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Accommodating Students with Specific Learning Disabilities in Math: Developing Fact Fluency

Rory Sneyd
Bridgewater State University

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Dr. J. Edward Carter, Thesis Advisor	Date:12/16/2022
Dr. Shannon Lockard, Thesis Advisor	Date:12/18/2022
Dr. Rachel Stahl, Committee Member	Date:12/15/2022
Dr. Ken Dobush, Committee Member	Date:12/17/2022

Abstract

Fact fluency is a crucial component for all students both with and without learning disabilities, as it lays the foundation for future success in all topics of math. For students who have a diagnosed Specific Learning Disability (SLD) in the area of math, fact fluency may be a difficult concept to grasp. To help students have a better understanding of fact fluency, there are multiple different tools that can be used. The purpose of this project is to research and outline some of the best ways to accommodate students who struggle with the concept of fact fluency in the area of mathematics. After creating a rubric based on the literature that outlines the criteria that teachers look for in online applications, the researcher rated each application. After conducting further analysis, the researcher discusses which applications should be highly recommended and why. Also, the researcher discusses different strategies that students can use to develop stronger skills with fact fluency, and then supports why these strategies are accurate. Lastly, the researcher discusses other engaging activities that help build a stronger background knowledge of fact fluency in the classroom based on the literature. By conducting this research, the investigator can justify the importance of fact fluency in mathematics.

Context

Since becoming an undergraduate student and studying Special Education with a concentration in moderate disabilities as well as mathematics, I have had the opportunity to observe students from multiple different grade levels in multiple different schools. Math has always been a top interest of mine, but for many of the students I observed, math is their least favorite subject. If there is one impact I want to have on students, it would definitely be to change this attitude about math. I want students to understand that math can be fun. One way I believe that I could change students' minds about math is by helping them create a better understanding of this content. For students with disabilities in math, it is often discovered that fact fluency is a major struggle. By researching different ways to teach students to have a more automatic fact fluency, I can use these ideas with students in my own future classroom. Through this investigation I have a much better understanding of how I can help my students become more confident in themselves about math! Lastly, because fact fluency is often overlooked with students that struggle with math, I will not only be able to use these ideas in my own classroom, but share these strategies among other teachers as well.

Background Information

According to Forbinger and Fuchs (2014), “fluency is the ability to find an answer quickly and effortlessly, either because the answer is memorized or because the individual has developed an efficient strategy for calculating the answer” (p. 154). The National Mathematics Advisory Panel (2008) stated that not attaining fluency with basic math skills such as addition, subtraction, multiplication, and division can lead to students having the inability to comprehend computational problems, as they lack the foundational skills (Hawkins et al, 2017). Also, students who have developed more fluent math facts have seen greater success in algebra at higher grade levels, which can also lead to success at the college level. This is why fact fluency is such an important component to a student’s early math career. Having knowledge of basic math skills such as addition, subtraction, multiplication, and division lays the foundation for many later topics in math.

According to Bay-Williams and King (2019), many school districts struggle to adopt one precise definition of what fluency with the basic mathematical computations actually is. These authors also state that acquiring this automaticity, also known as procedural fluency, requires a set plan to build “mathematical confidence and number sense” (p. 2). In order to develop this procedural fluency, there are three important pieces. The first component to achieve automaticity is being able to compute with accuracy: that is, being able to compute problems and achieve the correct answer. The second component to achieving procedural fluency is efficiency. Students should be able to come up with solutions in a reasonable time frame, including selecting the correct strategy to a given problem. Lastly, achieving procedural fluency requires the ability to not only find the appropriate strategy, but adapt the strategy to better fit a problem when necessary, and then generalize it to other like problems.

Since the early 2010s, the usage of mobile devices throughout education have evolved greatly, and there are a variety of different software applications that can be implemented by school districts that positively impact learning. According to the Intervention in School and Clinic Journal, many parents want to see their children use technology in a way that will help their education by improving their academic achievement (Ok, et al., 2015). When teachers and parents are deciding which applications to allow their students to use, inappropriate applications are often selected before surveying its educations attributes. Having a set idea of what characteristics to look for in math applications is a valuable asset to assisting a child's math fluency.

Despite fact fluency being a difficult topic for students with SLDs in math, researchers describe how teachers can introduce tricks that not only develop strategies for solving problems, but also help students understand what they are learning (Wadlington, E., & Wadlington, P. L., 2008). These tricks can help students see relationships between what they are learning and other concepts they have seen in the past. In regards to fact fluency, there are multiple tricks that can help students develop a better understanding. The book titled *Secrets of Mental Math* by Arthur Benjamin and Michael Sherman outlines some tricks anyone can use to make quick calculations. Some of these include tricks such as casting out nines and casting out elevens, which is a way to double check a solution to a problem using basic math facts (2007). There are multiple other tricks in this book such as multiplying by 11, and multiplying 3x2 equations. Another set of useful tricks are the divisibility rules. There is a set list of rules that can be used to check to see if a number is divisible by certain integers. These divisibility rules not only help students become more fluent with dividing, but also reinforce major math facts.

Having continued practice of the basic math facts is another important way to develop stronger fluency skills. In the journal article titled *Implementing an Effective Mathematics Fact Fluency Practice Activity, Riccomini* states that “most effective fluency-building practice requires both effective instructional design and implementation that is purposeful and targeted” (Riccomini, et al., 2017). In order to establish this, there are four important components to implement: modeling; numerous opportunities to respond to the question; feedback that supports the correct answer; and the amount of known facts being proportional to the unknown facts. There are a variety of activities that can support teaching these components when implementing effective instruction.

Math Applications for Fact Fluency

Literature Review

Multiple researchers have analyzed online applications that benefit students in the area of mathematics, but most of these applications are costly. However, there are still a variety of free applications students can take advantage of to support their math fluency. Although there are many different outlines of which categories are important when assessing an online application, there are many similarities as well. Based on the literature, the following categories are the most crucial when evaluating an online math application: A) Ease of Use, B) Data Collection/Progress Monitoring, C) Knowledge of the Productive Struggle in Math, D) Variety of Strategies, and E) Student Engagement. Explanations of each of these categories is outlined below.

The journal article titled *How to Find Good Apps: An Evaluation Rubric for Instructional Apps for Teaching Students with Learning Disabilities* provides a great overview of the important categories for assessing an application. According to the research, an important quality for creating a rubric to evaluate applications is Ease of Use. When a student has a diagnosed SLD, research shows that “apps should be easy and simple and allow students to receive assistance, either technical or instructional, and be fully accessible” (Ok, et al., 2015). Therefore, the app should be straightforward and easy to navigate for all students at different levels of achievement. Also, the objectives within the app should be clear and concise. Students should not need to dig through different menus or pages in the application to find items such as directions, objectives, or scores. Critical information to the application should be easily accessible.

The second category included in a rubric for assessing math applications is Data Collection and Progress Monitoring. This same journal states that “progress monitoring is used to alert the teacher and learner when the student is or is not meeting the instructional objective and or making sufficient progress towards some goal.” Therefore, progress monitoring is a way to provide continued evaluation of a student's progress throughout a specific topic. Providing a record of the student’s performance is important. Applications rated the highest employ some sort of scoring or point system used. Teachers cannot report on a student’s progress of a certain topic without specific data that informs them on how they are doing (Ok, et al., 2015).

Productive struggle is the process of helping students work through why a problem is incorrect, instead of giving the student the correct answer right away. According to El-ahwal and Shahin (2020), one way to understand math more thoroughly is by struggling at first. Doing this infers that the student actually learns and understands the topic. When students achieve this productive struggle in mathematics, students are making efforts towards goals and achievements that they did not believe they could achieve. Referring back to the Intervention in School and Clinic Journal, actually understanding a topic requires a deeper learning (Ok, et al., 2015). Students with learning disabilities need repeated consistent and sufficient practice opportunities in order to learn foundational math skills. When students continuously practice, “they begin to break intellectual boundaries by using and elaborating their skills and form more connections to learning.” Thus, applications that contain ideas of productive struggle to help connect deeper into learning abilities are more useful to math teachers.

Including a variety of strategies in order to solve problems is another crucial component when evaluating applications in math. Students with SLDs “learn best when provided a combination of explicit and strategic instruction” (2015). Therefore, in order to exhibit positive outcomes, providing strategies that specifically link to the student’s needs is important. Applications are rated higher when the app provides incremental steps for students.

Lastly, student engagement is criteria that should be evident throughout the whole application. Student engagement is a crucial component to support students’ motivation, given that it “guides direction and maintains persistence” (2015). Without encouraging motivation, students may be easily disheartened, and lose their incentive towards learning. If an application provides engagement in the beginning, but does not maintain students’ motivation, it tends to be rated poorly. Therefore, providing engaging, student-centered features that maintains student engagement is an important criteria for evaluating applications (2015).

Overall, these five categories provide important criteria when considering using apps for students.

Methodology for Application Analysis

To develop a deeper understanding of which math applications are effective for promoting fact fluency, the researcher followed a sequenced set of steps to conduct this investigation. First, a literature review was conducted to establish an evaluation rubric with a variety of criteria for scoring each characteristic of a good application for promoting math fluency. Next, using the evaluation rubric, each application was independently scored across five categories using a three-point scale, 3 representing strong features, 2 representing some, but limited features, and 1 representing little to no features. Once each of the five categories were scored, the ratings were tabulated into one aggregate score. Then, using the aggregate scores, an analysis of each applications ranking could be recorded. The evaluation rubric is posted below.

Evaluation Rubric for Applications in Math

	3	2	1
Ease of Use	Application is easy to understand, there are clear directions to all tasks, and students at multiple levels of achievement are able to navigate the application with no problems.	Application is somewhat easy to understand and the directions are not very clear, and students at all levels of achievement are able to navigate the application with minimal issues.	Application is difficult to understand and there are either no directions or the directions leave unanswered questions. Most or all students are unable to navigate through the application successfully.
Data Collection/ Progress Monitoring	Application uses a tool to track student's progress over a period of time using a point/score/grade system.	Application does not keep track of the student's data, but gives the student a score in some sort of point/score/grade system.	Application gives little to no feedback to students, and does not track the students' progress over time.
Knowledge of Productive Struggle in Math	Application gives examples to students and thoroughly gives step-by-step instruction to achieve the correct solution in problems students get incorrect.	Application gives examples to students, but does not give step-by-step instruction when a student gets a problem incorrect.	Application does not give students any examples and provides no instruction to incorrect answers.
Variety of Strategies	Application provides students with multiple tools and strategies to complete tasks.	Application has a limited number of tools and strategies that can assist students in completing tasks.	Application does not give students any tools or strategies in order to complete tasks.
Student Engagement	Application promotes student involvement by providing a variety of visual and auditory stimuli that keeps students focused on the task at hand.	Application provides either minimal visual and auditory stimuli that will distract students from the task at hand.	Application does not use any sort of visual or auditory stimuli to keep students engaged.

Results

Twenty applications were selected for analysis. Applications were chosen based on a few specific criteria. The first criteria included that the application was free to purchase, and there was no purchase necessary within the application in order to be able to use it. Also, the application had to allow for practice of at minimum two of the four basic math operations. If the app did not use addition and subtraction or multiplication and division, it must have had the option to practice all four operations. Applications without this feature were not included in this research. The last criteria was that these applications were suitable for use for students at the elementary or middle school level. Applications aimed towards high school students were not included. Once all twenty applications were chosen, each application was evaluated using the scoring rubric designed by the researcher.

Data Analysis

An aggregate summary of the application ratings across the five rubric categories is posted in the chart below. The chart organizes applications from the most highly rated applications to the least highly rated applications.

Name of App	Ease of Use	Data Collection/ Progress Monitoring	Knowledge of Productive Struggle	Variety of Strategies	Student Engagement	Total Rated Score
Mental Math	3	2	1	3	2	11
Fact Cards	3	2	2	2	2	11
MATHCards	3	1	2	1	3	10
Fluent	3	1	2	2	2	10
Math Brain Booster	3	2	1	1	2	9
Math	2	2	2	1	2	9
Math Tango	2	2	1	1	3	9
QMM	3	2	1	1	2	9
Math Games	2	3	2	1	1	9
Sushi Monster	2	1	1	1	3	8
Math Tracker	3	1	1	1	2	8
#LearnMaths	3	1	1	1	2	8
Tap Math	2	1	1	1	3	8
5 Dice	1	2	1	1	2	7
Arithmetic Brain	2	1	1	1	2	7
My Math App	2	1	1	1	2	7
Math App	2	1	1	1	2	7
SwunMath	1	1	1	2	2	7
Math	2	1	1	1	1	6
Addition and Subtraction for Kids	1	1	1	1	1	5

The results showed that the top four rated apps were Mental Math, Fact Cards, MATHCards, and Fluent. Although these are the results based on the evaluation rubric designed from the literature, it is important to acknowledge that the value of each application may vary depending on the specific needs of a specific student. For example, one student may feel as though an application was very easy to use, while another may have found it very difficult. Or, an application may provide multiple strategies for solving a problem, but it still may not be a strategy that works for the student. Therefore, despite an application being highly scored and recommended, it still may not be the right app for your student. Each application chosen should be strictly dependent on a student's needs. Also, some criteria may alter the scores based on the importance to the teacher, but this may not be a crucial component given this aspect. For example, a teacher may not be looking to collect data on a student and would rather have them focus on repeated practice, but this criteria would bring the apps score down. Thus, one aspect may highly change the scores, so it is all dependent on the teacher.

When looking at one of the highest rated applications Mental Math, the application was rated rather high in all areas except for knowledge of the productive struggle. This application provided flash cards for each operation (addition, subtraction, multiplication, and division), and also provided levels of difficulty for each. However, when an incorrect answer was submitted, the correct answer was given immediately, and then a new question was asked. This did not give students a chance to try the problem again, or give them any tips for solving the problem, hence why it was rated poorly in the area of the productive struggle. One important observation to point out was the text to speech setting, which when turned on read every problem to the user. Although not outlined in the criteria of the rubric, this may be an important aspect based on the student user.

Fact Cards was another top-rated application that was rated the same as Mental Math These applications were very similar, except for in the areas of Productive Struggle and Variety of Strategies. Unlike while using Mental Math, Fact Cards gave students the option to try a problem again when they got it wrong, rather than just moving to the next question. For Variety of Strategies, this application was rated lower than Mental Math, as it did not provide any difficulty level options. Despite these differences, these apps were rated the same. This is one example of how deciding which app to use is completely dependent on the needs of the student.

MATHCards was another application very similar in score to both Mental Math and Fact Cards, but it was rated lower. This application did not provide any data collection, and it did not provide different strategies for completing problems. However, this application was rated higher as far as student engagement, as the visuals were more geared towards the likes of students. The last application rated in the top four was Fluent. This app scored the same in the categories Ease of Use, Data Collection/Progress Monitoring, and Productive Struggle as MATHCards. On the other hand, Fluent was rated higher on Variety of Strategies due to the apps feature of having different levels of difficulty. MATHCards was rated higher on student engagement as it already provided options for students to pick for answers, while Fluent did not have the same feature.

Overall, most of these free apps included some features that teachers look for when picking out applications. But, it is completely dependent on the student, as well as the teacher, to decide which application is the right fit.

Divisibility Rules

Literature Review

As stated above, introducing students to tricks for finding solutions to math problems as well as checking solutions is a valuable concept that all students could learn from. According to the journal article by Elizabeth and Patrick Wadlington, students with disabilities in mathematics tend to exhibit two different learning styles when given instruction (2008). These are a quantitative learning style and a qualitive learning style. A student who shows a quantitative learning style is "usually good with language skills and concepts. They are sequentially oriented and like to take apart problems, solve each piece, and then reassemble those pieces." This type of students generally reveals strengths in calculation procedures, but finds difficulty with broader concepts and principles. A student that demonstrates a qualitative learning style typically learns mathematics the best when they are given techniques to solve problems with different patterns and relationships. The students with a qualitive learning style "are usually good at concepts such as backward counting, subtraction and division, estimation, fractions interpreted by visual models, and spatial relations between and manipulation of geometric shapes." Given this, integrating both learning styles is an efficient way to being successful in math. For a student with a SLD, math is a very complex language given its variety of symbols and complex vocabulary. For example, a student with the reading disability dyslexia may find difficulty when analyzing math pieces due to its complex language. Therefore, developing tricks for memorizing symbols and procedures is a crucial component that can substantially increase a student's ability in math (Wadlington, E., & Wadlington, P. L., 2008).

According to Shermer and Benjamin's book, there are multiple tips and tricks that can be used to enhance math fact fluency. Benjamin is convinced that any students with any learning style can understand the skill of rapid math calculations with repeated practice. Therefore, he gives a variety of techniques that people can use to make a great number of calculations. For example, one technique Sherman and Benjamin presented was a trick for multiplying by 11. This technique states to add the 2 digits of the number you are multiplying by 11, and then put this sum in between the digits you just added to get your answer. One example the authors mentioned was the multiplication problem 32x11. They explained "To solve this problem, simply add the digits, 3+2=5, put the 5 between the 3 and the 2, and there is your answer: 352" (Shermer and Benjamin, p. 1). This text also contains addition tricks for both 2-digit and 3-digit numbers. For multiplication of 2-digit numbers, it is helpful to break the numbers down into simpler and more manageable parts. Many people already do this when trying to solve simple math problems in their head, but it can be useful when there are larger number involved. Students with SLDs would benefit from understanding this trick given the fact that it breaks down the problem into simpler parts. Divisibility rules are another useful trick for students who are struggling with fact fluency. These rules break apart division and help see if one number is divisible by another. Ankit Patodi states that many students do not learn these divisibility rules due to the fact that many teachers do not understand the logic behind them (2021). Divisibility rules help build fact fluency of positive numbers by developing routine tricks that can be applied throughout a variety of calculations.

Proofs of Divisibility Rules

In order to better understand the divisibility rules, it is important to prove why each one works. The table below gives a summary of how to determine if an integer is divisible by a given number. These divisibility rules are outlined through the number 12, and they give quick and easy mental math checks that students can use to determine divisibility.

Number	Rule

2	An integer n is divisible by 2 if and only if the integer has an even number in the ones place.
3	An integer n is divisible by 3 if and only if the sum of the digits is divisible by 3.
4	Let n be an integer with at least 3 digits. Then n is divisible by 4 if and only if the number formed by the last 2 digits is divisible by 4.
5	An integer n is divisible by 5 if and only if the digit in the ones place is a 0 or 5.
6	An integer n is divisible by 6 if and only if the integer n has an even number in the ones place, and the sum of the digits is divisible by 3.
7	An integer n is divisible by 7 if and only if when you double the last digit and subtract it from the number created by the remaining digits, that number is divisible by 7.
8	Let n be an integer with more than 3 digits. n is divisible by 8 if and only if the number formed by the last 3 digits is divisible by 8.
9	An integer n is divisible by 9 if and only if the sum of all of the digits is also divisible by 9.
10	An integer n is divisible by 10 if and only if the digit in the ones place is a 0.
11	An integer n is divisible by 11 if and only if the alternating sum of the digits where the even digits are added and the odd digits are subtracted is divisible by 11.
12	An integer n is divisible by 12 if and only if the sum of the digits is divisible by 3, and the number formed by the last 2 digits is divisible by 4.

Theorem: An integer n is divisible by 2 if and only if the integer has an even number in the ones place.

Proof: Given an integer $n = d_m d_{m-1} \dots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, we can say that:

$$n = 10^m d_m + 10^{m-1} d_{m-1} + \cdots + 10^2 d_2 + 10^1 d_1 + 10^0 d_0$$

$$n \equiv 0 + 0 + \cdots + 0 + 0 + d_0 \pmod 2$$

$$n \equiv d_0 (\text{mod } 2)$$

Therefore, when $n = \sum_{i=0}^{m} 10^m d_m$, we can conclude that n is congruent to $d_0\,(\text{mod } 2)$.

=> First, assume that n is divisible by 2. Therefore, we can say that 2 divides n, implying that $n \equiv 0(\text{mod } 2)$. As stated earlier, we know that $n \equiv d_0\,(\text{mod } 2)$. Thus, we can say that $d_0(\text{mod } 2) \equiv 0(\text{mod } 2)$, and therefore the digit in the ones place (d_0) must be a multiple of 2 (or d_0 must be even).

<= Now assume that the digit in the ones place is even. Therefore, we can say that 2 divides d_0. This implies that $d_0 \equiv 0(\text{mod } 2)$. As stated earlier, we know that $n \equiv d_0(\text{mod } 2)$, and therefore by transitivity, $n \equiv 0(\text{mod } 2)$, so n is divisible by 2. Thus, when the digit in the ones place is even, n is divisible by 2.

Thus we see that an integer n is divisible by 2 if and only if the integer has an even number in the ones place.■

Theorem: An integer n is divisible by 3 if and only if the sum of the digits is divisible by 3.

Proof: Given an integer $n = d_m d_{m-1} \ldots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, we can say that:

$$n = 10^m d_m + 10^{m-1} d_{m-1} + \cdots + 10^2 d_2 + 10^1 d_1 + 10^0 d_0$$

$$n \equiv \big((1 \times d_m) + \cdots + (1 \times d_2) + (1 \times d_1) + (1 \times d_0)\big)(\text{mod } 3)$$

$$n \equiv (d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 3)$$

Therefore, n is congruent to the sum of the digits mod3.

=> Assume that n is an integer that is divisible by 3. Therefore, we can say that 3 divides n, meaning that $n \equiv 0(\text{mod } 3)$. As stated above, $n \equiv (d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 3)$, which implies that $(d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 3) \equiv 0(\text{mod } 3)$. Therefore, when n is divisible by 3, the sum of the digits is also divisible by 3.

<= Now assume that the sum of the digits of an integer n is divisible by 3. Therefore, we can say that 3 divides $(d_m + \cdots + d_2 + d_1 + d_0)$, which implies that $(d_m + \cdots + d_2 + d_1 + d_0) \equiv 0(\text{mod}3)$. Since $n \equiv (d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 3)$, we can say that $n \equiv 0(\text{mod } 3)$, meaning n is divisible by 3.

Therefore, an integer n is divisible by 3 if and only if the sum of the digits is divisible by 3.■

Theorem: Let n be an integer with at least 3 digits. n is divisible by 4 if and only if the number formed by the last 2 digits is divisible by 4.

Proof: Given an integer $n = d_m d_{m-1} \dots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, let $x = d_m \dots d_3 d_2$. Therefore, n can be rewritten as $n = 100x + d_1 d_0$.

=> First, assume that n is divisible by 4. That is, $n = 4k$ for some integer k. So,

$$4k = 100x + d_1 d_0$$
$$4k - 100x = d_1 d_0$$
$$4(k - 25x) = d_1 d_0$$

Therefore, since x and k are both integers, the number formed by the last 2 digits is divisible by 4.

<= Now, we want to assume that the last 2 digits of a number are divisible by 4. That is, we can say that $4l = d_1 d_0$ for some integer l. Therefore,

$$n = 100x + d_1 d_0 = 100x + 4l.$$

By basic algebra, we can say that

$$n = 4(25x + l).$$

Therefore, the integer n is divisible by 4.

Thus, when n is an integer with at least 3 digits, n is divisible by 4 if and only if the number formed by the last 2 digits is divisible by 4.■

Theorem: An integer n is divisible by 5 if and only if the digit in the ones place is a 0 or 5.

Proof: Given an integer $n = d_m d_{m-1} \dots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, we can say that:

$$n = 10^m d_m + 10^{m-1} d_{m-1} + \cdots + 10^2 d_2 + 10^1 d_1 + 10^0 d_0$$
$$n \equiv 0 + 0 + \cdots + 0 + 0 + d_0 (\text{mod } 5)$$
$$n \equiv d_0 (\text{mod } 5)$$

Therefore, when $n = \sum_{i=0}^{m} 10^m d_m$, we can conclude that n is congruent to $d_0 (\text{mod } 5)$.

=> First, assume that n is divisible by 5. Therefore, we can say that 5 divides n, implying that$n \equiv 0 (\text{mod } 5)$. As stated earlier, we know that $n \equiv d_0 (\text{mod } 5)$. Thus, we can say that $d_0 (\text{mod } 5) \equiv 0 (\text{mod } 5)$, and therefore the digit in the ones place, d_0, must be a 0 or a 5.

<= Now assume that the digit in the ones place is a 0 or a 5. Therefore, we can say that 5 divides d_0, implying that $d_0 \equiv 0 (\text{mod } 5)$. As stated earlier, we know that $n \equiv d_0 (\text{mod } 5)$, and therefore by transitivity, $n \equiv 0 (\text{mod } 5)$, so n is divisible by 5. Thus, when the digit in the ones place is aa 0 or a 5, n is divisible by 5.

Thus, an integer n is divisible by 5 if and only if the digit in the ones place is a 0 or a 5. ■

Theorem: An integer n is divisible by 6 if and only if the integer n has an even number in the ones place, and the sum of the digits is divisible by 3.

Proof: Assume that n is divisible by 6. Since 2 divides 6, and 6 divides n, we can say that 2 divides n. Also, we know that 3 divides 6, so therefore 3 divides n. Now assume that n is divisible by 2 and 3. Since 2 and 3 are relatively prime, we can say that 6 also divides n. Therefore, an integer n is divisible by 6 if and only if n is divisible by 2 and 3. By previous assertions, an integer n is divisible by 2 if and only if the integer has an even number in the ones pace. Also, an integer n is divisible by 3 if and only if the sum of the digits is divisible by 3. Therefore, an integer n is divisible by 6 if and only if the integer n has an even number in the ones place, and the sum of the digits is divisible by 3. ■

Theorem: An integer n is divisible by 7 if and only if when you double the last digit and subtract it from the number created by the remaining digits, the number is divisible by 7.

Proof: Given an integer $n = d_m d_{m-1} \dots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, let $x = d_m \dots d_3 d_2 d_1$ and let $y = d_0$. Therefore, we can say $n = 10x + y$. We want to show that 7 divides n if and only if 7 divides $x - 2y$.

=> First, assume that 7 divides n. Thus, given some integer k, we know that:

$$10x + y = 7k.$$

We want to show that $x - 2y = 7k$. By manipulating the original equation, we get:

$$5(10x + y) = 5(7k)$$

$$50x + 5y = 35k$$

$$x + 5y = 35k - 49x$$

$$x - 2y = 35k - 49x - 7y$$

$$x - 2y = 7(5k - 7x - y)$$

Since x, k, and y are all integers, it is clear than $x - 2y$ is divisible by 7. In other words, $d_m d_{m-1} \ldots d_2 d_1 - 2d_0$ is divisible by 7.

<= Now, we want to show that if 7 divides $x - 2y$, then 7 divides $n = 10x + y$. First, assume that 7 divides $x - 2y$. Therefore, we can say that:

$$x - 2y = 7l$$

For some integer l. By manipulating this equation, we get:

$$10(x - 2y) = 10(7l)$$

$$10x - 20y = 70l$$

$$10x = 70l + 20y$$

$$10x + y = 70l + 21y$$

$$10x + y = 7(10l + 3y)$$

Since l and y are both integers, we can say that 7 divides $10x + y$. In other words, n is divisible by 7.

Therefore, an integer n is divisible by 7 if and only if when you double the last digit and subtract it from the number created by the remaining digits, the number is divisible by 7. ■

Theorem: Let n be an integer with at least 3 digits. n is divisible by 8 if and only if the number formed by the last 3 digits is divisible by 8.

Proof: Given an integer $n = d_m d_{m-1} \ldots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, where n has more than 3 digits, let $x = d_m \ldots d_4 d_3$. Therefore, n can be written as $n = 1000x + d_2 d_1 d_0$.

=> First, assume that n is divisible by 8. That is, $n = 8k$ for some integer k. So,

$$8k = 1000x + d_2 d_1 d_0$$

$$8k - 1000x = d_2 d_1 d_0$$

$$8(k - 125x) = d_2 d_1 d_0$$

Since x and k are both integers, when a number is divisible by 8, the number formed by the last 3 digits of the number is divisible by 8.

<= Now, we want to assume that the number formed by the last 3 digits of the number is divisible by 8. That is, we can say $8l = d_2d_1d_0$ for some integer *l*. Therefore,

$$n = 1000x + d_2d_1d_0 = 1000x + 8l.$$

By basic algebra, we can say that

$$n = 8(125x + l).$$

Since *x* and *l* are both integers, we can say that *n* is a multiple of 8. Therefore, when the number formed by the last 3 digits of a number is divisible by 8, the entire number is divisible by 8. Thus, when *n* is an integer with at least 3 digits, *n* is divisible by 8 if and only if the number formed by the last 3 digits of the number is divisible by 8.■

Theorem: An integer n is divisible by 9 if and only if the sum of the digits is divisible by 9.

Proof: Given an integer $n = d_m d_{m-1} \ldots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, we can say that:

$$n = 10^m d_m + 10^{m-1} d_{m-1} + \cdots + 10^2 d_2 + 10^1 d_1 + 10^0 d_0$$

$$n \equiv \big((1 \times d_m) + \cdots + (1 \times d_2) + (1 \times d_1) + (1 \times d_0)\big)(\text{mod } 9)$$

$$n \equiv (d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 9)$$

Therefore, n is congruent to the sum of the digits mod 9.

=> Assume that n is an integer that is divisible by 9. Therefore, we can say that 9 divides n, meaning that $n \equiv 0(\text{mod } 9)$. As stated above, $n \equiv (d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 9)$, which implies that $(d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 9) \equiv 0(\text{mod } 9)$. Therefore, when n is divisible by 9, the sum of the digits is also divisible by 9.

<= Now assume that the sum of the digits of an integer n is divisible by 9. Therefore, we can say that 9 divides $(d_m + \cdots + d_2 + d_1 + d_0)$, which implies that $(d_m + \cdots + d_2 + d_1 + d_0) \equiv 0(\text{mod} 9)$. Since $n \equiv (d_m + \cdots + d_2 + d_1 + d_0)(\text{mod } 9)$, we can say that $n \equiv 0(\text{mod } 9)$, meaning n is divisible by 9.

Therefore, an integer n is divisible by 9 if and only if the sum of the digits is divisible by 9.■

Theorem: An integer n is divisible by 10 if and only if the digit in the ones place is a 0.

Proof: Given an integer $n = d_m d_{m-1} \ldots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$, we can say that:

$$n = 10^m d_m + 10^{m-1} d_{m-1} + \cdots + 10^2 d_2 + 10^1 d_1 + 10^0 d_0$$
$$n \equiv 0 + 0 + \cdots + 0 + 0 + d_0 (\text{mod } 10)$$
$$n \equiv d_0 (\text{mod } 10)$$

Therefore, n is congruent to $d_0(\text{mod } 10)$.

=> First, assume that n is divisible by 10. Therefore, we can say that 10 divides n, implying that$n \equiv 0(\text{mod } 10)$. As stated earlier, we know that $n \equiv d_0(\text{mod } 10)$. Thus, we can say that $d_0 \equiv 0(\text{mod } 10)$, and therefore the digit in the ones place, d_0, must be a 0.

<= Now assume that the digit in the ones place is a 0. Therefore, we can say that 10 divides d_0, implying that $d_0 \equiv 0(\text{mod } 10)$. As stated earlier, we know that $n \equiv d_0(\text{mod } 10)$, and therefore by transitivity, $n \equiv 0(\text{mod } 10)$, so n is divisible by 10. Thus, when the digit in the ones place is a 0, n is divisible by 10.

Thus, an integer n is divisible by 10 if and only if the digit in the ones place is a 0.■

Theorem: An integer *n* is divisible by 11 if and only if the alternating sum of the digits where the even digits are added and the odd digits are subtracted is divisible by 11.

Proof: Let *n* be an integer such that $n = d_m d_{m-1} \ldots d_2 d_1 d_0 = \sum_{i=0}^{m} 10^m d_m$. We know that when *m* is odd, $10^{\mathrm{m}} d_m = -1^m (\text{mod} 11)$. When *m* is even, $10^{\mathrm{m}} d_m = 1^m (\text{mod } 11)$. Therefore, we can write *n* as the alternating sum of the digits, written as:

$$n \equiv \sum_{i=0}^{m} (-1)^m d_m (\text{mod } 11) \equiv d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^m d_m (\text{mod } 11).$$

=> Assume that *n* is divisible by 11. Given this, we can say that 11 divides *n*, which implies that $n \equiv 0(\text{mod } 11)$. Since $n \equiv d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^m d_m(\text{mod } 11)$, by transitivity, we can conclude that $d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^m d_m \equiv 0(\text{mod } 11)$.Thus, the alternating sum of the digits in this case is divisible by 11.

<= Now, we want to assume that the alternating sum of the digits is divisible by 11. Thus, we can say that since 11 divides $d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^m d_m$, then $d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^m d_m \equiv 0(\text{mod } 11)$. Since $n \equiv d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^m d_m(\text{mod } 11)$, then we can conclude that $n \equiv 0(\text{mod } 11)$. Thus, *n* is divisible by 11.

Therefore, an integer *n* is divisible by 11 if and only if the alternating sum of the digits where the even digits are added and the odd digits are subtracted is divisible by 11.■

Theorem: An integer n is divisible by 12 if and only if the sum of the digits is divisible by 3, and the number formed by the last 2 digits is divisible by 4.

Proof: Assume that n is divisible by 12. Since 3 divides 12, and 12 divides n, we can say that 3 divides n. Also, we know that 4 divides 12, so therefore 4 divides n. Now assume that n is divisible by 3 and 4. Since 3 and 4 are relatively prime, we can say that 12 also divides n. Therefore, an integer n is divisible by 12 if and only if n is divisible by 3 and 4. By previous assertions, an integer n is divisible by 3 if and only if the sum of the digits is divisible by 3. Also, an integer n is divisible by 4 if and only if the number formed by the last 2 digits is divisible by 4. Therefore, an integer n is divisible by 12 if and only if the sum of the digits is divisible by 3, and the number formed by the last 2 digits is divisible by 4. ∎

Continued Practice

Literature Review

According to the book by Bay-Williams and Kling about mastering fact fluency, a student becomes fluent in a topic by a continued practice. An effective way to introduce math facts is not just through introducing and discussing the facts, but by using more interactive methods (2019). Baroody, et al explains that fluency is derived through three different stages: counting strategies, reasoning strategies, and mastery (Baroody, et al., 2009). Bay-Williams and Kling use these stages in order to discuss the different strategies that can be used in order to help students achieve proficient math fact fluency throughout the classroom.

Before a student can develop a deeper understanding of math facts there are three stages of learning required before the content is completely mastered. Stage one is the process of a student developing strategies to help them answer basic math facts. Even if these strategies are not very efficient, this beginning stage is important to developing automaticity during fact fluency. Stage two primarily uses the strategies developed from stage 1 to increase speed of the facts, and to hopefully no longer need these strategies. Many programs for fluency skip the second stage and head straight to the third, but doing this does not allow the student to become independently sufficient to perform these facts, and typically get pushed back to stage 1 after a few months of no continued practices. Stage three occurs when students obtain “highly efficient production of answers, either through quick strategy application or through recall” (Bay-Williams & Kling, p. 4-5). Continued practice using these three stages is a key to developing automatic math fact fluency.

There are multiple types of continued practice a student can use to improve their math facts. The most documented type of practice is through the use of in-person games (Bay-Williams & Kling, 2019). According to the literature, students typically have a better reaction and an increased motivation to do an activity when it is described as a game. The following questions should be used when deciding on game activities for students to develop fact fluency: “To what extend does the game...

1) Provide an opportunity to practice the subset of facts that the students are learning?
2) Appeal to the age of your students?
3) Employ visuals or tools (such as ten frames, quick looks, or arrays) to support strategy development?
4) Involve selecting from among derived fact strategies (for mastery-leveled games)?
5) Provide opportunities for discussion among students about their mathematical thinking?
6) Encourage individual accountability? (For example, are students solving their own facts or competing to solve the same fact? The former practice provides more “think time” and avoids opting out)
7) Remove time pressures?
8) Involve logic or strategic moves, enhancing the “fun factor?”
9) Offer opportunities for adaptation so that all students can experience appropriate challenge?
10) Lend itself to you being able to listen and watch in order to assess progress (p. 11)?”

A combination of these elements ensures a certain level of quality in each game that make sure that they help develop fluency and automaticity in math facts (Bay-Williams & Kling, 2019).

Analysis

Strategies for developing and deciding the best games to promote fact fluency are similar to the strategies used for evaluating the best apps for students. Both apps and games encourage visuals that will keep students engaged, but also promote fluency. Just like when using apps, when using games it is important to be able to assess the progress of students to make certain the game is having a positive effect on the student, and not causing a decline in their skills. The productive struggle that teachers look for in math applications, should also be encouraged when using games. When a student is held accountable for solving facts, it helps them apply strategies for solving problems. After repeated practice, students will eventually not need the strategies, and they will be fluent in their math facts. Lastly, just as teachers look for a variety of different strategies for solving problems when employing an app, so do teachers employ, math games for students with an ability to adapt to a student's specific needs. This is especially important for a student with an SLD in math.

Summary

This research examined multiple ways to help students with disabilities achieve fluency with their basic math facts. First, twenty free applications that support math fluency were evaluated for their potential efficiency for supporting students with disabilities with their math fluency. Next, the concept of tricks and strategies for students was introduced based on the information outlined by the literature. Then, each divisibility rule was proved to develop a deeper understanding of how and why these strategies work. The last section outlines the characteristics that teachers can look for when designing games for students with math fluency challenges. By conducting this multi-layered investigation for promoting math fluency, I have enhanced my understanding of how apps, strategies and games might contribute to my future proficiency as a special education teacher in math.

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