}}) = s\kappa_{\mathbb{\lambda}}(\tfrac{2}{3}\,\omega_{\mathbb{\lambda}}^3 + \varepsilon^2\omega_{\mathbb{\lambda}}^5) = s\,\tfrac{\lambda}{2}\,(1 + \tfrac{3}{2}\,\varepsilon^2\omega_{\mathbb{\lambda}}^2), \quad s \in \{-1, 1\}. \tag{D.25}$$

From (3.4), we get that $\lambda^{\uparrow}, \lambda^{\downarrow}$ are of the form $\frac{\lambda}{2} + O(\lambda^2)$. By Lemma B.12, we have that (B.12) holds. In particular, we have that $\omega_{\mathbb{\lambda}} \underset{\cdot}{=} o_{\mathbb{K}}(1)$, which helps verify that $\zeta^{\downarrow}_{\lambda}(c_1 + \omega_{\mathbb{\lambda}})$ and $\zeta^{\uparrow}_{\lambda}(c_1 - \omega_{\mathbb{\lambda}})$ are also of the form $\frac{\lambda}{2} + O(\lambda^2) \underset{\cdot}{=} \frac{\lambda}{2} + o_{\mathbb{K}}(\lambda\omega_{\mathbb{\lambda}}^2)$, where the last relation is based on $\lambda \underset{\cdot}{=} o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}^2)$ from (B.12). Here, we also used the above-mentioned asymptotics of $\lambda^{\uparrow}, \lambda^{\downarrow}$ and the definitions of $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ in (5.6). Then by (D.25) we have that

$$h'_{\mathbb{\lambda}}(c_1 + \omega_{\mathbb{\lambda}}) - \zeta^{\downarrow}_{\lambda}(c_1 + \omega_{\mathbb{\lambda}}) \underset{\cdot}{=} \tfrac{3}{4}\,\varepsilon^2\lambda\omega_{\mathbb{\lambda}}^2 + o_{\mathbb{K}}(\lambda\omega_{\mathbb{\lambda}}^2), \tag{D.26}$$

$$-h'_{\mathbb{\lambda}}(c_1 - \omega_{\mathbb{\lambda}}) - \zeta^{\downarrow}_{\lambda}(c_1 - \omega_{\mathbb{\lambda}}) \underset{\cdot}{=} \tfrac{3}{4}\,\varepsilon^2\lambda\omega_{\mathbb{\lambda}}^2 + o_{\mathbb{K}}(\lambda\omega_{\mathbb{\lambda}}^2). \tag{D.27}$$

As the right-hand sides of (D.26,D.27) are positive for $\lambda > 0$ small enough, the first part of the statement is proved, and the second one follows immediately from the definition of $\mathcal{A}_{\lambda}$ and from $\omega_{\mathbb{\lambda}} \underset{\cdot}{=} O_{\mathbb{K}}(\lambda^{1/3}) \underset{\cdot}{=} o_{\mathbb{K}}(1)$ as $\lambda \to 0^+$. □

D.9. Proof of Lemma B.13

Proof. Similarly as in the proof of Lemma 5.6, we consider a variable $s \in \{-1, 1\}$ saying which case is considered. Besides this variable, we introduce also the variable $\pi^s_{\mathbb{\lambda}}$ such that

$$(s, \pi^s_{\mathbb{\lambda}}) \in \{(1, \overline{\pi}_{\mathbb{\lambda}}), (-1, \underline{\pi}_{\mathbb{\lambda}})\}$$

and the following variables (without emphasizing their dependence on $s \in \{-1, 1\}$)

$$u_{\mathbb{\lambda}} \overset{\text{def}}{=} \frac{\pi^s_{\mathbb{\lambda}} - c_1}{\omega_{\mathbb{\lambda}}}, \qquad z_{\mathbb{\lambda}} \overset{\text{def}}{=} 1 - su_{\mathbb{\lambda}} = \begin{cases} 1 - (\overline{\pi}_{\mathbb{\lambda}} - c_1)/\omega_{\mathbb{\lambda}} & \text{if} \quad s = 1, \\ 1 + (\underline{\pi}_{\mathbb{\lambda}} - c_1)/\omega_{\mathbb{\lambda}} & \text{if} \quad s = -1. \end{cases} \tag{D.28}$$

In terms of these variables, we just have to show that

$$z_{\mathbb{\lambda}} = \varepsilon\omega_{\mathbb{\lambda}} + o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}) \quad \text{as} \quad \lambda \to 0^+. \tag{D.29}$$

Immediately from (5.7,5.8), we have that $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ are of the form $c_1 + O_{\mathbb{K}}(\omega_{\mathbb{\lambda}}) \underset{\cdot}{=} c_1 + o_{\mathbb{K}}(1)$, see (B.12) in Lemma B.12. Hence, by the definition of $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ in (5.6), we get that

$$h'_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}}) = -\zeta^{\uparrow}_{\lambda}(\underline{\pi}_{\mathbb{\lambda}}) \underset{\cdot}{=} -\tfrac{\lambda}{2} + O_{\mathbb{K}}(\lambda^2) \quad \text{and} \quad h'_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}}) = \zeta^{\downarrow}_{\lambda}(\overline{\pi}_{\mathbb{\lambda}}) \underset{\cdot}{=} \tfrac{\lambda}{2} + O_{\mathbb{K}}(\lambda^2). \tag{D.30}$$

From the definition of $h_{\mathbb{\lambda}}$ in (5.4) and from (D.30), we obtain that

$$\kappa_{\mathbb{\lambda}}\,\omega_{\mathbb{\lambda}}^3(u_{\mathbb{\lambda}} - \tfrac{1}{3}u_{\mathbb{\lambda}}^3 + \varepsilon^2 u_{\mathbb{\lambda}}^5\,\omega_{\mathbb{\lambda}}^2) = h'_{\mathbb{\lambda}}(\pi^s_{\mathbb{\lambda}}) \underset{\cdot}{=} s\,\tfrac{\lambda}{2} + O_{\mathbb{K}}(\lambda^2), \quad s \in \{-1, 1\}.$$

This can be rewritten, with the help of the relation $\frac{\lambda}{2} = \frac{2}{3}\,\kappa_{\mathbb{\lambda}}\,\omega_{\mathbb{\lambda}}^3$ from (5.3), into the form

$$\varepsilon^2 u_{\mathbb{\lambda}}^5\,\omega_{\mathbb{\lambda}}^2 \underset{\cdot}{=} s\tfrac{2}{3} + (\tfrac{1}{3}\,u_{\mathbb{\lambda}}^3 - u_{\mathbb{\lambda}} + O_{\mathbb{K}}(\lambda)) \underset{\cdot}{=} sz_{\mathbb{\lambda}}^2(1 - \tfrac{1}{3}\,z_{\mathbb{\lambda}}) + O_{\mathbb{K}}(\lambda), \quad s \in \{-1, 1\}.$$

Here, we used the relation $u_\lambda = s(1-z_\lambda)$ obtained from (D.28). By (5.7,5.8), we have that $1-z_\lambda = su_\lambda \in (0,1)$ holds for every $\lambda \in (0,\lambda_{5.6})$, and therefore also $z_\lambda \in (0,1)$. Hence, as $u_\lambda \in [-1,1]$ holds for $\lambda \in (0,\lambda_{5.6})$, as $\omega_\lambda \mathrel{=:} o_{\mathbb{K}}(1)$ by (B.12) and as $\lambda \mathrel{=:} o(1)$, we get that

$$z_\lambda \mathrel{=:} \sqrt{\frac{s\,\varepsilon^2 u_\lambda^5 \omega_\lambda^2}{1-z_\lambda/3}} + o_{\mathbb{K}}(1) \mathrel{=:} o_{\mathbb{K}}(1). \tag{D.31}$$

Then $u_\lambda = s(1-z_\lambda) \mathrel{=:} s + o_{\mathbb{K}}(1)$, and we obtain from the left-hand equality of (D.31) together with $z_\lambda \mathrel{=:} o_{\mathbb{K}}(1)$, stated also in (D.31), that (D.29) holds. □

D.10. Proof of Lemma B.14

Lemma D.2. As $\lambda \to 0^+$, we have that $\nabla \ln \kappa_\lambda \mathrel{=:} O_{\mathbb{K}}(\sqrt{\kappa_\lambda}), \nabla^2 \ln \kappa_\lambda \mathrel{=:} O_{\mathbb{K}}(\kappa_\lambda)$.

P r o o f. See (4.8,5.3). It is sufficient to show that the first and the second derivatives of

$$\xi_\lambda \overset{\text{def}}{=\!=} \xi_{\lambda,c} \overset{\text{def}}{=\!=} p/\kappa_\lambda = [c_1(1-c_1) - \tfrac{c_2}{c_4}]^2 + [\tfrac{c_3}{c_4}]^2 + \lambda^a$$

are of the order $O_{\mathbb{K}}(\sqrt{\xi_\lambda})$ and $O_{\mathbb{K}}(1)$, respectively, i. e., that (D.32) holds, as

$$\tfrac{\partial \ln \kappa_\lambda}{\partial c_i} = -\tfrac{\partial \ln \xi_\lambda}{\partial c_i} = -\tfrac{\partial \xi_\lambda}{\partial c_i}\,\tfrac{1}{\xi_\lambda} \quad \text{and} \quad \tfrac{\partial^2 \ln \kappa_\lambda}{\partial c_i \partial c_j} = -\tfrac{\partial^2 \xi_\lambda}{\partial c_i \partial c_j}\,\tfrac{1}{\xi_\lambda} + \tfrac{1}{\xi_\lambda^2}\,\tfrac{\partial \xi_\lambda}{\partial c_i}\,\tfrac{\partial \xi_\lambda}{\partial c_j}.$$

Obviously, $\kappa_\lambda > 0$ is large if and only if ξ_λ is small. We will verify that

$$\tfrac{\partial \xi_\lambda}{\partial c_i} \mathrel{=:} O_{\mathbb{K}}(\sqrt{\xi_\lambda}), \qquad \tfrac{\partial^2 \xi_\lambda}{\partial c_i \partial c_j} \mathrel{=:} O_{\mathbb{K}}(1) \quad \text{if} \quad i,j \in \{1,2,3,4\}. \tag{D.32}$$

First, note that the first and the second derivatives of ξ_λ are continuous functions in $c \in \mathbb{K}$ independent of $\lambda > 0$. As $\mathbb{K}$ is a compact subset of $\mathbb{R}^3 \times (0,\infty)$, we immediately have the second relation in (D.32). To obtain the first relation in (D.32), the same arguments can be used together with the chain rule and with the property $0 \le \frac{\partial \Xi}{\partial u} = 2u \le 2\sqrt{\Xi(u,v)}$ where $\Xi(u,v) \overset{\text{def}}{=\!=} u^2 + v^2 + \lambda^a$. □

In order to be able to find the derivatives of function $h_\lambda(x)$ defined in (5.4) up to the second order, we need the following lemma introducing accompanying functions $H_\lambda(x), \mathcal{H}_\lambda(x)$ defined in (D.38,D.39). Note that $\partial/\partial c_0$ stands for the derivative w.r.t. x.

Lemma D.3. Let $h_\lambda(x)$ be a function defined in (5.4). Then

$$\tfrac{\partial h_\lambda}{\partial c_i}(x) = h_\lambda'(x)\,\mathfrak{d}_i + H_\lambda(x) \cdot \tfrac{\partial}{\partial c_i} \ln \kappa_\lambda, \tag{D.33}$$

$$\tfrac{\partial h_\lambda'}{\partial c_i}(x) = h_\lambda''(x)\,\mathfrak{d}_i + H_\lambda'(x) \cdot \tfrac{\partial}{\partial c_i} \ln \kappa_\lambda, \tag{D.34}$$

$$\tfrac{\partial H_\lambda}{\partial c_i}(x) = H_\lambda'(x)\,\mathfrak{d}_i + \mathcal{H}_\lambda(x) \cdot \tfrac{\partial}{\partial c_i} \ln \kappa_\lambda \tag{D.35}$$

hold for $i \in \mathbf{5} = \{0,1,2,3,4\}$, and

$$h_\lambda'(x) = \kappa_\lambda\,[\omega_\lambda^2(x-c_1) - \frac{1}{3}\,(x-c_1)^3 + \varepsilon^2(x-c_1)^5], \tag{D.36}$$

$$H_\lambda'(x) = \tfrac{\kappa_\lambda}{3}\,[\omega_\lambda^2(x-c_1) - (x-c_1)^3 + 3\varepsilon^2(x-c_1)^5] \tag{D.37}$$

hold where

$$H_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} \tfrac{\kappa_{\mathbb{\lambda}}}{6}\,[\omega_{\mathbb{\lambda}}^2(x-c_1)^2 - \tfrac{1}{2}\,(x-c_1)^4 + \varepsilon^2(x-c_1)^6], \tag{D.38}$$

$$\mathcal{H}_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} \tfrac{\kappa_{\mathbb{\lambda}}}{6}\,[\,\tfrac{1}{3}\,\omega_{\mathbb{\lambda}}^2(x-c_1)^2 - \tfrac{1}{2}\,(x-c_1)^4 + \varepsilon^2(x-c_1)^6]. \tag{D.39}$$

P r o o f. Note that the statement of the lemma is trivial for $i = 0$. Hence, we may assume that $i \in \{1,2,3,4\}$. In this case, $\mathfrak{d}_i = -1_{[i=1]}$.

Formula (D.36) can be obtained immediately from (5.4), similarly (D.37) from (D.38), and (D.34) follows immediately from (D.33). Hence, it is enough to verify (D.33) and (D.35) without using (D.34). By the definition of $\omega_{\mathbb{\lambda}}$ in (5.3), we have that

$$\ln\lambda = \ln\tfrac{4}{3} + \ln\kappa_{\mathbb{\lambda}} + 3\ln\omega_{\mathbb{\lambda}}, \qquad \text{and so} \qquad \tfrac{\partial}{\partial c_i}\ln\omega_{\mathbb{\lambda}} = -\tfrac{1}{3}\cdot\tfrac{\partial}{\partial c_i}\ln\kappa_{\mathbb{\lambda}}. \tag{D.40}$$

From (5.4), we obtain that

$$\tfrac{\partial h_{\mathbb{\lambda}}}{\partial c_i}\,(x) = h'_{\mathbb{\lambda}}(x)\,\mathfrak{d}_i + h_{\mathbb{\lambda}}(x)\cdot\tfrac{\partial}{\partial c_i}\ln\kappa_{\mathbb{\lambda}} + \kappa_{\mathbb{\lambda}}\,\omega_{\mathbb{\lambda}}^2(x-c_1)^2\cdot\tfrac{\partial}{\partial c_i}\ln\omega_{\mathbb{\lambda}}. \tag{D.41}$$

Then (D.33) follows from (5.4,D.38,D.40,D.41). Similarly, we get that

$$\tfrac{\partial H_{\mathbb{\lambda}}}{\partial c_i}\,(x) = H'_{\mathbb{\lambda}}(x)\,\mathfrak{d}_i + H_{\mathbb{\lambda}}(x)\cdot\tfrac{\partial}{\partial c_i}\ln\kappa_{\mathbb{\lambda}} + 2\,\tfrac{\kappa_{\mathbb{\lambda}}}{6}\,\omega_{\mathbb{\lambda}}^2(x-c_1)^2\cdot\tfrac{\partial}{\partial c_i}\ln\omega_{\mathbb{\lambda}}, \tag{D.42}$$

and then we obtain (D.35) from (D.38,D.39,D.40,D.42). □

Remark D.4. By Lemmas B.12 and D.2, $\omega_{\mathbb{\lambda}} \mathrel{\underset{\cdot}{=}} O(\lambda^{1/3})$ and

$$\kappa_{\mathbb{\lambda}} \mathrel{\underset{\cdot}{=}} o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}^{-1}), \quad \tfrac{\partial\ln\kappa_{\mathbb{\lambda}}}{\partial c_i} \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\kappa_{\mathbb{\lambda}}^{1/2}), \quad \tfrac{\partial^2\ln\kappa_{\mathbb{\lambda}}}{\partial c_i\partial c_j} \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\kappa_{\mathbb{\lambda}}), \quad i,j\in\{1,2,3,4\}. \tag{D.43}$$

P r o o f. (of Lemma B.14) We restrict to $\lambda \in (0,\lambda_{5.6})$. By (5.7,5.8) and Lemma 5.6,

$$h'_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}}) = -\zeta_\lambda^{\uparrow}(\underline{\pi}_{\mathbb{\lambda}}), \quad h'_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}}) = \zeta_\lambda^{\downarrow}(\overline{\pi}_{\mathbb{\lambda}}), \qquad c \in \tilde{\mathbb{G}}_\lambda \supseteq \mathbb{K}. \tag{D.44}$$

As in the proof of Lemma B.13, we will further write $\pi_{\mathbb{\lambda}}^s$ instead of $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ together with the accompanying variable $s \in \{-1,1\}$ saying which case is considered, and we use $u_{\mathbb{\lambda}}$ defined by (D.28). First, note that $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ are here defined on $\tilde{\mathbb{G}}_\lambda$ so that

$$F_s(\pi_{\mathbb{\lambda}}^s, c) = 0 \qquad \text{where} \qquad \begin{cases} F_{-1}(x,c) \overset{\text{def}}{=\!=} h'_{\mathbb{\lambda}}(x) + \zeta_\lambda^{\uparrow}(x), \\ \;\;F_1(x,c) \overset{\text{def}}{=\!=} h'_{\mathbb{\lambda}}(x) - \zeta_\lambda^{\downarrow}(x). \end{cases} \tag{D.45}$$

We are going to use the Theorem on implicitly defined functions, and for this reason we are interested in $\partial F_s/\partial x$. By Lemma B.13, we have that

$$u_{\mathbb{\lambda}} = \tfrac{\pi_{\mathbb{\lambda}}^s - c_1}{\omega_{\mathbb{\lambda}}} \mathrel{\underset{\cdot}{=}} s(1-\varepsilon\omega_{\mathbb{\lambda}}) + o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}), \qquad \text{hence} \qquad 1-u_{\mathbb{\lambda}}^2 \mathrel{\underset{\cdot}{=}} 2\varepsilon\omega_{\mathbb{\lambda}} + o_{\mathbb{K}}(\omega_{\mathbb{\lambda}}) \tag{D.46}$$

as $\lambda \to 0^+$, and therefore, since $\lambda = \tfrac{4}{3}\kappa_{\mathbb{\lambda}}\omega_{\mathbb{\lambda}}^3$ holds by (5.3), we get that

$$h''_{\mathbb{\lambda}}(\pi_{\mathbb{\lambda}}^s) = \kappa_{\mathbb{\lambda}}\,\omega_{\mathbb{\lambda}}^2[1-u_{\mathbb{\lambda}}^2 + 5\varepsilon^2u_{\mathbb{\lambda}}^4\omega_{\mathbb{\lambda}}^2] \mathrel{\underset{\cdot}{=}} 2\varepsilon\kappa_{\mathbb{\lambda}}\,\omega_{\mathbb{\lambda}}^3 + o_{\mathbb{K}}(\lambda) \mathrel{\underset{\cdot}{=}} \tfrac{3}{2}\,\varepsilon\lambda + o_{\mathbb{K}}(\lambda), \tag{D.47}$$

$$H'_{\mathbb{\lambda}}(\pi_{\mathbb{\lambda}}^s) = \tfrac{\lambda}{4}\,[u_{\mathbb{\lambda}}(1-u_{\mathbb{\lambda}}^2) + 3\varepsilon^2u_{\mathbb{\lambda}}^5\omega_{\mathbb{\lambda}}^2] \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda\omega_{\mathbb{\lambda}}), \tag{D.48}$$

see (D.37,D.38). From (D.46), we have that $\varpi^s_{\lambda} \mathrel{\dot{=}} O_{\mathbb{K}}(1)$. Then we get from the definition of $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ in (5.6) that the values from (D.44) are of the order $O_{\mathbb{K}}(\lambda)$. Hence, as $\frac{\mathrm{d}}{\mathrm{d}x}\,\zeta^{\downarrow}_{\lambda}(x) = \zeta^{\downarrow}_{\lambda}(x)^2$ and $\frac{\mathrm{d}}{\mathrm{d}x}\,\zeta^{\uparrow}_{\lambda}(x) = -\zeta^{\uparrow}_{\lambda}(x)^2$, we obtain with the help of (D.44,D.45,D.47) that

$$\tfrac{\partial F_s}{\partial x}\,(\varpi^s_{\lambda}, c) \mathrel{\dot{=}} h''_{\lambda}(\varpi^s_{\lambda}) - h'_{\lambda}(\varpi^s_{\lambda})^2 \mathrel{\dot{=}} \tfrac{3}{2}\,\varepsilon\lambda + o_{\mathbb{K}}(\lambda) > 0 \tag{D.49}$$

holds for $\lambda > 0$ small enough. Let $\lambda_{B.14} \in (0, \lambda_{5.6})$ be such that the derivatives on the left-hand side of (D.49) are positive for $s \in \{-1, 1\}$ if $\lambda \in (0, \lambda_{B.14})$ and if $c \in \mathbb{K}$.

Let $\lambda \in (0, \lambda_{B.14})$ be fixed for a moment. By the Theorem on explicitly defined functions, $\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda}$ are C^2-functions defined locally uniquely by (D.45) on an open set $\mathbb{G}_{\lambda}$ such that $\mathbb{K} \subseteq \mathbb{G}_{\lambda} \subseteq \tilde{\mathbb{G}}_{\lambda}$, and we are allowed to use the chain rule in order to get the corresponding derivatives. Since $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ do not depend on c, the focus is on $\partial h'_{\lambda}/\partial c_i(\varpi^s_{\lambda})$. From Lemma D.3, namely from (D.34), we obtain that

$$\tfrac{\partial h'_{\lambda}}{\partial c_i}\,(\varpi^s_{\lambda}) = -h''_{\lambda}(\varpi^s_{\lambda})1_{[i=1]} + H'_{\lambda}(\varpi^s_{\lambda}) \cdot \tfrac{\partial}{\partial c_i}\ln\kappa_{\lambda}\,, \quad i \in \{1,2,3,4\}, \quad s \in \{-1,1\}. \tag{D.50}$$

From Remark D.4, we have that $\frac{\partial \ln\kappa_{\lambda}}{\partial c_i} \mathrel{\dot{=}} o_{\mathbb{K}}(\omega_{\lambda}^{-1/2}) \mathrel{\dot{=}} o_{\mathbb{K}}(\omega_{\lambda}^{-1})$, and then we obtain from (D.45,D.47,D.48,D.50) that

$$\tfrac{\partial F_s}{\partial c_i}(\varpi^s_{\lambda}, c) = \tfrac{\partial h'_{\lambda}}{\partial c_i}\,(\varpi^s_{\lambda}) \mathrel{\dot{=}} -\tfrac{3}{2}\,\varepsilon\lambda 1_{[i=1]} + o_{\mathbb{K}}(\lambda). \tag{D.51}$$

From the chain rule and (D.49,D.51), we have that

$$\frac{\partial \varpi^s_{\lambda}}{\partial c_i} = -\frac{\frac{\partial}{\partial c_i}F(\varpi^s_{\lambda}, c)}{\frac{\partial F_s}{\partial x}(\varpi^s_{\lambda}, c)} \mathrel{\dot{=}} \frac{\frac{3}{2}\,\varepsilon\lambda 1_{[i=1]} + o_{\mathbb{K}}(\lambda)}{\frac{3}{2}\,\varepsilon\lambda + o_{\mathbb{K}}(\lambda)} \mathrel{\dot{=}} 1_{[i=1]} + o_{\mathbb{K}}(1)$$

whenever $i \in \{1,2,3,4\}$. Obviously, the set $\mathbb{G}_{\lambda}$ can be chosen to be convex, for example, of the form $\{c \in \mathbb{R}^4; \operatorname{dist}(\mathbb{K}, c) < \delta\}$ for some small $\delta > 0$. To see that this choice can ensure that $\mathbb{G}_{\lambda} \subseteq \tilde{\mathbb{G}}_{\lambda}$, it is important to realize that the set $\tilde{\mathbb{G}}_{\lambda}$ is open and that $\mathbb{K}$ is its convex compact subset, which means that $\operatorname{dist}(\mathbb{K}, \mathbb{R}^4 \backslash \tilde{\mathbb{G}}_{\lambda}) > 0$. □

Remark D.5. The notation ϖ^s_{λ} introduced in the previous proof will be used also in the proof of Lemma D.7, and besides this notation, we will also use some results from the proof, namely that $H'_{\lambda}(\underline{\varpi}_{\lambda}), H'_{\lambda}(\overline{\varpi}_{\lambda})$ are of the order $O_{\mathbb{K}}(\lambda\omega_{\lambda})$, see (D.48).

Assumption D.6. From here, many times we need to restrict ourselves to $\lambda \in (0, \lambda_{B.14})$. Instead, we briefly write that ($\mathbb{A}$*D.6*) holds or that we assume ($\mathbb{A}$*D.6*).

D.11. Proof of Theorem B.16

Here, we assume ($\mathbb{A}$*D.6*).

P r o o f. By Lemma B.10, (B.8) holds. From Lemma B.14, we get that there is $\lambda_{B.14} \in (0, \lambda_{5.6})$ such that whenever $\lambda \in (0, \lambda_{B.14})$, there is an open convex set $\mathbb{G}_{\lambda} \supseteq \mathbb{K}$ such that $\overline{\varpi}, \underline{\varpi}$ defined by (5.7,5.8) are of type $C^2(\mathbb{G}_{\lambda})$. Then also the functions $g^{\downarrow}_{\lambda}, g^{\uparrow}_{\lambda}$ from (B.13,B.14) are of $C^2(\mathcal{A}_{\lambda} \times \mathbb{G}_{\lambda})$ if we ignore the corresponding restrictions in (B.13,B.14) that are primarily related to the definition of f_λ. Then the functions $f^{\uparrow}_{\mathbb{\lambda}} \stackrel{\text{def}}{=} g^{\uparrow}_{\mathbb{\lambda}} - h_{\mathbb{\lambda}}$, $f^{\downarrow}_{\mathbb{\lambda}} \stackrel{\text{def}}{=} g^{\downarrow}_{\mathbb{\lambda}} - h_{\mathbb{\lambda}}$ are also of type $C^2(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$. Note that

$$f^{\uparrow}_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}}) = 0 = f^{\downarrow}_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}}), \qquad \tfrac{\partial}{\partial x} f^{\uparrow}_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}}) = 0 = \tfrac{\partial}{\partial x} f^{\downarrow}_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}}), \quad c \in \mathbb{G}_\lambda,$$

where the equalities on the left follow from (B.13,B.14) and the ones on the right from (5.7,5.8). Then, by Lemma E.13 with $m = 5, u_0 = c \in \mathbb{G}_\lambda$ and $v : u \mapsto \underline{\pi}_{\lambda,u}$ and $u \mapsto \overline{\pi}_{\lambda,u}$, respectively, each point $(\underline{\pi}_{\mathbb{\lambda}}, c^{\mathsf{T}})^{\mathsf{T}}, (\overline{\pi}_{\mathbb{\lambda}}, c^{\mathsf{T}})^{\mathsf{T}}, c \in \mathbb{G}_\lambda$, has an open neighbourhood $\mathcal{O}$ and a sequence of functions $(f_n)_{n=1}^{\infty} \subseteq C^2(\mathcal{O})$ such that (E.27,E.28) hold with f replaced by $f^{\uparrow}_{\mathbb{\lambda}}$ or $f^{\downarrow}_{\mathbb{\lambda}}$ according to which of the two cases is considered. Then we can apply Lemma E.11 (iii) to function $f_{\mathbb{\lambda}}$ to obtain that (B.16) holds and that the coordinates of $\tilde{\nabla}^2 f_{\mathbb{\lambda}}$ are Borel measurable and locally bounded on $\mathcal{A}_\lambda \times \mathbb{G}_\lambda$. It is easily seen that once f_λ can play the role of g_∞ from Lemma E.11 (iii), we have that $f_\lambda \in C^1(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$. □

D.12. Proof of Lemma C.1

Here, we assume $(\mathbb{A}D.6)$.

Proof. First, we get from Notation B.15, (5.3,5.4),(5.7,5.8) and Lemma B.12 that

$$\mathfrak{m} \stackrel{\text{def}}{=} \sup\{|f_{\mathbb{\lambda}}(x)|; \underline{\pi}_{\mathbb{\lambda}} \le x \le \overline{\pi}_{\mathbb{\lambda}}\} \le \sup_{|x-c_1| \le \omega_{\mathbb{\lambda}}} |h_{\mathbb{\lambda}}(x)| \underset{\cdot}{=} O_{\mathbb{K}}(\lambda^{4/3}).$$

Further, as $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ are of $O_{\mathbb{K}}(1)$ by Lemma B.13 and as $\lambda^{\uparrow}, \lambda^{\downarrow}$ are of $O(\lambda)$, we obtain that

$$1_{[x \le \underline{\pi}_{\mathbb{\lambda}}]}[f_{\mathbb{\lambda}}(x) - f_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}})] = 1_{[x \le \underline{\pi}_{\mathbb{\lambda}}]} \ln \frac{1 + \lambda^{\uparrow} \underline{\pi}_{\mathbb{\lambda}}}{1 + \lambda^{\uparrow} x} \underset{\cdot}{=} O_{[-n,n]\times\mathbb{K}}(\lambda^{\uparrow}) \underset{\cdot}{=} O_{[-n,n]\times\mathbb{K}}(\lambda)$$
$$1_{[x \ge \overline{\pi}_{\mathbb{\lambda}}]}[f_{\mathbb{\lambda}}(x) - f_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}})] = 1_{[x \ge \overline{\pi}_{\mathbb{\lambda}}]} \ln \frac{1 + \lambda^{\downarrow} x}{1 + \lambda^{\downarrow} \overline{\pi}_{\mathbb{\lambda}}} \underset{\cdot}{=} O_{[-n,n]\times\mathbb{K}}(\lambda^{\downarrow}) \underset{\cdot}{=} O_{[-n,n]\times\mathbb{K}}(\lambda),$$

and therefore, $f_{\mathbb{\lambda}}(x) \underset{\cdot}{=} O_{[-n,n]\times\mathbb{K}}(\lambda), n \in \mathbb{N}$. Here, we used that

$$|f_{\mathbb{\lambda}}(x)| \le \mathfrak{m} + 1_{[x \le \underline{\pi}_{\mathbb{\lambda}}]}[f_{\mathbb{\lambda}}(x) - f_{\mathbb{\lambda}}(\underline{\pi}_{\mathbb{\lambda}})] + 1_{[x \ge \overline{\pi}_{\mathbb{\lambda}}]}[f_{\mathbb{\lambda}}(x) - f_{\mathbb{\lambda}}(\overline{\pi}_{\mathbb{\lambda}})] \underset{\cdot}{=} O_{[-n,n]\times\mathbb{K}}(\lambda).$$

Further, note that $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ attain only positive values on $\mathcal{A}_\lambda$, and therefore, $-\zeta^{\uparrow}_{\lambda}(x) < \zeta^{\downarrow}_{\lambda}(x)$ holds for every $x \in \mathcal{A}_\lambda$. If $x \in (-1/\lambda^{\uparrow}, \underline{\pi}_{\mathbb{\lambda}}]$, then $f'_{\mathbb{\lambda}}(x) = -\zeta^{\uparrow}_{\lambda}(x)$, and similarly, $f'_{\mathbb{\lambda}}(x) = \zeta^{\downarrow}_{\lambda}(x)$ holds if $x \in [\overline{\pi}_{\mathbb{\lambda}}, 1/\lambda^{\downarrow})$. Hence, in both considered cases, we have that (C.1) holds immediately. Let $\lambda \in (0, \lambda_{B.14})$. Note that $f_\lambda \in C^1(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$ holds by Theorem B.16. Let $x \in (\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}})$, then

$$f'_{\mathbb{\lambda}}(x) = h'_{\mathbb{\lambda}}(x) \in (-\zeta^{\uparrow}_{\lambda}(x), \zeta^{\downarrow}_{\lambda}(x))$$

holds by the definition of $\underline{\pi}_{\mathbb{\lambda}}, \overline{\pi}_{\mathbb{\lambda}}$ in Notation 5.7. The last part of the statement of the lemma follows immediately from the definition of the functions $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ in (5.6). □

D.13. Proof of Lemma C.2

It follows from Lemmas D.7 and D.8. Assume $(\mathbb{A}D.6)$.

Lemma D.7. Let f_{λ} be as in Notation B.15. If $i, j \in \{1,2,3,4\}$, then

$$1_{\mathcal{A}_\lambda \setminus [\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda})}(x) \tfrac{\partial}{\partial c_i} f_\lambda(x,c) \doteqdot O_{\mathbb{R}\times\mathbb{K}}(\lambda), \tag{D.52}$$

$$1_{\mathcal{A}_\lambda \setminus [\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda})}(x) \tfrac{\partial^2}{\partial c_i \partial c_j} f_\lambda(x_+,c) \doteqdot O_{\mathbb{R}\times\mathbb{K}}(\lambda). \tag{D.53}$$

The expression in (D.53) on the left is equal to zero if (i,j) or (j,i) is in $\{0\}\times\{1,2,3,4\}$.

Proof. As $\mathcal{A}_\lambda \setminus [\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda}] \ni x \mapsto f'_{\lambda}(x)$ does not depend on c, we get that $\frac{\partial}{\partial c_i} f'_\lambda(x,c) = 0$ holds for any $x \in \mathcal{A}_\lambda \setminus [\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda}]$ and $i \in \{1,2,3,4\}$. This gives that the expression in (D.53) on the left is zero if just one of the values i,j is zero. Further, we will consider only $i,j \in \{1,2,3,4\}$. As in the proofs of Lemmas B.13 and B.14, we write π^s_{λ} instead of $\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda}$ and we use the accompanying variable $s \in \{-1,1\}$ saying which case is considered. Further, we write $\zeta^{\langle s\rangle}_\lambda$ for function $\zeta^{\uparrow}_\lambda$ if $s=-1$ and for function $\zeta^{\downarrow}_\lambda$ if $s=1$. By Lemma B.14, we have that $\partial \pi^s_{\lambda}/\partial c_i$ exists and it is finite for $\lambda>0$ small enough. Then we get, with the help of the equalities in (5.7,5.8) on the right, that

$$\tfrac{\partial f_{\lambda}}{\partial c_i}(x) = \tfrac{\partial}{\partial c_i}\left[h_{\lambda}(\pi^s_{\lambda}) + s\textstyle\int_{\pi^s_{\lambda}}^x \zeta^{\langle s\rangle}_\lambda(y)\,\mathrm{d}y\right] \tag{D.54}$$

$$= \tfrac{\partial h_{\lambda}}{\partial c_i}(\pi^s_{\lambda}) + \tfrac{\partial \pi^s_{\lambda}}{\partial c_i}\left[h'_{\lambda}(\pi^s_{\lambda}) - s\zeta^{\langle s\rangle}_\lambda(\pi^s_{\lambda})\right] = \tfrac{\partial h_{\lambda}}{\partial c_i}(\pi^s_{\lambda}) \tag{D.55}$$

holds if $\lambda>0$ is small enough for

$$\begin{aligned} &x \in (-1/\lambda^{\uparrow}, \underline{\pi}_{\lambda}) \quad &&\text{if} \quad s=-1,\\ &x \in (\overline{\pi}_{\lambda}, 1/\lambda^{\downarrow}) \quad &&\text{if} \quad s=1. \end{aligned} \tag{D.56}$$

Note that $f_{\lambda} \in C^1(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$ by Theorem B.16. As $f_{\lambda} = h_{\lambda}$ holds on $(\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda})$, we get that $\frac{\partial}{\partial c_i} f_{\lambda}(\pi^s_{\lambda}) = \frac{\partial}{\partial c_i} h_{\lambda}(\pi^s_{\lambda})$. Hence, thanks to (D.54,D.55), in order to show (D.52), it is sufficient to verify that $\frac{\partial}{\partial c_i} h_{\lambda}(\pi^s_{\lambda}) \doteqdot O_{\mathbb{K}}(\lambda)$. By (D.33) in Lemma D.3 and (5.7,5.8), we have that

$$\tfrac{\partial h_{\lambda}}{\partial c_i}(\pi^s_{\lambda}) = -s\,\zeta^{\langle s\rangle}_\lambda(\pi^s_{\lambda})\,1_{[i=1]} + H_{\lambda}(\pi^s_{\lambda}) \cdot \tfrac{\partial}{\partial c_i}\ln\kappa_{\lambda}. \tag{D.57}$$

Note that Lemma B.13 gives that

$$\pi^s_{\lambda} \doteqdot c_1 + s\omega_{\lambda} + o_{\mathbb{K}}(\omega_{\lambda}). \tag{D.58}$$

From (D.58,D.38), we have that $H_{\lambda}(\pi^s_{\lambda}) \doteqdot O_{\mathbb{K}}(\lambda\omega_{\lambda})$, and from the definition of $\zeta^{\langle s\rangle}_\lambda$ combined with (D.58), we obtain that $\zeta^{\langle s\rangle}_\lambda(\pi^s_{\lambda}) \doteqdot O_{\mathbb{K}}(\lambda)$. Then we get from (D.43,D.57) that $\frac{\partial}{\partial c_i} h_{\lambda}(\pi^s_{\lambda}) \doteqdot O_{\mathbb{K}}(\lambda)$, and we obtain (D.52) from this and from (D.54,D.55). If x,s are as in (D.56), we get from (D.54,D.55) and (D.57) with the help of Lemma D.3, namely with the help of (D.35), that for $\lambda>0$ small enough

$$\tfrac{\partial^2 f_{\lambda}}{\partial c_i \partial c_j}(x) = \tfrac{\partial}{\partial c_j}\tfrac{\partial[h_{\lambda}}{\partial c_i}(\pi^s_{\lambda})] = H_{\lambda}(\pi^s_{\lambda})\,\tfrac{\partial^2 \ln\kappa_{\lambda}}{\partial c_i \partial c_j} + \mathcal{H}_{\lambda}(\pi^s_{\lambda})\,\tfrac{\partial \ln\kappa_{\lambda}}{\partial c_i}\,\tfrac{\partial \ln\kappa_{\lambda}}{\partial c_j} \tag{D.59}$$

$$- \tfrac{\partial\pi}{\partial c_j}\,\zeta^{\langle s\rangle}_\lambda(\pi^s_{\lambda})^2 1_{[i=1]} + \left(\tfrac{\partial \pi^s_{\lambda}}{\partial c_j} - 1_{[j=1]}\right) H'_{\lambda}(\pi^s_{\lambda})\,\tfrac{\partial \ln\kappa_{\lambda}}{\partial c_i}. \tag{D.60}$$

From (D.58) and (D.38,D.39), we get that $H_{\lambda}(\pi^s_{\lambda}), \mathcal{H}_{\lambda}(\pi^s_{\lambda})$ are of the order $O_{\mathbb{K}}(\lambda\omega_{\lambda})$. From Remarks D.4 and D.5, we obtain that $H'_{\lambda}(\pi^s_{\lambda})\frac{\partial \ln\kappa_{\lambda}}{\partial c_i} \doteqdot o_{\mathbb{K}}(\lambda)$. Further, as

$$\zeta^{\langle s\rangle}_\lambda(\pi^s_{\lambda})^2 \doteqdot O_{\mathbb{K}}(\lambda^2) \doteqdot O_{\mathbb{K}}(\lambda),$$

we obtain from the "moreover part" of Lemma B.14 and from (D.43) that the right-hand side of (D.59,D.60) is of $O_{\mathbb{K}}(\lambda)$. To obtain (D.53), it is enough to realize that the expression in (D.59,D.60) on the right does not depend on x if $s \in \{-1, 1\}$ is fixed. $\square$

Lemma D.8. Let $f_{\mathbb{\lambda}}$ be as in Notation B.15. We have the following asymptotic relations

$$1_{[\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}}]}(x)\nabla f_\lambda(x,c) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda), \tag{D.61}$$

$$1_{[\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}})}(x)[\nabla^2 f_\lambda(x_+,c) - h''_{\mathbb{\lambda}}(x)\,\mathfrak{d}\mathfrak{d}^{\intercal}] \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda\kappa_{\mathbb{\lambda}}^{1/2}). \tag{D.62}$$

P r o o f. Let $\lambda \in (0, \lambda_{B.14})$. Then $f_\lambda \in C^1(\mathcal{A}_\lambda \times \mathbb{G}_\lambda)$ by Theorem B.16. As $-\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$ are increasing functions on $\mathcal{A}_\lambda$, we get from Lemmas 5.6,C.1 that $[\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}}] \subseteq [-n,n]$ holds for some $n \in \mathbb{N}$ and then that for $x \in [\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}}]$

$$O_{\mathbb{K}}(\lambda) \mathrel{\underset{\cdot}{=}} -\zeta^{\uparrow}_{\lambda}(-n) \le -\zeta^{\uparrow}_{\lambda}(x) \le f'_{\mathbb{\lambda}}(x) \le \zeta^{\downarrow}_{\lambda}(x) \le \zeta^{\downarrow}_{\lambda}(n) \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda),$$

which verifies (D.61) in the 0th coordinate. Note that $f_{\mathbb{\lambda}} = h_{\mathbb{\lambda}}$ holds on $[\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}}]$. Then by (D.33) in Lemma D.3, we will have (D.61) once we verify that

$$1_{[\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}}]}(x)H_{\mathbb{\lambda}}(x)\,\tfrac{\partial \ln \kappa_{\mathbb{\lambda}}}{\partial c_i} \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda), \qquad i \in \{1,2,3,4\}. \tag{D.63}$$

From (D.37,D.38,D.39) in Lemma D.3, we get that $1_{[|x-c_1|\le\omega_{\mathbb{\lambda}}]}H'_{\mathbb{\lambda}}(x) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda)$,

$$1_{[|x-c_1|\le\omega_{\mathbb{\lambda}}]}H_{\mathbb{\lambda}}(x) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda\omega_{\mathbb{\lambda}}), \qquad 1_{[|x-c_1|\le\omega_{\mathbb{\lambda}}]}\mathcal{H}_{\mathbb{\lambda}}(x) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda\omega_{\mathbb{\lambda}}). \tag{D.64}$$

From the property in (D.64) on the left together with (D.43) in Remark D.4 and (5.7,5.8), we have that (D.63) holds. As $f_{\mathbb{\lambda}} = h_{\mathbb{\lambda}}$ holds on $(\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}})$, we get by (D.33,D.35) in Lemma D.3 that on $(\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}})$

$$\tfrac{\partial^2 f_{\mathbb{\lambda}}}{\partial c_i \partial c_j} = H_{\mathbb{\lambda}}\tfrac{\partial^2 \ln \kappa_{\mathbb{\lambda}}}{\partial c_i \partial c_j} + \mathcal{H}_{\mathbb{\lambda}}\tfrac{\partial \ln \kappa_{\mathbb{\lambda}}}{\partial c_i}\tfrac{\partial \ln \kappa_{\mathbb{\lambda}}}{\partial c_j} + h''_{\mathbb{\lambda}}\,\mathfrak{d}_i\mathfrak{d}_j + H'_{\mathbb{\lambda}}[\mathfrak{d}_j\tfrac{\partial \ln \kappa_{\mathbb{\lambda}}}{\partial c_i} + \mathfrak{d}_i\tfrac{\partial \ln \kappa_{\mathbb{\lambda}}}{\partial c_j}]. \tag{D.65}$$

As $\kappa_{\mathbb{\lambda}}$ does not depend on x, which is represented by the variable c_0 here, we have that $\tfrac{\partial}{\partial c_0}\ln\kappa_{\mathbb{\lambda}} = 0 = \tfrac{\partial^2}{\partial c_0 \partial c_i}\ln\kappa_{\mathbb{\lambda}}$ holds if $i \in \mathbf{5}$. From (D.43,D.64,D.65) and from the relation above (D.64), we have that

$$1_{(\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}})}(x)[\tfrac{\partial^2 f_{\mathbb{\lambda}}(x)}{\partial c_i \partial c_j} - h''_{\mathbb{\lambda}}(x)\,\mathfrak{d}_i\mathfrak{d}_j] \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda\kappa_{\mathbb{\lambda}}^{1/2}), \quad i,j \in \mathbf{5}. \tag{D.66}$$

From (D.66) we easily obtain (D.62) as the corresponding limit from the right exists. $\square$

D.14. Proof of Lemma C.4

P r o o f. Here, we assume ($\mathbb{A}D.6$). First, we will show that

$$\mathbb{I}_{\mathbb{\lambda}}(x)\mathbb{Y}_{\mathbb{\lambda}}(x) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda) \quad \text{where} \quad \mathbb{I}_{\mathbb{\lambda}}(x) \overset{\text{def}}{=\!=} 1_{[|x-c_1|\le\omega_{\mathbb{\lambda}}]} - 1_{[\underline{\varpi}_{\mathbb{\lambda}},\overline{\varpi}_{\mathbb{\lambda}})}(x). \tag{D.67}$$

By Lemma 5.6, there exists $n \in \mathbb{N}$ such that (5.5) holds whenever $\lambda \in (0, \lambda_{5.6})$, and then we get from Lemma C.1 that $1_{[|x-c_1|\le\omega_{\mathbb{\lambda}}]}f'_{\mathbb{\lambda}}(x) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda)$. By the definition of $f_{\mathbb{\lambda}}$

on $\mathcal{A}_\lambda \backslash [\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})$ and the definitions of $\zeta^{\uparrow}_{\lambda}, \zeta^{\downarrow}_{\lambda}$, we have that $f''_{\lambda}(x_+) = f'_{\lambda}(x)^2$ holds for $x \in \mathcal{A}_\lambda \backslash [\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})$, and then we get that

$$\mathbb{I}_{\lambda}(x) f''_{\lambda}(x_+) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda)^2 \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda). \tag{D.68}$$

Lemma B.13 gives that $\mathbb{I}_{\lambda}(x)[\omega_{\lambda} - |x - c_1|] \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\omega^2_{\lambda})$ as follows. Note that if $\mathbb{I}_{\lambda}(x) \neq 0$, then $c_1 - \omega_{\lambda} \le x \le \underline{\varpi}_{\lambda}$ or $\overline{\varpi}_{\lambda} \le x \le c_1 + \omega_{\lambda}$, and in the latter case, for example, we have from Lemma B.13 that $0 \ge |x - c_1| - \omega_{\lambda} = x - c_1 - \omega_{\lambda} \ge \varepsilon\omega^2_{\lambda} + o_{\mathbb{K}}(\omega^2_{\lambda})$. Here, we have used that $x \ge \overline{\varpi}_{\lambda} \ge c_1$ holds in this case, see (5.8). The remaining case could be treated similarly. Then by Lemma B.12

$$\mathbb{I}_{\lambda}(x) p c_4^2 [(x - c_1)^2 - \omega^2_{\lambda}] \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\omega^3_{\lambda}) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda), \tag{D.69}$$

as $c_4^2 \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(1)$. Further, we obtain from the definition of $\mathbb{D}_c(x)$ in (4.8) that

$$\mathbb{D}_c(x) - \mathbb{D}_c(c_1) = [\mathbb{S}(x, c_4) - \mathbb{S}(c_1, c_4)][\mathbb{S}(x, c_4) + \mathbb{S}(c_1, c_4) - 2c_2].$$

This gives that

$$1_{[|x-c_1| \le \omega_{\lambda}]} \mathbb{D}_c(x) \mathrel{\underset{\cdot}{=}} 1_{[|x-c_1| \le \omega_{\lambda}]} \mathbb{D}_c(c_1) + O_{\mathbb{R}\times\mathbb{K}}(\omega_{\lambda}) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(1). \tag{D.70}$$

Then we get (D.67) from (D.68,D.69,D.70). Second, from the definition of ω_{λ} in (5.3) and the definition of h_{λ} in (5.4), we have that

$$1_{[|x-c_1| \le \omega_{\lambda}]} \big\{ h''_{\lambda}(x) - \kappa_{\lambda}[\omega^2_{\lambda} - (x - c_1)^2] \big\} = 1_{[|x-c_1| \le \omega_{\lambda}]} 5\,\varepsilon^2 \kappa_{\lambda} (x - c_1)^4 \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda),$$

and as $f_{\lambda}(x) = h_{\lambda}(x)$ holds for $x \in [\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda}] \subseteq (c_1 - \omega_{\lambda}, c_1 + \omega_{\lambda})$, we obtain that

$$1_{[\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})}(x) f''_{\lambda}(x_+) \mathrel{\underset{\cdot}{=}} \kappa_{\lambda}[\omega^2_{\lambda} - (x - c_1)^2] 1_{[\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})}(x) + O_{\mathbb{R}\times\mathbb{K}}(\lambda). \tag{D.71}$$

From (5.3,D.70,D.71), we obtain that

$$1_{[\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})}(x) \{ f''_{\lambda}(x_+) \mathbb{D}_c(x) - \kappa_{\lambda}[\omega^2_{\lambda} - (x - c_1)^2] \mathbb{D}_c(c_1) \} \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\kappa_{\lambda}\omega^3_{\lambda}) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda).$$

Here, we used that $\kappa_{\lambda}\omega^3_{\lambda} = \frac{3}{4}\lambda$, see (5.3). In terms of $\mathbb{Y}_{\lambda}$, we have that

$$1_{[\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})}(x) \{ \mathbb{Y}_{\lambda}(x) + [\omega^2_{\lambda} - (x - c_1)^2][p c_4^2 - \kappa_{\lambda} \mathbb{D}_c(c_1)] \} \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda). \tag{D.72}$$

As $c_4^2 p / \kappa_{\lambda} = \mathbb{D}_c(c_1) + c_4^2 \lambda^a$ holds by (5.3), we have from (D.72) and from (5.7,5.8), saying that $[\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda}] \subseteq (c_1 - \omega_{\lambda}, c_1 + \omega_{\lambda})$, that

$$1_{[\underline{\varpi}_{\lambda}, \overline{\varpi}_{\lambda})}(x) \mathbb{Y}_{\lambda}(x) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda) + O_{\mathbb{R}\times\mathbb{K}}(\kappa_{\lambda}\omega^2_{\lambda}\lambda^a) \mathrel{\underset{\cdot}{=}} O_{\mathbb{R}\times\mathbb{K}}(\lambda) + \lambda^{1+a} O_{\mathbb{R}\times\mathbb{K}}(\omega^{-1}_{\lambda}). \tag{D.73}$$

If $a = \infty$, then $\lambda^{1+a} = 0$, and in this case, we obtain (C.4) with $\aleph = 1$ from (D.67,D.73). If $a = 2/7 \in (0, \frac{1}{2})$, we get, according to Remark D.1, that

$$\omega^{-1}_{\lambda} \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda^{-(a+1)/3}), \qquad \text{and therefore} \qquad \lambda^{1+a}\omega^{-1}_{\lambda} \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda^{\frac{2}{3}(a+1)}) \mathrel{\underset{\cdot}{=}} O_{\mathbb{K}}(\lambda^{\aleph}),$$

where $\aleph = \frac{2}{3}(a + 1) \le 1$. Also in this case, we obtain that (C.4) from (D.67,D.73). □

D.15. Proof of Lemma C.7

Here, we assume ($\mathbb{A}D.6$).

Lemma D.9. Put $\mathbb{F}_{\lambda}(x) \stackrel{\text{def}}{=\!=} f'_{\lambda}(x)x(1-x)+c_1-x$. Then

$$\inf_{|x-c_1|\geq \omega_{\lambda}} \mathbb{F}^2_{\lambda}(x) \gtrsim \min\{\mathbb{F}^2_{\lambda}(c_1+\omega_{\lambda}), \mathbb{F}^2_{\lambda}(c_1-\omega_{\lambda})\} + O_{\mathbb{K}}(\lambda) \gtrsim \omega^2_{\lambda} + O_{\mathbb{K}}(\lambda). \quad \text{(D.74)}$$

Proof. From Lemmas 5.6 and C.1, we obtain that $f'_{\lambda}(c_1 \pm \omega_{\lambda}) \eqsim O_{\mathbb{K}}(\lambda)$, and then, with the help of Lemma B.12, we get that

$$\mathbb{F}_{\lambda}(c_1 \pm \omega_{\lambda}) \eqsim \mp \omega_{\lambda} + O_{\mathbb{K}}(\lambda), \qquad \mathbb{F}^2_{\lambda}(c_1 \pm \omega_{\lambda}) \eqsim \omega^2_{\lambda} + O_{\mathbb{K}}(\lambda), \quad \text{(D.75)}$$

which gives the second relation in (D.74). The first relation in (D.74) is obtained from the first relation in (D.75) once we show that the function $\mathbb{F}_{\lambda}$ decreases on both intervals $(-1/\lambda^{\uparrow}, c_1-\omega_{\lambda}]$ and $[c_1+\omega_{\lambda}, 1/\lambda^{\downarrow})$. On these intervals, we have that

$$f''_{\lambda}(x) = f'_{\lambda}(x)^2 \in \{\zeta^2_{-\lambda^{\uparrow}}(x), \zeta^2_{\lambda^{\downarrow}}(x)\}, \quad \zeta_z(x) \stackrel{\text{def}}{=\!=} \tfrac{z}{1-zx}, \quad z \in \{-\lambda^{\uparrow}, \lambda^{\downarrow}\} \subseteq (-\infty, 1),$$

and then also that $\mathbb{F}'_{\lambda}(x) = [1+xf'_{\lambda}(x)][f'_{\lambda}(x)(1-x)-1] < 0$ since $1+x\zeta_z(x) = \frac{1}{1-zx} > 0$ and $1+(x-1)\zeta_z(x) = \frac{1-z}{1-zx} > 0$ hold whenever $x \in \mathcal{A}_{\lambda}, z \in (-\infty, 1)$. □

Proof. (of Lemma C.7) If $\lambda \in (0, \lambda_{B.14})$, we have (5.7,5.8) and, according to the definition of f_{λ}, that $f''_{\lambda}(x_+) = f'_{\lambda}(x)^2$ holds if $|x-c_1| > \omega_{\lambda}$. Then we get from Lemma C.2 that

$$1_{[|x-c_1|>\omega_{\lambda}]}[\mathbb{T}_{\lambda}(x,B,\mathcal{D}) - \mathfrak{F}_{\lambda}(x,B,\mathcal{D})] \eqsim O_{\mathbb{R}\times\mathbb{K}\times R_n\times R^2_n}(\lambda), \quad n \in \mathbb{N}, \quad \text{(D.76)}$$
$$\mathfrak{F}_{\lambda}(x,B,\mathcal{D}) \stackrel{\text{def}}{=\!=} \tfrac{p}{2} f'_{\lambda}(x)^2\mathcal{D}_{0,0} + f'_{\lambda}(x)B_0 + \tfrac{p}{2} c^2_4[(x-c_1)^2 - \omega^2_{\lambda}].$$

Further, on $\mathbb{V}_{\lambda}$ we have the equality in

$$1_{[|x-c_1|>\omega_{\lambda}]}\mathfrak{F}_{\lambda}(x,B,\mathcal{D}) = 1_{[|x-c_1|>\omega_{\lambda}]} c^2_4 \tfrac{p}{2} [\mathbb{F}^2_{\lambda}(x) - \omega^2_{\lambda}] \gtrsim O_{\mathbb{K}}(\lambda), \quad \text{(D.77)}$$

while the second relation follows from Lemma D.9. Then (C.6) follows from (D.76,D.77). Put

$$\mathcal{K}_n \stackrel{\text{def}}{=\!=} \{(x,c,B,\mathcal{D}) \in \mathbb{R}\times\mathbb{K}\times[-n,n]^{\mathbf{5}} \times [-n,n]^{\mathbf{5}\times\mathbf{5}}; (x,B,\mathcal{D}) \in \mathbb{V}_{\lambda}\}.$$

By Lemmas C.1 and C.2, we have that

$$\nabla f_{\lambda}(x)^{\intercal} B \eqsim O_{\mathcal{K}_n}(\lambda), \quad \tfrac{1-p}{2} \nabla f_{\lambda}(x)^{\intercal} \mathcal{D} \nabla f_{\lambda}(x) \eqsim O_{\mathcal{K}_n}(\lambda).$$

Then we obtain from Notation C.5 that

$$\mathbb{T}_{\lambda}(x,B,\mathcal{D}) \eqsim \tfrac{1}{2} \operatorname{tr}\{\nabla^2 f_{\lambda}(x_+)\mathcal{D}\} + \tfrac{p}{2} c^2_4[(x-c_1)^2 - \omega^2_{\lambda}] + O_{\mathcal{K}_n}(\lambda). \quad \text{(D.78)}$$

From (C.5), we have that $\mathbb{D}_c(x) = \operatorname{tr}\{\mathcal{D}\,\mathfrak{d}\mathfrak{d}^{\intercal}\}$ on $\mathbb{V}_{\lambda}$, and then we get from (C.2) that

$$\operatorname{tr}\{\nabla^2 f_{\lambda}(x_+)\mathcal{D}\} \eqsim O_{\mathcal{K}_n}(\lambda) + 1_{[\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda})}(x) \cdot [O_{\mathcal{K}_n}(\lambda\kappa^{1/2}_{\lambda}) + h''_{\lambda}(x)\mathbb{D}_c(x)]. \quad \text{(D.79)}$$

As $h''_{\lambda}(x) = f''_{\lambda}(x_+)$ holds for $x \in [\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda})$, it follows from (C.3,D.78,D.79) that

$$1_{[\underline{\pi}_{\lambda}, \overline{\pi}_{\lambda})}(x) \cdot [\mathbb{T}_{\lambda}(x,B,\mathcal{D}) - \tfrac{1}{2}\,\mathbb{Y}_{\lambda}(x)] \eqsim O_{\mathcal{K}_n}(\lambda\kappa^{1/2}_{\lambda}),$$

and then as $1_{[\underline{\pi}_{\lambda},\overline{\pi}_{\lambda})}(x) \leq 1_{[|x-c_1|\leq \omega_{\lambda}]}$ holds if $\lambda \in (0,\lambda_{5.6})$, we get from Lemma C.4 that

$$1_{[\underline{\pi}_{\lambda},\overline{\pi}_{\lambda})}(x)\cdot \mathbb{T}_{\lambda}(x,B,\mathcal{D}) \doteqdot O_{\mathcal{K}_n}(\lambda^{\aleph}) + O_{\mathcal{K}_n}(\lambda\kappa_{\lambda}^{1/2}). \tag{D.80}$$

By (D.78,D.79) and Lemma B.13, we get with $\mathbb{I}_{\lambda}(x) \stackrel{\text{def}}{=} 1_{[c_1-\omega_{\lambda},\underline{\pi}_{\lambda})\cup[\overline{\pi}_{\lambda},c_1+\omega_{\lambda}]}(x)$ that

$$\mathbb{I}_{\lambda}(x)\,\mathbb{T}_{\lambda}(x,B,\mathcal{D}) \doteqdot O_{\mathcal{K}_n}(\lambda) + \mathbb{I}_{\lambda}(x)\,\tfrac{p}{2}\,c_4^2[(x-c_1)^2-\omega_{\lambda}^2] \doteqdot O_{\mathcal{K}_n}(\lambda) \tag{D.81}$$

since $\omega_{\lambda} \doteqdot O_{\mathbb{K}}(\lambda^{1/3})$ by Lemma B.12. If we sum up (D.80,D.81), we will obtain that

$$1_{[|x-c_1|\leq \omega_{\lambda}]}\cdot \mathbb{T}_{\lambda}(x,B,\mathcal{D}) \doteqdot O_{\mathcal{K}_n}(\lambda^{\aleph}) + O_{\mathcal{K}_n}(\lambda\kappa_{\lambda}^{1/2}) \doteqdot O_{\mathcal{K}_n}(\lambda^{q}) \tag{D.82}$$

as follows. If $(q,a)=(1,\infty)$, we have that $\aleph=1$ and $\kappa_{\lambda} \doteqdot O_{\mathbb{K}}(1)$, and hence, (D.82) obviously holds in this case. If $(q,a)=(\frac{6}{7},\frac{2}{7})$, then $\kappa_{\lambda} \doteqdot O_{\mathbb{K}}(\lambda^{-a})$ holds by definition of κ_{λ} as $\mathbb{D}_c(c_1)\geq 0$, and therefore, $\lambda\kappa_{\lambda}^{1/2} \doteqdot O_{\mathbb{K}}(\lambda^{1-\frac{a}{2}})$. Further, $\aleph=\frac{2}{3}(a+1)$ and

$$\aleph \wedge (1-\tfrac{a}{2}) = [\tfrac{2}{3}(a+1)] \wedge (1-\tfrac{a}{2}) = \tfrac{6}{7} = q$$

holds for the optimal choice $a=\frac{2}{7}$. Thus, $O_{\mathcal{K}_n}(\lambda^{\aleph}) + O_{\mathcal{K}_n}(\lambda\kappa_{\lambda}^{1/2}) \doteqdot O_{\mathcal{K}_n}(\lambda^{q})$. □

D.16. Proof of Lemma B.17

Here, we assume ($\mathbb{A}D.6$).

P r o o f. By (B.5) in Lemma B.7 and by the definition of F in (4.1), we get that

$$(\ln \tfrac{\mathcal{W}}{\mathcal{W}_0})^{(c)} \stackrel{\text{as}}{=} \textstyle\int \sigma_s^2(\theta_s\pi_s - \frac{\pi_s^2}{2})\,\mathrm{d}s + \int[\sigma\pi\,\mathrm{d}W - \zeta_{\lambda}^{\uparrow}(\pi)\,\mathrm{d}\pi^{\uparrow(c)} - \zeta_{\lambda}^{\downarrow}(\pi)\,\mathrm{d}\pi^{\downarrow(c)}]. \tag{D.83}$$

By Theorem B.16, $f_{\lambda} \in C^1(\mathcal{A}_{\lambda}\times\mathbb{G}_{\lambda})$ and

$$f_{\lambda}(G)^{(c)} \stackrel{\text{as}}{=} f_{\lambda}(G_0) + \textstyle\int[\nabla f_{\lambda}(G)^{\intercal}\mathrm{d}G^{(c)} + \frac{1}{2}\,\mathrm{tr}\{\tilde{\nabla}^2 f_{\lambda}(G)\,\mathrm{d}\langle\!\langle G^{(c)}\rangle\!\rangle\}]. \tag{D.84}$$

From Lemma B.10, we obtain that

$$G^{(c)} \stackrel{\text{as}}{=} G_0 + \textstyle\int \mathfrak{a}_s^G\,\mathrm{d}s + \mathfrak{m}^G + 1_{\{0\}}(\pi^{\uparrow(c)} - \pi^{\downarrow(c)}), \quad \mathfrak{a}^G \in \mathbb{PM}_b^{\mathbf{5}}, \mathfrak{m}^G \in \mathbb{CM}_l^{\mathbf{5}}. \tag{D.85}$$

From (B.9,D.85), we get that $\langle\!\langle G^{(c)}\rangle\!\rangle \stackrel{\text{as}}{=} \langle\!\langle \mathfrak{m}^G\rangle\!\rangle \stackrel{\text{as}}{=} \int \mathfrak{n}_s^G\,\mathrm{d}s$, and then the equality in (B.17) follows from equalities (D.83,D.84,D.85). Since the considered strategy is strictly λ-admissible, we have that $\nabla f_{\lambda}(G) \in \mathbb{PM}_b^{\mathbf{5}}$ and $\tilde{\nabla}^2 f_{\lambda}(G) \in \mathbb{PM}_b^{\mathbf{5}\times\mathbf{5}}$. Then as $\sigma\pi \in \mathbb{PM}_b$ and $W \in \mathbb{CM}_l, \mathfrak{m}^G \in \mathbb{CM}_l^{\mathbf{5}}$, we obtain that also $\mathfrak{m}^V \in \mathbb{CM}_l$. Since $\mathfrak{a}^G \in \mathbb{PM}_b^{\mathbf{5}}, \mathfrak{n}^G \in \mathbb{PM}_b^{\mathbf{5}\times\mathbf{5}}$ and $\sigma,\theta,\pi \in \mathbb{PM}_b$, we get that also $\mathfrak{a}^V \in \mathbb{PM}_b$. The property $\mathfrak{D}^V \in \mathbb{CFV}$ is obvious. □

D.17. Proof of Lemma C.8

Here, we assume ($\mathbb{A}D.6$).

P r o o f. Since we restrict to $\lambda \in (0,\lambda_{B.14})$ and $\lambda_{B.14} < \lambda_{5.6}$ according to Lemma B.14, we get from Lemma 5.6 that there exists $n \in \mathbb{N}$ such that $|c_1 \pm \omega_{\lambda}| \leq n$ holds whenever

$c \in \mathbb{K}$. Further, consider K defined as $k_n \in \mathbb{N}$ from Lemma B.10. By Lemma C.7, there exist $K_{C.8} \in (0,\infty)$ and $\lambda_{C.8} \in (0, \lambda_{B.14})$ such that for every $\lambda \in (0, \lambda_{C.8})$

$$1_{[|x-c_1|>\omega_\lambda]}(1_{\mathbb{V}_\lambda}\mathbb{T}_\lambda)(x,B,\mathcal{D}) \geq -K_{C.8}\lambda \geq -K_{C.8}\lambda^q \tag{D.86}$$

whenever $c \in \mathbb{K}$ and $B \in R_K, \mathcal{D} \in R_K^2$ and that

$$1_{[|x-c_1|\leq\omega_\lambda]}|(1_{\mathbb{V}_\lambda}\mathbb{T}_\lambda)(x,B,\mathcal{D})| \leq K_{C.8}\lambda^q \tag{D.87}$$

if we additionally assume that the coordinates of $B, \mathcal{D}$ in (D.87) are within $[-K,K]$. Let $\lambda \in (0,\lambda_{C.8})$ be fixed from here, and let (φ,ψ) be a λ-admissible strategy. By the choice of k_n from Lemma B.10, $1_{[|\pi|\leq n]}\mathfrak{a}^G, 1_{[|\pi|\leq n]}\mathfrak{n}^G$ attain values within $[-K,K]^{\mathbf{5}}$ and $[-K,K]^{\mathbf{5}\times\mathbf{5}}$, respectively, where $\mathfrak{a}^G \in \mathbb{PM}^{\mathbf{5}}$ and $\mathfrak{n}^G \in \mathbb{PM}^{\mathbf{5}\times\mathbf{5}}$ from Lemma B.10 attain values in R_K and R_K^2, respectively, and they are associated with the strategy (φ,ψ). Since $(\pi_t, \mathfrak{a}_t^G, \mathfrak{n}_t^G) \in \mathbb{V}_{\lambda,\mathfrak{C}_t}, t \in [0,\infty)$, we obtain from (D.86,D.87) that (C.7) holds, see Notation C.5 and (5.11,B.18). □

D.18. Proof of Theorem B.18

Remark D.10. (a) Let (φ,ψ) be a pure jump strategy. Then $\varphi^{\uparrow(c)} = 0 = \varphi^{\downarrow(c)}$ and see the definition of $\pi^\uparrow, \pi^\downarrow$ in (B.2,B.3) in order to agree that also $\pi^{\uparrow(c)} = 0 = \pi^{\downarrow(c)}$.

(b) Let (φ,ψ) be an admissible strategy. Whenever $t \in (0,\infty)$, we have that

$$\Delta\varphi_t > 0 \quad \Rightarrow \quad \Delta\mathcal{W}_t^\uparrow = \Delta\mathcal{W}_t + \lambda^\uparrow S_t \Delta\varphi_t = S_t(1+\lambda^\uparrow)\Delta\varphi_t + \Delta\psi_t = 0,$$

and similarly also that $\Delta\varphi_t < 0 \Rightarrow \Delta\mathcal{W}_t^\downarrow = 0$. Then by (3.10), we have that

$$\Delta \ln \mathcal{W}_t = \ln \tfrac{1+\lambda^\uparrow\pi_{t-}}{1+\lambda^\uparrow\pi_t} \cdot 1_{[\Delta\varphi_t>0]} + \ln \tfrac{1-\lambda^\downarrow\pi_{t-}}{1-\lambda^\downarrow\pi_t} \cdot 1_{[\Delta\varphi_t<0]}, \quad t \in (0,\infty). \tag{D.88}$$

Proof. (of Theorem B.18) Here, we consider the restriction ($\mathbb{A}D.6$). First, we show that $V \in \mathfrak{c}\mathbb{S}$ is a continuous process. Let $t \in (0,\infty)$. It follows from the definition of f_λ and of a (λ-admissible) $[\Theta - \varpi\,(\underline{\pi},\bar{\pi})\,\Theta + \varpi]$-strategy that

$$\Delta f_\lambda(G_t) = f_\lambda(\pi_t, \mathfrak{C}_t) - f_\lambda(\pi_{t-}, \mathfrak{C}_t) = \begin{cases} \ln \frac{1+\lambda^\uparrow\pi_{t-}}{1+\lambda^\uparrow\pi_t} & \text{if } \pi_{t-} \leq \pi_t \\ \ln \frac{1-\lambda^\downarrow\pi_{t-}}{1-\lambda^\downarrow\pi_t} & \text{if } \pi_{t-} \geq \pi_t. \end{cases}$$

As $\operatorname{sign}(\Delta\pi_t) = \operatorname{sign}(\Delta\varphi_t)$, Remark D.10 (b) gives that $\Delta V_t = \Delta \ln \mathcal{W}_t - \Delta f_\lambda(G_t) = 0$.

Lemma C.8 gives $K_{B.18} \stackrel{\text{def}}{=} K_{C.8} \in (0,\infty)$ and $\lambda_{B.18} \stackrel{\text{def}}{=} \lambda_{C.8}$ such that (C.7) holds if $\lambda \in (0,\lambda_{C.8})$. In particular, we have that $|\nu - \mathfrak{a}^V| \leq K_{B.18}\lambda^q$ holds for such λ as $|\pi_t - \Theta_t| < \varpi_t$ holds for every $t \in [0,\infty)$ by the definition of a $[\Theta - \varpi\,(\underline{\pi},\bar{\pi})\,\Theta + \varpi]$-strategy. We are going to show (B.21). Since $\lambda < \lambda_{B.18} = \lambda_{C.8} < \lambda_{5.6}$, we have that the considered strategy is strictly λ-admissible, see Remark 5.11. Since the considered strategy is pure jump, we have that $\pi^{\uparrow(c)}, \pi^{\downarrow(c)} = 0$ by Remark D.10, and consequently also that $\mathfrak{D}^V = 0$, see (B.20). As V is a continuous process by the first part of the proof, we get from Lemma B.17 and from $\mathfrak{D}^V = 0$ that (B.21) holds, and we get from the strong law of large numbers for Brownian martingales that $\frac{1}{t}\,\mathfrak{m}_t^V \stackrel{\text{as}}{\to} 0$ as $t \to \infty$, and then (B.22) follows from already proved parts of the statement. □

D.19. Proof of Theorem 5.12

Here, we assume ($\mathbb{A}D.6$).

P r o o f. By Lemma 5.6, there exists $n \in \mathbb{N}$ such that G attains values in $[-n, n] \times \mathbb{K}$ if $\lambda \in (0, \lambda_{5.6})$. From Lemma C.1 (i), we get that there exist $K_{5.12} \in (K_{B.18}, \infty)$ and $\lambda_{5.12} \in (0, \lambda_{B.18})$ such that $2|f_\lambda(G)| \leq K_{5.12}\lambda \leq K_{5.12}\lambda^q$ holds whenever $\lambda \in (0, \lambda_{5.12})$. Let $\lambda \in (0, \lambda_{5.12})$ and τ be an integrable stopping time. By Theorem B.18

$$|\ln \tfrac{\mathcal{W}_\tau}{\mathcal{W}_0} - \textstyle\int_0^\tau \nu_s \,\mathrm{d}s - \mathfrak{m}^V_\tau| \overset{\text{as}}{\leq} K_{B.18}\lambda^q \tau + |f_\lambda(G_\tau) - f_\lambda(G_0)| \overset{\text{as}}{\leq} K_{5.12}\lambda^q(1+\tau). \tag{D.89}$$

As $\mathfrak{m}^V \in \mathbb{CM}_l$ and τ is an integrable stopping time, we have that $\mathbb{E}[\mathfrak{m}^V_\tau] = 0$, as stated already in Notation 2.4. Then we get (5.12) from (D.89). □

D.20. Proof of Theorem B.19

Here, we assume ($\mathbb{A}D.6$).

P r o o f. Put $K_{B.19} \overset{\text{def}}{=} K_{C.8}, \lambda_{B.19} \overset{\text{def}}{=} \lambda_{C.8}$. Let $\lambda \in (0, \lambda_{C.8})$. By Lemma C.1 (ii), (C.1) holds if $\lambda \in (0, \lambda_{B.14}), x \in \mathcal{A}_\lambda$ and $c \in \mathbb{K}$. This gives first that $f'_\lambda(G_t) \in [-\zeta^\uparrow_\lambda(\pi_t), \zeta^\downarrow_\lambda(\pi_t)]$, which ensures that the proces $\mathfrak{D}^V$ from (B.20) is non-increasing, and second, that also

$$\Delta f_\lambda(G_t) = \textstyle\int_{\pi_{t-}}^{\pi_t} f'_\lambda(x, \mathfrak{C}_t)\,\mathrm{d}x \geq \begin{cases} -\int_{\pi_{t-}}^{\pi_t} \zeta^\uparrow_\lambda(x)\,\mathrm{d}x = -\ln \frac{1+\lambda^\uparrow \pi_t}{1+\lambda^\uparrow \pi_{t-}} & \text{if} \quad \pi_{t-} \leq \pi_t, \\ +\int_{\pi_{t-}}^{\pi_t} \zeta^\downarrow_\lambda(x)\,\mathrm{d}x = -\ln \frac{1-\lambda^\downarrow \pi_t}{1-\lambda^\downarrow \pi_{t-}} & \text{if} \quad \pi_{t-} \geq \pi_t, \end{cases} \tag{D.90}$$

$t \in (0, \infty)$. From (D.90) and from (D.88) in Remark D.10, we get that

$$\Delta V_t = \Delta \ln \mathcal{W}_t - \Delta f_\lambda(G_t) \leq 0, \;\; t \in (0, \infty), \qquad \text{where} \qquad V = \ln \mathcal{W} - f_\lambda(G). \tag{D.91}$$

As the considered strategy is strictly λ-admissible, we have that $\mathfrak{m}^V \in \mathbb{CM}_l$ by (B.19) in Lemma B.17. Further, we obtain from Lemmas B.17 and C.8 that

$$\mathbb{CM}_l \ni \tilde{\mathfrak{m}}^V \overset{\text{def}}{=} V^{(c)} - V_0 - \mathfrak{D}^V - \textstyle\int \mathfrak{a}^V_s \,\mathrm{d}s \overset{\text{as}}{=} \mathfrak{m}^V, \qquad \mathfrak{a}^V \leq \nu + K_{C.8}\lambda^q. \tag{D.92}$$

Then we get from (D.91,D.92) and from $-\mathfrak{D}^V \in \mathbb{CI}$ that $-V + \tilde{\mathfrak{m}}^V + \int[\nu_s + K_{C.8}\lambda^q]\,\mathrm{d}s \in \mathring{\mathbb{C}}\mathbb{I}$, i. e., (B.23) holds. □

D.21. Proof of Theorem 5.13

P r o o f. By Lemma 5.6, there exists $n \in \mathbb{N}$ such that (5.5) holds whenever $\lambda \in (0, \lambda_{5.6})$. From Lemma C.1, we obtain that there exist $K_{C.1} \in (0, \infty), \lambda_{C.1} \in (0, \lambda_{B.14}) \subseteq (0, \lambda_{5.6})$ such that $|f_{\lambda}(x)| \leq K_{C.1}\lambda$ holds whenever $\lambda \in (0, \lambda_{C.1}), c \in \mathbb{K}$ and $|x| \leq n$. Then we can put $K_{5.13} \overset{\text{def}}{=} K_{5.12} + K_{B.18} + K_{B.19} + 2K_{C.1}$. Note that if $\mathfrak{X} \subseteq \mathcal{A}_\lambda$ and $\lambda \in (0, \lambda_{C.1})$,

$$K_{\mathfrak{X}} \overset{\text{def}}{=} \sup_{x \in \mathfrak{X}} \sup_{c \in \mathbb{K}} |f_\lambda(x, c)| \leq K_{C.1}\lambda + \sup_{x \in \mathfrak{X}} \max\{|\ln(1 + \lambda^\uparrow x)|, |\ln(1 - \lambda^\downarrow x)|\}, \tag{D.93}$$

see Notation B.15. In this proof, we restrict to $\lambda \in (0, \lambda_{5.12} \wedge \lambda_{B.18} \wedge \lambda_{B.19} \wedge \lambda_{C.1})$.

1. Initially, we assume that the strategy (φ, ψ) with the wealth process $\mathcal{W}$ is strictly λ-admissible, i. e., the corresponding position π attains values in a compact set $\mathfrak{X} \subseteq \mathcal{A}_\lambda$. As $\mathfrak{C}$ attains values in $\mathbb{K}$, we have that $|f_\lambda(G)| \leq K_{\mathfrak{X}}$ holds with $G \stackrel{\text{def}}{=} (\pi, \mathfrak{C}^\intercal)^\intercal$. By Theorem 5.12, we have that (5.12) holds. By Theorem B.19, we get that

$$\ln \mathcal{W} - f_\lambda(G) - \int [\nu_s + K_{B.19}\lambda^q]\, \mathrm{d}s \in \mathbb{CM}_l \ominus \mathbb{CI}. \tag{D.94}$$

In particular, as any $\mathfrak{m} \in \mathbb{CM}_l$ satisfies $\frac{1}{t}\mathfrak{m}_t \stackrel{\text{as}}{\to} 0$ as $t \to \infty$, we have that

$$\limsup_{t\to\infty} \tfrac{1}{t}[\ln \mathcal{W}_t - \textstyle\int_0^t \nu_s\, \mathrm{d}s] \stackrel{\text{as}}{\leq} K_{B.19}\lambda^q. \tag{D.95}$$

As we assume that $\mathcal{W}_0 = \hat{\mathcal{W}}_0$, we obtain from (5.12,D.94) that

$$\mathbb{E}[\ln \mathcal{W}_\tau - \ln \hat{\mathcal{W}}_\tau] \leq 2K_{\mathfrak{X}} + K_{B.19}\lambda^q\, \mathbb{E}[\tau] + K_{5.12}(1 + \mathbb{E}[\tau])\lambda^q.$$

Hence, (5.15) holds according to (D.93), and then (5.14) follows immediately. Finally, we get from (D.95) and Theorem B.18 that (5.13) holds as follows

$$\limsup_{t\to\infty} \tfrac{1}{t}\,(\ln \mathcal{W}_t - \ln \hat{\mathcal{W}}_t) \stackrel{\text{as}}{\leq} (K_{B.19} + K_{B.18})\,\lambda^q \leq K_{5.13}\lambda^q.$$

2. Let $\epsilon > 0$. Then there exists $\tilde{\lambda} \in (0, \lambda)$ such that

$$\sup_{c\in\mathbb{K}} |\omega^2_{\lambda,c} - \omega^2_{\tilde{\lambda},c}| \leq \epsilon,$$

see Definition 5.3. By Remark 3.9, there exists a strictly $\tilde{\lambda}$-admissible strategy with the wealth process $\tilde{\mathcal{W}} \geq \mathcal{W}$ such that $\tilde{\mathcal{W}}_0 = \mathcal{W}_0 = \hat{\mathcal{W}}_0$. Then, as in (D.95), we get that

$$\limsup_{t\to\infty} \tfrac{1}{t}[\ln \tilde{\mathcal{W}}_t - \textstyle\int_0^t \tilde{\nu}_s\, \mathrm{d}s] \leq K_{B.19}\tilde{\lambda}^q \qquad \text{where} \qquad \tilde{\nu} \stackrel{\text{def}}{=} \tfrac{\sigma^2}{2}(\theta^2 - \tilde{\varpi}^2)$$

and where $\tilde{\varpi} \stackrel{\text{def}}{=} (\omega_{\tilde{\lambda},\mathfrak{c}_t})_{t\geq 0}$. Then again, with the help of Theorem B.18, we obtain that

$$\limsup_{t\to\infty} \tfrac{1}{t}[\ln \tilde{\mathcal{W}}_t - \ln \hat{\mathcal{W}}_t] \leq K_{5.13}\tilde{\lambda}^q + \limsup_{t\to\infty} \tfrac{1}{t}\textstyle\int_0^t |\tilde{\nu}_s - \nu_s|\, \mathrm{d}s \leq K_{5.13}\lambda^q + \tfrac{\epsilon}{2} \sup_{c\in\mathbb{K}} c_4^2.$$

As the last term can be made arbitrary small by the choice of $\epsilon > 0$, we finally obtain from the inequality $\mathcal{W} \leq \tilde{\mathcal{W}}$ that (5.13) holds. The relation (5.14) can be obtained in the same way. □

D.22. Proof of Theorem 5.9

Proof. Without loss of generality, we may assume that $\mathfrak{w} = 1$. First, we put $\Phi_0 \stackrel{\text{def}}{=} \mathfrak{p}/S_0$, $\Psi_0 \stackrel{\text{def}}{=} 1 - \mathfrak{p}$, $\tau_0 \stackrel{\text{def}}{=} 0$. Once Φ_n, Ψ_n, τ_n is defined for $n \in \mathbb{N}_0$, we set

$$\tau_{n+1} \stackrel{\text{def}}{=} \tau_n^\uparrow \wedge \tau_n^\downarrow \qquad \text{where} \qquad \tau_n^\uparrow \stackrel{\text{def}}{=} \inf\{t \geq \tau_n; \mathfrak{a}_t^\uparrow(\Psi_n + \Phi_n S_t) = \Phi_n S_t\}, \tag{D.96}$$

and analogously, we consider $\tau_n^\downarrow$ with $\mathfrak{a}^\uparrow$ replaced by $\mathfrak{a}^\downarrow$ in (D.96). Further, put

$$A_n \stackrel{\text{def}}{=} \begin{cases} \mathfrak{a}^\uparrow_{\tau_n} & \text{on } C_n^\uparrow \\ \mathfrak{a}^\downarrow_{\tau_n} & \text{on } C_n^\downarrow \end{cases}, \quad B_n \stackrel{\text{def}}{=} \begin{cases} \mathfrak{b}^\uparrow_{\tau_n} & \text{on } C_n^\uparrow \\ \mathfrak{b}^\downarrow_{\tau_n} & \text{on } C_n^\downarrow \end{cases}, \quad \Lambda_n(x) \stackrel{\text{def}}{=} \begin{cases} 1 + \lambda^\uparrow x & \text{on } C_n^\uparrow \\ 1 - \lambda^\downarrow x & \text{on } C_n^\downarrow \end{cases}, \tag{D.97}$$

on the set $C_n \overset{\text{def}}{=\!=} [\tau_n < \infty]$ where $C_n^\uparrow \overset{\text{def}}{=\!=} C_n \cap [\tau_n = \tau_{n-1}^\uparrow]$ and $C_n^\downarrow \overset{\text{def}}{=\!=} C_n \cap [\tau_n = \tau_{n-1}^\downarrow]$ are disjoint sets as $C_n^\uparrow \cap C_n^\downarrow \subseteq [\tau_n < \infty, a_{\tau_n}^\uparrow = a_{\tau_n}^\downarrow] = \emptyset$, $n \in \mathbb{N}$. On the set C_n, we define Φ_n, Ψ_n by the following equations

$$(\Psi_{n-1} + \Phi_{n-1} S_{\tau_n})\, \Lambda_n(A_n) = (\Psi_n + \Phi_n S_{\tau_n})\, \Lambda_n(B_n), \tag{D.98}$$

$$\Psi_n - \Psi_{n-1} = -\Lambda_n(1)\, S_{\tau_n}(\Phi_n - \Phi_{n-1}), \tag{D.99}$$

and we set them to zero on $\Omega \backslash C_n$. It follows by induction that $C_n \subseteq [(\Phi_n, \Psi_n) \neq (0,0)]$, $n \in \mathbb{N}_0$. From (D.96,D.97), we obtain the first equality in

$$[\tau_n < \infty] \subseteq [A_n \Psi_{n-1} + (A_n - 1)\Phi_{n-1} S_{\tau_n} = 0 = B_n \Psi_n + (B_n - 1)\Phi_n S_{\tau_n}], \tag{D.100}$$

and the second equality in (D.100) follows from the first one combined with (D.98,D.99).

1. Our first task is to show that $\tau \overset{\text{def}}{=\!=} \lim_n \tau_n = \infty$. Let $\omega \in \Omega$ be such that $\tau(\omega) < \infty$. Without loss of generality, we may assume that $\Omega = \{\omega\}$, and then we can obviously omit the argument ω. Since $(A_{n+1}, B_n)_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}^2$, it has at least one mass point, say $(\mathfrak{a}, \mathfrak{b}) \in \{\mathfrak{a}_\tau^\uparrow, \mathfrak{a}_\tau^\downarrow\} \times \{\mathfrak{b}_\tau^\uparrow, \mathfrak{b}_\tau^\downarrow\}$. Since we assume that $\mathfrak{a}_\tau^\uparrow < \mathfrak{b}_\tau^\uparrow < \mathfrak{b}_\tau^\downarrow < \mathfrak{a}_\tau^\downarrow$, we have that $\mathfrak{a} \neq \mathfrak{b}$. Hence, at least one of them is non-zero. For example, if $\mathfrak{b} \neq 0$, then for infinitely many n we have that $B_n \neq 0$, and for those n, with the help of (D.100), we can express

$$\Psi_n = (B_n^{-1} - 1)\Phi_n S_{\tau_n}. \tag{D.101}$$

Note that if (D.101) holds, then $\Phi_n \neq 0$ as the case $\Phi_n = \Psi_n = 0$ cannot happen. Then for all n such that (D.101) holds, we have, according to (D.100), that

$$0 = A_{n+1}(B_n^{-1} - 1) S_{\tau_n} + (A_{n+1} - 1) S_{\tau_{n+1}},$$

which gives (after passing $n \to \infty$ along a suitable subsequence) that

$$0 = [\mathfrak{a}(\mathfrak{b}^{-1} - 1) + (\mathfrak{a} - 1)] S_\tau.$$

Since $S_\tau \in (0, \infty)$, we have that $\mathfrak{a}(1 - \mathfrak{b}) = (1 - \mathfrak{a})\mathfrak{b}$, i. e., $\mathfrak{a} = \mathfrak{b}$, which is a contradiction with $\mathfrak{a} \neq \mathfrak{b}$. The case $\mathfrak{a} \neq 0$ can be shown to lead to a contradiction in the same way. Hence, $\tau = \infty$.

2. Our second task is to show that the strategy (φ, ψ), defined below, is admissible. Put

$$\varphi \overset{\text{def}}{=\!=} \textstyle\sum_{n=0}^\infty \Phi_n 1_{[\tau_n, \tau_{n+1})}, \quad \psi \overset{\text{def}}{=\!=} \textstyle\sum_{n=0}^\infty \Psi_n 1_{[\tau_n, \tau_{n+1})}, \quad \mathcal{W} \overset{\text{def}}{=\!=} \psi + \varphi S. \tag{D.102}$$

In order to show that the strategy (φ, ψ) is self-financing, it is enough to verify that

$$\Delta \psi_t = -S_t^\uparrow (\Delta\varphi_t)^+ + S_t^\downarrow (\Delta\varphi_t)^-, \quad t \in (0, \infty), \tag{D.103}$$

cf. (3.6), but (D.103) follows from (D.99), see (3.3). Its wealth process $\mathcal{W}$ starts from $\mathcal{W}_0 = \Psi_0 + \Phi_0 S_0 = 1$. The ask and the bid wealth processes $\mathcal{W}^\uparrow, \mathcal{W}^\downarrow$ from (3.8) are rcll, and they start from $\mathcal{W}_0^\uparrow = 1 + \lambda^\uparrow \mathfrak{p} > 0$ and $\mathcal{W}_0^\downarrow = 1 - \lambda^\downarrow \mathfrak{p} > 0$, respectively. We are going to show, by induction, that

$$\mathcal{W} \geq \mathcal{W}^\uparrow \wedge \mathcal{W}^\downarrow > 0 \quad \text{on} \quad [\tau_n, \tau_{n+1}), \quad n \in \mathbb{N}. \tag{D.104}$$

Note that the first inequality in (D.104) holds immediately and that we already have the second inequality at zero. What we assume in the nth step of the induction is that

$$[\tau_n < \infty] \subseteq [\mathcal{W}^{\uparrow}_{\tau_n} \wedge \mathcal{W}^{\downarrow}_{\tau_n} > 0] \cap [\mathfrak{a}^{\uparrow}_{\tau_n} \mathcal{W}_{\tau_n} < \Phi_n S_{\tau_n} < \mathfrak{a}^{\downarrow}_{\tau_n} \mathcal{W}_{\tau_n}], \tag{D.105}$$

$n \in \mathbb{N}_0$. Recall that $\tau_0 = 0$. To verify (D.105) for $n = 0$, it remains to realize that $\Phi_0 S_0 = \mathfrak{p} \in (\mathfrak{a}^{\uparrow}_0, \mathfrak{a}^{\downarrow}_0)$. Let $\omega \in \Omega$ be fixed. For brevity of the notation, we will omit this symbol, and we set $m \stackrel{\text{def}}{=} n+1$. Obviously, we may assume that $\tau_n < \infty$. As $\mathcal{W}, S, \mathfrak{a}^{\uparrow}, \mathfrak{a}^{\downarrow}$ are continuous on $[\tau_n, \tau_m)$, we get from (D.105) and the definition of τ_m in (D.96) that

$$\mathfrak{a}^{\uparrow}\mathcal{W} < \Phi_n S < \mathfrak{a}^{\downarrow}\mathcal{W} \quad \text{on} \quad [\tau_n, \tau_m). \tag{D.106}$$

If we use that $\mathcal{W}$ is continuous on $[\tau_n, \tau_m)$ and that $\mathcal{W}_{\tau_n} \geq \mathcal{W}^{\uparrow}_{\tau_n} \wedge \mathcal{W}^{\downarrow}_{\tau_n} > 0$ again, we obtain from (D.106) that the case $\inf_{[\tau_n,\tau_m)} \mathcal{W} \leq 0$ implies that $\Phi_n = 0$, which gives that $\mathcal{W} = \Psi_n = \mathcal{W}_{\tau_n} > 0$ on $[\tau_n, \tau_m)$. Here, we used that $0 < S \in \mathbb{CA}$. Hence, we have that

$$\inf_{[\tau_n,\tau_m)} \mathcal{W} > 0. \tag{D.107}$$

From (D.106,D.107), we get that

$$\Phi_n S/\mathcal{W} \in [\mathfrak{a}^{\uparrow}, \mathfrak{a}^{\downarrow}] \subseteq \mathcal{A}_\lambda = (-1/\lambda^{\uparrow}, 1/\lambda^{\downarrow}) \quad \text{on} \quad [\tau_n, \tau_m) \tag{D.108}$$

and that the same holds for the limit at time τ_m from the left if $\tau_m < \infty$. Then we obtain, with the help of (D.107), that

$$0 < \Phi_n S + \mathcal{W}/\lambda^{\uparrow} = \mathcal{W}^{\uparrow}/\lambda^{\uparrow}, \qquad 0 < -\Phi_n S + \mathcal{W}/\lambda^{\downarrow} = \mathcal{W}^{\downarrow}/\lambda^{\downarrow}, \tag{D.109}$$

holds on $[\tau_n, \tau_m)$ and that the same holds for the limits at τ_m from the left if $\tau_m < \infty$. In particular, (D.104) holds for the considered $n \in \mathbb{N}$. For the rest of the proof, we may assume that $\tau_m < \infty$. From (D.98) with n replaced by m, we obtain that

$$\mathcal{W}_{\tau_m} = \mathcal{W}_{\tau_m -} \cdot \Lambda_m(A_m)/\Lambda_m(B_m) > 0.$$

From (D.100) with n replaced by m, we have that $\mathcal{W}_{\tau_m} B_m = \Phi_m S_{\tau_m}$, which gives that the relations in (D.108) hold with n replaced by m at time τ_m as $B_m \in [\mathfrak{b}^{\uparrow}_{\tau_m} \mathfrak{b}^{\downarrow}_{\tau_m}] \subseteq (\mathfrak{a}^{\uparrow}_{\tau_m} \mathfrak{a}^{\downarrow}_{\tau_m})$. Then we have that also (D.109) holds with n replaced by m at time τ_m. Hence, we have that (D.105) holds with n replaced by $m = n + 1$. This completes the proof of (D.104), and then we have that (φ, ψ) is an admissible strategy.

From (D.102,D.106,D.107), we obtain that the position $\pi = \varphi S/\mathcal{W} \in (\mathfrak{a}^{\uparrow}, \mathfrak{a}^{\downarrow})$ holds at every $(t, \omega) \in [0, \infty) \times \Omega$. Hence, it remains to show that (5.10) holds for each $t \in (0, \infty)$. Let $t \in (0, \infty), \omega \in \Omega$ and $m \in \mathbb{N}$ be such that $t = \tau_m(\omega)$. Put $n \stackrel{\text{def}}{=} m - 1 \in \mathbb{N}_0$. From (D.100), we get that

$$\pi_{t-} = \tfrac{\Phi_n S_t}{\mathcal{W}_{t-}} = \tfrac{\Phi_n S_t}{\Psi_n + \Phi_n S_t} = A_m, \qquad \pi_t = \tfrac{\Phi_m S_t}{\mathcal{W}_t} = \tfrac{\Phi_m S_t}{\Psi_m + \Phi_m S_t} = B_m,$$

which together with (D.97) gives (5.10). □

E. TRUE APPENDIX

In this section, $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ is an arbitrary complete filtration on a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$. If $\mathscr{B} \subseteq \mathscr{A}$ is a σ-algebra, we denote by $\mathbb{L}(\mathscr{B})$ the set of all $\mathscr{B}$-measurable real valued random variables.

Lemma E.1. If $X \in \mathbb{BI}_b(\mathscr{F})^m, m \in \mathbb{N}$, and $f \in C^2(\mathbb{R}^m)$, then $Y \stackrel{\text{def}}{=} f(X) \in \mathbb{BI}_b(\mathscr{F})$.

P r o o f. For brevity, we will omit the reference to the filtration $\mathscr{F}$ in the notation. By assumption, $X \in \mathbb{CA}_b^m \cap \mathbb{CS}_l^m$ and there is $a \in \mathbb{PM}_b^m$ such that

$$M \stackrel{\text{def}}{=} X - X_0 - \int a_s \, ds \in \mathbb{CM}_l^m. \tag{E.1}$$

Then as $f \in C^2(\mathbb{R}^m)$, we get, with the help of the Itô Lemma, that

$$Y \in \mathbb{CA}_b \cap \mathbb{CS}, \quad \nabla f(X) \in \mathbb{CA}_b^m \subseteq \mathbb{PM}_b^m, \quad \nabla^2 f(X) \in \mathbb{CA}_b^{m\times m} \subseteq \mathbb{PM}_b^{m\times m}. \tag{E.2}$$

Since $X \in \mathbb{CS}_l^m$, there exists $n \in \mathbb{N}$ such that the processes $\langle X^{(i)}\rangle, i \leq m$, are n-Lipschitz. Then we get that all components of $\langle\!\langle X\rangle\!\rangle$ are n-Lipschitz processes as follows

$$|\langle X^{(i)}, X^{(j)}\rangle_t - \langle X^{(i)}, X^{(j)}\rangle_s| \stackrel{\text{as}}{\leq} (\langle X^{(i)}\rangle_t - \langle X^{(i)}\rangle_s)^{1/2}(\langle X^{(j)}\rangle_t - \langle X^{(j)}\rangle_s)^{1/2} \stackrel{\text{as}}{\leq} n|t-s|,$$

$s, t \in [0, \infty)$. Hence, there exists $b \in \mathbb{PM}_b^{m\times m}$ such that $\langle\!\langle X\rangle\!\rangle \stackrel{\text{as}}{=} \int b_s \, ds$. From (E.1,E.2) and the Itô rule, we get that

$$\mathbb{CM}_l \ni L \stackrel{\text{def}}{=} Y - Y_0 - \int c_s \, ds \stackrel{\text{as}}{=} \int \nabla f(X)^{\intercal} dM, \tag{E.3}$$

$$c \stackrel{\text{def}}{=} \nabla f(X)^{\intercal} a + \tfrac{1}{2} \operatorname{tr}\{\nabla^2 f(X)\, b\} \in \mathbb{PM}_b. \tag{E.4}$$

To get that $Y \in \mathbb{CA}_b \cap \mathbb{CS}_l$ from the first relation in (E.2), it is enough to realize that

$$\langle Y\rangle \stackrel{\text{as}}{=} \langle f(X)\rangle \stackrel{\text{as}}{=} \langle L\rangle \stackrel{\text{as}}{=} \int g_s \, ds \quad \text{where} \quad g \stackrel{\text{def}}{=} \nabla f(X)^{\intercal} b \, \nabla f(X) \in \mathbb{PM}_b,$$

and then $Y \in \mathbb{BI}_b$ follows from (E.3,E.4). We used that $Y - \int c_s \, ds = L + Y_0 \in \mathbb{CM}$ as Y_0 is a bounded $\mathscr{F}_0$-measurable random variable and as $\mathbb{CM}_l \subseteq \mathbb{CM}$. □

Lemma E.2. Let $T \in \overset{\leftarrow}{\mathbb{C}}\mathbb{I}(\mathscr{F})$ be an increasing process with $T_0 = 0, \lim_{t\to\infty} T_t = \infty$. Then $\tau \stackrel{\text{def}}{=} (\tau_s)_{s\geq 0} \in \mathbb{CI}(\tilde{\mathscr{F}})$ starts from $\tau_0 = 0$ and $\lim_{s\to\infty} \tau_s = \infty$ where

$$\tau_s \stackrel{\text{def}}{=} \inf\{t \geq 0; T_t \geq s\}, \qquad \tilde{\mathscr{F}} \stackrel{\text{def}}{=} (\mathscr{F}_{\tau_s})_{s\geq 0}. \tag{E.5}$$

Moreover, $[\tau_s \leq t] = [T_t \geq s] \in \mathscr{F}_t$ and $[\tau_s < t] = [s < T_{t-}]$ if $s, t \in [0, \infty)$ where $T_{0-} \stackrel{\text{def}}{=} 0$. In particular,

$$[t = \tau_s] = [T_{t-} \leq s \leq T_t], \qquad s, t \in [0, \infty). \tag{E.6}$$

P r o o f. Since the process T is $\mathscr{F}$-adapted, we have that $[T_t \geq s] \in \mathscr{F}_s, s \in [0, \infty)$. Further, we get immediately from the definition of τ_s in (E.5) that $[T_t \geq s] \subseteq [\tau_s \leq t]$. On the other hand, again from the definition of τ_s, we obtain that

$$[\tau_s \leq t] = \cap_{r>t}[\tau_s < r] \subseteq \cap_{r>t}[s \leq T_r] \subseteq [s \leq T_{t+}] = [s \leq T_t]$$

as T is a non-decreasing right-continuous process. Hence,

$$[\tau_s \le t] = [T_t \ge s] \in \mathscr{F}_t, \quad s,t \in [0,\infty), \tag{E.7}$$

which means that τ_s is an $\mathscr{F}$-stopping time whenever $s \in [0,\infty)$, and as τ is obviously a non-decreasing process, we get that $\tilde{\mathscr{F}}$ from (E.5) is a well-defined filtration. Further, as T is an increasing process, we get for every $t \in (0,\infty)$ that

$$[s < T_{t-}] = \cup_{r<t}[s \le T_r] = \cup_{r<t}[\tau_s \le r] = [\tau_s < t], \quad [s < T_{0-}] = \emptyset = [\tau_s < 0]. \tag{E.8}$$

It follows immediately from the definition in (E.5) that τ is a non-decreasing process (with $\tau_0 = 0$ and $\lim_{s\to\infty} \tau_s = \infty$ since T starts from $T_0 = 0$ and has locally bounded trajectories as it is an increasing process defined on $[0,\infty)$). In order to show that τ is a continuous process, assume the contrary. Let $s \in [0,\infty), \omega \in \Omega$ be such that $\tau_{s-}(\omega) < r < t < \tau_{s+}(\omega)$ holds for some $r,t \in [0,\infty)$. Then we get from (E.7,E.8) that

$$\begin{aligned} &\omega \in [t < \tau_{s+}] \subseteq \cap_{u>s}[t < \tau_u] = \cap_{u>s}[T_t < u] = [T_t \le s] \\ &\omega \in [r > \tau_{s-}] \subseteq \cap_{u<s}[r > \tau_u] = \cap_{u<s}[u < T_{r-}] \subseteq [s \le T_{r-}] \quad \text{if} \quad s \in (0,\infty). \end{aligned}$$

Obviously, also if $s = 0$, we have that $[r > \tau_{s-}] = [r > 0] \subseteq [0 \le T_{r-}] = [s \le T_{r-}]$. Thus, in all cases, we have the following contradiction $\omega \in [T_t \le s] \cap [s \le T_{r-}] \cap [T_{r-} < T_t]$ as the process T is increasing by assumption. Finally, $[\tau_s \le t] \in \mathscr{F}_{t\wedge\tau_s} \subseteq \mathscr{F}_{\tau_s} = \tilde{\mathscr{F}}_s$ holds if $t \in [0,\infty)$, and $[\tau_s \le t] = \emptyset \in \tilde{\mathscr{F}}_0 \subseteq \tilde{\mathscr{F}}_s, t \in (-\infty,0)$, which gives that $\tau_s \in \mathbb{L}(\tilde{\mathscr{F}}_s), s \in [0,\infty)$, i. e., τ is an $\tilde{\mathscr{F}}$-adapted process. ☐

Lemma E.3. In the context of Lemma E.2, let $X \in \mathbb{CA}(\mathscr{F})$, then $\tilde{X} \overset{\text{def}}{=} (X_{\tau_s})_{s\ge0} \in \mathbb{CA}(\tilde{\mathscr{F}})$ and the same holds with $\mathbb{CA}$ replaced by $\mathbb{CI}, \mathbb{CFV}, \mathbb{CM}_{loc}$, respectively.

Proof. 1. By Lemma E.2, τ_s are $\mathscr{F}$-stopping times, and as $X \in \mathbb{CA}(\mathscr{F})$, we get that $\tilde{X}_s = X_{\tau_s} \in \mathbb{L}(\mathscr{F}_{\tau_s}) = \mathbb{L}(\tilde{\mathscr{F}}_s), s \in [0,\infty)$. Since τ is a continuous process by the same lemma, we have that $\tilde{X}(\omega) = X(\omega) \circ \tau(\omega)$ is a continuous function whenever $\omega \in \Omega$. Hence, $\tilde{X} \in \mathbb{CA}(\tilde{\mathscr{F}})$.

2. If $X \in \mathbb{CI}(\mathscr{F})$, then obviously $\tilde{X} \in \mathbb{CI}(\tilde{\mathscr{F}})$. If $X \in \mathbb{CFV}(\mathscr{F})$, then there are $Y, Z \in \mathbb{CI}(\mathscr{F})$ such that $X = Y - Z$ and then $\tilde{X} = \tilde{Y} - \tilde{Z} \in \mathbb{CFV}(\tilde{\mathscr{F}})$ as $\tilde{Y} \overset{\text{def}}{=} (Y_{\tau_s})_{s\ge0} \in \mathbb{CI}(\tilde{\mathscr{F}})$ and $\tilde{Z} \overset{\text{def}}{=} (Z_{\tau_s})_{s\ge0} \in \mathbb{CI}(\tilde{\mathscr{F}})$.

3. Let $X \in \mathbb{CM}(\mathscr{F})$ be bounded. Then $\tilde{X} \in \mathbb{CA}_b(\tilde{\mathscr{F}})$ and we obtain from the Optional Sampling Theorem, see Theorem 1.3.22 in [24], that $\tilde{X} \in \mathbb{CM}(\tilde{\mathscr{F}})$ as follows

$$\mathbb{E}[\tilde{X}_s|\tilde{\mathscr{F}}_u] \overset{\text{as}}{=} \mathbb{E}[X_{\tau_s}|\mathscr{F}_{\tau_u}] \overset{\text{as}}{=} X_{\tau_u} = \tilde{X}_u. \qquad 0 \le u \le s < \infty.$$

4. Let $X \in \mathbb{CM}_{loc}(\mathscr{F})$. Then $M^{(n)} \overset{\text{def}}{=} (Y_{u\wedge\nu_n})_{u\ge0} \in \mathbb{CM}(\mathscr{F}), n \in \mathbb{N}$, are bounded where

$$\nu_n \overset{\text{def}}{=} \{t \ge 0; |Y_t| \ge n\}, \quad Y \overset{\text{def}}{=} X - X_0. \quad \text{Put} \quad \tilde{\nu}_n \overset{\text{def}}{=} T_{\nu_n-}.$$

Since ν_n are $\mathscr{F}$-stopping times, we obtain from Lemma E.2 that

$$[\tilde{\nu}_n \le s] = [T_{\nu_n-} \le s] = [\nu_n \le \tau_s] \in \mathscr{F}_{\tau_s} = \tilde{\mathscr{F}}_s, \quad s \in [0,\infty),$$

i. e., $\tilde{\nu}_n$ are $\tilde{\mathscr{F}}$-stopping times, and from (E.6), we obtain that $\tau_{T_{t-}} = t, t \in [0,\infty)$, which gives that $\nu_n = \tau_{\tilde{\nu}_n}$. Since Y is a continuous process, we have that $\nu_n \uparrow \infty$, which ensures that $\lim_n \tilde{\nu}_n = \lim_{t\to\infty} T_{t-} = \infty$. Obviously, $\tilde{\nu}_n \le \tilde{\nu}_{n+1}, n \in \mathbb{N}$. From the previous step of the proof, we obtain that $\tilde{M}^{(n)} \stackrel{\text{def}}{=} (M^{(n)}_{\tau_s})_{s\ge 0} \in \mathbb{CM}(\tilde{\mathscr{F}})$. Since $\nu_n = \tau_{\tilde{\nu}_n}$, we have that

$$\tilde{X}_{s\wedge\tilde{\nu}_n} - \tilde{X}_0 = X_{\tau_s} \wedge \nu_n - X_0 = M^{(n)}_{\tau_s}, \quad s \in [0,\infty),$$

and then we get that $\tilde{X} \in \mathbb{CM}_{loc}(\tilde{\mathscr{F}})$ holds by definition. □

Corollary E.4. Let $m \in \mathbb{N}, X \in \mathbb{CM}_{loc}(\mathscr{F})^m$ and $\tilde{X} \stackrel{\text{def}}{=} (X_{\tau_s})_{s\ge 0}$. Then

$$\widetilde{\langle\!\langle X \rangle\!\rangle} \stackrel{\text{def}}{=} (\langle\!\langle X \rangle\!\rangle_{\tau_s})_{s\ge 0} \stackrel{\text{as}}{=} \langle\!\langle \tilde{X} \rangle\!\rangle. \tag{E.9}$$

P r o o f. By Lemma E.3, $\tilde{X} \in \mathbb{CM}_{loc}(\tilde{\mathscr{F}})^m$. By the Doob-Mayer Theorem

$$Y \stackrel{\text{def}}{=} XX^{\intercal} - \langle\!\langle X \rangle\!\rangle \in \mathbb{CM}_{loc}(\mathscr{F})^{m\times m}, \quad \tilde{X}\tilde{X}^{\intercal} - \langle\!\langle \tilde{X} \rangle\!\rangle \in \mathbb{CM}_{loc}(\tilde{\mathscr{F}})^{m\times m},$$

and Lemma E.3 gives again that $\tilde{Y} \stackrel{\text{def}}{=} \tilde{X}\tilde{X}^{\intercal} - \widetilde{\langle\!\langle X \rangle\!\rangle} \in \mathbb{CM}_{loc}(\tilde{\mathscr{F}})^{m\times m}$. Then $\widetilde{\langle\!\langle X \rangle\!\rangle} - \langle\!\langle \tilde{X} \rangle\!\rangle \in \mathbb{CM}_{loc}(\tilde{\mathscr{F}})^{m\times m} \cap \mathbb{CFV}(\tilde{\mathscr{F}})^{m\times m}$ starts from zero, which gives the equality in (E.9). □

Lemma E.5. In the context of Lemma E.2, let both $X \in \overset{\circ\bullet}{\mathbb{C}}\mathbb{A}(\mathscr{F})^m$ and $(X_{t-})_{t\ge 0}$ attain values in an open convex set $\mathbb{G} \subseteq \mathbb{R}^m$ where $m \in \mathbb{N}$ and $X_{0-} \stackrel{\text{def}}{=} X_0$. If

$$\forall\, t \in (0,\infty) \qquad [\Delta T_t = 0] \subseteq [\Delta X_t = 0], \tag{E.10}$$

then $\tilde{X} \stackrel{\text{def}}{=} \hat{X} - \bar{X} \in \mathbb{CA}(\tilde{\mathscr{F}})^m$ attains values in $\mathbb{G}$ and $X = (\tilde{X}_{T_t})_{t\ge 0}$ where

$$\hat{X} \stackrel{\text{def}}{=} (X_{\tau_s})_{s\ge 0} \quad \bar{X}_s \stackrel{\text{def}}{=} \textstyle\sum_{t>0} 1_{[T_{t-}\le s<T_t]}(T_t - s)\Delta X_t/\Delta T_t, \quad s \in [0,\infty). \tag{E.11}$$

(i) If $X \in \overset{\circ\bullet}{\mathbb{C}}\mathbb{FV}(\mathscr{F})^m$, then $\tilde{X} \in \mathbb{CFV}(\tilde{\mathscr{F}})^m$. (ii) If $X \in \overset{\circ\bullet}{\mathbb{C}}\mathbb{S}(\mathscr{F})^m$, then $\tilde{X} \in \mathbb{CS}(\tilde{\mathscr{F}})^m$.

P r o o f. By assumption $X_t, X_{t-} \in \mathbb{G}, t \in [0,\infty)$. Since $\mathbb{G}$ is a convex set, we get from the definition of $\tilde{X}$ that $\tilde{X}_s(\omega) \in co\{X_t(\omega), X_{t-}(\omega)\} \in \mathbb{G}$ holds with $t = \tau_s(\omega), \omega \in \Omega$, see (E.11) and (E.6) in Lemma E.2. Here, *co* stands for the convex hull of a set. Hence, $\tilde{X}$ attains values in $\mathbb{G}$. As T is an increasing function, we get from (E.6,E.11) that

$$[s = T_t] \subseteq [\bar{X}_s = 0, \tau_s = t] \subseteq [\tilde{X}_s = \hat{X}_s = X_t], \quad s,t \in [0,\infty),$$

which verifies that $X = (\tilde{X}_{T_t})_{t\ge 0}$. As $X \in \overset{\circ\bullet}{\mathbb{C}}\mathbb{A}(\mathscr{F})^m \subseteq \mathbb{PM}(\mathscr{F})^m$ and as τ_s is an $\mathscr{F}$-stopping time by Lemma E.2, we get that $\hat{X}_s = X_{\tau_s} \in \mathbb{L}(\mathscr{F}_{\tau_s})^m = \mathbb{L}(\tilde{\mathscr{F}}_s)^m$, $s \in [0,\infty)$. Further, as $T \in \overset{\circ\bullet}{\mathbb{C}}\mathbb{A}(\mathscr{F})$, we have by (E.6) that for any $A_t \in \mathscr{F}_t$

$$\forall\, r \in [0,\infty) \quad A_t \cap [T_{t-} \le s < T_t] \cap [\tau_s \le r] = A_t \cap [T_{t-} \le s < T_t] \cap [t \le r] \in \mathscr{F}_r,$$

which means that $A_t \cap [T_{t-} \le s < T_t] \in \mathscr{F}_{\tau_s} = \tilde{\mathscr{F}}_s$. As $(T_t - s)\Delta X_t/\Delta T_t \in \mathbb{L}(\mathscr{F}_t)^m$, $t \in (0,\infty)$, we obtain from this that $\bar{X}_s \in \mathbb{L}(\tilde{\mathscr{F}}_s)^m$, and as $\hat{X}_s \in \mathbb{L}(\tilde{\mathscr{F}}_s)^m$, we have that $\tilde{X}_s = \hat{X}_s - \bar{X}_s \in \mathbb{L}(\tilde{\mathscr{F}}_s)^m, s \in [0,\infty)$, i. e., $\tilde{X}$ is $\tilde{\mathscr{F}}$-adapted.

Since X is an rcll process by assumption and as $\tau \in \mathbb{CI}(\tilde{\mathscr{F}})$ holds by Lemma E.2, we immediately get that $\hat{X} = (X_{\tau_s})_{s\geq 0}$ is also an rcll process. As $\cup_t[T_{t-} < s \leq T_t] \subseteq [\Delta\hat{X}_s = 0]$, we have that $\Delta\hat{X}_s = \Delta X_t$ if there exists $t \in (0,\infty)$ such that $s = T_{t-}$, and it is zero otherwise. More precisely, $\Delta\hat{X}_s = \sum_t 1_{[s=T_{t-}]}\Delta X_t, s \in (0,\infty)$.

Since X is an rcll process, we have that

$$\Delta X_r \to 0 \quad \text{as} \quad r \to t^+ \quad \text{if } t \in [0,\infty), \quad \text{and also as} \quad r \to t^- \quad \text{if} \quad t \in (0,\infty). \tag{E.12}$$

The process $\bar{X}$ is obviously continuous on $\cup_t(T_{t-}, T_t)$. From (E.11,E.12), we obtain that it is continuous from the right at every $T_t, t \in [0,\infty)$, and that it has the limit zero from the left at every $T_{t-}, t \in (0,\infty)$, since T is an increasing rcll function. Hence, $\bar{X}$ is an rcll process, and we get from (E.11) that $\Delta\bar{X}_s = \sum_{t>0} 1_{[s=T_{t-}]}\Delta X_t = \Delta\hat{X}_s, s \in (0,\infty)$. This means that the rcll process $\tilde{X} = \hat{X} - \bar{X}$ is continuous as $\Delta\tilde{X}_s = \Delta\hat{X}_s - \Delta\bar{X}_s = 0$, $s \in (0,\infty)$, i. e., $\tilde{X} \in \mathbb{CA}(\tilde{\mathscr{F}})^m$.

(i) If $X \in \overset{\bullet}{\mathfrak{C}}\mathbb{I}(\mathscr{F})^m$, then we obtain that $\tilde{X} \in \mathbb{CA}(\tilde{\mathscr{F}})^m$ have also non-decreasing coordinates as follows. Let $\omega \in \Omega$ and $0 \leq u < s$ be fixed, and put $0 \leq r \overset{\text{def}}{=\!=} \tau_u(\omega) \leq t \overset{\text{def}}{=\!=} \tau_s(\omega)$. (a) If $r = t$, we get from (E.11) and from $X \in \overset{\bullet}{\mathfrak{C}}\mathbb{I}(\mathscr{F})^m$ that $\bar{X}_u(\omega) \geq \bar{X}_s(\omega)$, and consequently, $\tilde{X}_u = X_r - \bar{X}_u \leq X_t - \bar{X}_s = \tilde{X}_s$ holds at ω. Note that inequalities between two elements of $\mathbb{R}^m$ are considered coordinatewise here. (b) If $r < t$, we get from (a),(E.11) and from $X \in \overset{\bullet}{\mathfrak{C}}\mathbb{I}(\mathscr{F})^m$ that $\tilde{X}_u \leq \tilde{X}_{T_r} = X_r \leq X_{t-} = \tilde{X}_{T_{t-}} \leq \tilde{X}_s$ at ω.

If $X \in \overset{\bullet}{\mathfrak{C}}\mathbb{FV}(\mathscr{F})^m$, then there are $Y, Z \in \overset{\bullet}{\mathfrak{C}}\mathbb{I}(\mathscr{F})^m$ such that $X = Y - Z$. Then we obtain from the arguments mentioned just above to Y, Z that $\tilde{Y} \overset{\text{def}}{=\!=} \hat{Y} - \bar{Y}$ and $\tilde{Z} \overset{\text{def}}{=\!=} \hat{Z} - \bar{Z}$ are in $\overset{\bullet}{\mathfrak{C}}\mathbb{I}(\tilde{\mathscr{F}})^m$ where $\hat{Y}, \hat{Z}$ and $\bar{Y}, \bar{Z}$ come from (E.11) with every X replaced Y and Z, respectively. Finally, we get that $\tilde{X} = \tilde{Y} - \tilde{Z} \in \overset{\bullet}{\mathfrak{C}}\mathbb{FV}(\tilde{\mathscr{F}})^m$.

(ii) Let $X \in \overset{\bullet}{\mathfrak{C}}\mathbb{S}(\mathscr{F})$. Then there are $Y \in \overset{\bullet}{\mathfrak{C}}\mathbb{FV}(\mathscr{F})^m, Z \in \mathbb{CM}_{loc}(\mathscr{F})^m$ s.t. $X = Y + Z$, and we get from Lemma E.3 that $\tilde{Z} \overset{\text{def}}{=\!=} (Z_{\tau_s})_{s\geq 0} \in \mathbb{CM}_{loc}(\tilde{\mathscr{F}})^m$ and from the previous part of the proof that $\tilde{Y} \overset{\text{def}}{=\!=} \tilde{X} - \tilde{Z} \in \mathbb{CFV}(\tilde{\mathscr{F}})^m$, which gives that $\tilde{X} \in \mathbb{CS}(\tilde{\mathscr{F}})^m$. □

Corollary E.6. In the context of Lemma E.5 (ii), $\widetilde{\langle\!\langle X^{(c)}\rangle\!\rangle} \overset{\text{def}}{=\!=} (\langle\!\langle X^{(c)}\rangle\!\rangle_{\tau_s})_{s\geq 0} \overset{\text{as}}{=\!=} \langle\!\langle \tilde{X}\rangle\!\rangle$.

Proof. By assumption, there exists $M \in \mathbb{CM}_{loc}(\mathscr{F})^m$ such that $X - M \in \overset{\bullet}{\mathfrak{C}}\mathbb{FV}(\mathscr{F})^m$. Then $X^{(c)} - M \in \mathbb{CFV}(\mathscr{F})^m$, and we have that $\langle\!\langle X^{(c)}\rangle\!\rangle \overset{\text{as}}{=\!=} \langle\!\langle M\rangle\!\rangle$. By Lemma E.5, $\tilde{X} - \tilde{M} \in \mathbb{CFV}(\tilde{\mathscr{F}})^m$ where $\tilde{M} \overset{\text{def}}{=\!=} (M_{\tau_s})_{s\geq 0}$, which gives that $\langle\!\langle \tilde{X}\rangle\!\rangle \overset{\text{as}}{=\!=} \langle\!\langle \tilde{M}\rangle\!\rangle$. Then it is enough to use Corollary E.4 to get that $\langle\!\langle \tilde{X}\rangle\!\rangle \overset{\text{as}}{=\!=} \langle\!\langle \tilde{M}\rangle\!\rangle \overset{\text{as}}{=\!=} (\langle\!\langle M\rangle\!\rangle_{\tau_s})_{s\geq 0} \overset{\text{as}}{=\!=} (\langle\!\langle X^{(c)}\rangle\!\rangle_{\tau_s})_{s\geq 0}$. □

Notation E.7. Further in this appendix, if $X, \tilde{X}$ are as in Lemma E.5, we simply write $\tilde{X} \overset{\text{def}}{=\!=} \mathfrak{T}(X)$. Note that Corollary E.6 says that $\mathfrak{T}(\langle\!\langle X^{(c)}\rangle\!\rangle) \overset{\text{as}}{=\!=} \langle\!\langle \mathfrak{T}(X)\rangle\!\rangle$ if we extend this notation also for matrix valued processes and that $\mathfrak{T}(X) = (X_{\tau_s})_{s\geq 0}$ if $X \in \mathbb{CA}(\mathscr{F})$.

Remark E.8. (Dominated convergence) If $X \in \mathbb{CS}(\mathscr{F})$ and if $Y, Y^{(n)} \in \mathbb{PM}(\mathscr{F})$ are such that $|Y^{(n)}| \leq Y, n \in \mathbb{N}$, and that $\int Y \,\mathrm{d}X$ is well-defined, then by [18, part III, 2.1.19]

$$Y^{(n)} \to Y^{(1)} \quad \Rightarrow \quad \int_0^t Y^{(n)} \,\mathrm{d}X \to \int_0^t Y^{(1)} \,\mathrm{d}X \tag{E.13}$$

in probability as $n \to \infty$ whenever $t \in [0,\infty)$. Note that $\int Y \,\mathrm{d}X$ is well-defined (in the Itô sense) if there exist $A^{\uparrow}, A^{\downarrow} \in \mathbb{CI}(\mathscr{F})$ such that $X - A^{\uparrow} + A^{\downarrow} \in \mathbb{CM}_{loc}(\mathscr{F})$ and that

$\int_0^t |Y|(\mathrm{d}A^\uparrow + \mathrm{d}A^\downarrow) + \int_0^t Y^2 \,\mathrm{d}\langle X\rangle < \infty$ almost surely whenever $t \in [0,\infty)$. Further, note that the conclusion in (E.13) holds also if $Z \stackrel{\text{def}}{=} 1_{\{(t,\omega); Y^{(n)} \not\to Y^{(1)}\}} \in \mathbb{PM}_b(\mathscr{F})$ is such that $\int_0^\infty Z(\mathrm{d}A^\uparrow + \mathrm{d}A^\downarrow + \mathrm{d}\langle X\rangle) \stackrel{\text{as}}{=} 0$ since then we may consider $(1-Z)Y^{(n)}$ instead of $Y^{(n)}$.

Lemma E.9. In the context of Lemma E.2, let $X \in \tilde{\mathbb{C}}\mathbb{S}(\mathscr{F}), Y \in \tilde{\mathbb{C}}\mathbb{A}(\mathscr{F})$ and let $[\Delta T_t = 0] \subseteq [\Delta X_t = \Delta Y_t = 0]$ hold for every $t \in (0,\infty)$. Then for each $t \in (0,\infty)$

$$\int_0^t Y \mathrm{d}X^{(c)} \stackrel{\text{as}}{=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \tilde{Y} \mathrm{d}\tilde{X} \quad \text{where} \quad \tilde{\mathbb{H}} \stackrel{\text{def}}{=} [0,\infty) \setminus \cup_{r>0} (T_{r-}, T_r] \tag{E.14}$$

and where $\mathbb{CS}(\tilde{\mathscr{F}}) \ni \tilde{X} \stackrel{\text{def}}{=} \mathfrak{T}(X)$ and $\mathbb{CA}(\tilde{\mathscr{F}}) \ni \tilde{Y} \stackrel{\text{def}}{=} \mathfrak{T}(Y)$. See Notation E.7.

P r o o f. 1. First, we will show that (the last equality in the following holds)

$$\widetilde{X}^{(c)}_s \stackrel{\text{def}}{=} X^{(c)}_{\tau_s} = X_{\tau_s} - \sum_{t \in (0,\tau_s]} \Delta X_t = \tilde{X}_{T_{\tau_s}} - \sum_{t \in (0,\tau_s]} (\tilde{X}_{T_t} - \tilde{X}_{T_{t-}}) \stackrel{\text{as}}{=} X_0 + \int_0^s 1_{\tilde{\mathbb{H}}} \,\mathrm{d}\tilde{X}, \tag{E.15}$$

$s \in [0,\infty)$. Note that $X = (\tilde{X}_{T_t})_{t\geq 0}$ holds by Lemma E.5, which ensures the middle equality in (E.15). Note also that the point (ii) of Lemma E.5 applied to $X^{(c)}$ says that $\widetilde{X}^{(c)} \in \mathbb{CS}(\tilde{\mathscr{F}})$. Further, $1_{\tilde{\mathbb{H}}}$ is an $\tilde{\mathscr{F}}$-progressive process as $1 - 1_{\tilde{\mathbb{H}}}$ is a sum of processes of type $1_{(T_{r-},T_r]}$ that are left-continuous and they can be easily shown to be $\tilde{\mathscr{F}}$-adapted with the help of Lemma E.2. Finally, since $(s, T_{\tau_s}]$ and $\tilde{\mathbb{H}}$ from (E.14) are disjoint random sets, we get with the help of Remark E.8 that

$$\int_0^s 1_{\tilde{\mathbb{H}}} \,\mathrm{d}\tilde{X} \stackrel{\text{as}}{=} \int_0^{T_{\tau_s}} 1_{\tilde{\mathbb{H}}} \,\mathrm{d}\tilde{X} \stackrel{\text{as}}{=} \tilde{X}_{T_{\tau_s}} - \tilde{X}_0 - \textstyle\sum_{0<t\leq\tau_s} (\tilde{X}_{T_t} - \tilde{X}_{T_{t-}}),$$

which gives the desired equality. Here, keep in mind that $s \leq T_{\tau_s} \in \mathbb{L}(\mathscr{F}_{\tau_s}) = \mathbb{L}(\tilde{\mathscr{F}}_s)$ as τ_s is an $\mathscr{F}$-stopping time and as $T \in \tilde{\mathbb{C}}\mathbb{I}(\mathscr{F})$ is an $\mathscr{F}$-progressive process.

2. Let Y be bounded, and let $t \in (0,\infty)$ be fixed. Then we obtain from (E.15) that

$$\int_0^{T_t} \tilde{Y} \,\mathrm{d}\widetilde{X}^{(c)} \stackrel{\text{as}}{=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \tilde{Y} \,\mathrm{d}\tilde{X} \stackrel{\text{as}}{=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \tilde{Y} \,\mathrm{d}\widetilde{X}^{(c)}. \tag{E.16}$$

Note that $Y = (\tilde{Y}_{T_u})_{u\geq 0}$ by Lemma E.5. Since at every $\omega \in \Omega$ we have that

$$\widetilde{Y}^{(n)}_s \stackrel{\text{def}}{=} \textstyle\sum_{k=0}^\infty Y_{kt/n} 1_{[T_{kt/n} \leq s < T_{(k+1)t/n}]} \to \tilde{Y}_s, \quad n \to \infty,$$

up to countably many $s \in \tilde{\mathbb{H}}$, we obtain from Remark E.8 and (E.16) that

$$\int_0^{T_t} 1_{\tilde{\mathbb{H}}} \tilde{Y} \,\mathrm{d}\tilde{X} \stackrel{\text{as}}{=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \tilde{Y} \,\mathrm{d}\widetilde{X}^{(c)} = \mathbb{P}\lim_{n\to\infty} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \widetilde{Y}^{(n)} \,\mathrm{d}\widetilde{X}^{(c)}. \tag{E.17}$$

As $X^{(c)} = (\widetilde{X}^{(c)}_{T_u})_{u\geq 0}$ holds by Lemma E.5, we obtain from (E.15) and Remark E.8 that

$$\int_0^{T_t} 1_{\tilde{\mathbb{H}}} \widetilde{Y}^{(n)} \,\mathrm{d}\widetilde{X}^{(c)} \stackrel{\text{as}}{=} \int_0^{T_t} \widetilde{Y}^{(n)} \,\mathrm{d}\widetilde{X}^{(c)} \stackrel{\text{as}}{=} \textstyle\sum_{k=0}^{n-1} Y_{kt/n} (X^{(c)}_{(k+1)t/n} - X^{(c)}_{kt/n}) \to \int_0^t Y \mathrm{d}X^{(c)} \tag{E.18}$$

in probability as $n \to \infty$. Then (E.14) follows from (E.17,E.18).

3. In general, put $Y^{[n]} \overset{\text{def}}{=\!=} (-n) \vee Y \wedge n \in \overset{\bullet}{\mathbb{C}}\mathbb{A}(\mathscr{F})$ and $\tilde{Y}^{[n]} \overset{\text{def}}{=\!=} \mathfrak{T}(Y^{[n]}) \to \tilde{Y} = \mathfrak{T}(Y)$ as $n \to \infty$, and use step 2 of this proof and Remark E.8 to get that

$$\int_0^t Y \mathrm{d}X^{(\mathrm{c})} \overset{\text{as}}{=\!=} \mathbb{P}\lim_{n\to\infty} \int_0^t Y^{[n]}\,\mathrm{d}X^{(\mathrm{c})} \overset{\text{as}}{=\!=} \mathbb{P}\lim_{n\to\infty} \int_0^{T_t} 1_{\tilde{\mathbb{H}}}\,\tilde{Y}^{[n]}\,\mathrm{d}\tilde{X} \overset{\text{as}}{=\!=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}}\,Y\,\mathrm{d}\tilde{X}.$$

Note that for fixed $\omega \in \Omega$ and $t \in [0,\infty)$, we have that $(Y_r^{[n]}(\omega))_{r\le t} = (Y_r(\omega))_{r\le t}$ holds for $n \in \mathbb{N}$ large enough, which ensures that also $(\tilde{Y}_s^{[n]}(\omega))_{s\le T_t} = (\tilde{Y}_s(\omega))_{s\le T_t}$ holds for such n, and this ensures the pointwise convergence $\tilde{Y}^{[n]} \to \tilde{Y}$ as $n \to \infty$. □

Notation E.10. If $x \in \mathbb{R}^{m\times n}$ and $m, n \in \mathbb{N}$, we denote $||x|| \overset{\text{def}}{=\!=} \sqrt{\operatorname{tr}\{xx^{\intercal}\}}$.

Lemma E.11. (i) Let $m \in \mathbb{N}$ and let both $X \in \overset{\bullet}{\mathbb{C}}\mathbb{F}\mathbb{V}(\mathscr{F})^m$ and $(X_{t-})_{t>0}$ attain values in an open convex set $\mathbb{G} \subseteq \mathbb{R}^m$. If $g \in C^1(\mathbb{G})$, then $Y \overset{\text{def}}{=\!=} g(X) \in \overset{\bullet}{\mathbb{C}}\mathbb{F}\mathbb{V}(\mathscr{F})$ and

$$Y^{(\mathrm{c})} \overset{\text{as}}{=\!=} Y_0 + \int \nabla g(X)^{\intercal}\mathrm{d}X^{(\mathrm{c})}. \tag{E.19}$$

(ii) If $g \in C^2(\mathbb{G})$, then (i) holds with $\overset{\bullet}{\mathbb{C}}\mathbb{F}\mathbb{V}$ replaced by $\overset{\bullet}{\mathbb{C}}\mathbb{S}$ and with (E.19) replaced by

$$Y^{(\mathrm{c})} \overset{\text{as}}{=\!=} Y_0 + \int \nabla g(X)^{\intercal}\mathrm{d}X^{(\mathrm{c})} + \tfrac{1}{2}\int \operatorname{tr}\{\nabla^2 g(X)\,\mathrm{d}\langle\!\langle X^{(\mathrm{c})}\rangle\!\rangle\}. \tag{E.20}$$

(iii) The point (ii) holds also with g replaced by $g_\infty \in C^1(\mathbb{G})$ and with $\nabla^2 g$ by $\widetilde{\nabla}^2 g_\infty : \mathbb{G} \to \mathbb{R}^{m\times m}$ if each $x \in \mathbb{G}$ has an open neighbourhood $\mathcal{O} \subseteq \mathbb{G}$ and a sequence of $C^2(\mathcal{O})$-functions $(g_n)_{n=1}^{\infty}$ such that both $g_n \to g_\infty$ and $\nabla g_n \to \nabla g_\infty$ uniformly on $\mathcal{O}$ and that

$$\widetilde{\nabla}^2 g_\infty = \lim\nolimits_{n\to\infty} \nabla^2 g_n \ \text{ on } \ \mathcal{O}, \qquad \limsup\nolimits_{n\to\infty} \sup\nolimits_{\mathcal{O}} ||\nabla^2 g_n|| < \infty.$$

Note that here, the coordinates of $\widetilde{\nabla}^2 g_\infty$ are Borel measurable and locally bounded on $\mathbb{G}$.

Proof. (i) By Lemma E.5, there exists $\tilde{X} \in \mathbb{C}\mathbb{F}\mathbb{V}(\tilde{\mathscr{F}})^m$ with values in $\mathbb{G}$ such that $X = (\tilde{X}_{T_t})_{t\ge0}$ holds with $T_t \overset{\text{def}}{=\!=} t + \sum_{s\in(0,t]} ||\Delta X_s||$, where $\tilde{\mathscr{F}}$ comes from (E.5). Then we have that $\tilde{Y} \overset{\text{def}}{=\!=} g(\tilde{X}) \overset{\text{as}}{=\!=} \tilde{Y}_0 + \int \nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X}$, which gives that

$$\Delta Y_t = g(X_t) - g(X_{t-}) = g(\tilde{X}_{T_t}) - g(\tilde{X}_{T_{t-}}) \overset{\text{as}}{=\!=} \int_{T_{t-}}^{T_t} \nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X}, \quad t \in (0,\infty),$$

similarly as $Y_t - Y_0 = g(\tilde{X}_{T_t}) - g(\tilde{X}_0) \overset{\text{as}}{=\!=} \int_0^{T_t} \nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X}$, and then we have that

$$Y_t - Y_0 - \sum\nolimits_{s\in(0,t]} \Delta Y_s \overset{\text{as}}{=\!=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X} \overset{\text{as}}{=\!=} \int_0^t \nabla g(X)^{\intercal}\,\mathrm{d}X^{(\mathrm{c})} \tag{E.21}$$

where $\tilde{\mathbb{H}}$ and the last equality come from Lemma E.9.

(ii) Take the proof of (i) preceding (E.21) and replace $\mathbb{F}\mathbb{V}$ by $\mathbb{S}$ and $\nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X}$ by $\nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X} + \frac{1}{2}\operatorname{tr}\{\nabla^2 g(\tilde{X})\,\mathrm{d}\langle\!\langle \tilde{X}\rangle\!\rangle\}$ and use Remark E.8 to get that

$$\begin{aligned} Y_t - Y_0 - \sum\nolimits_{s\in(0,t]} \Delta Y_s &\overset{\text{as}}{=\!=} \int_0^{T_t} 1_{\tilde{\mathbb{H}}} \,[\nabla g(\tilde{X})^{\intercal}\mathrm{d}\tilde{X} + \tfrac{1}{2}\operatorname{tr}\{\nabla^2 g(\tilde{X})\,\mathrm{d}\langle\!\langle \tilde{X}\rangle\!\rangle\}] && \text{(E.22)} \\ &\overset{\text{as}}{=\!=} \int_0^t \nabla g(X)^{\intercal}\,\mathrm{d}X^{(\mathrm{c})} + \tfrac{1}{2}\int_0^t \operatorname{tr}\{\nabla^2 g(X)\,\mathrm{d}\langle\!\langle X^{(\mathrm{c})}\rangle\!\rangle\}], && \text{(E.23)} \end{aligned}$$

where the last equality again follows from Lemma E.9 and Corollary E.6. Here, we used the Itô rule for $\tilde{X} \in \mathbb{CS}(\tilde{\mathscr{F}})^m$ attaining values in $\mathbb{G}$, see Corollary 15.20 in [22]. Note that the assumption from there that the considered filtration is right-continuous is not essential.

(iii,a) Our first goal is to show that $\nu \overset{\text{def}}{=\!=} \inf\{t \geq 0; L_t \neq 0\} \overset{\text{as}}{=} \infty$ where

$$L \overset{\text{def}}{=\!=} g_\infty(\tilde{X}) - g_\infty(\tilde{X}_0) - \int \nabla g_\infty(\tilde{X})\, \mathrm{d}\tilde{X} - \tfrac{1}{2}\int \operatorname{tr}\{\widetilde{\nabla}^2 g_\infty(\tilde{X})\, \mathrm{d}\langle\!\langle \tilde{X} \rangle\!\rangle\}. \tag{E.24}$$

As $\tilde{X}$ attains values in $\mathbb{G}$ and as any open cover of $\mathbb{G}$ has a countable sub-cover, it is enough to verify that

$$\forall\, x \in \mathbb{G} \;\; \exists\, \delta > 0 \;\; \mathbb{P}(\nu < \infty, ||\tilde{X}_\nu - x|| < \delta) = 0. \tag{E.25}$$

Let $x \in \mathbb{G}$ be fixed and let $\mathcal{O}$ and $g_n \in C^2(\mathcal{O})$ be as in the statement of the lemma. Further, let $L^{(n)}$ be processes defined similarly as L in (E.24) but with g_∞ replaced by g_n and $\widetilde{\nabla}^2 g_\infty$ by $\nabla^2 g_n$. By the Itô rule used in the point (ii), we have that $L^{(n)}$ is a constant zero. Let $\delta > 0$ be such that 2δ-neighbourhood of x is a subset of $\mathcal{O}$, and put

$$\tilde{\nu} \overset{\text{def}}{=\!=} \inf\{t \geq \nu; ||\tilde{X}_t - x|| \geq \delta\}.$$

From the equality $L^{(n)} \overset{\text{as}}{=} 0$ and Remark E.8, we obtain that

$$0 \overset{\text{as}}{=} \lim_{n\to\infty} 1_A[L^{(n)}_{\nu\vee t\wedge\tilde{\nu}} - L^{(n)}_\nu] \overset{\text{as}}{=} 1_A[L_{\nu\vee t\wedge\tilde{\nu}} - L_\nu] \quad \text{where} \quad A \overset{\text{def}}{=\!=} [\nu < \infty, ||\tilde{X}_\nu - x|| < \delta] \in \tilde{\mathscr{F}}_\nu,$$

whenever $t \in [0, \infty)$. Since L is a continuous process starting from $L_0 = 0$, we get by the definition of ν that $[\nu < \infty] \subseteq [L_\nu = 0]$ and that the case $\mathbb{P}(A \cap [\nu < \tilde{\nu}]) > 0$ would lead to a contradiction with the definition of ν combined with the equalities almost surely written above. On the other hand, as $\tilde{X}$ is also a continuous process, we have that $A \subseteq [\nu < \tilde{\nu}]$. Hence, the set A is $\mathbb{P}$-null, and this is what we wanted to show in (E.25).

(b) The point (iii) can be obtained similarly as (ii) with the help of (iii,a). □

Corollary E.12. Let $X, Y \in \overset{\circ\!\bullet}{\mathbb{C}}\mathbb{FV}(\mathscr{F})$. Then also $XY \in \overset{\circ\!\bullet}{\mathbb{C}}\mathbb{FV}(\mathscr{F})$ and

$$(XY)^{(\mathrm{c})} \overset{\text{as}}{=} X_0 Y_0 + \int Y\, \mathrm{d}X^{(\mathrm{c})} + \int X\, \mathrm{d}Y^{(\mathrm{c})}. \tag{E.26}$$

Moreover, if $X, (X_{t-})_{t>0}$ attain only positive values, then also $X^{-1}, \ln X \in \overset{\circ\!\bullet}{\mathbb{C}}\mathbb{FV}(\mathscr{F})$ and

$$(X^{-1})^{(\mathrm{c})} \overset{\text{as}}{=} X_0^{-1} - \int X^{-2}\, \mathrm{d}X^{(\mathrm{c})}, \qquad (\ln X)^{(\mathrm{c})} \overset{\text{as}}{=} \ln X_0 + \int X^{-1}\, \mathrm{d}X^{(\mathrm{c})}.$$

P r o o f. Consider $g(x, y) = xy, x^{-1}, \ln x$ on $\mathbb{R}^2$ or on $(0, \infty) \times \mathbb{R}$ and use Lemma E.11. □

Lemma E.13. Let v be a C^2-function in a neighbourhood of $u_0 \in \mathbb{R}^{m-1}, m \in \mathbb{N}$. Let

(A1) f be a C^2-function in a neighbourhood of $w_0 \overset{\text{def}}{=\!=} \binom{x_0}{u_0}$ where $x_0 \overset{\text{def}}{=\!=} v(u_0)$,

(A2) $f(v(u), u) = 0, \frac{\partial f}{\partial x}(v(u), u) = 0$ hold for u from a neighbourhood of u_0.

Then there exist an open set $\mathcal{O} \ni w_0$ and a sequence of functions $(f_n)_{n=1}^{\infty} \subseteq C^2(\mathcal{O})$ s.t.

$$f_n \to 1_F f,\ \nabla f_n \to 1_F \nabla f \quad \text{uniformly on } \mathcal{O}, \tag{E.27}$$

$$\nabla^2 f_n \to 1_F \nabla^2 f \quad \text{on } \mathcal{O}, \qquad \limsup_{n\to\infty} \sup_{\mathcal{O}} ||\nabla^2 f_n|| < \infty \tag{E.28}$$

hold as $n \to \infty$ with $F \stackrel{\text{def}}{=} \{\binom{x}{u}; x \geq v(u)\}$. In particular, $1_F f \in C^1(\mathcal{O})$.

P r o o f. By (A1) there exists an open set $\mathcal{O}_0 \ni w_0$ such that $f \in C^2(\mathcal{O}_0)$, and we get from (A2) that there exists an open neighbourhood $\mathfrak{O}$ of u_0 such that $v \in C^2(\mathfrak{O})$ and that

$$\binom{v(u)}{u} \in \mathcal{O}_0, \quad f(v(u),u) = \frac{\partial f}{\partial x}(v(u),u) = 0 \quad \text{hold whenever} \quad u \in \mathfrak{O}. \tag{E.29}$$

Obviously, we may assume that $\mathfrak{O}$ is chosen so small that $||\nabla v||, ||\nabla^2 v||$ are bounded on $\mathfrak{O}$. Let us consider $\mathfrak{s} \in C^2(\mathbb{R})$ with values in $[0,1]$ defined by

$$\mathfrak{s}(x) \stackrel{\text{def}}{=} 1_{(-1,\infty)}(x) + 1_{(-1,0)}(x)\big(x - \tfrac{\sin(2\pi x)}{2\pi}\big) = \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x \leq -1. \end{cases} \tag{E.30}$$

Put $\mathcal{O}_1 \stackrel{\text{def}}{=} \mathcal{O}_0 \cap (\mathbb{R} \times \mathfrak{O})$ and consider $\xi_n \in C^2(\mathcal{O}_1)$ and $f_n \in C^2(\mathcal{O}_1)$ defined as follows

$$\xi_n(x,u) \stackrel{\text{def}}{=} \mathfrak{s}(n[x - v(u)]), \quad f_n(x,u) \stackrel{\text{def}}{=} \xi_n(x,u)\, f(x,u). \tag{E.31}$$

Note that $\frac{||\nabla \xi_n||}{n}, \frac{||\nabla^2 \xi_n||}{n^2}$ are bounded on $\mathcal{O}_1$ and that

$$0 \leq \xi_n - 1_F \leq I_n \quad \text{on } \mathcal{O}_1 \quad \text{where} \quad I_n(x,u) \stackrel{\text{def}}{=} 1_{[-1/n < x - v(u) < 0]} \to 0, \quad n \to \infty. \tag{E.32}$$

From (E.29), we have that $\frac{\mathrm{d}}{\mathrm{d}u_i} f(v(u),u) = 0$ if $u \in \mathfrak{O}, i < m$. Then since $f \in C^2(\mathcal{O}_1)$ and $v \in C^2(\mathfrak{O})$, we obtain from the chain rule and from (E.29) that

$$\nabla_u f(v(u),u) = -\frac{\partial f}{\partial x}(v(u),u)\, \nabla_u v(u) = 0 \in \mathbb{R}^{m-1} \quad \text{if} \quad u \in \mathfrak{O}. \tag{E.33}$$

Then we get from (E.29,E.33) that

$$f(v(u),u) = 0, \quad \nabla f(v(u),u) = 0 \in \mathbb{R}^m \quad \text{if} \quad u \in \mathfrak{O}. \tag{E.34}$$

Since $f \in C^2(\mathcal{O}_1)$ and $\mathcal{O}_1 \ni w_0$ is an open set, there exists $\epsilon > 0$ such that

$$\{w \in \mathbb{R}^m; ||w - w_0|| \leq \epsilon\} \subseteq \mathcal{O}_1, \quad \sup\{||\nabla^2 f(w)||; ||w - w_0|| \leq \epsilon\} \leq \tfrac{1}{\epsilon}. \tag{E.35}$$

Let $\delta \in (0,\epsilon)$ be so small that $||\binom{v(u)}{u} - w_0|| \leq \epsilon$ and $u \in \mathfrak{O}$ whenever $||u - u_0|| \leq \delta$. Then we get from (E.34,E.35) and from the definition of I_n in (E.32) that

$$\sup_{||w-w_0||\leq\delta} ||(I_n \nabla f)(w)|| \leq \tfrac{1}{n\delta}, \quad \sup_{||w-w_0||\leq\delta} |(I_n f)(w)| \leq \tfrac{1}{n^2\delta}. \tag{E.36}$$

Since $||\nabla \xi_n||/n, ||\nabla^2 \xi_n||/n^2$ are bounded on $\mathcal{O}_1$, we have from (E.36) and (E.31) that there exist $K_1, K_2, K_3 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$

$$|f_n - 1_F f| = |(\xi_n - 1_F) f| \leq |I_n f| \leq \tfrac{K_1}{n^2}$$

$$||\nabla f_n - 1_F \nabla f|| \leq ||(\xi_n - 1_F)\nabla f|| + ||\nabla \xi_n|| \cdot |I_n f| \leq \tfrac{K_1 ||\nabla \xi_n||}{n^2} + ||I_n \nabla f|| \leq \tfrac{K_2}{n}$$

$$||\nabla^2 f_n - 1_F \nabla^2 f|| \leq J_n \stackrel{\text{def}}{=} I_n(||\nabla^2 f|| + 2||\nabla \xi_n|| \cdot ||\nabla f|| + ||\nabla^2 \xi_n|| \cdot |f|) \leq K_3$$

hold on $\{w \in \mathbb{R}^m; ||w - w_0|| \leq \delta\}$. Moreover, since $I_n = I_n^2 \to 0$ as $n \to \infty$ according to (E.32), we have that also $J_n(w) \to 0$ as $n \to \infty$ whenever $||w - w_0|| \leq \delta$. □

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Petr Dostál, Institute of Information Theory and Automation, The Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 08 Praha 8 and Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 186 75 Praha 8. Czech Republic.
e-mail: dostal@utia.cas.cz, dostal@karlin.mff.cuni.cz, dostal@matfyz.cz