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Mathematica Bohemica, Vol. 148 (2023), No. 1, 11–34

Persistent URL: <http://dml.cz/dmlcz/151524>

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EXISTENCE AND STABILITY RESULTS OF NONLINEAR
HIGHER-ORDER WAVE EQUATION WITH
A NONLINEAR SOURCE TERM AND A DELAY TERM

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Received August 21, 2020. Published online February 24, 2022.
Communicated by Michela Eleuteri

Abstract. We consider the initial-boundary value problem for a nonlinear higher-order nonlinear hyperbolic equation in a bounded domain. The existence of global weak solutions for this problem is established by using the potential well theory combined with Faedo-Galarkin method. We also established the asymptotic behavior of global solutions as $t \rightarrow \infty$ by applying the Lyapunov method.

Keywords: nonlinear higher-order hyperbolic equation; nonlinear source term; global existence

MSC 2020: 35B40, 35L75, 35L05

1. INTRODUCTION

In this paper we consider the following coupled problem of the nonlinear higher-order hyperbolic equation with nonlinear source term and delay term:

$$(1.1) \quad \begin{cases} u_{tt}(x, t) + \mathcal{A}u(x, t) + \mu_1 g_1(u_t(x, t)) \\ \quad + \mu_2 g_2(u_t(x, t - \tau)) = a|u|^{p-2}u & \text{in } \Omega \times]0, \infty[, \\ D^\alpha u(x, t) = 0, \quad |\alpha| \leq m - 1 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \\ u_t(x, t - \tau) = f_0(x, t - \tau) & \text{in } \Omega \times]0, \tau[, \end{cases}$$

where $\mathcal{A} = (-\Delta)^m$, $m \geq 1$, is a natural number, $\mu_1, \mu_2 > 0$ and $p > 1$ is a real number, Ω is a bounded domain in \mathbb{R}^n , $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, Δ is

the Laplace operator in \mathbb{R}^n , $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $D = \partial_i^\alpha / \partial x_i^{\alpha_i} = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n})$, $x = (x_1, x_2, \dots, x_n)$, a , μ_1 and μ_2 are positive real numbers, g_1 and g_2 are two functions, $\tau > 0$ is a time delay, and the initial data (u_0, u_1, f_0) are in a suitable function space.

When $m = 1$, Liu and Zuazua (see [6], [15]) considered the equation

$$(1.2) \quad \begin{cases} u_{tt}(x, t) - \Delta u(x, t) + a_0 u_t(x, t) + a u_t(x, t - \tau) = 0 & \text{in } \Omega \times]0, \infty[, \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega. \end{cases}$$

It is well-known, in the absence of delay ($a = 0$, $a_0 > 0$), that this system is exponentially stable. In the presence of delay ($a > 0$), Nicaise and Pignotti (see [9]) examined system (1.2) and proved, under the assumption that the weight of the feedback with delay is smaller than the one without delay (i.e., $0 < a < a_0$), that the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is unstable.

In the case for $m = 1$, Benaissa and Louhibi (see [2]) studied the following problem:

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) + \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(u_t(x, t - \tau)) = 0 & \text{in } \Omega \times]0, \infty[, \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \\ u_t(x, t - \tau) = f_0(x, t - \tau) & \text{in } \Omega \times]0, \tau[. \end{cases}$$

They showed global existence of weak solutions using the Faedo-Galerkin method, and obtained general stability estimates by introducing multiplier method and general weighted integral inequalities.

For the initial-boundary value problem of a single higher order nonlinear hyperbolic equation

$$(1.3) \quad \begin{cases} u_{tt}(x, t) + \mathcal{A}u(x, t) + a|u_t|^{r-2}u_t = b|u|^{p-2}u & \text{in } \Omega \times]0, \infty[, \\ D^\alpha u(x, t) = 0, \quad |\alpha| \leq m - 1 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \end{cases}$$

Nakao (see [7]) has used Galerkin's method to present the existence and uniqueness of the bounded solutions, almost periodic solutions to problem (1.3) as the dissipative term is a linear function νu_t . Nakao and Kuwahara (see [8]) study decay estimates of global solutions to problem (1.3) with the degenerate dissipative term $a(x)u_t$ by using a different inequality.

In the case of $m \geq 1$ and $\mu_1 = \mu_2 = 0$, problem (1.1) becomes the following initial-boundary value problem:

$$(1.4) \quad \begin{cases} u_{tt}(x, t) + (-\Delta)^m u(x, t) = a|u|^{p-2}u & \text{in } \Omega \times]0, \infty[, \\ D^\alpha u(x, t) = 0, \quad |\alpha| \leq m-1 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega. \end{cases}$$

Brenner and von Wahl (see [3]) proved the existence and uniqueness of classical solutions to (1.4) in a Hilbert space. Moreover, Ye (see [14]) proved that this system is stable polynomial type with decay rates depending on the smoothness of initial data. Pecher in [11] investigated the existence and uniqueness of Cauchy problem for the equation in (1.4) by use of the potential well method due to Payne and Sattinger (see [10]). Wang in [12] showed that the scattering operators map a band in H^s into H^s if the nonlinearities have critical or subcritical powers in H^s .

Yanbing et al. in [13] studied solutions to

$$(1.5) \quad \begin{cases} u_{tt}(x, t) + \Delta^2 u(x, t) - \Delta u(x, t) - \alpha \Delta u_t(x, t) = f(u) & \text{in } \Omega \times]0, \infty[, \\ \Delta u(x, t) = u(x, t) = 0, \quad |\alpha| \leq m-1 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \end{cases}$$

and proved a global well-posedness result, asymptotic behavior and finite time blow up for a strongly damped nonlinear wave equation.

In this article, we prove the global existence of solutions for problem (1.1) by applying the potential well theory and Faedo-Galerkin method. Meanwhile, we study the asymptotic behavior of global solutions by the Lyapunov method.

This article is organized as follows: in the next section, we give some preliminaries and main results. Then Section 3 contains the proofs of the global existence and general decay results.

2. PRELIMINARIES AND MAIN RESULTS

To state and prove our result, we use the following assumptions:

(A1) $g_1: \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing function of class C^1 and $H: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying

$$(2.1) \quad \begin{aligned} & H(0) = 0 \text{ and } H \text{ is linear on } [0, \varepsilon] \text{ or} \\ & H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon] \text{ such that} \\ & c_1' |s| \leq |g_1(s)| \leq c_1 |s| \text{ if } |s| \geq \varepsilon, \\ & s^2 + g_1^2(s) \leq H^{-1}(sg_1(s)) \text{ if } |s| \leq \varepsilon, \end{aligned}$$

where H^{-1} denotes the inverse function of H and ε, c_1, c_1' are positive constants.

(A2) $g_2: \mathbb{R} \rightarrow \mathbb{R}$ is an odd nondecreasing function of class $C^1(\mathbb{R})$ such that there exist $c_2, \alpha_1, \alpha_2 > 0$,

$$(2.2) \quad |g_2'(s)| \leq c_2,$$

$$(2.3) \quad \alpha_1 s g_2(s) \leq G(s) \leq \alpha_2 s g_1(s),$$

where $G(s) = \int_0^s g_2(r) dr$.

(A3) $\alpha_2 \mu_2 < \alpha_1 \mu_1$.

(A4) $m \geq 1$ is a natural number, p satisfies $2 \leq p < \infty$ if $n \leq 2m$ and $2 \leq p \leq 2(n-m)/(n-2m)$ if $n > 2m$.

Lemma 2.1. *Let q be a real number with $2 \leq q < \infty$ if $n \leq 2m$ and $2 \leq q \leq 2n/(n-2m)$ if $n > 2m$. Then there is a constant C_s depending on Ω and q such that*

$$\|u\|_q \leq C_s \|\mathcal{A}^{1/2}u\|_2 \quad \forall u \in H_0^m(\Omega).$$

Remark 2.2. Let us denote by Φ^* the conjugate function of the differentiable convex function Φ , i.e.,

$$\Phi^*(s) = \sup_{t \in \mathbb{R}^+} (st - \Phi(t)).$$

Then Φ^* is the Legendre transform of Φ , which is given by (see Arnold [1], pages 61–62)

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)] \quad \text{if } s \in (0, \Phi'(r)],$$

and Φ^* satisfies the generalized Young inequality

$$(2.4) \quad AB \leq \Phi^*(A) + \Phi(B) \quad \text{if } A \in (0, \Phi'(r)], B \in (0, r].$$

We introduce, as in Nicaise and Pignotti (see [9]), the new variable

$$z(x, \varrho, t) = u_t(x, t - \varrho\tau), \quad x \in \Omega, \varrho \in (0, 1), t > 0.$$

Then we have

$$(2.5) \quad \tau z_t(x, \varrho, t) + z_\varrho(x, \varrho, t) = 0 \quad \text{in } \Omega \times (0, 1) \times (0, \infty).$$

Therefore, problem (1.1) is equivalent to

$$(2.6) \quad \begin{cases} u_{tt}(x, t) + \mathcal{A}u(x, t) + \mu_1 g_1(u_t(x, t)) \\ \quad + \mu_2 g_2(z(x, 1, t)) = a|u|^{p-2} & \text{in } \Omega \times]0, \infty[, \\ \tau z_t(x, \varrho, t) + z_\varrho(x, \varrho, t) = 0 & \text{in } \Omega \times]0, 1[\times]0, \infty[, \\ D^\alpha u(x, t) = 0, \quad |\alpha| \leq m-1 & \text{on } \partial\Omega \times [0, \infty[, \\ z(x, 0, t) = u_t(x, t) & \text{on } \Omega \times [0, \infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \\ z(x, \varrho, 0) = f_0(x, -\varrho\tau) & \text{in } \Omega \times]0, 1[. \end{cases}$$

We first define the following functionals:

$$(2.7) \quad \begin{aligned} I(t) &= I(u(t)) = \|\mathcal{A}^{1/2}u\|_2^2 - a\|u\|_p^p, \\ J(t) &= J(u(t)) = \frac{1}{2}\|\mathcal{A}^{1/2}u\|_2^2 - \frac{a}{p}\|u\|_p^p. \end{aligned}$$

We denote the total energy by

$$(2.8) \quad \begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\mathcal{A}^{1/2}u(t)\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx - \frac{a}{p}|u|_p^p \\ &= \frac{1}{2}\|u_t\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx + J(u(t)), \end{aligned}$$

where ξ is a positive constant such that

$$\tau \frac{\mu_2(1 - \alpha_1)}{\alpha_1} < \xi < \tau \frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}$$

and

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\mathcal{A}^{1/2}u_0\|_2^2 + \xi \int_\Omega \int_0^1 G(f_0(x, -\varrho\tau)) \, d\varrho \, dx - \frac{a}{p}|u_0|_p^p.$$

Then for problem (2.6) we can define the stable set as

$$\mathcal{W} = \{u \setminus u \in H_0^m(\Omega), I(u) > 0\} \cup \{0\}.$$

We give an explicit formula for the derivative of the energy.

Lemma 2.3. *Let (u, z) be a solution to problem (2.6). Then the energy functional defined by (2.8) satisfies*

$$(2.9) \quad \begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right) \int_\Omega u_t(x, t)g_1(u_t(x, t)) \, dx \\ &\quad - \left(\frac{\xi\alpha_1}{\tau} - \mu_2(1 - \alpha_1)\right) \int_\Omega z(x, 1, t)g_2(z(x, 1, t)) \, dx \\ &\leq 0. \end{aligned}$$

Proof. By multiplying the first equation in (2.6) by u_t , integrating over Ω and using integration by parts, we obtain

$$(2.10) \quad \begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|\mathcal{A}^{1/2} u(t)\|_2^2 - \frac{a}{p} \|u(t)\|_p^p \right) \\ + \mu_1 \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx \\ + \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) \, dx = 0. \end{aligned}$$

We multiply the second equation in (2.6) by $\xi g_2(z)$, we integrate the result over $\Omega \times (0, 1)$ to obtain

$$\begin{aligned} \xi \int_{\Omega} \int_0^1 z_t(x, \varrho, t) g_2(z(x, \varrho, t)) \, d\varrho \, dx &= -\frac{\xi}{\tau} \int_{\Omega} \int_0^1 z_{\varrho}(x, \varrho, t) g_2(z(x, \varrho, t)) \, d\varrho \, dx \\ &= -\frac{\xi}{\tau} \int_{\Omega} \int_0^1 \frac{\partial}{\partial \varrho} (G(z(x, \varrho, t))) \, d\varrho \, dx \\ &= -\frac{\xi}{\tau} \int_{\Omega} (G(z(x, 1, t)) - G(z(x, 0, t))) \, dx. \end{aligned}$$

Hence

$$(2.11) \quad \xi \frac{d}{dt} \int_{\Omega} \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx = -\frac{\xi}{\tau} \int_{\Omega} G(z(x, 1, t)) \, dx + \frac{\xi}{\tau} \int_{\Omega} G(u_t(x, t)) \, dx.$$

Combining (2.10) and (2.11), we obtain

$$\begin{aligned} E'(t) &= -\mu_1 \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx - \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) \, dx \\ &\quad - \frac{\xi}{\tau} \int_{\Omega} G(z(x, 1, t)) \, dx + \frac{\xi}{\tau} \int_{\Omega} G(u_t(x, t)) \, dx, \end{aligned}$$

and recalling (2.3), we obtain

$$(2.12) \quad \begin{aligned} E'(t) &\leq - \left(\mu_1 - \frac{\xi \alpha_2}{\tau} \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx \\ &\quad - \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) \, dx - \frac{\xi}{\tau} \int_{\Omega} G(z(x, 1, t)) \, dx. \end{aligned}$$

From the definition of G and by using Remark 2.2, we obtain

$$G^*(s) = s g_2^{-1}(s) - G(g_2^{-1}(s)) \quad \forall s \geq 0.$$

Hence

$$\begin{aligned} G^*(g_2(z(x, 1, t))) &= z(x, 1, t) g_2(z(x, 1, t)) - G(z(x, 1, t)) \\ &\leq (1 - \alpha_1) z(x, 1, t) g_2(z(x, 1, t)). \end{aligned}$$

Using (2.3) and (2.4) with $A = g_2(z(x, 1, t))$ and $B = u_t(x, t)$, from (2.12) we obtain

$$\begin{aligned}
E'(t) &\leq - \left(\mu_1 - \frac{\xi \alpha_2}{\tau} \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx - \frac{\xi}{\tau} \int_{\Omega} G(z(x, 1, t)) \, dx \\
&\quad + \mu_2 \int_{\Omega} (G(u_t(x, t)) + G^*(g_2(z(x, 1, t)))) \, dx \\
&\leq - \left(\mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2 \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx \\
&\quad - \left(\frac{\xi \alpha_1}{\tau} - \mu_2 (1 - \alpha_1) \right) \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) \, dx \\
&\leq 0.
\end{aligned}$$

□

Theorem 2.4 (Local existence). *Assume that (A1)–(A4) hold. If $u_0 \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, $u_1 \in H_0^m(\Omega)$ and $f_0 \in H_0^m(\Omega, H^m(0, 1))$ satisfy the compatibility condition $f(\cdot, 0) = u_1$, then there exists $T > 0$ such that problem (1.1) has a unique local solution $u(t)$ which satisfies*

$$u \in \mathcal{C}([0, \infty); H_0^m(\Omega)), \quad u_t \in \mathcal{C}([0, \infty); L^2(\Omega)).$$

Now we have the existence of a global solution.

Theorem 2.5. *Let*

$$u_0 \in H^{2m}(\Omega) \cap \mathcal{W}, \quad u_1 \in H_0^m(\Omega) \cap L^2(\Omega) \quad \text{and} \quad f_0 \in H_0^m(\Omega, H^m(0, 1))$$

satisfy the compatibility condition $f(\cdot, 0) = u_1$. Assume that (A1)–(A4) hold. Then (1.1) admits a global weak solution $u(x, t)$ such that

$$\begin{aligned}
u &\in L^\infty([0, \infty); H^{2m}(\Omega) \cap H_0^m(\Omega)), \quad u_t \in L^\infty([0, \infty); H_0^m(\Omega) \cap L^2(\Omega)), \\
u_{tt} &\in L^2([0, \infty); L^2(\Omega)).
\end{aligned}$$

Also we have a uniform decay rates for the energy.

Theorem 2.6. *Assume that (A1)–(A4) hold. Then there exist positive constants w_1, w_2, w_3 and ε_0 such that the solution of (1.1) satisfies*

$$E(t) \leq w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t \geq 0,$$

where

(2.13)

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} \, ds \quad \text{and} \quad H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \varepsilon'], \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon']. \end{cases}$$

3. PROOFS OF MAIN RESULTS

Proof of Theorem 2.5. Throughout this section we assume $u_0 \in H^{2m}(\Omega) \cap \mathcal{W}$, $u_1 \in H_0^m(\Omega) \cap L^2(\Omega)$ and $f_0 \in H_0^m(\Omega, H^m(0, 1))$. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w^1, w^2, \dots, w^k\}$, where the set $\{w^k, k \in \mathbb{N}\}$ is a basis of $H^{2m}(\Omega) \cap H_0^m(\Omega)$. Now, we define for $1 \leq j \leq k$, the sequence $\varphi^j(x, \varrho)$ as follows:

$$\varphi^j(x, 0) = w^j.$$

Then, we may extend $\varphi^j(x, 0)$ by $\varphi^j(x, \varrho)$ over $L^2(\Omega \times (0, 1))$ so that $(\varphi^j)_j$ forms a base of $L^2(\Omega, H^m(0, 1))$ and denote by Z_k the space generated by $\{\varphi^k\}$. We construct approximate solutions (u^k, z^k) , $k = 1, 2, 3, \dots$, in the form

$$u^k(t) = \sum_{j=1}^k c^{jk}(t) w^j(x), \quad z^k(t) = \sum_{j=1}^k d^{jk}(t) \varphi^j,$$

where c^{jk} and d^{jk} ($j = 1, 2, \dots, k$) are determined by the ordinary differential equations

$$(3.1) \quad (u_{tt}^k(t), w^j) + (\mathcal{A}^{1/2} u^k(t), \mathcal{A}^{1/2} w^j) + \mu_1(g_1(u_t^k), w^j) + \mu_2(g_2(z^k(\cdot, 1)), w^j) = a(|u^k|^{p-2} u^k, w^j),$$

$$(3.2) \quad z^k(x, 0, t) = u_t^k(x, t),$$

$$(3.3) \quad u^k(0) = u_0^k = \sum_{j=1}^k (u_0, w^j) w^j \rightarrow u_0 \quad \text{in } H^{2m}(\Omega) \cap \mathcal{W} \text{ as } k \rightarrow \infty,$$

$$(3.4) \quad u_t^k(0) = u_1^k = \sum_{j=1}^k (u_1, w^j) w^j \rightarrow u_1 \quad \text{in } H_0^m(\Omega) \cap L^2(\Omega) \text{ as } k \rightarrow \infty,$$

and

$$(3.5) \quad (\tau z_t^k + z_\varrho^k, \varphi^j) = 0, \quad 1 \leq j \leq k,$$

$$(3.6) \quad z^k(\varrho, 0) = z_0^k = \sum_{j=1}^k (f_0, \varphi^j) \varphi^j \rightarrow f_0 \quad \text{in } H_0^m(\Omega, H^m(0, 1)) \text{ as } k \rightarrow \infty.$$

By virtue of the theory of ordinary differential equations, the systems (3.1)–(3.6) have a unique local solution which is extended to a maximal interval $[0, T_k]$ (with $0 < T_k < \infty$) by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that $u^k(t)$ is from the class C^2 . In the next step, we obtain a priori

estimates for the solution, so that it can be extended outside $[0, T_k]$ to obtain one solution defined for all $t > 0$. In order to use a standard compactness argument for the limiting procedure, it suffices to derive some a priori estimates for (u^k, z^k) .

First estimate. Since the sequences u_0^k , u_1^k and z_0^k converge, the standard calculations, using (3.1)–(3.6), similar to those used to derive (2.9), yield

$$E^k(t) - E^k(0) \leq -\beta_1 \int_0^t \int_{\Omega} u_t^k g_1(u_t^k) dx ds - \beta_2 \int_0^t \int_{\Omega} z^k(x, 1, s) g_2(z^k(x, 1, s)) dx ds,$$

where $\beta_1 = \mu_1 - \xi \alpha_2 / \tau - \mu_2 \alpha_2$ and $\beta_2 = \xi \alpha_1 / \tau - \mu_2 (1 - \alpha_1)$. So we obtain

$$E^k(t) + \beta_1 \int_0^t \int_{\Omega} u_t^k g_1(u_t^k) dx ds + \beta_2 \int_0^t \int_{\Omega} z^k(x, 1, s) g_2(z^k(x, 1, s)) dx ds \leq E^k(0),$$

where

$$\begin{aligned} E^k(t) &= \frac{1}{2} \|u_t^k\|_2^2 + \xi \int_{\Omega} \int_0^1 G(z^k(x, \varrho, t)) d\varrho dx + J(u^k(t)), \\ J(u^k(t)) &= \frac{1}{2} \|\mathcal{A}^{1/2} u^k\|_2^2 - \frac{a}{p} \|u^k\|_p^p \end{aligned}$$

and

$$E^k(0) = \frac{1}{2} \|u_1^k\|_2^2 + \xi \int_{\Omega} \int_0^1 G(z^k(x, \varrho, 0)) d\varrho dx + \frac{1}{2} \|\mathcal{A}^{1/2} u_0^k\|_2^2 - \frac{a}{p} \|u_0^k\|_p^p \leq C_1.$$

For some C_1 independent of k we obtain the first estimate:

$$\begin{aligned} \|u_t^k\|_2^2 + \int_{\Omega} \int_0^1 G(z^k(x, \varrho, t)) d\varrho dx + J(u^k(t)) + \int_0^t \int_{\Omega} u_t^k g_1(u_t^k) dx ds \\ + \int_0^t \int_{\Omega} z^k(x, 1, s) g_2(z^k(x, 1, s)) dx ds \leq C_1. \end{aligned}$$

These estimates imply that the solution (u^k, z^k) exists globally in $[0, \infty[$.

$$(3.7) \quad u^k \text{ is bounded in } L_{\text{loc}}^{\infty}(0, \infty, H_0^m(\Omega)),$$

$$(3.8) \quad u_t^k \text{ is bounded in } L_{\text{loc}}^{\infty}(0, \infty, L^2(\Omega)),$$

$$(3.9) \quad G(z^k(x, \varrho, t)) \text{ is bounded in } L_{\text{loc}}^{\infty}(0, \infty, L^1(\Omega \times (0, 1))),$$

$$(3.10) \quad u_t^k(t) g_1(u_t^k(t)) \text{ is bounded in } L^1(\Omega \times (0, T)),$$

$$(3.11) \quad z^k(x, 1, t) g_2(z^k(x, 1, t)) \text{ is bounded in } L^1(\Omega \times (0, T)).$$

Second estimate. First, we estimate $u_{tt}^k(0)$. Taking $t = 0$ in (3.1), we obtain

$$(u_{tt}^k(0), w^j) + (\mathcal{A}^{1/2}u^k(0), \mathcal{A}^{1/2}w^j) + \mu_1(g_1(u_{tt}^k(0)), w^j) \\ + \mu_2(g_2(z^k(\cdot, 1)(0), w^j)) = a(|u^k(0)|^{p-2}u^k(0), w^j),$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k ,

$$(u_{tt}^k(0), u_{tt}^k(0)) + (\mathcal{A}u^k(0), u_{tt}^k(0)) + \mu_1(g_1(u_{tt}^k(0)), u_{tt}^k(0)) \\ + \mu_2(g_2(z^k(\cdot, 1)(0), u_{tt}^k(0))) = a(|u^k(0)|^{p-2}u^k(0), u_{tt}^k(0)).$$

Using Hölder's inequality, we have

$$\|u_{tt}^k(0)\| \leq \|\mathcal{A}u^k(0)\| + \mu_1\|g_1(u_{tt}^k)\| + \mu_2\|g_2(z_1^k)\| + a\| |u_0^k|^{p-2}u_0^k \|.$$

Since $g_1(u_1^k)$, $g_2(z_1^k)$ are bounded in $L^2(\Omega)$, (3.3), (3.4) and (3.6) yield

$$\|u_{tt}^k(0)\| \leq C,$$

where C is a positive constant independent of k .

Now, differentiating (3.1) with respect to t ,

$$(u_{ttt}^k(t), w^j) + (\mathcal{A}u_t^k(t), w^j) + \mu_1(u_{tt}^k g_1'(u_t^k), w^j) \\ + \mu_2(z_t^k g_2'(z^k(\cdot, 1)), w^j) = a(p-1)(|u^k|^{p-2}u_t^k, w^j),$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k ,

$$(3.12) \quad \frac{1}{2} \frac{d}{dt} (\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2) + \mu_1 \int_{\Omega} (u_{tt}^k(t))^2 g_1'(u_t^k(t)) \, dx \\ + \mu_2 \int_{\Omega} u_{tt}^k(t) z_t^k(x, 1, t) g_2'(z^k(x, 1, t)) \, dx \\ = a(p-1) \int_{\Omega} |u^k(t)|^{p-2} u_t^k(t) u_{tt}^k(t) \, dx.$$

We have from Hölder's inequality

$$a(p-1) \int_{\Omega} |u^k(t)|^{p-2} u_t^k(t) u_{tt}^k(t) \, dx \leq a(p-1) \|u^k(t)\|_{2(p-1)}^{p-2} \|u_t^k(t)\|_{2(p-1)} \|u_{tt}^k(t)\|_2,$$

where

$$\frac{p-2}{2(p-1)} + \frac{1}{2(p-1)} + \frac{1}{2} = 1.$$

Using Lemma 2.1, Young's inequality and (3.7), we have

$$\begin{aligned}
(3.13) \quad & a(p-1) \int_{\Omega} |u^k(t)|^{p-2} u_t^k(t) u_{tt}^k(t) \, dx \\
& \leq a(p-1) C_s^{p-1} \|\mathcal{A}^{1/2} u^k(t)\|_2^{p-2} \|\mathcal{A}^{1/2} u_t^k(t)\|_2 \|u_{tt}^k(t)\|_2 \\
& \leq C(\varepsilon) \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \varepsilon \|u_{tt}^k(t)\|_2^2.
\end{aligned}$$

Differentiating (3.5) with respect to t , we obtain

$$(\tau z_{tt}^k + z_{t\varrho}^k, \varphi^j) = 0.$$

Multiplying by d_t^{jk} and summing over j from 1 to k , it follows that

$$\frac{\tau}{2} \frac{d}{dt} \|z_t^k\|_2^2 + \frac{1}{2} \frac{d}{d\varrho} \|z_t^k\|_2^2 = 0.$$

Integrating over $(0, 1)$ with respect to ϱ , we obtain

$$(3.14) \quad \frac{\tau}{2} \frac{d}{dt} \int_0^1 \|z_t^k\|_2^2 \, d\varrho + \frac{1}{2} \|z_t^k(x, 1, t)\|_2^2 - \frac{1}{2} \|u_{tt}^k(x, t)\|_2^2 = 0.$$

Taking the sum of (3.12) and (3.14), we obtain

$$\begin{aligned}
(3.15) \quad & \frac{1}{2} \frac{d}{dt} \left(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \tau \int_0^1 \|z_t^k\|_2^2 \, d\varrho \right) \\
& \quad + \mu_1 \int_{\Omega} (u_{tt}^k(t))^2 g_1'(u_t^k(t)) \, dx + \frac{1}{2} \|z_t^k(x, 1, t)\|_2^2 \\
& = a(p-1) \int_{\Omega} |u^k(t)|^{p-2} u_t^k(t) u_{tt}^k(t) \, dx \\
& \quad - \mu_2 \int_{\Omega} u_{tt}^k(t) z_t^k(x, 1, t) g_2'(z^k(x, 1, t)) \, dx + \frac{1}{2} \|u_{tt}^k(x, t)\|_2^2.
\end{aligned}$$

Using (2.2) and Young's inequality, we conclude

$$(3.16) \quad \int_{\Omega} |u_{tt}^k(t)| |z_t^k(x, 1, t)| |g_2'(z^k(x, 1, t))| \, dx \leq \varepsilon \|z_t^k(x, 1, t)\|_2^2 + \frac{c_2^2}{4\varepsilon} \|u_{tt}^k\|_2^2.$$

A combination of (3.13), (3.15) and (3.16) then yields

$$\begin{aligned}
(3.17) \quad & \frac{1}{2} \frac{d}{dt} \left(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \tau \int_0^1 \|z_t^k\|_2^2 \, d\varrho \right) \\
& \quad + \mu_1 \int_{\Omega} (u_{tt}^k(t))^2 g_1'(u_t^k(t)) \, dx + \left(\frac{1}{2} - \varepsilon \right) \|z_t^k(x, 1, t)\|_2^2 \\
& \leq \left(\varepsilon + \frac{c_3^2}{4\varepsilon} + \frac{1}{2} \right) \|u_{tt}^k\|_2^2 + C(\varepsilon) \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2,
\end{aligned}$$

integrating over $[0, t]$ for all $t \in [0, T]$ with arbitrary fixed T ,

$$\begin{aligned}
(3.18) \quad & \frac{1}{2}(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \tau\|z_t^k(x, \varrho, t)\|_{L^2(\Omega \times (0,1))}^2) \\
& + \mu_1 \int_0^t \int_{\Omega} (u_{tt}^k(t))^2 g_1'(u_t^k(t)) \, dx \, dt + \left(\frac{1}{2} - \varepsilon\right) \int_0^t \|z_t^k(x, 1, t)\|_2^2 \, dt \\
& \leq \frac{1}{2}(\|u_{tt}^k(0)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(0)\|_2^2 + \tau\|z_t^k(x, \varrho, 0)\|_{L^2(\Omega \times (0,1))}^2) \\
& + \left(\varepsilon + \frac{c_3^2}{4\varepsilon} + \frac{1}{2}\right) \int_0^t \|u_{tt}^k\|_2^2 \, dt + C(\varepsilon) \int_0^t \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 \, dt.
\end{aligned}$$

Then from (3.18), after choosing ε small enough and using Gronwall's lemma, we obtain

$$\begin{aligned}
& \|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \tau\|z_t^k(x, \varrho, t)\|_{L^2(\Omega \times (0,1))}^2 \\
& + \mu_1 \int_0^t \int_{\Omega} (u_{tt}^k(t))^2 g_1'(u_t^k(t)) \, dx \, dt + \left(\frac{1}{2} - \varepsilon\right) \int_0^t \|z_t^k(x, 1, t)\|_2^2 \, dt \leq M
\end{aligned}$$

for all $t \in [0, T]$, where M is a positive constant independent of $k \in \mathbb{N}$. Therefore, we conclude that

$$(3.19) \quad u_{tt}^k \text{ is bounded in } L_{\text{loc}}^{\infty}(0, \infty, L^2(\Omega)),$$

$$(3.20) \quad u_t^k \text{ is bounded in } L_{\text{loc}}^{\infty}(0, \infty, H_0^m(\Omega)),$$

$$(3.21) \quad z_t^k \text{ is bounded in } L_{\text{loc}}^{\infty}(0, \infty, L^2(\Omega \times (0, 1))).$$

Third estimate. Replacing w^j by $\mathcal{A}w^j$ in (3.1), multiplying by c_t^{jk} and summing over j from 1 to k , it follows that

$$\begin{aligned}
(3.22) \quad & \frac{1}{2} \frac{d}{dt} (\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \|\mathcal{A}u^k(t)\|_2^2) + \mu_1 \int_{\Omega} |\mathcal{A}^{1/2}u_t^k(t)|^2 g_1'(u_t^k) \, dx \\
& + \mu_2 \int_{\Omega} \mathcal{A}^{1/2}z^k(x, 1, t) \mathcal{A}^{1/2}u_t^k g_2'(z^k(x, 1, t)) \, dx \\
& = a \int_{\Omega} \mathcal{A}^{1/2}(|u^k|^{p-2}u^k) \mathcal{A}^{1/2}u_t^k \, dx.
\end{aligned}$$

Replacing φ^j by $\mathcal{A}\varphi^j$ in (3.5), multiplying by d^{jk} and summing over j from 1 to k , it follows that

$$\frac{\tau}{2} \frac{d}{dt} \|\mathcal{A}^{1/2}z_t^k\|_2^2 + \frac{1}{2} \frac{d}{d\varrho} \|\mathcal{A}^{1/2}z_t^k\|_2^2 = 0.$$

We integrate over $(0, 1)$ to find

$$(3.23) \quad \frac{\tau}{2} \frac{d}{dt} \int_0^1 \|\mathcal{A}^{1/2}z^k(t)\|_2^2 \, d\varrho + \frac{1}{2} \|\mathcal{A}^{1/2}z_t^k(x, 1, t)\|_2^2 - \frac{1}{2} \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 = 0.$$

Combining (3.22), (3.23), using (2.2), Cauchy-Schwarz and Young's inequalities produce the estimate

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \|\mathcal{A} u^k(t)\|_2^2 + \int_0^1 \|\mathcal{A}^{1/2} z^k(x, \varrho, t)\|_2^2 d\varrho \right) \\
& \quad + \mu_1 \int_{\Omega} |\mathcal{A}^{1/2} u_t^k(t)|^2 g_1'(u_t^k) dx \\
& \quad + \left(\frac{1}{2} - \varepsilon \right) \|\mathcal{A}^{1/2} z_t^k(x, 1, t)\|_2^2 \\
& \leq C(\varepsilon) \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + a \int_{\Omega} \mathcal{A}^{1/2} (|u^k|^{p-2} u^k) \mathcal{A}^{1/2} u_t^k dx.
\end{aligned}$$

Integrating the last inequality over $(0, t)$, we have

$$\begin{aligned}
& \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \|\mathcal{A} u^k(t)\|_2^2 + \int_0^1 \|\mathcal{A}^{1/2} z^k(x, \varrho, t)\|_2^2 d\varrho \\
& \quad + 2\mu_1 \int_0^t \int_{\Omega} |\mathcal{A}^{1/2} u_t^k(s)|^2 g_1'(u_t^k) dx dt \\
& \quad + 2 \left(\frac{1}{2} - \varepsilon \right) \int_0^t \|\mathcal{A}^{1/2} z_t^k(x, 1, s)\|_2^2 dt \\
& \leq A^k(0) + C(\varepsilon) \int_0^t \|\mathcal{A}^{1/2} u_t^k(s)\|_2^2 dt \\
& \quad + c \int_0^t \|\mathcal{A}^{1/2} u^k(s)\|^{p-1} \|\mathcal{A}^{1/2} u_t^k(s)\| dt \\
& \leq A^k(0) + C(\varepsilon) \int_0^t \|\mathcal{A}^{1/2} u_t^k(s)\|_2^2 dt \\
& \quad + \frac{c}{2} \int_0^t \|\mathcal{A}^{1/2} u^k(s)\|^{2(p-1)} dt + \frac{c}{2} \int_0^t \|\mathcal{A}^{1/2} u_t^k(s)\| dt \\
& \leq A^k(0) + \max \left\{ \left(C(\varepsilon) + \frac{c}{2} \right), C(E^k(0))^p \right\} \\
& \quad \times \int_0^t (\|\mathcal{A}^{1/2} u_t^k(s)\|_2^2 + \|\mathcal{A}^{1/2} u^k(s)\|^2) dt,
\end{aligned}$$

where

$$A^k(0) = \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \|\mathcal{A} u^k(t)\|_2^2 + \|\mathcal{A}^{1/2} z^k(x, \varrho, t)\|_{L^2(\Omega \times (0,1))}^2,$$

and using Gronwall's lemma, we have

$$(3.24) \quad \|\mathcal{A}^{1/2} u_t^k(t)\|_2^2 + \|\mathcal{A} u^k(t)\|_2^2 + \int_0^1 \|\mathcal{A}^{1/2} z^k(x, \varrho, t)\|_2^2 d\varrho \leq A^k(0) e^{cT}$$

for all $t \in \mathbb{R}_+$, therefore we conclude that

$$(3.25) \quad u^k \text{ is bounded in } L_{\text{loc}}^\infty(0, \infty, H^{2m}(\Omega)),$$

$$(3.26) \quad z^k \text{ is bounded in } L_{\text{loc}}^\infty(0, \infty, H_0^m(\Omega, L^2(0, 1))).$$

Passing to the limit. Applying Dunford-Petti's theorem, we conclude from (3.7)–(3.11), (3.19)–(3.21) and (3.25)–(3.26), after replacing the sequences u^k and z^k by subsequences if necessary, that

$$(3.27) \quad u^k \rightharpoonup u \text{ weak-star in } L^\infty(0, \infty; H^{2m}(\Omega)),$$

$$(3.28) \quad u_t^k \rightharpoonup u_t \text{ weak-star in } L^\infty(0, \infty; H_0^m(\Omega)),$$

$$(3.29) \quad u_{tt}^k \rightharpoonup u_{tt} \text{ weak-star in } L^\infty(0, \infty; L^2(\Omega)),$$

$$(3.30) \quad g_1(u_t^k) \rightharpoonup \chi \text{ weak-star in } L^2(\Omega \times (0, T)),$$

$$(3.31) \quad z^k \rightharpoonup z \text{ weak-star in } L^\infty(0, \infty, H_0^m(\Omega, L^2(0, 1))),$$

$$(3.32) \quad z_t^k \rightharpoonup z_t \text{ weak-star in } L^\infty(0, \infty, L^2(\Omega \times (0, T))),$$

$$(3.33) \quad g_2(u_t^k) \rightharpoonup \psi \text{ weak-star in } L^2(\Omega \times (0, T)).$$

On the other hand, from Aubin-Lions theorem (see Lions [5]), we deduce that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ such that

$$(3.34) \quad u^m \rightarrow u \text{ strongly in } L^2(0, T, L^2(\Omega)),$$

$$(3.35) \quad u_t^m \rightarrow u_t \text{ strongly in } L^2(0, T, L^2(\Omega)),$$

which implies

$$u^m \rightarrow u \text{ almost everywhere in } \mathcal{B}$$

and

$$(3.36) \quad u_t^m \rightarrow u_t \text{ almost everywhere in } \mathcal{B}.$$

Hence

$$(3.37) \quad |u^m|^{p-2}u^m \rightarrow |u|^{p-2}u \text{ almost everywhere in } \mathcal{B},$$

where $\mathcal{B} = \Omega \times (0, T)$.

$$\int_{\mathcal{B}} (|u^m|^{p-2}u^m)^{p/(p-1)} dx dt \leq \int_{\mathcal{B}} |u^m|^p dx dt \leq C \| \mathcal{A}^{1/2}u^m \|_{L^2(\mathcal{B})}^p,$$

using (3.7) we obtain

$$(3.38) \quad \|u^m\|_{L^{p/(p-1)}(\mathcal{B})} \leq C.$$

Thus, using (3.37), (3.38) and Lions lemma, we derive

$$(3.39) \quad |u^m|^{p-2}u^m \rightharpoonup |u|^{p-2}u \text{ weakly in } L^{p/(p-1)}(\mathcal{B}),$$

and

$$z^m \rightarrow z \text{ strongly in } L^2(0, T, L^2(\Omega)),$$

which implies

$$z^m \rightarrow z \text{ almost everywhere in } \mathcal{B}.$$

Lemma 3.1. *For each $T > 0$, $g_1(u_t), g_2(z(x, 1, t)) \in L^1(\mathcal{B})$ and $\|g_1(u_t)\|_{L^1(\mathcal{B})} \leq K$, $\|g_2(z(x, 1, t))\|_{L^1(\mathcal{B})} \leq K$, where K is a constant independent of t .*

Proof. By (A1) and (3.36), we have

$$\begin{aligned} g_1(u_t^m(x, t)) &\rightarrow g_1(u_t(x, t)) \text{ almost everywhere in } \mathcal{B}, \\ 0 \leq u_t^k(x, t)g_1(u_t^m(x, t)) &\rightarrow u_t(x, t)g_1(u_t(x, t)) \text{ almost everywhere in } \mathcal{B}. \end{aligned}$$

Hence, by (3.10) and Fatou's lemma, we have

$$(3.40) \quad \int_0^T \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx \, dt \leq K_1 \quad \text{for } T > 0.$$

Now, we can estimate $\int_0^T \int_{\Omega} |g_1(u_t(x, t))| \, dx \, dt$. By Cauchy-Schwarz inequality and using (3.40), we have

$$\begin{aligned} \int_0^T \int_{\Omega} |g_1(u_t(x, t))| \, dx \, dt &\leq c|\mathcal{B}|^{1/2} \left(\int_0^T \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx \, dt \right)^{1/2} \\ &\leq c|\mathcal{B}|^{1/2} K_1^{1/2} \equiv K. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_0^T \int_{\Omega} |g_2(z(x, 1, t))| \, dx \, dt &\leq c|\mathcal{B}|^{1/2} \left(\int_0^T \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx \, dt \right)^{1/2} \\ &\leq c|\mathcal{B}|^{1/2} K_1^{1/2} \equiv K. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. $g_1(u_t^k) \rightarrow g_1(u_t)$ in $L^1(\Omega \times (0, T))$, $g_2(z^k) \rightarrow g_2(z)$ in $L^1(\Omega \times (0, T))$.

Proof. Let $E \subset \Omega \times [0, T]$ and set

$$E_1 = \left\{ (x, t) \in E : |g_1(u_t^k(x, t))| \leq \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E . If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$,

$$\int_E |g_1(u_t^k)| \, dx \, dt \leq c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right) \right)^{-1} \int_{E_2} |u_t^k g_1(u_t^k)| \, dx \, dt.$$

By applying (3.10) we deduce that $\sup_k \int_E |g_1(u_t^k)| \, dx \, dt \rightarrow 0$ as $|E| \rightarrow 0$. From Vitali's convergence theorem we deduce that

$$g_1(u_t^k) \rightarrow g_1(u_t) \text{ in } L^1(\Omega \times (0, T)).$$

Similarly, we have

$$g_2(z^k) \rightarrow g_2(z) \text{ in } L^1(\Omega \times (0, T)).$$

This completes the proof of Lemma 3.2. □

Hence

$$(3.41) \quad g_1(u_t^k) \rightharpoonup g_1(u_t) \text{ weak in } L^2(\Omega \times (0, T)),$$

$$(3.42) \quad g_2(z^k) \rightharpoonup g_2(z) \text{ weak in } L^2(\Omega \times (0, T)).$$

By multiplying (3.1) by $\theta(t) \in \mathcal{D}(0, T)$ and by integrating over $(0, T)$, it follows that

$$(3.43) \quad \begin{aligned} & \int_0^T (u_t^k(t), w^j) \theta'(t) \, dt + \int_0^T (\mathcal{A}^{1/2} u^k(t), \mathcal{A}^{1/2} w^j) \theta(t) \, dt \\ & \quad + \mu_1 \int_0^T (g_1(u_t^k), w^j) \theta(t) \, dt \\ & \quad + \mu_2 \int_0^T (g_2(z^k(\cdot, 1)), w^j) \theta(t) \, dt \\ & = \int_0^T (|u^k|^{p-2} u^k, w^j) \theta(t) \, dt, \end{aligned}$$

and by multiplying (3.3) by $\theta(t) \in \mathcal{D}(0, T)$ and integrating over $(0, T) \times (0, 1)$, it follows that

$$(3.44) \quad \int_0^T \int_0^1 (\tau z_t^k + z_\varrho^k, \varphi^j) \theta(t) \, dt \, d\varrho = 0.$$

The convergences of (3.27)–(3.33), (3.39), (3.41) and (3.42) are sufficient to pass to the limit in (3.43) and (3.44) to obtain

$$\begin{aligned} \int_0^T (u_t^k(t), w)\theta'(t) dt + \int_0^T (\mathcal{A}^{1/2}u^k(t), \mathcal{A}^{1/2}w)\theta(t) dt + \mu_1 \int_0^T (g_1(u_t^k), w)\theta(t) dt \\ + \mu_2 \int_0^T (g_2(z^k(\cdot, 1)), w)\theta(t) dt = \int_0^T (|u^k|^{p-2}u^k, w)\theta(t) dt \end{aligned}$$

and

$$\int_0^T \int_0^1 (\tau z_t + z_\varrho, \varphi)\theta(t) dt d\varrho = 0.$$

By integrating, we have

$$\int_0^T (u_{tt} + \mathcal{A}u^k(t) + \mu_1 g_1(u_t) + \mu_2 g_2(z(\cdot, 1)), w)\theta(t) dt = \int_0^T |u|^{p-2}u\theta(t) dt.$$

This completes the proof of Theorem 2.5. \square

4. ASYMPTOTIC BEHAVIOR

Proof of Theorem 2.6. In this section, we prove the energy decay result by constructing a suitable Lyapunov functional.

Now we define the following functional

$$(4.1) \quad L(t) = NE(t) + N_1 F_1(t) + F_2(t),$$

where

$$(4.2) \quad F_1(t) = \int_{\Omega} uu_t dx,$$

$$(4.3) \quad F_2(t) = \int_{\Omega} \int_0^1 e^{-2\tau\varrho} G(z(x, \varrho, t)) d\varrho dx$$

and we also need the following lemma:

Lemma 4.1. *Let (u, z) be a solution of problem (2.6). Then there exist two positive constants λ_1, λ_2 such that*

$$(4.4) \quad \lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0,$$

for N sufficiently large and N_1 a positive real number to be chosen appropriately later.

Proof. Let $L(t) = N_1 F_1(t) + F_2(t)$,

$$|L(t)| \leq N_1 \int_{\Omega} |uu_t| dx + \int_{\Omega} \int_0^1 e^{-2\tau\rho} G(z(x, \rho, t)) d\rho dx.$$

By Young's inequality, Lemma 2.1, (2.8) and the fact that $e^{-2\tau\rho} \leq 1$ for all $\rho \in [0, 1]$, we obtain

$$|L(t)| \leq \frac{N_1}{2} \|u_t\|_2^2 + \frac{N_1 C_s}{2} \|A^{1/2} u\|_2^2 + \int_{\Omega} \int_0^1 G(z(x, \rho, t)) d\rho dx \leq cE(t).$$

Consequently, $|L(t) - NE(t)| \leq cE(t)$, which yields

$$(N - c)E(t) \leq L(t) \leq (N + c)E(t).$$

Choosing N large enough, we obtain estimate (4.4). \square

Lemma 4.2. *Let (u, z) be a solution of (2.6). Then the functional $F_1(t)$ defined by (4.2) satisfies for any $\eta > 0$ the estimate*

$$(4.5) \quad F_1'(t) \leq \|u_t\|_2^2 + a\|u\|_p^p - (1 - \eta C_s^2(\mu_1 + \mu_2)) \|A^{1/2} u\|_2^2 \\ + \frac{\mu_1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 dx + \frac{\mu_2 c_2}{4\eta} \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx.$$

Proof. Taking the derivative of $F_1(t)$ with respect to t and using the first equation of (2.6), we obtain

$$(4.6) \quad F_1'(t) = \|u_t\|_2^2 + \int_{\Omega} u_{tt} u dx = \|u_t\|_2^2 + a\|u\|_p^p - \|A^{1/2} u\|_2^2 \\ - \mu_1 \int_{\Omega} u g_1(u_t(x, t)) dx - \mu_2 \int_{\Omega} u g_2(z(x, 1, t)) dx.$$

Now, we estimate the terms on the right hand side of (4.6) using Young's inequality and Lemma 2.1 and we obtain

$$(4.7) \quad \int_{\Omega} u g_1(u_t) dx \leq \eta C_s^2 \|A^{1/2} u\|_2^2 + \frac{1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 dx,$$

$$(4.8) \quad \int_{\Omega} u g_2(z(x, 1, t)) dx \leq \eta C_s^2 \|A^{1/2} u\|_2^2 + \frac{1}{4\eta} \int_{\Omega} |g_2(z(x, 1, t))|^2 dx.$$

From (2.2), (4.8) becomes

$$(4.9) \quad \int_{\Omega} u g_2(z(x, 1, t)) dx \leq \eta C^2 \|A^{1/2} u\|_2^2 + \frac{c_2}{4\eta} \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx.$$

Estimate (4.5) follows by substituting (4.7) and (4.9) into (4.6). \square

Lemma 4.3. *Let (u, z) be the solution to (2.6). Then the functional $F_2(t)$ defined by (4.3) satisfies the estimate*

$$(4.10) \quad F_2'(t) \leq -2e^{-2\tau} \int_{\Omega} \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx \\ - \frac{\alpha_1 e^{-2\tau}}{\tau} \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) \, dx + \frac{\alpha_2}{\tau} \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx.$$

Proof. By differentiating (4.3) with respect to t and using (2.3) and (2.5), we obtain

$$\begin{aligned} F_2'(t) &= -\frac{1}{\tau} \int_{\Omega} \int_0^1 e^{-2\tau\varrho} \frac{\partial}{\partial \varrho} G(z(x, \varrho, t)) \, d\varrho \, dx \\ &= -\frac{1}{\tau} \int_{\Omega} \int_0^1 \frac{\partial}{\partial \varrho} (e^{-2\tau\varrho} G(z(x, \varrho, t))) \, dx + 2\tau e^{-2\tau\varrho} G(z(x, \varrho, t)) \, d\varrho \, dx \\ &= -\frac{1}{\tau} \int_{\Omega} (e^{-2\tau} G(z(x, 1, t)) - G(u_t(x, t))) \, dx \\ &\quad - 2 \int_{\Omega} \int_0^1 e^{-2\tau\varrho} G(z(x, \varrho, t)) \, d\varrho \, dx \\ &= -\frac{1}{\tau} \int_{\Omega} e^{-2\tau} G(z(x, 1, t)) \, dx + \frac{1}{\tau} \int_{\Omega} G(u_t(x, t)) \, dx \\ &\quad - 2 \int_{\Omega} \int_0^1 e^{-2\tau\varrho} G(z(x, \varrho, t)) \, d\varrho \, dx \\ &= -2F_2(t) + \frac{1}{\tau} \int_{\Omega} G(u_t(x, t)) \, dx - \frac{e^{-2\tau}}{\tau} \int_{\Omega} G(z(x, 1, t)) \, dx \\ &\leq -2F_2(t) + \frac{\alpha_2}{\tau} \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx \\ &\quad - \frac{\alpha_1 e^{-2\tau}}{\tau} \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) \, dx. \end{aligned}$$

Since $-e^{-2\tau\varrho}$ is an increasing function, we have $-e^{-2\tau\varrho} \leq -e^{-2\tau}$ for all $\varrho \in [0, 1]$, we deduce

$$-2F(t) \leq -2e^{-2\tau} \int_{\Omega} \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx.$$

The proof of Lemma 4.3 is complete. \square

Lemma 4.4. *Let (u, z) be a solution of (2.6) and assume that (A1)–(A4) hold. Then the functional defined by (4.1) satisfies*

$$(4.11) \quad L'(t) \leq -mE(t) + C_1(\|u_t\|_2^2 + \|g_1(u_t)\|_2^2)$$

for positive constants m and C_1 .

Proof. By differentiating (4.1) and recalling (2.9), (4.5) and (4.10), we obtain

$$\begin{aligned}
(4.12) \quad L'(t) &= NE'(t) + N_1 F'_1(t) + F'_2(t) \\
&\leq - \left(N\beta_1 - \frac{\alpha_2}{\tau} \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) \, dx \\
&\quad - \left(N\beta_2 - N_1 \frac{\mu_2 c_2}{4\eta} + \frac{\alpha_1 e^{-2\tau}}{\tau} \right) \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) \, dx \\
&\quad - 2e^{-2\tau} \int_{\Omega} \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx + N_1 \frac{\mu_1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 \, dx \\
&\quad + N_1 \|u_t\|_2^2 + N_1 a \|u\|_p^p - N_1 (1 - \eta C_s^2(\mu_1 + \mu_2)) \| \mathcal{A}^{1/2} u \|_2^2.
\end{aligned}$$

We choose N large enough such that

$$N\beta_1 - \frac{\alpha_2}{\tau} > 0 \quad \text{and} \quad N\beta_2 - N_1 \frac{\mu_2 c_2}{4\eta} > 0.$$

Thus, (4.12) becomes

$$\begin{aligned}
L'(t) &\leq - 2e^{-2\tau} \int_{\Omega} \int_0^1 G(z(x, \varrho, t)) \, d\varrho \, dx - N_1 (1 - \eta C^2(\mu_1 + \mu_2)) \| \mathcal{A}^{1/2} u \|_2^2 \\
&\quad + N_1 a \|u\|_p^p + N_1 \|u_t\|_2^2 + N_1 \frac{\mu_1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 \, dx.
\end{aligned}$$

We choose η small enough so that $1 - \eta C^2(\mu_1 + \mu_2) > 0$. Noting by

$$m = \min \left\{ 2N_1 (1 - \eta C^2(\mu_1 + \mu_2)), \frac{2e^{-2\tau}}{\xi} \right\}$$

and choosing N_1 small enough so that $pN_1 \leq m$, we obtain

$$\begin{aligned}
L'(t) &\leq -mE(t) + \frac{m}{2} \|u_t\|_2^2 + N_1 \|u_t\|_2^2 + N_1 \frac{\mu_1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 \, dx \\
&\leq -mE(t) + c \left(\|u_t\|_2^2 + \int_{\Omega} |g_1(u_t)|^2 \, dx \right).
\end{aligned}$$

This completes the proof of Lemma 4.4. □

As in Komornik (see [4]), we consider the following partition of Ω :

$$\Omega_1 = \{x \in \Omega : |u_t| \geq \varepsilon\}, \quad \Omega_2 = \{x \in \Omega : |u_t| \leq \varepsilon\}.$$

By using (2.1), we have

$$(4.13) \quad \int_{\Omega_1} |u_t|^2 \, dx + \int_{\Omega_1} |g_1(u_t)|^2 \, dx \leq (c'_1 + c_1) \int_{\Omega_1} u_t g_1(u_t) \, dx \leq -\mu_1 E'(t)$$

and

$$\int_{\Omega_2} |u_t|^2 \, dx + \int_{\Omega_2} |g_1(u_t)|^2 \, dx \leq \int_{\Omega_2} H^{-1}(u_t g_1(u_t)) \, dx.$$

Case 1. H is linear on $[0, \varepsilon']$. In this case, one can easily check that there exists $\mu_2 > 0$ such that

$$(4.14) \quad \int_{\Omega_2} |u_t|^2 dx + \int_{\Omega_2} |g_1(u_t)|^2 dx \leq -\mu_2 E'(t).$$

Substitution of (4.13) and (4.14) into (4.11) gives

$$(4.15) \quad (L(t) + \mu E(t))' \leq m H_2(E(t)), \quad \text{where } \mu = C_1(\mu_1 + \mu_2).$$

Case 2. $H'(0) > 0$ and $H'' > 0$ on $]0, \varepsilon']$. We define

$$I_1(t) = \frac{1}{|\Omega_2|} \int_{\Omega_2} u_t g(u_t) dx$$

and use Jensen's inequality and the concavity of H^{-1} to obtain

$$H^{-1}(I_1(t)) \geq \tilde{C} \int_{\Omega_2} H^{-1}(u_t g(u_t)) dx.$$

By using (2.1), we obtain

$$(4.16) \quad \int_{\Omega_2} (|u_t|^2 + |g_1(u_t)|^2) dx \leq \int_{\Omega_2} H^{-1}(u_t g_1(u_t)) dx \\ \leq \tilde{C} H^{-1}(I_1(t)) \leq \tilde{C} H^{-1}(-C_2 E'(t)).$$

Combining (4.11), (4.13) and (4.16), we get

$$(4.17) \quad (L(t) + C_1 \mu_1 E(t))' \leq -m E(t) + \tilde{C} H^{-1}(-C_2 E'(t)).$$

By recalling that $E' \leq 0$, $H' > 0$, $H'' > 0$ on $(0, \varepsilon]$ and making use of (4.17), we obtain

$$\begin{aligned} & (H'(\varepsilon_0 E(t))(L(t) + C_1 \mu_1 E(t)) + \tilde{C} C_2 E(t))' \\ &= \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C_1 \mu_1 E(t)) \\ & \quad + H'(\varepsilon_0 E(t))(L(t) + C_1 \mu_1 E(t))' + \tilde{C} C_2 E'(t) \\ & \leq -m H'(\varepsilon_0 E(t)) E(t) + \tilde{C} C_2 E'(t) \\ & \quad + \tilde{C} H'(\varepsilon_0 E(t)) H^{-1}(-C_2 E'(t)), \end{aligned}$$

by using Remark 2.2 with H^* , the convex conjugate of H in the sense of Young, we obtain

$$\begin{aligned}
(4.18) \quad & (H'(\varepsilon_0 E(t))(L(t) + C_1 \mu_1 E(t)) + \tilde{C} C_2 E(t))' \\
& \leq -m H'(\varepsilon_0 E(t)) E(t) + \tilde{C} H^*(H'(\varepsilon_0 E(t))) \\
& \leq -m H'(\varepsilon_0 E(t)) E(t) + \tilde{C} \varepsilon_0 H'(\varepsilon_0 E(t)) E(t) \\
& \leq -C_3 H'(\varepsilon_0 E(t)) E(t) \\
& = -C_3 H_2(E(t)).
\end{aligned}$$

Let

$$\tilde{L}(t) = \begin{cases} L(t) + \mu E(t) & \text{if } H \text{ is linear on } [0, \varepsilon], \\ H'(\varepsilon_0 E(t)) \{L(t) C + C_1 \mu_1 E(t)\} + \tilde{C} C_2 E(t) & \text{if } H'(0) > 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon]. \end{cases}$$

From (4.15) and (4.18), it follows

$$(4.19) \quad \frac{d}{dt} \tilde{L}(t) \leq -C_3 H_2(E(t)) \quad \forall t \geq t_0.$$

From Lemma 4.1, we have that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$ for some positive constants $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$,

$$(4.20) \quad \tilde{\varepsilon}_1 E(t) \leq \tilde{L}(t) \leq \tilde{\varepsilon}_2 E(t).$$

Let

$$(4.21) \quad L(t) = \frac{1}{\tilde{\varepsilon}_2} \tilde{L}(t).$$

Then we observe from (4.19) and (4.21) that

$$L'(t) \leq -\frac{C_3}{\tilde{\varepsilon}_2} H_2(E(t)) \leq -\frac{C_3}{\tilde{\varepsilon}_2} H_2\left(\frac{1}{\tilde{\varepsilon}_2} \tilde{L}(t)\right) = -\frac{C_3}{\tilde{\varepsilon}_2} H_2(L(t)).$$

Then

$$\frac{L'(t)}{H_2(L(t))} \leq -\frac{C_3}{\tilde{\varepsilon}_2}.$$

By recalling (2.13), we deduce $H_2(t) = -1/H_1'(t)$, hence

$$L'(t) H_1'(L(t)) \geq \frac{C_3}{\tilde{\varepsilon}_2}.$$

A simple integration over $(0, t)$ yields

$$H_1(L(t)) \geq H_1(L(0)) + \frac{C_3}{\tilde{\varepsilon}_2} t.$$

Exploiting the fact that H_1^{-1} is decreasing, we infer

$$L(t) \leq H_1^{-1} \left(\frac{C_3}{\varepsilon_2} t + H_1(L(0)) \right).$$

Consequently, the equivalence of L , \tilde{L} and E yields the estimate

$$E(t) \leq w_3 H_1^{-1}(w_1 t + w_2).$$

This completes the proof of Theorem 2.6. □

Acknowledgements. For any decision, the authors would like to thank the anonymous referees and the handling editor for their careful reading and for relevant remarks/suggestions leading to improving the paper.

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