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IMPROVED INFERENCE FOR THE GENERALIZED PARETO DISTRIBUTION UNDER LINEAR, POWER AND EXPONENTIAL NORMALIZATION

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We discuss three estimation methods: the method of moments, probability weighted moments, and L-moments for the scale parameter and the extreme value index in the generalized Pareto distribution under linear normalization. Moreover, we adapt these methods to use for the generalized Pareto distribution under power and exponential normalizations. A simulation study is conducted to compare the three methods on the three models and determine which is the best, which turned out to be the probability weighted moments. A new computational technique for improving fitting quality is proposed and tested on two real-world data sets using the probability weighted moments. We looked back at various maximal data sets that had previously been addressed in the literature and for which the generalized extreme value distribution under linear normalization had failed to adequately explain them. We use the suggested procedure to find good fits.

Keywords: generalized Pareto distribution, generalized extreme value distribution, method of moments, probability weighted moments, L-moments, linear-power-exponential normalization

Classification: 62F10, 62F03

1. INTRODUCTION

Extreme Value Theory (EVT) has risen to prominence as one of the most important statistical areas in applied sciences. EVT dates back to Gnedenko [19], who identified the only three possible limiting types of distribution functions (DFs) of the linearly normalized maximum $X_{n:n}$ (or minimum $X_{1:n}$) of i.i.d random variables (RVs) $X_1, X_2, \ldots, X_n$ with a continuous DF F (say). These limit DFs can be incorporated using the von Mises–Jenkinson form

$$G_\gamma(x;\mu,\sigma) = \exp\left[-\left[1-\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{\frac{1}{\gamma}}\right], \qquad 1-\gamma\left(\frac{x-\mu}{\sigma}\right) > 0, \tag{1}$$

where $\gamma \in \mathbb{R}$ is a shape parameter known as the extreme value index (EVI) and plays an important role in the empirical studies of extreme events, while μ and $\sigma > 0$ are

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the location and scale parameters, respectively. The DF $G_\gamma(x;\mu,\sigma)$, which is known as the generalized extreme value distribution under linear normalization (GEVL), includes the Gumbel, Frèchet, and Weibull types according as $\gamma = 0$ (interpreted as $\gamma \to 0$), $\gamma > 0$ and $\gamma < 0$, respectively. In order to broaden the class of the limit laws in EVT to solve more approximation problems, Pancheva [24] introduced power normalization $\left|\frac{X_{n:n}}{\alpha_n}\right|^{\frac{1}{\beta_n}} \mathcal{S}(X_{n:n})$, where $\mathcal{S}(x) = \mathrm{sign}(x) = -1, 0, 1$, according as $x < 0, x = 0, x > 0$, respectively. Another incentive to use power normalization is the possibility of achieving a higher rate of convergence (cf. Barakat et al [4]). The DF F is said to belong to the p-max domain of attraction of a non-degenerate DF H under power normalization if for some $\alpha_n > 0, \beta_n > 0$,

$$P\left(\left|\frac{X_{n:n}}{\alpha_n}\right|^{\frac{1}{\beta_n}} \mathcal{S}(M_n) \le x\right) = F^n(\alpha_n |x|^{\beta_n} \mathcal{S}(x)) \xrightarrow[n]{w} H(x), \tag{2}$$

where " $\xrightarrow[n]{w}$ " denotes the weak convergence, as $n \to \infty$. There are only six possible continuous p-types of H in (2), which are known as the p-max stable laws. These laws fulfil the stability relation $H^n(\alpha_n |x|^{\beta_n} \mathcal{S}(x)) = H(x)$, $x \in \mathbb{R}$, for every $n \ge 1$, where $\alpha_n > 0$ and $\beta_n > 0$ are some appropriate constants. Here, two DFs, F and G, are of the same p-type if we can find $\alpha > 0$ and $\beta > 0$, for which $F(x) = G(\alpha |x|^{\beta} \mathcal{S}(x))$, for all x. Consequently, any non-degenerate DF H is a p-max stable if and only if, for every $n \ge 1$, the DFs H and H^n are of the same p-type. Nasri–Roudsari [23] summarized the p-max stable laws by the two von Mises forms

$$H_{1;\gamma}(x;a,b) = \exp[-(1-\gamma \log ax^b)^{\frac{1}{\gamma}}],\ x > 0,\ 1-\gamma\log ax^b > 0, \tag{3}$$

$$H_{2;\gamma}(x;a,b) = \exp[-(1-\gamma \log a(-x)^b)^{\frac{1}{\gamma}}],\ x < 0,\ 1-\gamma\log a(-x)^b > 0. \tag{4}$$

If (2) holds with $H(x) = H_{i;\gamma}(x;a,b)$, $i \in \{1,2\}$, we say that the DF F is in the power max-domain of attraction of $H_{i;\gamma}(x;a,b)$. Each of the families (3) and (4) is called generalized extreme value distribution under power normalization (GEVP). For more details about the EVT under the power normalization and its use in the modeling of extreme events, we refer to Nasri-Roudsari [23] and Barakat et al. ([5, 6, 7, 12, 9, 3, 10, 11]). Once again, in order to broaden the scope of the applications of EVT, Ravi and Mavitha [26] extended EVT under exponential normalization $\mathcal{T}_n(x) = \mathcal{T}_{u_n,v_n}(x) = \exp\{u_n(|\log|x||)^{v_n}$ $\mathcal{S}(\log|x|)\}\mathcal{S}(x)$,$u_n$, $v_n > 0$. Under the exponential transformation, two DFs F and G are of the same e-type if $F(x) = G(\exp\{(u(|\log|x||)^v\ \mathcal{S}(\log|x|))\}\mathcal{S}(x)) = G(\mathcal{T}_{u,v}(x))$, for some constants $u > 0$, $v > 0$. In this case, a non-degenerate DF $\mathcal{E}(.)$ is said to be an e-max-stable law if there exist $u_n > 0$ and $v_n > 0$, such that

$$\begin{aligned} P(\mathcal{T}_n^{-}(X_{n:n})\} \le x) &= P\left(\left\{\left[\exp\left(\left(\frac{|\log|X_{n;n}||}{u_n}\right)^{1/v_n} \mathcal{S}(\log|X_{n:n}|)\right)\right]\right\}\mathcal{S}(X_{n:n}) \le x\right) \\ &= P(X_{n:n} \le \mathcal{T}_n(x)) = F^n(\mathcal{T}_n(x)) \xrightarrow[n]{w} \mathcal{E}(x). \end{aligned} \tag{5}$$

If (5) is satisfied, we say that the DF F belongs to the e-max-domain of attraction of $\mathcal{E}$. Ravi and Mavitha [26] showed that the DF $\mathcal{E}$ is a limit in (5) if and only if, for every $n \geq 1$, the DFs $\mathcal{E}$ and $\mathcal{E}^n$ are of the same e-type (for more details about the exponential transformation, see Barakat et al.[2]). Ravi and Mavitha [26] showed that the possible continuous limit laws in (5) are 12 types and they attract more DFs than the p-max-stable laws. This effectively implies that while the linear and power models may fail to fit the given extreme data, the exponential model succeeds. Barakat et al. [8] summarized these e-max stable DFs by the following von Mises forms:

$$\left.\begin{array}{ll} W_{1;\gamma}(x;a,b)=\exp[-(1-\gamma\log(a(\log x)^b))^{\frac{1}{\gamma}}], & 1-\gamma\log(a(\log x)^b)>0, \\ W_{2;\gamma}(x;a,b)=\exp[-(1-\gamma(-\log(a(-\log x)^b)))^{\frac{1}{\gamma}}], & 1-\gamma(-\log(a(-\log x)^b))>0, \\ W_{3;\gamma}(x;a,b)=\exp[-(1-\gamma\log(a(-\log(-x))^b))^{\frac{1}{\gamma}}], & 1-\gamma\log(a(-\log(-x)))>0, \\ W_{4;\gamma}(x;a,b)=\exp[-(1-\gamma(-\log(a(\log(-x))^b)))^{\frac{1}{\gamma}}], & 1-\gamma(-\log(a(\log(-x))^b)>0, \end{array}\right\} \tag{6}$$

where $\gamma \in \mathbb{R}$ is a shape parameter and when $\gamma = 0$, $W_{i;\gamma}(x;a,b)$ is defined as usual by $\lim_{\gamma\to 0} W_{i;\gamma}(x;a,b)$, $i = 1,2,3,4$. Each DF in (6) is called generalized extreme value distribution under exponential normalization (GEVE), denoted by GEVE(γ, a, b).
The generalized extreme value DFs defined in (1), (3), (4), and (6) provide prevailing parametric approach for modeling extreme events, which is known as the block maxima (BM). Its application consists of partitioning a data set into blocks of equal length and fitting the GEV(L, P, E) to the set of block maxima. The peaks over threshold (POT) strategy is a variation of the BM approach (see Davison and Smith [17]). The techniques for resolving exceedances over a high threshold are crucial for hydrology, environmental science, and other fields of study. In POT approach, we use the observations above an appropriate threshold (see Coles [15]). The POT approach in the linear case is based on the generalized Pareto distribution (GPD) under linear normalization (denoted by GPDL) introduced in the pioneering papers by Balkema and de Haan ([1]) and Pickands ([25]). The GPDL is the limit distribution of scaled excesses over high thresholds, which can be written as

$$V_\gamma(x;\mu,\sigma) = 1+\log G_\gamma(x;\mu,\sigma) = \left\{\begin{array}{ll} 1-\left(1-\gamma\left(\frac{x-\mu}{\sigma}\right)\right)^{\frac{1}{\gamma}}, & 1-\gamma\left(\frac{x-\mu}{\sigma}\right)>0, x>0, \text{ if } \gamma\neq 0, \\ 1-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right), & \left(\frac{x-\mu}{\sigma}\right)>0, \text{ if } \gamma = 0. \end{array}\right. \tag{7}$$

The GPD under power normalization (denoted by GPDP) was derived by Barakat et al. ([5]) for each of the models (3) and (4), respectively, by

$$\left.\begin{array}{l} C_{1;\gamma}(x;a,b) = 1+\log H_{1;\gamma}(x;a,b), \\ C_{2;\gamma}(x;a,b) = 1+\log H_{2;\gamma}(x;a,b). \end{array}\right\} \tag{8}$$

The GPD under exponential normalization (denoted by GPDE) was derived by Barakat et al. ([8]) for each of the models (6) by

$$\left.\begin{array}{l} Q_{1;\gamma}(x;a,b) = 1+\log W_{1;\gamma}(x;a,b)\ , \\ Q_{2;\gamma}(x;a,b) = 1+\log W_{2;\gamma}(x;a,b), \\ Q_{3;\gamma}(x;a,b) = 1+\log W_{3;\gamma}(x;a,b), \\ Q_{4;\gamma}(x;a,b) = 1+\log W_{4;\gamma}(x;a,b). \end{array}\right\} \tag{9}$$

The GPDs have been widely used in the extreme value framework. The success of any of the GPDL, GPDP and GPDE when it is used to fit real data sets depends basically on the parameter estimation process. Several methods exist in the literature for estimating the GPDL parameters. Mostly, the estimation is performed by the method of moments (MOM), the probability weighted moments (PWM) and the L-moments (LM). In this paper, we limit ourselves to estimate the parameters of GPD (L, P, E) by the approaches MOM, PWM, and LM with comparison via a simulation study. Moreover, we suggest a new computational technique to improve the parameter estimation process. This technique is applied to two real-world data sets.

The paper is structured as follows: in Section 2, we discuss the methods MOM, PWM, and LM for the parameter scale and EVI in the GPDL. Meanwhile, we are adapting these methods to use in the GPDP and GPDE. In Section 3 , we conduct a simulation study to compare the three methods and determine the best. It turns out that the best method is PWM. A new computational technique is suggested in Section 4 to improve the quality of fitting and in Section 5 is applied to two real-world data sets via the PWM. In Section 6 , we revisited some maximum data sets that had previously been studied by Barakat et al. ([8]), where the EVT under linear, power, and exponential normalization did not succeed to describe these data sets via the BM method, where the maximum likelihood estimate (MLE) was used. After applying the suggested method to these MLEs in GEVL, we reveal that the GEVL could fit these data sets.

2. METHODS OF ESTIMATION

2.1. The MOM method

The ancient and direct MOM has been widely used to estimate the parameters of the two-parameter GPDL (7), with $\mu = 0$, as well as many other univariate continuous distributions.

The MOM for GPDL. The MOM estimates of the parameters γ and σ are easily obtained by using the expressions for the mean and variance of the RV $X \sim V_\gamma(x; 0, \sigma)$, which are $E(X) = \frac{\sigma}{1+\gamma}$, $\gamma > -1$ and $Var(X) = \frac{\sigma^2}{(1+\gamma)^2(1+2\gamma)}$, $\gamma > -\frac{1}{2}$ (cf. de Zea Bermudez and Kotz, [18]). Thus, the MOM estimates of γ and σ are given by $\hat{\gamma} = \frac{1}{2}\big(\frac{\bar{x}^2}{s^2} - 1\big)$ and $\hat{\sigma} = \frac{1}{2}\bar{x}\big(\frac{\bar{x}^2}{s^2} + 1\big)$, where $\bar{x}$ and s^2 stand for the sample mean and variance, respectively.

The MOM for GPDP. Clearly, if $X \sim C_{1;\gamma}(x; a, b)$, then $Z \sim V_\gamma(x; \mu, \sigma)$, where $Z = \log X$, $\sigma = \frac{1}{b}$ and $\mu = \frac{-\log a}{b}$. Therefore, by applying the transformation $z = \log x$ on the items in the sample that we have (see Remark 2.1), the MOM estimates of the parameters γ and b, when $a = 1$, can be easily obtained by using the corresponding MOM estimates for γ and σ, when $\mu = 0$, in the GPDL, as $\hat{\gamma} = \frac{1}{2}\big(\frac{\bar{z}^2}{s^2} - 1\big)$ and $\hat{\sigma} = \frac{1}{2}\bar{z}\big(\frac{\bar{z}^2}{s^2} + 1\big)$, where $\bar{z}$ and s^2 are the transformed sample mean and variance, respectively. The MOM estimates for the parameters in $C_{2;\gamma}(x; 1, b)$ can be obtained by a similar way (by using the transformation $z = \log|x|$) (see Remark 2.1).

The MOM for GPDE. Let $X \sim Q_{1;\gamma}(x; a, b)$. Further, let $Z = \log\log X$ (note that in $Q_{1;\gamma}(x; a, b)$, we have $x > 1$). Then, $Z \sim V_\gamma(x; \mu, \sigma)$, where $\sigma = \frac{1}{b}$ and $\mu = \frac{-\log a}{b}$. Therefore, by applying the transformation $z = \log\log x$ on the items in the sample that we have, the MOM estimates of the parameters γ and b, when $a = 1$, can be easily obtained by using the corresponding MOM estimates for γ and σ, when $\mu = 0$, in the GPDL, as $\hat{\gamma} = \frac{1}{2}\big(\frac{\bar{z}^2}{s^2} - 1\big)$ and $\hat{\sigma} = \frac{1}{2}\bar{z}\big(\frac{\bar{z}^2}{s^2} + 1\big)$, where $\bar{z}$ and s^2 are the transformed sample mean and variance, respectively. The MOM estimates for the parameters in $Q_{i;\gamma}(x; 1, b),\ i = 2, 3, 4$, can be obtained by a similar way (see Remark 2.1).

Remark 2.1. For any DF F, Christoph and Falk [16] demonstrated that the F's upper tail behavior indicates whether F belongs to the domain of attraction of the limit law (3), in which case $C_{1;\gamma}(x; a, b)$ would apply, or that of (4), in which case $C_{2;\gamma}(x; a, b)$ would apply. The right endpoint of F should be positive in the first case and negative in the second. As a result, only positive data (or negative data) larger than the chosen threshold can be used to apply the modeling under power normalization using $C_{1;\gamma}(x; a, b)$ (or using $C_{2;\gamma}(x; a, b)$). Simply put, this means that the appropriate threshold should be established so that the sample located after it only contains positive or negative values (see, Barakat et al. [10]). Additionally, in order to apply the models $Q_{i;\gamma}(x; a, b), i = 1, 2, 3, 4$, the appropriate threshold should be selected so that the sample lies after that threshold in either $(1, +\infty)$, or $(0, 1)$, or $(-1, 0)$, or $(-\infty, -1)$, according to $Q_{1;\gamma}$, or $Q_{2;\gamma}$, or $Q_{3;\gamma}$, or $Q_{4;\gamma}$ (see Barakat et al.,[8]). The associated necessary transformations are $\log\log x$, $\log \mid \log x \mid$, $\log \mid \log \mid x \mid\mid$, and $\log\log \mid x \mid$.

2.2. The PWM method

The PWM method was first proposed in the statistical literature in the early 1970s and is now widely utilized in hydrological applications, inter alia. The PWM of an RV X with DF F is defined as

$$M_{p,r,s} = E[X^p F^r(X)(1 - F(X))^s] = \int_0^1 x^p(F)F^r(1 - F)^s\, \mathrm{d}F, \tag{10}$$

where p, r and s are real numbers. These moments are especially handy for any distribution F that has a simple quantile function $x(F)$ (a simple inverse) as observed by Greenwood et al. ([20]). Many a time, PWM are much more simple in expressing the parameters of a distribution than by the ordinary moments about the origin of order $p > 0$, $M_{p,0,0} = E[X^p]$. Consequently, for several distributions it is most useful to consider either the moments

$$\alpha_s = M_{1,0,s} = E[X(1 - F(X))^s], \tag{11}$$

or

$$\beta_r = M_{1,r,0} = E[XF^r(X)], \tag{12}$$

for $s, r \geq 0$, where s and r are preferably chosen to be small. Since the order p is set to be 1 in the above formulas, the PWM will depend directly on the observations. For more details about the PWM, see Castillo et al. ([14]) and de Zea Bermudez and Kotz ([18]).

The PWM for GPDL. By using the PWM for $V_\gamma(x; 0, \sigma)$ given in (7), we get

$$\alpha_s = E[X(1 - F(X))^s] = \frac{\sigma}{(s + 1)(s + 1 + \gamma)},\ \gamma > -1, s = 0, 1, 2. \tag{13}$$

Thus, the following expressions can easily be obtained for the GPDL parameters: $\gamma = \frac{\alpha_0}{\alpha_0-2\alpha_1} - 2$ and $\sigma = \frac{2\alpha_0\alpha_1}{\alpha_0-2\alpha_1}$, where α_0 and α_1 are given in (13). The quantities α_0 and α_1 are then replaced by appropriate sample estimates

$$a_s = \frac{1}{n}\sum_{i=1}^{n} x_{i:n}(1-p_{i:n})^s, \ s = 0, 1. \tag{14}$$

Similarly to those based on the "regular" moments, the associated estimators based on the PWM are also consistent (see Landwehr et al.,[22]). Obviously, the estimate of α_0 is simply the sample mean $\bar{x}$. The plotting positions, $p_{i:n}$, imply that $1-p_{i:n}$ estimates the survival function $1-F$. The expressions for $p_{i:n}$ is available in the literature, such as $p_{i:n} = \frac{i-\nu}{n}$, $0 \le \nu \le 1$, or $p_{i:n} = \frac{i-\nu}{n+1-2\nu}$ $-0.5 < \nu < 0.5$.

The PWM for GPDP and GPDE. Again, if $X_1 \sim C_{1;\gamma}(x; a, b)$ and $X_2 \sim Q_{1;\gamma}(x; a, b)$, then $Z_i \sim V_\gamma(x; \mu, \sigma)$, $i = 1, 2$, where $Z_1 = \log X_1$, $Z_2 = \log\log X_2$, $\sigma = \frac{1}{b}$, and $\mu = \frac{-\log a}{b}$. Therefore, by applying the transformation $z_1 = \log x_1$, or $z_2 = \log\log x_2$ on the items in the given sample and by using the PWM for GPDL given in (13), we have

$$\alpha_s = E[Z_i(1-F(Z_i))^s] = \frac{\sigma}{(s+1)(s+1+\gamma)}, \ \gamma > -1, \ i = 1, 2, s = 0, 1, 2.$$

Consequently, $\gamma = \frac{\alpha_0}{\alpha_0-2\alpha_1} - 2$ and $\sigma = \frac{2\alpha_0\alpha_1}{\alpha_0-2\alpha_1}$, where the quantities α_0 and α_1 are then replaced by transformed sample estimates given by (14). The PWM estimates for the parameters in $C_{2;\gamma}(x; 1, b)$, and $Q_{i;\gamma}(x; 1, b)$, $i = 2, 3, 4$, can be obtained by a similar way (by using the transformations $z = \log|x|$ and $z = \log|(\log|x|)|$, respectively).

2.3. The LM method

The LM are expectations of certain linear combinations of order statistics (OSs). They can be defined for any RV whose mean exists and they are used in many aspects of the statistical inference. The LM of an i.i.d. random sample $X_1, X_2, \ldots, X_n$ (drawn from a DF F) with the associated OSs $X_{1:n} < X_{1:n} < \cdots < X_{1:n}$, are defined by (cf. Hosking, [21] and de Zea Bermudez and Kotz, [18]).

$$L_r = \frac{1}{r}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}E(X_{r-i:r}), \ r = 1, 2, \ldots. \tag{15}$$

The expectation of an OS $X_{j:r}$, $1 \le j \le r$, may be written as

$$E(X_{j:r}) = \frac{r!}{(j-1)!(r-j)!}\int_0^1 x(F)F^{j-1}(1-F)^{r-j}\,\mathrm{d}F = \frac{M_{1,j-1,r-j}}{B(j, r-j+1)}, \tag{16}$$

where $B(.,.)$ is the usual Beta function and $M_{1,j-1,r-j}$ is given in (10). Expression (16) clarifies the relationship between the PWM and LM. On the other side, the natural estimator of L_r based on an observed sample of data is a linear combination of the ordered data values, i. e., an L-statistic.

The LM for GPDL, GPDP, and GPDE. We require L_1 and L_2 to estimate the parameters σ and γ in $V_\gamma(x;0,\sigma)$ presented in (7). In light of (15), L_1 and L_2 are given by

$$L_1 = E(X_{1:1}) = E(X) = \int_0^1 x(F)\,\mathrm{d}F = \beta_0,$$

$$L_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) = E(X) = \int_0^1 x(F)(2F-1)\,\mathrm{d}F = 2\beta_1 - \beta_0, \qquad (17)$$

where β_0 and β_1 are the PWM given in (12). On the other hand, we can easily show that $L_1 = \frac{\sigma}{1+\gamma}$ and $L_2 = \frac{\sigma}{(1+\gamma)(2+\gamma)}$. Moreover, the values of $\beta_r,\ r = 0,1$, in the expression (17) can be assessed by the analogous estimates to those presented in (14), $b_r = \frac{1}{n}\sum_{i=1}^{n} x_{i:n}p_{i:n}^{r}, r = 0,1$, respectively. Therefore, the LM estimates of the parameters γ and σ in $V_\gamma(x;0,\sigma)$ are $\hat{\gamma} = \frac{\hat{L}_1 - 2\hat{L}_2}{\hat{L}_2}$ and $\hat{\sigma} = \frac{\hat{L}_1(\hat{L}_1 - \hat{L}_2)}{\hat{L}_2}$, respectively, where $\hat{L}_i, i = 1,2$, are defined by replacing $\beta_r,\ r = 0,1$ by $b_r, r = 0,1$, in the expressions of $L_i,\ i = 1,2$. The same estimates for γ and σ in $C_{i;\gamma}(x;1,b)$, $i = 1,2$, and $Q_{i;\gamma}(x;1,b)$, $i = 1,2,3,4$, can be obtained by using the appropriate transformations given in the preceding two subsections, where the corresponding estimates $b_r, r = 0,1$ are calculated from the order transformed data.

3. SIMULATION STUDY

In this section, extensive simulation studies are conducted to investigate the estimation of the scale and shape parameters in the GPDL (see Table 1), GPDP (see Table 2) and GPDE (see Table 3) via the three estimation-methods MOM, PWM and LM. The simulation study is performed using MATLAB to compare the three estimation-methods based on mean square error (MSE). The following algorithms are used for this study:

An algorithm for the implemented simulation study

1. Generate a random sample of size 10,000 from each of
   $V_\gamma(x;\mu,\sigma)$, for
   $(\gamma,\mu,\sigma) = (0.3,0,4), (0.4,0,2), (0.4,0,2.1), (0.1,0,4), (0.1,0,0.125), (0.4,0,0.125)$,
   $C_{1;\gamma}(x;a,b)$, for
   $(\gamma,a,b) = (0.3,1,4), (0.4,1,2), (0.4,1,2.1), (0.1,1,4), (0.1,1,0.125), (0.4,1,0.125)$,
   and $Q_{1;\gamma}(x;a,b)$, for
   $(\gamma,a,b) = (0.3,1,4),\ (0.4,1,2),\ (0.4,1,2.1),\ (0.1,1,4),\ (0.1,1,0.125),\ (0.4,1,0.125)$.

2. Compute the estimates of the parameters in the three models according to the methods MOM, PWM and LM, which are explained in Section 2, where we took $\nu = 1$ in the application of the methods PWM and LM.

3. Repeat the above two steps 10,000 times. compute the average value of each estimate over these repetitions, as well as the corresponding MSE.

A cursory glance at the Tables 1 , 2 and 3 reveals that the PWM is the best method among the three studied ones. This easy-to-reach decision is based on the closeness between the actual and estimated values and related MSEs. Nonetheless, the PWM's performance is merely acceptable rather than exceptional. In the next section, we suggest a new computational method to improve the estimate of any unknown parameters in any parametric model via any estimation method. We call this method the Scan Method (abbreviated by SM).

sample size (n= 10,000)		average estimate value		MSE	
GPDL	Method	γ	σ	γ	σ
$V_{0.3}(.;0,4)$	MOM	-0.1884789	1.647756	9.544465e-05	0.002213221
	PWM	0.2850895	2.648459	8.892915e-08	0.0007306651
	LM	-4.322255	-4.322255	0.008546096	0.0007256639
$V_{0.4}(.;0,2)$	MOM	1.912345	2.389422	0.0009148744	6.065977e-05
	PWM	0.3651341	1.077717	4.862538e-07	0.0003402421
	LM	-4.400939	4.232211	0.009219605	0.001993106
$V_{0.4}(.;0,2.1)$	MOM	1.687912	2.316379	0.0006634868	1.872798e-05
	PWM	0.3937979	1.190255	1.538619e-08	0.0003310541
	LM	-4.424131	4.282871	0.009308896	0.00190597
$V_{0.1}(.;0,4)$	MOM	-0.4278824	1.810782	0.0001114639	0.001917071
	PWM	0.1135899	3.557115	7.387385e-08	7.845881e-05
	LM	-4.070404	6.179731	0.006956906	0.001900491
$V_{0.1}(.;0,0.125)$	MOM	73.42157	7.359994	2.150421	0.02093805
	PWM	0.1079126	0.1060756	2.504399e-08	1.432533e-07
	LM	-4.127745	3.230458	0.007149532	0.003857547
$V_{0.4}(.,0,0.125)$	MOM	616.6948	31.67153	151.9277	0.3980735
	PWM	0.3400918	0.06815604	1.435596e-06	1.292494e-06
	LM	-4.3314	3.382229	0.008954459	0.004243815

Tab. 1. Parameters estimation for $V_\gamma(x;0,\sigma)$.

sample size (n= 10,000)		average estimate value		MSE	
GPDP	Method	γ	b	γ	b
$C_{1;0.3}(.;1,4)$	MOM	20.28448	0.1372226	0.1597519	0.00596842
	PWM	0.5944732	1.853886	3.468579e-05	0.001842322
	LM	-4.643726	0.2503658	0.009776171	0.005623903
$C_{1;0.4}(.;1,2)$	MOM	29.09076	0.1086893	0.3292638	0.001430823
	PWM	0.6845182	1.933409	3.238024e-05	1.773734e-06
	LM	-4.705233	0.2491176	0.01042536	0.001226236
$C_{1;0.4}(.;1,2.1)$	MOM	29.08513	0.1086822	0.3291346	0.001586138
	PWM	0.649924	2.058392	2.498479e-05	6.925014e-07
	LM	-4.643596	0.2533144	0.01017515	0.001364099
$C_{1;0.1}(.;1,4)$	MOM	9.82774	0.2165181	0.03785157	0.005725894
	PWM	0.5094372	1.584648	6.705554e-05	0.00233357
	LM	-4.478134	0.2553607	0.008383723	0.005608929
$C_{1;0.1}(.;1,0.125)$	MOM	9.828673	0.2164211	0.03785883	3.34313e-06
	PWM	0.4971071	1.615896	6.307763e-05	0.0008891083
	LM	-4.457775	0.2577836	0.008309326	7.052598e-06
$C_{1;0.4}(.;1,0.125)$	MOM	29.03589	0.1088759	0.3280056	1.039946e-07
	PWM	0.6625963	1.933384	2.758273e-05	0.001308101
	LM	-4.674203	0.2510573	0.01029902	6.356179e-06

Tab. 2. Parameters estimation for $C_{1;\gamma}(x;1,b)$.

sample size (n= 10,000)		average estimate value		MSE	
GPDE	Method	γ	b	γ	b
$Q_{1;0.3}(.;1,4)$	MOM	13.55799	0.2287066	0.07030975	0.005689061
	PWM	0.3325494	2.520431	4.237846e-07	0.0008756502
	LM	-4.291154	0.2783529	0.008431477	0.005540263
$Q_{1;0.4}(.;1,2)$	MOM	9.871948	0.2451046	0.03588712	0.001231863
	PWM	0.3710689	1.897615	3.348027e-07	4.193111e-06
	LM	-4.41126	0.2635345	0.009259287	0.001206125
$Q_{1;0.4}(.;1,2.1)$	MOM	10.13077	0.2439227	0.03787513	0.001378009
	PWM	0.4013737	1.954396	7.5478e-10	8.480189e-06
	LM	-4.413785	0.2641721	0.00926901	0.001348106
$Q_{1;0.1}(.;1,4)$	MOM	13.48869	0.2250917	0.07170281	0.005699973
	PWM	0.3026536	2.475476	1.642739e-05	0.0009296693
	LM	-4.326141	0.2752273	0.007836291	0.005549573
$Q_{1;0.1}(.;1,0.125)$	MOM	62.5719	0.09625728	1.561095	3.304576e-07
	PWM	0.5752778	3.693667	9.03556e-05	0.005094154
	LM	-4.579735	0.2665614	0.008759967	0.008759967
$Q_{1;0.4}(.;1,0.125)$	MOM	62.75106	0.1157773	1.555062	3.402318e-08
	PWM	0.4828451	5.21012	2.745327e-06	0.01034338
	LM	-4.486404	0.2759981	0.009550777	9.120165e-06

Tab. 3. Parameters estimation for $Q_{1;\gamma}(x;1,b)$.

4. THE SM AND REAL DATA MODELING

4.1. The SM

The suggested method relies on the fact that in any modeling problem, especially in the extreme modeling problem, our main focus is not the estimation problem itself but to pick up a suitable family (with estimated parameters) to describe the given data. The estimation stage is only a preliminary stage. Therefore, we can consider any obtained estimates of the unknown parameters of any DF with known type as initial estimates, which may be improved by carrying out the Kolmogorov-Smirnov (K-S) test to check the fitting of this DF with several values around these initial values and choose the model with estimated values that yields the best fitting. According to this idea, the aimed improvement has not pertained to the estimated values, but it pertains to the chosen model's fitting quality to the given data. In the K-S test, we have four functions $[H, P, KSSTAT, CV]$. H is equal to 0 or 1, P is the p-value, $KSSTAT$ is the maximum difference between the data and the fitting curve, and CV is a critical value. Moreover,

- we accept H_0, if $H = 0$, $KSSTAT \leq CV$ and $P >$ level of significance and
- we reject H_0, if $H = 1$, $KSSTAT > CV$ and $P \leq$ level of significance.

However, as the model parameters are estimated from fitted data, P-values as well as the decision indicator H are just relative. That is why the P-values are not shown in the subsequent tables (Tables 6 , 9 , 11 , 13). Therefore, if any model could fit a data set, then its fitting quality depends on how small $KSSTAT$ is. Specifically, the model with estimated values that yields the best fitting should have minimum $KSSTAT$ value. Simultaneously, the fit quality can be observed from graphical comparison of empirical and model DFs. Below, we present an algorithm that describes the SM for two unknown parameters θ_1 and θ_2 in any DF $F(.;\theta_1,\theta_2)$. The estimations of the unknown parameters are assumed to be implemented by any estimation method such as the MOM, PWM, LM, and Maximum Likelihood Method. Throughout the following algorithm, we denote the given estimation method by M (say).

The SM algorithm for two parameters

- Apply the method M on a given data to get two initial estimates θ_{01} and θ_{02} for θ_1 and θ_2, respectively.
- Choose two intervals $(C_{1\theta_1}, C_{2\theta_1})$ and $(C_{1\theta_2}, C_{2\theta_2})$ (see Remark 4.2) such that $\theta_{0i} \in (C_{1\theta_i}, C_{2\theta_i}),\ i = 1, 2$. Moreover, divide the interval $(C_{1\theta_i}, C_{2\theta_i})$ by uniformly-disseminated dividers $\theta_{ti} = C_{1\theta_i} + t\ell_i$, $t = 0, 1, \ldots, n_i$, and $\theta_{n_i i} = C_{2\theta_i}$ $i = 1, 2$. (As an illustrative example, if $\theta_{0i} = 0.22$, we may choose $(C_{1i}, C_{2i}) = (0, 0.5)$, while $\ell_i = 0.01,\ i = 1, 2$, the dividers will be $0, 0.01, 0.02, \ldots, 0.5$, and the divider-number is $n_1 = n_2 = 51$).
- Check the fitting of the models $F(.;\theta_{i_1 1}, \theta_{i_2 2})$, $i_t \in \{0, 1, \ldots, n_t\},\ t = 1, 2$.
- Confine the models that fitted the given data and choose the one with the lowest $KSSAT$.

Remark 4.1. Clearly, the preceding algorithm can be simply modified to include more than one unknown parameter. Nonetheless, the extended algorithm will take longer to implement.

Remark 4.2. In the case that the model $F(.;\theta_{01},\theta_{02})$ fitted the given data (according a preliminary test before applying the SM method), we can choose a narrow scan interval. Otherwise, we choose relatively a larger scan interval. Moreover, if no model according to this larger scan interval fits the given data set, then we enlarge the scan interval or take the decision that the model is not suitable for this data.

5. REAL EXTREME DATA MODELING BY USING THE SM

In this section we consider two examples for real extreme data sets and use the suggested SM to model them by GPDL ($V_\gamma(.;0,\sigma)$), GPDP ($C_{1;\gamma}(.;1,b)$), and GPDE ($Q_{1;\gamma}(.;1,b)$) via the PWM.

Example 5.1. In this example, we use Spot Crude Oil Price of West Texas Intermediate monthly data set, from 01/01/1946 to 01/3/2021. This data set is available through the web site

`https://fred.stlouisfed.org/series/WTISPLC`

Table 4 shows the summary statistics for this data collection. Table 5 displays the PWM estimates of the parameters of the models under the title "PWM original", also it displays the new estimates, which corresponding to the best fitting quality via the application of the SM, under title "PWM modified". Finally, in Table 5, we display $(C_{1,\theta_i}, C_{2,\theta_i}, \ell_i, n_i), i = 1,2$, (the characterizations of SM) for each estimate of the parameters $\theta_1 = \gamma$ and $\theta_2 = \sigma,$ or b, in each of the considered models. Table 6 exhibits the fitting result of the models $V_\gamma(.;0,\sigma)$, $C_{1;\gamma}(.;1,b)$, and $Q_{1;\gamma}(.;1,b))$ before applying the suggested method SM and after its application. Finally, Figures 1 , 3 and 5 depict the data set against the estimated models GPDL, GPDP, and GPDE via PWM, while Figures 2 , 4 and 6 depict the data set against the estimated models GPDL, GPDP, and GPDE via PWM after applying the SM. Tables 5 , 6 and the given figures show that, in all cases the models could fit the data set, but the quality of the fitting is improved after applying the SM. Moreover, due to this study the GPDE (namely $Q_{1;0.96}(.;1,3.57)$) is the best favorable model for the given data set, followed by the GPDP (namely $C_{1;0.79}(.;1,1)$), and finally followed by the GPDL model (namely $V_{0.3}(.;0,40)$).

Descriptive statistics for the data set							
n	minimum	maximum	median	mean	SD	skewness	kurtosis
903	1.17	133.93	17.514	25.4860	27.515	1.442	1.455

Tab. 4. Summary statistics.

Parameters estimation for the model GPDL		
Method	$\gamma(=\theta_1, i=1)$	$\sigma\ (=\theta_2, i=2)$
PWM original	0.3756743	42.26858
PWM modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	0.3000 (-0.5,1,0.1,16)	40 (35,50,0.1,16)
Parameters estimation for the model GPDP		
Method	$\gamma(=\theta_1, i=1)$	$b\ (=\theta_2, i=2)$
PWM original	0.8595887	0.9465367
PWM modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	0.7900 (-3,1,0.01,401)	1 (0.1,3,0.1,30)
Parameters estimation for the model GPDE		
Method	$\gamma(=\theta_1, i=1)$	$b\ (=\theta_2, i=2)$
PWM original	0.9982003	3.461584
PWM modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	0.9600 (-1,1,0.01,201)	3.571429 (0.1,1,0.01,91)

Tab. 5. Parameters estimation.

Fitting data set by GPDL			
Method	H	$KSSTAT$	Decision
PWM original	0	0.0512	accept
PWM modified	0	0.0507	accept
Fitting data set by GPDP			
Method	H	$KSSTAT$	Decision
PWM original	0	0.0434	accept
PWM modified	0	0.0432	accept
Fitting data set by GPDE			
Method	H	$KSSTAT$	Decision
PWM original	0	0.0418	accept
PWM modified	0	0.0394	accept

Tab. 6. K-S test.

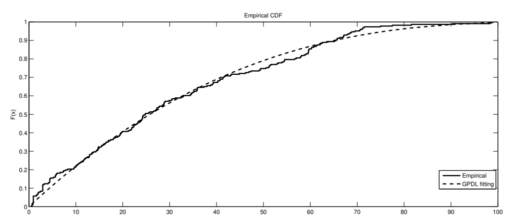

Fig. 1. Fitted GPDL with PWM.

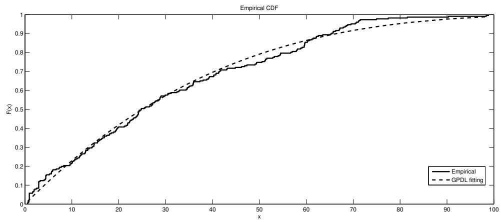

Fig. 2. Fitted GPDL with PWM via SM.

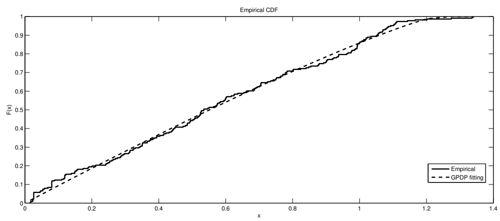

Fig. 3. Fitted GPDP with PWM,

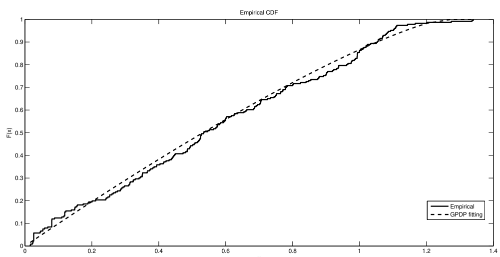

Fig. 4. Fitted GPDP with PWM via SM.

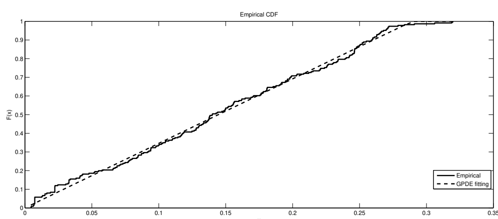

Fig. 5. Fitted GPDE with PWM.

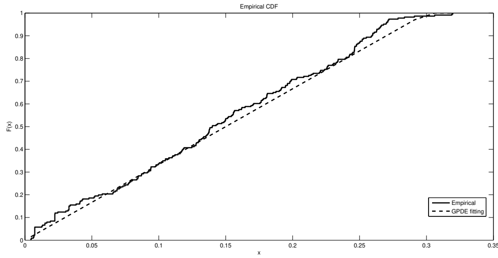

Fig. 6. Fitted GPDE with PWM via SM.

Example 5.2. In this example, we consider the Consumer Price Index: OECD Groups: All Items Non-Food and Non-Energy for the Group of Seven (DISCONTINUED) monthly data set, from 01/01/1971 to 01/09/2017. This data set is available through the web site

`https://fred.stlouisfed.org/series/CPGRLE01G7M659N`

The summary statistics for this data set is given in Table 7. Table 8 shows the original PWM estimates of the parameters of the models (PWM original), as well as the revised estimates (PWM modified), which correspond to the best fitting quality with the application of the SM. Finally, for each estimate of the parameters $\theta_1 = \gamma$ and $\theta_2 = \sigma$, or b, in each of the examined models, we display $(C_{1,\theta_i}, C_{2,\theta_i}, \ell_i, n_i), i = 1, 2,$ (the characterizations of SM) in Table 8. Table 9 shows the fitting results of the models before and after applying the proposed technique SM. Figures 7 and 9 show the data set in comparison to the estimated models GPDL and GPDP via PWM (the model GPDE failed to fit this data set), whereas Figures 8 , 10 , and 11 show the data set in comparison to the estimated models GPDL, GPDP, and GPDE via PWM after applying the SM. Tables 8 , 9, and the accompanying figures reveal that, in all cases, the models were able to fit the data set, with the exception of the model GPDE with initial estimate values (PWM), which failed to do so. After applying the SM, the fitting quality improves, and the model GPDE is able to fit the data set. Furthermore, according to this study, the GPDL ($V_{0.8}(.; 0, 6.3)$) is the most appropriate model for the provided data set, followed by the GPDE ($Q_{1;1.4}(.; 1, 1.587)$), and finally the GPDP model ($C_{1;1.2}(.; 1, 0.91)$).

Descriptive statistics for the data set							
n	minimum	maximum	median	mean	SD	skewness	kurtosis
561	0.7	13.4	2.6	3.984	3.032	1.212	0.550

Tab. 7. Summary statistics.

Parameters estimation for the model GPDL		
Method	$\gamma(=\theta_1, i=1)$	$\sigma\ (=\theta_2, i=2)$
PWM original	0.8444403	6.439844
PWM modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	0.8000 (-1,1,0.1,21)	6.3000 (5,7,0.1,21)
Parameters estimation for the model GPDP		
Method	$\gamma(=\theta_1, i=1)$	$b\ (=\theta_2, i=2)$
PWM original	1.372516	0.8506022
PWM modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	1.2000 (-3,15,0.1,181)	0.90900 (0.0001,3,0.1,30)
Parameters estimation for the model GPDE		
Method	$\gamma(=\theta_1, i=1)$	$b\ (=\theta_2, i=2)$
PWM original	1.862181	1.43906
PWM modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	1.4000 (-1,3,0.2,21)	1.58730 (0.1,3,0.01,291)

Tab. 8. Parameters estimation.

Fitting data set by GPDL			
Method	H	$KSSTAT$	Decision
PWM original	0	0.0731	accept
PWM modified	0	0.0701	accept
Fitting data set by GPDP			
Method	H	$KSSTAT$	Decision
PWM original	0	0.0993	accept
PWM modified	0	0.0810	accept
Fitting data set by GPDE			
Method	H	$KSSTAT$	Decision
PWM original	1	0.2057	reject
PWM modified	0	0.0792	accept

Tab. 9. K-S test.

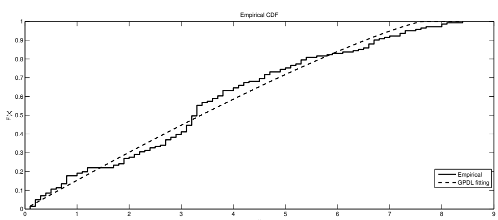

Fig. 7. Fitted GPDL with PWM.

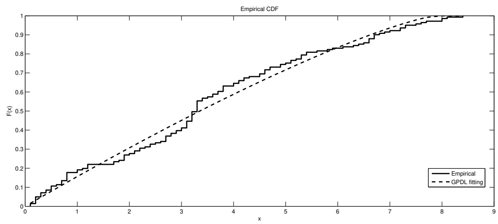

Fig. 8. Fitted GPDL with PWM via SM.

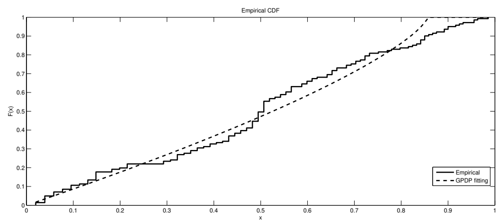

Fig. 9. Fitted GPDP with PWM.

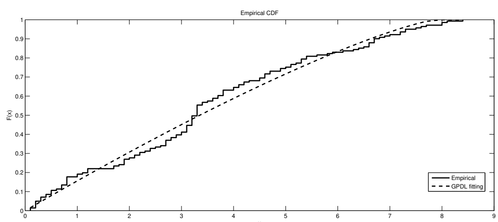

Fig. 10. Fitted GPDP with PWM via SM.

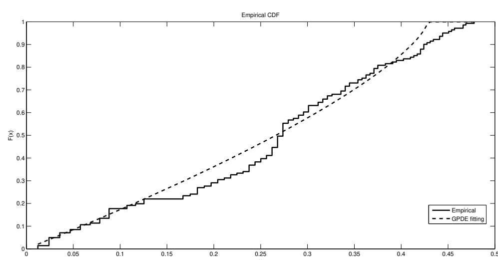

Fig. 11. Fitted GPDE with PWM via SM.

6. APPLICATION OF THE SM TO THE THREE-PARAMETER GEVL VIA THE BM AND MLE

Air pollution is a major environmental and social concern. Furthermore, it is a complex subject with numerous obstacles in terms of management and pollution reduction (cf. Barakat et al., ([6, 12, 11]), BuHamra,([13]), and Coles, ([15])). In Barakat et al. ([8]), a comparison study of the different extreme models was conducted using two real data sets (LB6 and GR4) of air pollutants, each of which contains the maximum data of the three pollutants mentioned, nitric oxide (NO), nitrogen dioxide (NO2), and particulate matter diameter less than 10 mm (PM10). In Barakat et al. ([8]), the three-parameter model GEVL $G_\gamma(.;\mu,\sigma)$ failed to fit these data sets using the BM technique and the MLE method to estimate the unknown parameters γ,σ, and μ. We apply the suggested approach SM to these MLEs in GEVL and find that the model GEVL is able to fit these data sets after this operation is accomplished via Tables $10-13$ and Figures $12-14$, as we did in the previous section.

Estimate parameters of the GEVL, $G_\gamma(x;\mu,\sigma)$, via BM approach for LB6				
	Pollutant	$\gamma(=\theta_1, i=1)$	$\sigma(=\theta_2, i=2)$	$\mu(=\theta_3, i=3)$
MLE original	NO	0.5794	16.6815	21.5258
MLE modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	NO	0.5000 (0.4,2,0.1,17)	18 (15,20,1,6)	22 (20,25,0.1,51)

Tab. 10. Parameters estimation of the GEVL for LB6.

Fitting data of LB6 by the GEVL $G_\gamma(x;\mu,\sigma)$				
	Pollutant	H	$KSSTAT$	Decision
MLE original	NO	1	0.0347	reject H_0
MLE modified	NO	0	0.0242	accept H_0

Tab. 11. K-S test for the maximum data from LB6.

Estimate parameters of the GEVL $G_\gamma(x;\mu,\sigma)$ via BM approach for GR4				
	Pollutant	$\gamma(=\theta_1, i=1)$	$\sigma(=\theta_2, i=2)$	$\mu(=\theta_3, i=3)$
MLE original	NO	1.0065	5.6233	6.1006
MLE modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	NO	1.1000 (0.1,10,1,10)	6 (1,10,1,10)	6.2000 (6,10,0.1,41)
MLE original	NO_2	-0.0537	14.9302	28.8916
MLE modified $(C_{1,\theta_i}, C_{2\theta_i}, \ell_i, n_i)$	NO_2	-0.0500 (-0.05,10,1,11)	16 (5,20,1,16)	28.7000 (25,30,0.1,51)

Tab. 12. Parameters estimation of the GEVL for GR4

Fitting data of GR4 by the GEVL $G_\gamma(x;\mu,\sigma)$				
	Pollutant	H	$KSSTAT$	Decision
MLE original	NO	1	0.0398	reject H_0
MLE modified	NO	0	0.0287	accept H_0
MLE original	NO_2	1	0.0370	reject H_0
MLE modified	NO_2	0	0.0241	accept H_0

Tab. 13. K-S test for the maximum data from GR4.

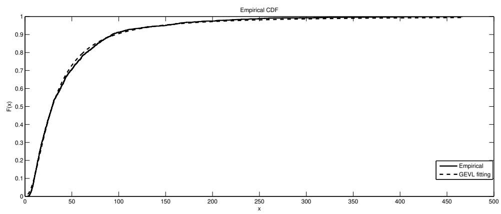

Fig. 12. Fitted GEVL with MLE forNO in LB6 via SM.

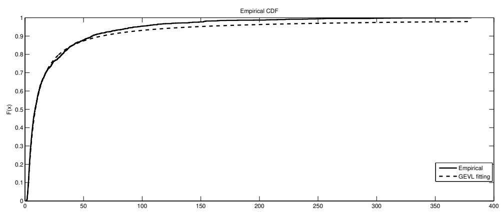

Fig. 13. Fitted GEVL with MLE for NO in GR4 via SM.

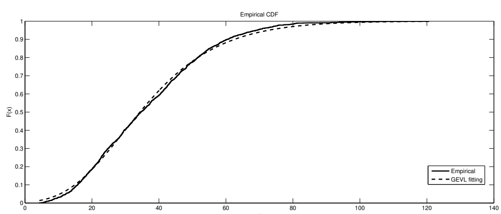

Fig. 14. Fitted GEVL with MLE for NO_2 in GR4 via SM.

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