

Tomáš Madaras; Mária Šurimová  
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*Mathematica Bohemica*, Vol. 148 (2023), No. 1, 105–115

Persistent URL: <http://dml.cz/dmlcz/151530>

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## HOMOGENEOUS COLOURINGS OF GRAPHS

TOMÁŠ MADARAS, MÁRIA ŠURIMOVÁ, Košice

Received January 11, 2021. Published online April 28, 2022.

Communicated by Riste Škrekovski

*Abstract.* A proper vertex  $k$ -colouring of a graph  $G$  is called  $l$ -homogeneous if the number of colours in the neighbourhood of each vertex of  $G$  equals  $l$ . We explore basic properties (the existence and the number of used colours) of homogeneous colourings of graphs in general as well as of some specific graph families, in particular planar graphs.

*Keywords:* proper colouring; homogeneous colouring; planar graph; triangulation

*MSC 2020:* 05C15

## 1. INTRODUCTION

Throughout this paper, we consider connected graphs without loops nor multiple edges. The notation and terminology used here are taken from the book [15]. A proper vertex  $k$ -colouring of a graph  $G$  is a mapping  $\varphi: V(G) \rightarrow \{1, \dots, k\}$  such that, for any pair of adjacent vertices  $x$  and  $y$ ,  $\varphi(x) \neq \varphi(y)$  holds. The *vertex palette* of a vertex  $x$  of  $G$  with respect to a colouring  $\varphi$  is the set  $\{\varphi(y): xy \in E(G)\}$  and the *colour degree* of  $x$  with respect to  $\varphi$  is the cardinality of its vertex palette.

The properly coloured graphs may show great variability of possible colour degrees. For example, it is easy to see that, for each graph and each its proper colouring, there exist two vertices of the same colour degree, and one may construct, for each positive integer  $n$ , an  $n$ -vertex graph and its proper colouring such that all colour degrees, except of two, are different. It is enough to take so called antiregular or quasiperfect graphs defined first in [2], and colour each vertex differently.

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This work was supported by the Slovak Research and Development Agency under the contract No. APVV-19-0153. This work was also supported by the Science Grant Agency project VEGA 1/0368/16, and partially supported by DAAD, Germany (as part of BMBF) and by the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57447800.

The other end of possible colour degrees spectra consists of graphs which can be properly coloured so that their vertices have the same colour degree, say  $l$ ; a graph with this property is called *l-homogeneous*. The set of all positive integers  $l$  for which a graph  $G$  is  $l$ -homogeneous forms the *palette of homogeneity* of  $G$  denoted by  $\text{pal}(G)$ . If  $\text{pal}(G) \neq \emptyset$ , then we set  $p^-(G) = \min \text{pal}(G)$  and  $p^+(G) = \max \text{pal}(G)$ . The graphs whose palette of homogeneity is an integer interval (possibly degenerate one) have a *full homogeneity palette*, and if  $\text{pal}(G) = [1, \delta(G)]$ , then  $G$  is said to be *completely homogeneous*. Furthermore, for  $l \in \text{pal}(G)$ , the set of positive integers  $k$  such that there exists a proper  $l$ -homogeneous  $k$ -colouring of  $G$  forms the *scale of l-homogeneity* of  $G$ , denoted by  $\text{scal}_l(G)$ , with  $\chi_l^-(G) = \min \text{scal}_l(G)$  and  $\chi_l^+(G) = \max \text{scal}_l(G)$ .

The properly homogeneously coloured graphs can be viewed as a possible extension of bipartite graphs, and, also, as special colour-bounded hypergraphs introduced in [3], where, for each hyperedge, a lower and an upper bound for the number of used colours is prescribed. Given a graph  $G$  and a positive integer  $l$ , consider a colour-bounded hypergraph  $\mathcal{H} = (X, \mathcal{E}, \mathbf{s}, \mathbf{t})$  ( $\mathbf{s}, \mathbf{t}$  are positive integer vectors with  $|E(G)| + |V(G)|$  entries) where the vertex set  $X = V(G)$ ,  $\mathcal{E}$  is the union of  $E(G)$  and the set of all closed neighbourhoods of all vertices of  $G$ ; for every  $e \in \mathcal{E}$ ,  $|e| = 2$ , set  $s_e = t_e = 2$  and for every other  $e \in \mathcal{E}$ , set  $s_e = t_e = l + 1$ . Now, proper  $l$ -homogeneous colourings of  $G$  correspond to feasible vertex colourings of  $\mathcal{H}$  (meaning that, for every hyperedge  $e$  of  $\mathcal{H}$ , its number of colours equals  $s_e = t_e$ ). It seems, however, that, despite of ongoing research of colour-bounded hypergraphs (see, for example, [14] and references therein), the homogeneous colourings of graphs have not been studied in detail.

The graphs which are not homogeneously colourable are easy to find: it is enough to take any graph  $G$  with  $\omega(G) > \delta(G) + 1$  or an odd wheel with at least six vertices. On the other hand, if  $G$  is a  $d$ -regular graph, then (as mentioned in [9])  $p^+(G) = d$  and  $\text{scal}_d(G) = [\chi^{(2)}(G), |V(G)|]$  where  $\chi^{(2)}(G)$  is the 2-distance chromatic number of  $G$  (that is, the minimum number of colours such that, in a corresponding proper colouring of  $G$ , any two vertices which are adjacent or have a common neighbour receive distinct colours; for the survey of results, see [11]). Moreover, for  $d = 3$  and  $G$  bipartite, it was proved in [9] that  $2 \in \text{pal}(G)$  and  $\chi_2^-(G) \leq 6$  (the upper bound 6 might be possibly decreased to 4 and the inequality  $\chi_2^-(G) \leq 4$  holds also for bipartite  $G$  with  $d \geq 4$ ), yielding that cubic graphs have a full homogeneity palette. This does not hold in general for regular graphs of higher degree—the octahedron graph  $K_{2,2,2}$  is the smallest regular graph whose homogeneity palette is not full, and the 9-Paley graph is similarly the smallest odd-order regular graph with this property (this can be checked by a lengthy case analysis, which is omitted here, or by computer).

The aim of this paper is to further explore homogeneous graph colourings of graphs in general as well as graphs of particular families, bringing several estimates on  $\chi_l^-$  and  $\chi_l^+$ . The results obtained are contained in Section 2. The final section includes remarks on homogeneous colourings which are not necessarily proper.

## 2. THE RESULTS

We start with several observations on homogeneous colourings of particular graphs. By exploring the 2-distance chromatic number of cycles (see [11]), we conclude that, for an  $n$ -cycle  $C_n$ ,  $\text{pal}(C_n) = \{2\}$  if  $n$  is odd and  $\{1, 2\}$  otherwise; in addition,

$$\text{scal}_2(C_n) = \begin{cases} \{5\} & \text{if } n = 5, \\ [3, n] & \text{if } n \equiv 0 \pmod{3}, \\ [4, n] & \text{otherwise.} \end{cases}$$

**Theorem 1.** *For the complete  $k$ -partite graph  $G \cong K_{n_1, n_2, \dots, n_k}$  with  $n_1 \leq \dots \leq n_k$ ,*

$$\text{pal}(G) = \{(k-1)i : i = 1, \dots, n_1\}$$

and, for each  $p \in \text{pal}(G)$ ,  $\text{scal}_p(G) = \{k/p(k-1)\}$ .

*Proof.* Let  $V_1, \dots, V_k$  be parts of  $G$ ,  $|V_i| = n_i$  for  $i = 1, \dots, k$ , and let  $\varphi$  be a regular  $t$ -colouring of  $G$  with colour classes  $U_1, \dots, U_t$ ; since  $G$  is a complete multipartite graph, the partition  $\{U_1, \dots, U_t\}$  is a refinement of the partition  $\{V_1, \dots, V_k\}$ . If there exist  $i \neq j$  such that the parts  $V_i, V_j$  contain  $r$  and  $s$  sets from  $\{U_1, \dots, U_t\}$  with  $r \neq s$ , then the colour degrees of vertices from  $V_i$  differ from the colour degrees of vertices from  $V_j$ . Thus, for an  $l$ -homogeneous colouring  $\varphi$  of  $G$ , each part  $V_i$  contains the same number of colour classes of  $\varphi$  which yields that  $l = (k-1)i$  for some  $i = 1, \dots, n_1$  and  $\text{scal}_l(G) = \{k \cdot i\} = \{kl/(k-1)\}$ .  $\square$

By this result, the palette of homogeneity of the octahedron graph  $K_{2,2,2}$  is not full; it can be easily checked that, among homogeneously colourable graphs with this property, the octahedron graph is the smallest one.

Concerning an  $n$ -vertex wheel  $W_n$  and  $k \geq 4$ , any of its proper  $k$ -colourings induces  $k-1 \geq 3$  colours on its rim, and a 4-colouring of  $W_n$  which induces a 2-homogeneous 3-colouring on the rim is possible only for  $n \equiv 4 \pmod{6}$ . Thus, we obtain

**Theorem 2** ([1]). *Let  $W_n$  be an  $n$ -vertex wheel. Then*

- (a) for  $n \equiv 0, 2 \pmod{6}$ ,  $W_n$  is not homogeneously colourable,
- (b)  $\text{pal}(W_n) = \{2\}$ ,  $\text{scal}_2(W_n) = \{3\}$  for  $n \equiv 3, 5 \pmod{6}$ ,
- (c)  $\text{pal}(W_n) = \{3\}$ ,  $\text{scal}_3(W_n) = \{4\}$  for  $n \equiv 4 \pmod{6}$ ,
- (d)  $\text{pal}(W_n) = \{2, 3\}$ ,  $\text{scal}_2(W_n) = \{3\}$ ,  $\text{scal}_3(W_n) = \{4\}$  for  $n \equiv 1 \pmod{6}$ .

**Theorem 3.** *Let  $D_n$  be the graph of an  $n$ -sided prism. Then  $\{2, 3\} \subseteq \text{pal}(D_n)$  with  $D_n$  being completely homogeneous for even  $n$ . In addition,*

- (a)  $\text{scal}_3(D_3) = \text{scal}_3(D_6) = \{6\}$ ,
- (b)  $\text{scal}_3(D_n) = [5, 2n]$  for  $n \not\equiv 0 \pmod{4}$ ,  $n \neq 3, 6$ ,
- (c)  $\text{scal}_3(D_n) = [4, 2n]$  for  $n \equiv 0 \pmod{4}$ ,
- (d)  $\text{scal}_2(D_n) = [3, n]$  for  $n \equiv 0 \pmod{3}$ ,
- (e)  $\text{scal}_2(D_n) = [4, n]$  for  $n \not\equiv 0 \pmod{3}$ .

**Proof.** Let  $u_1 \dots u_n u_1, v_1 \dots v_n v_1$  be facial  $n$ -cycles of  $D_n$  with  $u_i$  being adjacent to  $v_i$  for  $i = 1, \dots, n$ . The results on the scale of 3-homogeneity of  $D_n$  follow from observations on the distance-2-chromatic number of  $D_n$  (see, for example, [7], Section 4). For  $k \in \text{scal}_2(C_n)$ , a 2-homogeneous  $k$ -colouring of  $D_n$  can be obtained as follows: taking a 2-homogeneous  $k$ -colouring of  $u_1 \dots u_n u_1, v_{i+1}$  (index modulo  $n$ ) is coloured with the colour of  $u_i$ . A 2-homogeneous 4-colouring of  $D_5$  assigns colour 1 to  $u_4, v_5$ , colour 2 to  $v_1, u_3, u_5$ , colour 3 to  $u_2, v_3$  and colour 4 to  $u_1, v_2, v_4$ .

Consider now a 2-homogeneous 3-colouring of  $D_n$ . Observe that all the three colours are then used on  $u_1, u_2, v_1, v_2$ . Without loss of generality, let  $u_1, v_2$  be coloured by 1,  $v_1$  by 2 and  $u_2$  by 3. Then  $u_3$  has colour 2 and  $v_3$  colour 3. By repeating this argument, we obtain that the restricted colouring of the cycle  $u_1 \dots u_n u_1$  obeys the repeating pattern 132..., implying that  $n$  is divisible by 3.

Finally, consider a 2-homogeneous  $k$ -colouring of  $D_n$  for  $k \geq 4$ . Then, for every vertex of  $D_n$ , exactly two of its neighbours have the same colour. First, consider the case that, for some  $i \in \{1, n\}$ ,  $u_{i-1}$  and  $u_{i+1}$  (indices modulo  $n$ ) have the same colour, say 1,  $u_i$  is coloured by 2 and  $v_i$  by 3. Then a new colour 4 is used on a neighbour—say  $v_{i+1}$ —of  $v_i$ , and, subsequently,  $v_{i-1}$  is coloured by 2 or 4. This yields that  $v_{i+2}$  is coloured by 1 or 3,  $u_{i+2}$  is coloured by 2 or 4 and  $v_{i-2}$  is coloured by 1 or 3. So, if  $u_{i-1}, u_{i+1}$  have the same colour, they belong to a set of 9 vertices using just 4 colours. In the other case, the same colour is used for two other neighbours of  $u_i$ , that is,  $v_i$  and, say,  $u_{i+1}$ . These observations imply that the number  $k$  of the used colours does not exceed the half of the number of vertices of  $D_n$ .  $\square$

Following the observation on 2-distance colouring from the introduction, we further explore relations between homogeneous colourings and other graph colourings. Recall that, for a positive integer  $k$ , a *dynamic  $k$ -colouring* of a graph  $G$  is a proper vertex  $k$ -colouring such that the colour degree of every non-pendant vertex is at least two; the smallest  $k$  for which  $G$  has a dynamic  $k$ -colouring is the *dynamic chromatic number*  $\chi_d(G)$  (see [12]). Now, if we take a non-bipartite graph  $G$  with  $\text{pal}(G) \neq \emptyset$ , we have  $l \geq 2$  for every  $l \in \text{pal}(G)$ . Thus every  $l$ -homogeneous colouring of  $G$  is also its dynamic colouring, hence we obtain

**Lemma 4.** *Let  $G$  be a non-bipartite graph with  $\text{pal}(G) \neq \emptyset$ . Then, for every  $l \in \text{pal}(G)$ ,  $\chi_d(G) \leq \chi_l^-(G)$ .*

Another connection to recently studied colouring concepts concerns so called WORM-colourings, defined in [8] in the following way: Given graphs  $R$ ,  $M$ , and  $G$ , an  $(R, M)$ -WORM colouring of  $G$  is a vertex colouring such that no subgraph of  $G$  isomorphic to  $R$  is rainbow (meaning that its vertices are coloured with different colours) and no subgraph of  $G$  isomorphic to  $M$  is monochromatic. If  $G$  admits at least one  $(R, M)$ -WORM colouring, then  $W_{R,M}^-(G)$  denotes the minimum number of colours in an  $(R, M)$ -WORM colouring of  $G$ . Now, if  $G$  admits an  $l$ -homogeneous colouring, then no star of  $G$  with more than  $l$  leafs is rainbow; thus, we have

**Lemma 5.** *Let  $G$  be a graph and  $l \in \text{pal}(G)$ . Then  $W^-(K_{1,l+1}, K_2) \leq \chi_l^-(G)$ .*

A necessary condition for the  $l$ -homogeneous colourability brings

**Lemma 6.** *If  $l \in \text{pal}(G)$ , then, for each vertex  $v$  of  $G$ , the induced subgraph  $G[N(v)]$  is  $l$ -colourable.*

The converse is not true, as seen on the non-homogeneous wheel  $W_6$  whose local induced subgraphs (5-cycle and 3-path) are 3-colourable. Another example is the circulant graph on 12 vertices with the generating set  $\{1, 3, 4\}$ : its local induced subgraphs are isomorphic to a 6-cycle with an extra edge joining a pair of antipodal vertices; hence, they are bipartite, but the whole circulant graph is easily checked not to be 2-homogeneous.

**Theorem 7.** *Let  $l \in \text{pal}(G)$ . Then  $\chi_l^+(G) \leq \lfloor \frac{l}{\delta(G)} |V(G)| \rfloor$ .*

*Proof.* Consider the sum of colour degrees of vertices of  $G$  with respect to an  $l$ -homogeneous  $k$ -colouring of  $G$ ; this sum equals  $l|V(G)|$ . On the other hand, for every  $j \in \{1, \dots, k\}$  a fixed vertex  $v$  of colour  $j$  contributes  $\deg(v)$  to the sum of colour degrees; hence this sum is at least  $k\delta(G)$  and the result follows.  $\square$

Given a positive integer  $t$ , a set  $S \subset V(G)$  is *distance- $t$  independent* in  $G$ , if the distance of every pair of vertices of  $S$  in  $G$  is at least  $t$ .

**Lemma 8.** *Let  $G$  be a  $k$ -regular graph and  $l \in \text{pal}(G)$ . Then*

$$\chi_l^+(G) \leq \max_{S \subset V(G)} \left\{ |V(G)| + |S| \left( 1 - \left\lceil \frac{k}{l} \right\rceil \right) \right\}$$

where the maximum is taken over all maximal distance-3 independent sets of  $G$ .

**Proof.** Having an  $l$ -homogeneous  $t$ -colouring of  $G$  and a vertex  $x$ , at least  $\lceil \frac{k}{t} \rceil$  neighbours of  $x$  have the same colour. Using this argument on each vertex of a maximal distance-3 independent set  $S$ , we obtain that  $t \leq |S| + |S| + (|V(G)| - |S| - |S| \cdot \lceil \frac{k}{t} \rceil) = |V(G)| + |S|(1 - \lceil \frac{k}{t} \rceil)$ .  $\square$

**Corollary 9.** For a  $k$ -regular graph of diameter 2 and  $l \in \text{pal}(G)$ ,  $\chi_l^+(G) \leq |V(G)| + 1 - \lceil \frac{k}{l} \rceil$ .

Next, we turn our attention to planar graphs and sufficient conditions for their homogeneous colourability.

**Theorem 10.** Let  $G$  be a connected planar graph of minimum degree at least 2 and girth at least 16. Then  $2 \in \text{pal}(G)$  and  $\chi_2^-(G) \leq 5$ .

**Proof.** By [10],  $G$  contains a path consisting of three 2-valent vertices. We proceed by induction on the number of vertices. Assume first that  $G$  contains a cycle  $C = x_1x_2 \dots x_px_1$ ,  $p \geq 16$ , such that  $x_2, \dots, x_p$  are 2-valent and  $\deg(x_1) \geq 3$ . If  $\deg(x_1) = 3$ , consider the graph  $G' = G - \{x_2, \dots, x_p, x_1, y_1, \dots, y_q\}$ ,  $q \geq 1$  where  $P = y_1 \dots y_q$  is a path such that  $\deg(y_q) \geq 3$  and the other vertices of  $P$  are 2-valent (since  $G$  is finite, such a path exists); if  $\deg(x_1) \geq 4$ , consider the graph  $G' = G - \{x_2, \dots, x_p\}$ . Then  $G' \subset G$  is connected planar of minimum degree at least 2 and girth at least 11, thus, it is 2-homogeneously colourable using at most five colours. Now, in the second case, a 2-homogeneous 4-colouring of  $C$  can be chosen such that the colour of  $x_1$  and its neighbours on  $C$  match the colours of  $x_1$  and its neighbours in  $G'$ . A similar argument is used also in the first case: we can extend a 2-homogeneous colouring of  $G'$  to all vertices of  $P$ , and then suitably choose a 2-homogeneous 4-colouring of  $C$  to match the colours (the details are left to the reader).

If the above situation does not occur in  $G$ , then there exists a path  $x_1x_2 \dots x_{p-1}x_p$  with  $p \geq 5$  such that  $x_1$  and  $x_p$  are of degree at least 3 and  $x_2, \dots, x_{p-1}$  are 2-valent. Consider the graph  $G' = G - \{x_2, \dots, x_{p-1}\}$ ; by induction, every component of  $G'$  is 2-homogeneously colourable using at most five colours. Without loss of generality, we consider now several possibilities for colours of  $x_1$ ,  $x_p$  and their vertex palettes in  $G'$ :

*Case 1:*  $x_1$  and  $x_p$  have colour 1.

*Case 1.1:* The vertex palette of  $x_1$  and  $x_p$  is  $\{2, 3\}$ .

*Case 1.1.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 1.

*Case 1.1.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 3.

*Case 1.1.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 3 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 2.

*Case 1.2:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{2, 3\}$  and  $\{2, 4\}$ , respectively.

*Case 1.2.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 1.

*Case 1.2.2:* If  $p = 3k + 2$ , then colour  $x_{p-1}$  with 4 and, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i-2}$  with 1.

*Case 1.2.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 2 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 4.

*Case 1.3:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{2, 3\}$  and  $\{4, 5\}$ , respectively.

*Case 1.3.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 2 and  $x_{3i-2}$  with 1.

*Case 1.3.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 4.

*Case 1.3.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 4 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 5.

*Case 2:*  $x_1$  is coloured with 1 and  $x_p$  with 2.

*Case 2.1:* The vertex palette of  $x_1$  and  $x_p$  is  $\{3, 4\}$ .

*Case 2.1.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 2.

*Case 2.1.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 5,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 4.

*Case 2.1.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 3 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 1.

*Case 2.2:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{2, 3\}$  and  $\{3, 4\}$ , respectively.

*Case 2.2.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 2.

*Case 2.2.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 4.

*Case 2.2.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 3 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 2,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 4.

*Case 2.3:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{2, 3\}$  and  $\{1, 3\}$ , respectively.

*Case 2.3.1:* If  $p = 3k + 1$ , then colour  $x_{p-1}$  with 3,  $x_{p-2}$  with 5 and, for every  $i = 1, \dots, k - 2$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 1.

*Case 2.3.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 1.

*Case 2.3.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 3 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 1.

*Case 2.4:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{2, 3\}$  and  $\{1, 4\}$ , respectively.

*Case 2.4.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 2.



*Case 2.4.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 1.

*Case 2.4.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 4 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 2 and  $x_{3i+1}$  with 1.

*Case 2.5:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{3, 4\}$  and  $\{3, 5\}$ , respectively.

*Case 2.5.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 5,  $x_{3i-1}$  with 4 and  $x_{3i+1}$  with 2.

*Case 2.5.2:* If  $p = 3k + 2$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 3,  $x_{3i-1}$  with 4 and  $x_{3i+1}$  with 5.

*Case 2.5.3:* If  $p = 3k$ , then colour  $x_{p-1}$  with 3 and, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 2,  $x_{3i-1}$  with 3 and  $x_{3i+1}$  with 4.

*Case 2.6:* The vertex palettes of  $x_1$  and  $x_p$  are  $\{2, 3\}$  and  $\{4, 5\}$ , respectively.

*Case 2.6.1:* If  $p = 3k + 1$ , then, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  with 4,  $x_{3i-1}$  with 3 and  $x_{3i-2}$  with 1.

*Case 2.6.2:* If  $p = 3k + 2$ , then colour  $x_{p-1}$  with 4 and, for every  $i = 1, \dots, k$ , colour  $x_{3i}$  by 3,  $x_{3i-1}$  by 2 and  $x_{3i-2}$  by 1.

*Case 2.6.3:* If  $p = 3k$ , then colour  $x_{p-1}$  by 4 and, for every  $i = 1, \dots, k - 1$ , colour  $x_{3i}$  by 3,  $x_{3i-1}$  by 2 and  $x_{3i+1}$  by 1.  $\square$

We conjecture that the upper bound 5 can be decreased to 4; as seen on cycles of lengths non-divisible by 3, this bound then would be best possible. Also, we conjecture that 2-homogeneity holds also for planar graphs of girth smaller than 16. On the other hand, observe that the graph obtained from the graph of 3-cube by deleting a vertex is triangle-free, but not 2-homogeneous.

**Theorem 11.** *Let  $G$  be a connected outerplanar graph with  $\delta(G) = 2$  such that the sizes of all inner faces are divisible by 3. Then  $2 \in \text{pal}(G)$  and  $\chi_2^-(G) = 3$ .*

*Proof.* By induction on the number  $f$  of inner faces of  $G$ . The result clearly holds for cycles  $C_{3k}$ , hence assume  $f > 1$ . Let  $xy$  be a pendant edge in the weak dual  $G^*$  of  $G$  with a pendant vertex  $x$ , and let  $\alpha_x = v_1v_2 \dots v_kv_1$  be a face of  $G$  that corresponds to  $x$  (the labelling of vertices of  $\alpha_x$  is chosen in such a way that the edge  $v_1v_2$  of  $G$  corresponds to the edge  $xy$  in  $G^*$ ). Then the graph  $G - \{v_3, \dots, v_k\}$  is outerplanar and has  $f - 1$  inner faces of proper sizes, thus, by induction, it is 2-homogeneously colourable using three colours; without loss of generality, let  $v_1$  and  $v_2$  have colours 1 and 2 in  $G'$ , respectively. Now, the 3-colouring of  $G$  extended from  $G'$  by colouring  $v_{3i}$  with 3,  $v_{3i+1}$  with 1 and  $v_{3i-1}$  with 2 is easily seen to be 2-homogeneous.  $\square$

Note that, among plane graphs which satisfy the assumptions of Theorem 11, there are so called catacondensed hexagonal systems (for the definition, see [5]). Note also that the outerplanarity condition in Theorem 11 is essential, as seen from the following example: for  $k \geq 1$ , take a cycle  $C_{9k-6} = v_1v_2 \dots v_{9k-6}v_1$  and add a new vertex  $v$  with new edges  $vv_1, vv_{3k-1}, vv_{6k-3}$ . If the resulting graph would be 2-homogeneous using three colours, then two neighbours of  $v$ —say,  $v_1$  and  $v_{3k-1}$ —have the same colour 1. Provided that  $v$  has colour, say, 3, the colours of  $v_2, \dots, v_{3k-3}$  are necessarily 2, 3, 1,  $\dots$ , 2, 3, but then  $v_{3k-2}$  cannot be coloured with one of 1, 2, 3.

In the following, by an *internally even near-triangulation* we mean a plane multigraph with exactly one non-triangular face (the outerface) such that all the vertices which are not incident with the outerface have even degree.

**Theorem 12.** *Let  $G$  be a plane triangulation. Then  $2 \in \text{pal}(G)$  if and only if  $G$  is Eulerian; in addition,  $\text{scal}_2(G) = \{3\}$ .*

**Proof.** If  $G$  is Eulerian, then it is 3-colourable by the Heawood Theorem. Every vertex  $v$  of  $G$  is the center of an even wheel in  $G$  and every 3-colouring of  $G$  yields two colours on the neighbours of  $v$ .

Conversely, let  $G$  be 2-homogeneous. Take a vertex  $v \in V(G)$ . Then every neighbour of  $v$  in  $G$  has colour  $a$  or  $b$  from  $[1, k]$ , and, since  $v$  is a center of a wheel in  $G$ , we obtain that  $v$  has even degree. Thus  $G$  is Eulerian.

Now, assume that there exists an Eulerian triangulation  $G$  with  $\text{scal}_2(G) \neq \{3\}$  (without loss of generality, assume that  $G$  contains a wheel  $W$  with the center coloured by 3 and other vertices coloured by 1 and 2, respectively).

Let  $H$  be a connected subgraph of  $G$  induced by vertices of colours 1, 2 and 3 such that  $W \subseteq H$ . Since  $G$  is 2-homogeneously coloured, all inner faces of  $H$  are triangles. Furthermore, there exists a vertex  $v \in V(G) \setminus V(H)$  of colour  $c \geq 4$  adjacent to two adjacent vertices  $u, w$  of  $H$ ; but then the neighbours of  $u, w$  in  $G$  are coloured with more than two colours, a contradiction.  $\square$

**Theorem 13.** *Let  $G$  be a plane near-triangulation with a non-triangular face  $\alpha$ . Then  $2 \in \text{pal}(G)$  if and only if  $G$  is internally even.*

**Proof.** In a plane near-triangulation  $G$ , every vertex is the center of a wheel or a fan. From the assumption of 2-homogeneity of  $G$  we obtain that no vertex of odd degree is incident only with triangular faces, hence it must be internal.

Now, let  $G$  be an internally even plane near-triangulation. By [4], Lemma 1,  $G$  is 3-colourable, and since every vertex of  $G$  is the center of an even wheel or a fan, any 3-colouring of  $G$  is 2-homogeneous.  $\square$

Note that analogous results do not hold for plane graphs with more than one non-triangular face, see the graph of 4-gonal pyramid with a new 2-valent vertex connecting two nonadjacent 3-valent vertices, which has two non-triangular faces and is not 2-homogeneous.

**Theorem 14.** *Let  $G$  be a plane triangulation with all vertices of odd degree. Then  $3 \in \text{pal}(G)$  and  $\chi_3^-(G) = 4$ .*

*Proof.* By the Four Colour Theorem and Heawood Theorem,  $\chi(G) = 4$ . Now, taking any 4-colouring  $c$  of  $G$ , the neighbours of any vertex  $x$  induce an odd wheel coloured, under  $c$ , in such a way that the neighbours of  $x$  have three colours.  $\square$

Note that Theorem 14 does not hold for non-triangulations with all vertices of odd degree, see the odd wheel  $W_6$  (which is not homogeneous at all). Also, Theorem 14 does not hold if vertices of even degree are allowed in triangulations: the graph of a 5-gonal bipyramid is not 3-homogeneous.

### 3. CONCLUDING REMARKS

New kinds of questions arise when relaxing the requirement of the homogeneous colouring to be proper: here trivially, each graph is 1-homogeneous, but, to decide whether a graph is 2-homogeneous using just two colours is NP-complete, as proved in [13] (the corresponding colouring is known as role R6-colouring, see also [6] for the applications in social network analysis). Various differences between general and regular homogeneous colourings are illustrated at the graph  $G$  on Figure 1: it has no role R6-colouring (the central vertex and one of its neighbours have the same colour, then the contradiction comes after colouring the neighbours of gray vertices), but is 2-homogeneous using three colours; however, 2-homogeneity cannot be achieved by a regular 3-colouring (two neighbours of the central vertex must have the same colour, which is then assigned also to the middle vertex of the outerpath), albeit four colours suffice.

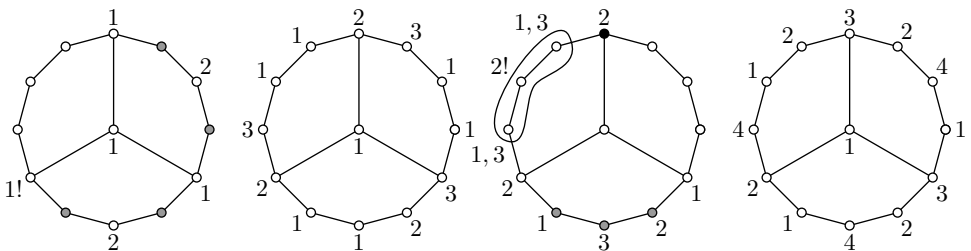


Figure 1. Homogeneous 2-colourings of  $G$ .

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*Authors' address: Tomáš Madaras, Mária Šurimová, Pavol Jozef Šafárik University, Šrobárova 2, 041 80 Košice, Slovakia, e-mail: tomas.madaras@upjs.sk, maria.surimova@student.upjs.sk.*