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MASTER

ACTUARIAL SCIENCE

MASTER'S FINAL WORK

DISSERTATION

OPTIMAL SURPLUS REINSURANCE

SIMÃO PEDRO GOMES GUEDES

OCTOBER - 2022

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SUPERVISION:
PROF. ALEXANDRA BUGALHO DE MOURA
PROF. JOÃO MANUEL DE SOUSA ANDRADE E SILVA

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"The future does not belong to the faint-hearted, but to the brave."
— Former US President, Ronald Reagan

"And whatever you do, whether in word or deed, do it all in the name of the Lord Jesus, giving thanks to God the Father through him."
— Colossians 3:17, Holy Bible (NIV)

ABSTRACT

In this work, we seek to optimise surplus reinsurance, which is a type of proportional reinsurance where proportions vary with the sum insured, based on a retention line and on a reinsurance policy limit. Optimisation is done on a portfolio which is based on real-life fire insurance policies.

We perform this optimisation by resorting to three different optimality criteria, which are based on the expected utility, the standard deviation and the Value-at-Risk of the insurer's wealth. Where we found it to be possible, expressions were derived which can be solved to find the optimal reinsurance contract. However, the difficulty in solving them analytically leads us to optimise them numerically. Where we could not derive such expressions, we resorted to simulations in order to find the optimal reinsurance contract. Such optimisations are carried out under different reinsurance commissions and different premium principles for the primary policies.

Our findings suggest at least four possible conclusions. Firstly, none of the three optimality criteria used is clearly better than the other ones, although the expected utility criterion tends to yield more risk-averse results. Secondly, usage of the standard deviation as the primary premium principle generally provides better results — with higher expected values by comparison to the standard deviations, as well as more favourable Values-at-Risk — than the expected value and the variance principles. Thirdly, our results highlight the importance that the reinsurance commission provide a reasonable cover for the expenses of the insurer, in order to produce acceptable outcomes. Lastly and perhaps more interestingly, we conclude that the optimal surplus reinsurance contract will, at least quite frequently, feature a reinsurance policy limit low enough to not fully cover the highest risks.

KEYWORDS: Reinsurance; Optimal reinsurance; Proportional reinsurance; Surplus reinsurance; Numerical optimisation; Simulation.

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RESUMO

Neste trabalho, procura-se optimizar o resseguro de *surplus*, que é um tipo de resseguro proporcional em que as proporções variam com o capital seguro, com base numa linha de retenção e num limite à apólice de resseguro. A optimização é efectuada numa carteira baseada em apólices reais de seguro de incêndio.

Realiza-se esta optimização com recurso a três diferentes critérios de optimalidade, baseados na utilidade esperada, no desvio-padrão e no Value-at-Risk da riqueza da seguradora. Quando nos foi possível, obtivemos expressões que podem ser resolvidas para encontrar o contrato de resseguro óptimo. Contudo, a dificuldade de as resolver analiticamente leva-nos a optimizá-las numericamente. Sempre que não conseguimos obter tais expressões, recorremos a simulações para encontrar o contrato de resseguro óptimo. Tais optimizações são efectuadas sob diferentes comissões de resseguro e diferentes princípios de prémio para as apólices primárias.

Os nossos resultados sugerem pelo menos quatro conclusões possíveis. Em primeiro lugar, nenhum dos três critérios de optimalidade empregues é claramente melhor do que os outros, embora o da utilidade esperada tenha tendência para produzir resultados mais avessos ao risco. Em segundo lugar, o uso do desvio-padrão como princípio do prémio para as apólices primárias costuma apresentar resultados melhores — com maiores valores esperados por comparação com os desvios-padrão, bem como Values-at-Risk mais favoráveis — do que os princípios do valor esperado e da variância. Em terceiro lugar, os nossos resultados salientam a importância de que a comissão de resseguro cubra razoavelmente as despesas da seguradora, a fim de gerar resultados admissíveis. Finalmente e talvez mais interessante, concluímos que o contrato de resseguro de *surplus* óptimo terá, pelo menos bastante frequentemente, um limite à apólice de resseguro suficientemente baixo para que não cubra totalmente os riscos mais elevados.

PALAVRAS-CHAVE: Resseguro; Resseguro óptimo; Resseguro proporcional; Resseguro de *surplus*; Optimização numérica; Simulação.

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1. Introduction

Reinsurance is defined as a contract between an insurer and a reinsurer whereby the insurer (also called 'reinsured' or 'cedant') cedes part of its risk to the reinsurer, which agrees to indemnify it to the insurer in exchange for a reinsurance premium. The underlying principle is the same as the one which exists for insurance (also called 'primary insurance' here, to help distinguish it), the difference being that the policyholder is now an insurer.

Insurers are interested in signing reinsurance contracts mainly because it helps them prevent high losses, which reduces the volatility of the risk they face and, therefore, helps stabilise their profits. In general, the less risk an insurer retains, the more stable its outcome is expected to be. On the other hand, assuming the insurance business is profitable on average, it is not in the insurer's interest to reinsure too much of its risk — in the limiting case, the insurer would cede all of its risk, at which point it would cease to be an insurer and become simply an intermediary. These remarks serve to illustrate the importance of studying the problem of optimising reinsurance, that is, of deciding how much risk the insurer should cede and in what way.

Reinsurance contracts are divided broadly into two categories, containing four standard types of reinsurance (which can exist on their own or be combined with one another):

- Proportional reinsurance:
    - Quota-share reinsurance;
    - Surplus reinsurance;
- Non-proportional reinsurance:
    - Excess of loss reinsurance;
    - Stop-loss reinsurance.

Non-proportional reinsurance contracts establish a value — called a 'retention line', or simply 'retention' or 'line' — above which the reinsurance comes into force, covering the portion of claims which exceeds the line.[1] This cover, however, is usually limited to

[1] The difference between the two types of non-proportional reinsurance is that, in excess of loss, the line applies to each policy individually, whilst, in stop-loss, it applies to the sum of the claims of all policies.

a multiple of the retention, above which the responsibility returns to the insurer; in this case, it is said that the insurer has bought k lines of reinsurance, where $k+1$ is the ratio between the reinsurance policy limit and the retention. It follows that a contract of this type with a reinsurance policy limit high enough to cover all primary policy limits will guarantee that the insurer shall never have to pay more than the retention line. Pricing these forms of reinsurance works in the same way as pricing primary insurance. Such forms are not within the scope of this work.

Quota-share reinsurance is easier to understand: the reinsured and reinsurer agree on a cession percentage, at which all claims will be proportionally indemnified by the reinsurer, usually under no other restrictions. Typically, this proportion is also used to price the contract: the reinsurance premium equals that same cession percentage times the total primary premium received (minus a commission). Proportionality is a key feature.

Surplus reinsurance is an alternative form of proportional reinsurance which draws from non-proportional reinsurance the concept of a retention line. Under this type of contract, all claims of a policy are reinsured proportionally, as with quota-share reinsurance, but the proportion varies with the primary policy limit:[2] for all policies whose limit exceeds the retention, whenever there is a claim, the insurer retains a proportion corresponding to the ratio between the retention and the policy limit, whilst the rest is ceded; for the other policies, nothing is ceded. This has the effect of retaining more of the smaller risks and less of the higher ones, as with non-proportional reinsurance. Such contracts will typically be subject to a reinsurance policy limit, in the same way as with non-proportional reinsurance. Pricing will usually be done by using, for each policy, the same proportion as that which was defined as the ceded percentage (minus a commission), in the same way as with quota-share reinsurance; the difference is that now, instead of a single proportion, a different proportion is applied to each policy and the total premium is calculated by then adding all the individual results.

Albrecher *et al.* (2017) consider surplus reinsurance to be an improvement over quota-share reinsurance. Both have the proportionality feature in common — which brings them advantages such as being administratively simple to implement in practice,

[2] According to Albrecher *et al.* (2017), the policy limit is, in some cases, replaced by the Probable Maximum Loss associated with each policy.

having the same effect as an increase in the solvency capital and reducing the risk of reinsurance moral hazard.[3] Yet surplus reinsurance, due to its higher cession percentages for larger risks, has the additional advantage of keeping small risks mostly or fully under the scope of the insurer, which is preferable if such risks are profitable on average and not prone to generating severe losses; in other words, reinsurance will come into force only for sufficiently large risks — as in non-proportional reinsurance —, which are the ones the insurer most wants to reinsure. The authors claim that surplus reinsurance is common in several non-life lines of business, such as fire, property, accident, engineering and marine insurance.

Interestingly, we have found that most papers mentioning surplus reinsurance do not include the reinsurance policy limit in their definitions or analyses. In doing so, they work with the implicit assumption that said limit is always greater than the highest primary policy limit. We do not know why this seems to be common practice. If it is due to the belief that the optimal surplus reinsurance contract will always fulfil that condition, our results will challenge that conjecture.

Variants of these standard proportional reinsurance contracts also exist: they are the variable quota-share reinsurance (where the portfolio is broken into segments of risk and a different proportion is applied to each segment) and the surplus reinsurance with a table of lines (where, similarly, a different retention line is used for each segment of risk), as defined by Glineur and Walhin (2006). These will not be studied in this work. In fact, the four standard types of reinsurance presented above should not be regarded as an exhaustive list, since reinsurance can theoretically be defined in any way the signing parties want.

Finally, there are also alternative and unusual definitions of surplus reinsurance, provided by Verbeek (1966) and El Attar *et al*. (2018), which strangely do not respect the feature

[3] Moral hazard is mainly a concern in non-proportional reinsurance. The authors explain that, in such contracts, when a claim is already known to be larger than the retention line, the insurer can become negligent when settling it. The reasoning is that, since any marginal increase on the value falls exclusively on the reinsurer, the insurer has little or no incentive to determine the claim's exact value (they recall that settling claims more rigorously is expensive). Within the non-proportional types, this risk is greater in the case of stop-loss reinsurance, where the only thing that matters is whether the sum of all claims exceeds the retention line or not. Meanwhile, in proportional reinsurance, since the insurer is always responsible for a fixed percentage of the claim (regardless of its value), it is in its interest to find exactly the true value of any claim.

of proportionality which all other works we have found recognise to be an essential trait of this type of contract. These other definitions will be explained in more detail in the following chapter but shall be ignored for the purposes of this work.

As we shall see in the following chapter, surplus reinsurance is rarely treated in literature about the optimal reinsurance problem. Our motivation to study it arises from this fact, together with our perception — following contacts with the Portuguese insurer which has given us data for this work — that surplus reinsurance may not be as uncommon in actuarial practice as its scarcity in actuarial literature might lead us to believe. We expect, therefore, to provide a positive contribution by presenting that which, to the best of our knowledge, is the first work which optimises a reinsurance contract specifically of the surplus type, based on real data.

2. STATE OF THE ART

2.1. On the optimal reinsurance problem in general

The optimal reinsurance problem has been studied in the actuarial academic literature for decades. Although the problem can be studied from the viewpoint of the reinsurer, or even from the viewpoint of both parties at once, most scholars usually analyse it from the cedant's perspective — that is, they seek the reinsurance contract which optimises a given function of the insurer's retained risk or final wealth.

The earliest works on optimal reinsurance were produced by de Finetti (1940), who focused on proportional reinsurance and whose findings will be explained in the next section, and by Borch (1960), who proved that stop-loss is the optimal type of reinsurance, in the sense that, for a given net reinsurance premium, it minimises the variance of the retained claims.

Since then, a significant amount of research has been conducted on this topic: each new paper will resort to a different optimality criterion or work under different assumptions, which is why new contributions with different results are regularly presented. For example, Centeno (1985) showed that — between a standard quota-share contract, an excess of loss contract and a combination of the two —, if the premium for the excess of loss contract is determined through either the expected value or the standard deviation principles, the excess of loss type is always optimal, in the sense that it minimises the skewness coefficient of the retained claims, under constraints on its expected value and standard deviation. Some authors seek the optimal type of reinsurance among all conceivable reinsurance contracts (not limited to the standard ones), as long as these respect the basic condition (to avoid moral hazard) that the ceded amounts never decrease if the claim size increases. Within this broad set of contracts, Denuit and Vermandele (1998) proved that stop-loss reinsurance is optimal, in the sense that it minimises the retained risk in the stop-loss order,[4] by comparison to all other types of reinsurance with the same expected retention and the same commission, assuming an expected value reinsurance premium principle; if stop-loss is not available, the same thing was proven

[4] Risk X is said to be smaller than risk Y in the stop-loss order if, for any given d greater than zero, $E(\max\{X - d; 0\})$ is smaller than $E(\max\{Y - d; 0\})$ — in other words, if the stop-loss reinsurance pure premium of X is smaller than that of Y, under any given retention line.

for excess of loss reinsurance, under the same criterion and hypotheses. In another work, Guerra and Centeno (2008) proved that stop-loss reinsurance is optimal, in the sense that it maximises the expected utility of the insurer's wealth, according to the exponential utility function, under an expected value reinsurance premium principle.

It is noteworthy that stop-loss reinsurance — and, when not so, excess of loss reinsurance — arises frequently as an optimal solution, even under very different optimisation criteria. This remark of recurring results in favour of non-proportional reinsurance makes analyses of proportional reinsurance less common, a fact which will be noted and discussed in the following section.

For a deeper insight into optimal reinsurance research, we recommend the paper by Centeno and Simões (2009), who compiled several classical and more recent results, or the book by Albrecher *et al*. (2017), for an exhaustive regard at the concept of reinsurance.

2.2. On the optimal surplus reinsurance problem

However, when reinsurance is restricted to the surplus type, the problem of its optimisation becomes very infrequent in the available literature. When this form of reinsurance does arise, it is, to the best of our knowledge, always analysed by comparison to other types of reinsurance (as we will see in the following paragraphs); we have found no papers where surplus reinsurance is optimised on its own. This applies to theoretical and to practical works on reinsurance.

The earliest theoretical work we can find on proportional reinsurance is that of de Finetti (1940), who obtained a method which gives the optimal cession percentages, for each risk, in order to minimise the variance of the insurer's profit after imposing a certain minimum value for the expected profit. This presumes a generalised type of proportional contract,[5] where each risk can have different cession proportions with no restriction. In a later work, Glineur and Walhin (2006) take de Finetti's result, present a simpler proof for it, and extend it to the more specific types of variable quota-share reinsurance and of

[5] We recall that, in the available literature, we have found four types of proportional reinsurance: standard quota-share (one cession proportion common to the whole portfolio); variable quota-share (one cession proportion for each segment of risk); standard surplus (cession proportions are calculated for each risk based on a common line); and surplus with a table of lines (cession proportions are calculated based on one line for each segment of risk). The result of de Finetti (1940) implies a generalised proportional contract, not bound by the rules of any of these four types.

surplus reinsurance with a table of lines. By applying their results to a numerical example, they prove that neither of these two types of reinsurance is always optimal over the other.

We can also find a few papers working from a practical viewpoint. Lampaert and Walhin (2005) seek to explain the lack of works on proportional reinsurance by recalling that non-proportional reinsurance is known to be optimal, a statement which they substantiate by citing the aforementioned findings of Denuit and Vermandele (1998). The authors comment, however, that this does not mean that, in practice, proportional reinsurance is never used. Rather, they state that non-proportional reinsurance has two main disadvantages: a difficulty in correctly pricing contracts and — especially for stop-loss reinsurance — a higher risk of moral hazard (as seen in the Introduction). Both such issues lead to high loadings, wherefore the authors argue that proportional reinsurance also has its place in the actuarial framework. Using the return on risk-adjusted capital (RORAC)[6] as the optimality criterion, they find the optimal reinsurance contract for their set of data under each of the four possible forms of proportional reinsurance and compare the results obtained across them. They conclude that, on their data, the best form is either standard surplus reinsurance or surplus reinsurance with a table of lines. Between these two options, the optimal choice depends on the determination of the table of lines: according to the authors, practitioners traditionally use one of two methods — called the 'inverse claim probability' and the 'inverse rate' —, trusting that either can provide a better result than a standard surplus contract. However, the authors prove this belief to be wrong, as, on their data, both such tables are sub-optimal when compared to standard surplus. Then, they build an alternative table of lines based on the aforementioned method of de Finetti (1940), as expanded by Glineur and Walhin (2006),[7] and this table proves to be optimal against all other proportional reinsurance contracts.

Other works have taken similar optimisation approaches, based on comparisons of different kinds of reinsurance. El-Bolkiny *et al.* (2018) make a more limited analysis across three types of proportional reinsurance (standard quota-share, variable quota-share and standard surplus) where, rather than finding the optimal values within each of them,

[6] The RORAC is defined as the difference between the retained premium and the expected value of the retained claims, divided by the difference between the capital requirement and the retained premium.
[7] Although the paper of Glineur and Walhin (2006) was published after that of Lampaert and Walhin (2005), the latter cites the former as an 'unpublished manuscript'.

they provide unexplained *ad hoc* values and simply compare their RORACs. Meanwhile, Verlaak and Beirlant (2003) work on compound forms of reinsurance, where multiple combinations of reinsurance types[8] are explored, and derive equations which can be solved to optimise such contracts. Concerning proportional reinsurance, these authors note that the available literature focuses almost exclusively on quota-share, which does not reflect common business practice, where surplus reinsurance is common and, for some contracts, it is the only type of proportional reinsurance in force. Veprauskaite and Sherris (2012) study three forms of proportional reinsurance: quota-share, surplus and a combination of the two. Assuming that a given total difference between sum insured and sum reinsured (which they call 'risk appetite') remains equal, they select the contract, within each of the three types, which respects this hypothesis. Then, they use a blend of three optimality criteria — the difference between retained premia and retained claims, the ratio between retained premia and retained claims, and the ratio between the variance and the mean of the retained claims — to determine which of the three contracts is optimal, on real data about life insurance. They conclude that quota-share reinsurance tends to be optimal when the variance of claim amounts is small, but otherwise the best choice is usually either surplus reinsurance or a combination.

As stated in the Introduction, none of the aforementioned papers consider the possibility that there be a policy limit under surplus reinsurance — which implies the assumption that this reinsurance policy limit, if it exists, is always higher than all primary policy limits (at least in any optimal contract). The only exception we have found is the work of El-Bolkiny *et al*. (2018), although, as stated earlier, they give the value for this limit *ad hoc* (as they do for the retention line), without seeking to optimise it. Our work will explicitly try to optimise the retention as well as the policy limit, without making this implicit assumption for which we can find no justification.

To the best of our knowledge, not many more academic works on optimal reinsurance mention surplus reinsurance. This lack of other theoretical works developed about it can be seen, for instance, in the paper of Centeno and Simões (2009), who gather a number of theoretical results obtained broadly within the optimal reinsurance problem, not one of

[8] In particular, they compare seven combinations: excess of loss after surplus; excess of loss after quota-share; stop-loss after quota-share; quota-share after stop-loss; quota-share after excess of loss; quota-share before surplus; and quota-share after surplus.

which mentions surplus reinsurance. More recently, Albrecher *et al*. (2017), when in the context of optimising reinsurance, only discuss surplus reinsurance briefly to report on the findings of Glineur and Walhin (2006) and of Lampaert and Walhin (2005).

It must be noted that some other papers do mention optimisation of surplus reinsurance, but present for it definitions which conflict with the one which is generally accepted. This is the case of El Attar *et al*. (2018), who apply the cession proportion only to the portion of the risk which exceeds the retention line, and of Verbeek (1966), who presents a definition that instead matches excess of loss reinsurance. Both such definitions are ignored here, because they are unusual and because they break the proportionality which is generally recognised to be a defining feature of surplus reinsurance. The fact that the most common definition is the one used here can be seen in all other papers mentioned above, as well as in the books by Albrecher *et al*. (2017) and by Clark (2014). We note that, of these two books, the latter is the one which provides the most explicit definition when it comes to what happens to payments (premia and claims) above the reinsurance policy limit, whereas the former mentions the possibility of a limit but does not go in detail.

Our work, therefore, will seek to provide a contribution by finding the optimal reinsurance contract of the surplus type, based on real data provided to us by a Portuguese insurer. As explained above, to the best of our knowledge, this is the first such work on surplus reinsurance.

3. THEORETICAL APPROACH

3.1. Definition of surplus reinsurance

As stated in the Introduction, surplus reinsurance is a kind of reinsurance contract whereby risk is reinsured proportionally, but there are different proportions for each primary policy. These proportions are calculated based on a fixed retention line which is common to all sums insured, and are subject to a reinsurance policy limit: for sums insured smaller than the retention, there is no reinsurance; for those greater than the retention but smaller than the limit, the retention percentage is the ratio between the line and the sum insured; for sums insured greater than the limit, their excess over the limit is also taken into account in the ratio for the retention percentage, in addition to the line. Thus, there is no reinsurance for small risks (that is, those not exceeding the retention line), whilst, for the other ones, if no policies exceed the limit, the reinsured percentage increases with the risk. For any policies exceeding the limit, the reinsurance effect begins to be lost and the cession percentage falls as the risk increases.

Mathematically, for each policy *i*, the retained risk under surplus reinsurance is:

$$Y_i = \begin{cases} \min\{X_i, V_i\} & V_i \leq M \\ \dfrac{M}{V_i}\min\{X_i, V_i\} & M < V_i \leq L \\ \dfrac{V_i - L + M}{V_i}\min\{X_i, V_i\} & V_i > L \end{cases}, \tag{1}$$

where:

- X_i is the random variable representing the underlying insured risk (without the ceiling at the sum insured);
- V_i is the sum insured,[9] that is, the primary policy limit;
- M is the retained line of the reinsurance policy; and
- L is the reinsurance policy limit (whence *L minus M* is the maximum amount payable by the reinsurer), normally a multiple of M.

[9] The terms 'sum insured', 'capital insured' and 'policy limit' are treated as synonyms and will be used interchangeably hereafter.

Naturally, the ceded risk belonging to that policy shall be: $Z_i = \min\{X_i, V_i\} - Y_i$.

Under proportional reinsurance, the reinsurance premium is usually defined as being proportional to the primary premium, with the same proportion as that of the reinsurance of claims, minus a commission which is aimed at covering partially the insurer's expenses. Thus, for each policy *i*, the reinsurance premium is:

$$P_{R,i} = \begin{cases} 0 & V_i \leq M \\ \left(1 - \dfrac{M}{V_i}\right)(1-c)\,P_i & M < V_i \leq L \\ \dfrac{L-M}{V_i}(1-c)\,P_i & V_i > L \end{cases}, \tag{2}$$

where:

- P_i is the primary policy's premium; and
- c is the reinsurance commission.

We note that the constant commission is a simplification. In reality, the commission often varies to compensate for changes in the reinsurer's loss ratio. This is done through special features established in the reinsurance contract, such as sliding scales, profit commissions and loss corridors. For a detailed explanation of what these are and how they work, we recommend the book by Clark (2014).

Thus, each policy is associated with a primary premium P_i, a reinsurance premium $P_{R,i}$ and a retained risk Y_i (which is the random variable). With this, we define a new random variable, which is the wealth generated by this policy. This 'wealth' is called like so for simplicity, but it should not be understood as the final capital of the insurer; rather, it is more accurately defined as the profit of the insurer, within the line of business under consideration, after the unit period of time (which we assume to be one year, by default).

Having said so, we define the wealth generated by policy *i* (written as W_i) as its primary premium — after deducting the expenses connected with it (which, also for simplicity, we assume will be a constant proportion of the premium) —, minus its reinsurance premium, minus its retained claims. Mathematically, this becomes:

$$W_i = P_i(1-d) - P_{R,i} - Y_i =$$

$$= \begin{cases} P_i(1-d) - \min\{X_i, V_i\} & V_i \le M \\ P_i\left((1-d) - \left(1 - \frac{M}{V_i}\right)(1-c)\right) - \frac{M}{V_i}\min\{X_i, V_i\} & M < V_i \le L \\ P_i\left((1-d) - \frac{L-M}{V_i}(1-c)\right) - \frac{V_i - L + M}{V_i}\min\{X_i, V_i\} & V_i > L \end{cases},$$

(3)

where d represents the insurer's expenses — which should not be lower than c, in order to prevent the possibility of obtaining a risk-free profit by reinsuring everything.

In order to add all the existing n policies to obtain the final characteristics of the contract, we can simply define the variables without indices to be the sums of their corresponding indexed variables, as:

$$P = \sum_{i=1}^{n} P_i \quad ; \qquad P_R = \sum_{i=1}^{n} P_{R,i} \quad ; \qquad X = \sum_{i=1}^{n} X_i \quad ;$$

$$Y = \sum_{i=1}^{n} Y_i \quad ; \qquad Z = \sum_{i=1}^{n} Z_i \quad ; \qquad W = \sum_{i=1}^{n} W_i \quad .$$

Finally, we obtain a full expression for the total wealth yielded by all policies:

$$\begin{aligned} W &= P(1-d) - P_R - Y = \\ &= \sum_{V_i \le M} \left(P_i(1-d) - \min\{X_i, V_i\}\right) + \\ &+ \sum_{M < V_i \le L} \left(P_i\left((1-d) - \left(1 - \frac{M}{V_i}\right)(1-c)\right) - \frac{M}{V_i}\min\{X_i, V_i\}\right) + \\ &+ \sum_{V_i > L} \left(P_i\left((1-d) - \left(\frac{L-M}{V_i}\right)(1-c)\right) - \frac{V_i - L + M}{V_i}\min\{X_i, V_i\}\right) \quad . \end{aligned}$$

(4)

We recall that we are assuming, for simplicity, that there is a single surplus reinsurance contract for all policies. A real-life insurer may purchase multiple reinsurance contracts, which can be of the same type and can be from different reinsurers to diversify risk. This may be used to increase the insurer's protection. In particular, we note that, if a second surplus contract (with the same constant commission) is bought on top of the first one such that the *M* of the second one equals the *L* of the first one, this has the same effect as buying one single larger contract with the *M* of the first one and the *L* of the second one.

3.2. Optimality criteria

Finding the optimal surplus reinsurance contract means finding the *M* and the *L* which optimise a given optimality criterion. In our approach, we will select for each such criterion a function of the insurer's total wealth (*W*).

For this purpose, we will use three different criteria to find the optimal reinsurance contract, which — roughly described — shall be: 1) maximising the expected utility; 2) minimising the standard deviation; and 3) optimising the Value-at-Risk.

3.2.1. Expected utility

For the first criterion, we resort to Utility Theory. This is a common approach in reinsurance optimisation, carried out, for instance, by Guerra and Centeno (2008). For this purpose, they choose the exponential utility function, defined as:

$$U(W) = \frac{1 - e^{-\beta W}}{\beta} \quad , \tag{5}$$

where β is the risk-aversion coefficient.

This function has two advantages: it incorporates the idea of risk-aversion, due to its concave shape; and, as proven by the same authors, the problem of maximising it is deeply connected to the problem of maximising the adjustment coefficient (a relevant risk measure, because, according to the Lundberg inequality, it establishes an upper bound for the probability of ruin). We are going to use it, for the same reasons.

We seek to maximise its expected value, that is, $E\big(U(W)\big)$, which is the same as minimising $E\big(e^{-\beta W}\big) = E\big(T(W)\big)$. First, we determine for it an explicit expression for each policy *i*, which, based on **(3)**, shall be:

$$E\big(T(W_i)\big) = E\big(e^{-\beta W_i}\big) =$$

$$= \begin{cases} e^{-\beta P_i(1-d)} E\big(e^{\beta \min\{X_i,V_i\}}\big) & V_i \le M \\ e^{-\beta P_i\left((1-d)-\left(1-\frac{M}{V_i}\right)(1-c)\right)} E\left(e^{\beta \frac{M}{V_i} \min\{X_i,V_i\}}\right) & M < V_i \le L \\ e^{-\beta P_i\left((1-d)-\frac{L-M}{V_i}(1-c)\right)} E\left(e^{\beta \frac{V_i-L+M}{V_i} \min\{X_i,V_i\}}\right) & V_i > L \end{cases} =$$

$$= \begin{cases} e^{-\beta P_i(1-d)} \left(\int_0^{V_i} e^{\beta x_i} f_{X_i}(x_i)dx_i + e^{\beta V_i} S_{X_i}(V_i)\right) & V_i \le M \\ e^{-\beta P_i\left((1-d)-\left(1-\frac{M}{V_i}\right)(1-c)\right)} \left(\int_0^{V_i} e^{\beta \frac{M}{V_i} x_i} f_{X_i}(x_i)dx_i + e^{\beta M} S_{X_i}(V_i)\right) & M < V_i \le L \\ e^{-\beta P_i\left((1-d)-\frac{L-M}{V_i}(1-c)\right)} \left(\int_0^{V_i} e^{\beta \frac{V_i-L+M}{V_i} x_i} f_{X_i}(x_i)dx_i + e^{\beta (V_i-L+M)} S_{X_i}(V_i)\right) & V_i > L \end{cases},$$

(6)

where:

- $f_{X_i}(x_i)$ is the probability density function of X_i; and
- $S_{X_i}(x_i)$ is the survival function of X_i.

Now, we want to minimise the expected value of the function *T(·)* applied to the sum of the wealth values generated by all policies, as shown in **(4)**. If we assume all policies are independent, this expected value becomes:

$$E\big(T(W)\big) = E\big(e^{-\beta W}\big) = E\big(e^{-\beta \sum_{i=1}^{n} W_i}\big) = E\left(\prod_{i=1}^{n} e^{-\beta W_i}\right) = \prod_{i=1}^{n} E\big(e^{-\beta W_i}\big) =$$

$$= \prod_{V_i \le M} e^{-\beta P_i(1-d)} \left(\int_0^{V_i} e^{\beta x_i} f_{X_i}(x_i)dx_i + e^{\beta V_i} S_{X_i}(V_i)\right) \times$$

$$\times \prod_{M < V_i \le L} e^{-\beta P_i\left((1-d)-\left(1-\frac{M}{V_i}\right)(1-c)\right)} \left(\int_0^{V_i} e^{\beta \frac{M}{V_i} x_i} f_{X_i}(x_i)dx_i + e^{\beta M} S_{X_i}(V_i)\right) \times$$

$$\times \prod_{V_i > L} e^{-\beta P_i\left((1-d)-\frac{L-M}{V_i}(1-c)\right)} \left(\int_0^{V_i} e^{\beta \frac{V_i-L+M}{V_i} x_i} f_{X_i}(x_i)dx_i + e^{\beta (V_i-L+M)} S_{X_i}(V_i)\right).$$

(7)

Because differentiating these functions with respect to M and to L (even for only one policy) is very difficult to do analytically, we have taken a numerical approach.

However, because these exponentials will mostly be very small numbers (that is, positive but close to zero), their product will be too small for the computer to reach an accurate optimisation. Therefore, we will instead minimise its natural logarithm (*ln*) — which we can legitimately do because the logarithm is an increasing monotonic function —, which shall be:

$$\begin{aligned}
\ln E\big(T(W)\big) &= \ln \prod_{i=1}^{n} E\big(e^{-\beta W_i}\big) = \sum_{i=1}^{n} \ln E\big(e^{-\beta W_i}\big) = \\
&= \sum_{V_i \le M} \ln\left(e^{-\beta P_i (1-d)} \left(\int_0^{V_i} e^{\beta x_i} f_{X_i}(x_i)\,dx_i + e^{\beta V_i} S_{X_i}(V_i) \right)\right) + \\
&+ \sum_{M < V_i \le L} \ln\left(e^{-\beta P_i\left((1-d)-\left(1-\frac{M}{V_i}\right)(1-c)\right)} \left(\int_0^{V_i} e^{\beta \frac{M}{V_i} x_i} f_{X_i}(x_i)\,dx_i + e^{\beta M} S_{X_i}(V_i) \right)\right) + \\
&+ \sum_{V_i > L} \ln\left(e^{-\beta P_i\left((1-d)-\frac{L-M}{V_i}(1-c)\right)} \left(\int_0^{V_i} e^{\beta \frac{V_i - L + M}{V_i} x_i} f_{X_i}(x_i)\,dx_i + \right.\right. \\
&\left.\left. + e^{\beta (V_i - L + M)} S_{X_i}(V_i) \right)\right) \quad .
\end{aligned} \tag{8}$$

The numerical results are presented and discussed later.

3.2.2. Standard deviation

The method of maximising expected utility has a disadvantage in that utility is a vague and merely theoretical concept, with an unclear practical interpretation. Furthermore, it requires selecting an arbitrary value for β as a risk-aversion coefficient.

An alternative form of considering risk-aversion so as to solve this problem is by measuring dispersion indicators of the insurer's risk under each possible reinsurance contract, instead of assigning arbitrary utility values to said risk, and attempting to optimise that. Two common measures of variability of a given random variable are the variance and the standard deviation. Using them is also frequent in the literature about optimal reinsurance; in fact, the earliest papers we know, by de Finetti (1940) and Borch (1960), are good examples of the usage of the variance to optimise reinsurance.

In this section, we will use the standard deviation σ_X (with X being any given random variable) — which has the advantage, over the variance, that it is measured in the same unit as the risk itself —, but we will have to convert to the variance in intermediate calculations in order to resort to its properties.

It is in the interest of the insurer to minimise the variability of the risk it takes on, because a high variability makes predictions difficult and increases the probability of severely high losses. Therefore, we might think about seeking to simply minimise the standard deviation. However, this would lead to a trivial solution whereby all risk would be reinsured — in other words, there would be no risk and, therefore, the standard deviation would be reduced to zero —, which is unreasonable because the insurer would essentially cease to be an insurer and because this strategy would result in a guaranteed loss, as the reinsurance commission (c) is smaller than the insurer's expenses (d) with the risk.

To overcome this problem, de Finetti (1940) and Borch (1960) fix a specific desired expected value and find the contract which fulfils that condition and minimises the variance. This approach, of course, has the disadvantage of having to select an arbitrary desired expected value. Meanwhile, Veprauskaite and Sherris (2012), in one of the three criteria they define, opt for minimising the ratio between the variance and the expected value of the retained claims. This method is more similar to the one we shall use.

We consider that what the insurer actually wants is to minimise the standard deviation while maximising the expected value. But a reduction in the standard deviation caused by an increase in the amount of risk reinsured will reduce the expected value of the wealth, assuming the risk is profitable. Therefore, the insurer wants to strike a balance between the two goals. This leads us to the strategy of maximising the ratio between the expected value and the standard deviation. Unlike the authors cited above, we recall that we prefer the standard deviation instead of the variance.

However, even this criterion has a flaw: for negative expected values, it ceases to be risk-averse and becomes risk-loving (because the ratio would then be maximised with a higher standard deviation) — imprudent behaviour for an insurer. Therefore, to ensure that the premise of risk-aversion is kept across all admissible solutions, the expected value of the wealth must have the riskless strategy wealth deduced from it in the ratio, to keep the numerator always positive. The idea is that any value of wealth higher than that which would be achieved with no risk is a gain, but we must accept a given amount of risk to reach this expected gain; therefore, we want to gain as much as possible, by comparison with the riskless scenario, whilst accepting as little volatility as possible. Essentially, this is the concept of the Sharpe ratio, as used in the theory of optimisation of financial investments — presented by Sharpe (1994) —, applied to reinsurance: the risk-free rate is replaced by the riskless scenario wealth, and the other variables remain the same.

Thus, the function to be maximised under this criterion is:

$$\frac{E(W)-W_{riskless}}{\sigma_W}=\frac{E(W)-P(c-d)}{\sigma_W} \quad . \tag{9}$$

The riskless strategy wealth is, of course, the one which is obtained by reinsuring everything, that is, by setting $M=0$ and $L\geq \max\{V_i\}$. It is in this case that the insurer will face no risk and the standard deviation is zero.

We now need to determine formulae for the calculation of the expected value and of the standard deviation of the wealth. For each policy *i*, considering the function of its generated wealth given in **(3)**, the expected value is:

$$E(W_i)=\begin{cases} P_i(1-d)-E(\min\{X_i,V_i\}) & V_i\leq M \\ P_i\left((1-d)-\left(1-\dfrac{M}{V_i}\right)(1-c)\right)-\dfrac{M}{V_i}E(\min\{X_i,V_i\}) & M<V_i\leq L \\ P_i\left((1-d)-\dfrac{L-M}{V_i}(1-c)\right)-\dfrac{V_i-L+M}{V_i}E(\min\{X_i,V_i\}) & V_i>L \end{cases} , \tag{10}$$

where:

$$E(\min\{X_i,V_i\})=\int_0^{V_i} x_i f_{X_i}(x_i)dx_i+V_i S_{X_i}(V_i) \quad . \tag{11}$$

Meanwhile, the formula for the variance of the wealth generated by the same policy *i*, also based on **(3)**, is:

$$Var(W_i) = \begin{cases} Var(\min\{X_i, V_i\}) & V_i \leq M \\ \dfrac{M^2}{V_i^2} Var(\min\{X_i, V_i\}) & M < V_i \leq L \\ \dfrac{(V_i - L + M)^2}{V_i^2} Var(\min\{X_i, V_i\}) & V_i > L \end{cases},$$

(12)

where:

$$\begin{aligned} Var(\min\{X_i, V_i\}) &= E((\min\{X_i, V_i\})^2) - E^2(\min\{X_i, V_i\}) = \\ &= E(\min\{X_i^2, V_i^2\}) - E^2(\min\{X_i, V_i\}) = \\ &= \int_0^{V_i} x_i^2 f_{X_i}(x_i) dx_i + V_i^2 S_{X_i}(V_i) - \left(\int_0^{V_i} x_i f_{X_i}(x_i) dx_i \right)^2 - \\ &- 2 \int_0^{V_i} x_i f_{X_i}(x_i) dx_i \, V_i S_{X_i}(V_i) - V_i^2 S_{X_i}^2(V_i) \ . \end{aligned}$$

(13)

Finally, summing the wealth for all policies as in **(4)** and applying to it the results **(10)** and **(12)**, assuming all policies are independent, we obtain that the expected value of the total wealth is:

$$\begin{aligned} E(W) &= E\left(\sum_{i=1}^{n} W_i\right) = \sum_{i=1}^{n} E(W_i) = \\ &= \sum_{V_i \leq M} \big(P_i(1-d) - E(\min\{X_i, V_i\})\big) + \\ &+ \sum_{M < V_i \leq L} \left(P_i \left((1-d) - \left(1 - \frac{M}{V_i}\right)(1-c) \right) - \frac{M}{V_i} E(\min\{X_i, V_i\}) \right) + \\ &+ \sum_{V_i > L} \left(P_i \left((1-d) - \left(\frac{L-M}{V_i}\right)(1-c) \right) - \frac{V_i - L + M}{V_i} E(\min\{X_i, V_i\}) \right) \end{aligned}$$

(14)

and its variance is:

$$
\begin{aligned}
Var(W) &= Var\left(\sum_{i=1}^{n} W_i\right) = \sum_{i=1}^{n} Var(W_i) = \\
&= \sum_{V_i \leq M} Var(\min\{X_i, V_i\}) + \sum_{M<V_i \leq L} \frac{M^2}{V_i^2} Var(\min\{X_i, V_i\}) + \\
&+ \sum_{M<V_i \leq L} \frac{(V_i - L + M)^2}{V_i^2} Var(\min\{X_i, V_i\}) \quad .
\end{aligned}
\tag{15}
$$

As in the first criterion, optimising this analytically is very difficult, perhaps even impossible. We have to do it numerically.

3.2.3. Value-at-Risk

Another optimisation criterion we can use is based on another very important risk measure used in the actuarial profession: the Value-at-Risk (written as *VaR*). It is sometimes used in reinsurance optimisation, for example, by Cai and Tan (2007) and by Cai *et al*. (2008), who also use the related Tail-Value-at-Risk (also called Conditional Tail Expectation).

We recall that the Value-at-Risk at α of a random variable (written as VaR_α) — where α is a value, usually expressed in percentage, between 0 and 1 — is the number such that the probability that the random variable takes a value no greater than it is α. Essentially, it is the quantile α of the distribution being considered. In other words, for the usual cases and assuming a continuous distribution:

$$
VaR_\alpha(X) = r \Leftrightarrow P(X \leq r) = \alpha \quad ,
\tag{16}
$$

where:

- $P(\cdot)$ is the probability of an event; and
- X, in this context, is any given random variable.

Usually, the α is very high, close to 100%, because the random variable being studied measures actuarial losses and the point of the VaR is to provide a notion of how high a loss can realistically be, in the most extreme scenarios (on the right tail of the loss probability distribution). Naturally, having fixed a given α, insurers would like to have a VaR as low as possible. Common values for the α would be 95%, 99% or 99.5%.

Meanwhile, the Tail-Value-at-Risk (TVaR) is defined as the mean of all values above the Value-at-Risk. It is generally regarded as having two advantages over the VaR: it gives information as to what happens above the VaR, until the end of the distribution tail; and it is a coherent risk measure.[10] Nevertheless, it has the notably significant disadvantage that it is very difficult to calculate in practice. Possibly for this reason, the EU-wide Solvency II rules stipulate the usage of the VaR and set its α at 99.5%. Thus, for simplification, we shall also use the VaR in this work and, drawing inspiration from Solvency II, we shall set it at the same α of 99.5%.

However, in this case, we would like to measure the risk of the wealth, as we have been doing above, and not specifically of the claims. It does not change the reasoning, other than that it now requires us to think of the Value-at-Risk in reverse (on the left tail), setting it at 0.5% instead and seeking to maximise it. Note that:

$$VaR_{0.5\%}(W) = P(1-d) - P_R - VaR_{99.5\%}(Y) \quad , \tag{17}$$

because Y is the only random component of W, as given in **(4)**.

Please note, however, that, for multiple policies with different claim distributions, it is very difficult — perhaps even impossible — to obtain an explicit equation for the Value-at-Risk of the sum of all claims, even assuming that they are independent. This means that the optimisation according to this criterion will have to be done numerically, not because it is difficult to do it analytically, but because there is no analytical expression to optimise. We will, instead, simulate numerically the VaR under different reinsurance contracts and find the one which optimises it.

As stated earlier, the insurer would like the Value-at-Risk of the wealth to be as high as possible. However, similarly to when we sought to minimise the standard deviation, doing

[10] A coherent risk measure must, among other criteria, be sub-additive: if two risks are added, the risk measure of the sum must not be higher than the sum of the risk measures of the original risks (it may be lower, though, because diversifying a portfolio tends to reduce risk). The Value-at-Risk has been proven not to fulfil this condition, whereas the Tail-Value-at-Risk does. For more on coherent risk measures, we recommend Kaas *et al*. (2008).

only this would lead us back to the trivial solution of reinsuring everything — although, this time, concluding this is not immediate, because reinsuring less (that is, accepting more risk) has two opposing effects on the $VaR_{0.5\%}(W)$ [see **(17)**]: it increases the $VaR_{99.5\%}(Y)$ (because the risk is higher), but it also decreases the reinsurance premium (P_R). Determining which of these two effects is stronger requires us to recall that we are working under proportional reinsurance:[11] looking back at **(1)** and **(2)**, we see that, for each policy i, any change in the amount of reinsurance causes a proportional change in $P_{R,i}$ and in the retained risk Y_i; and, since the VaR is a positively homogeneous measure — meaning that $VaR(kY_i) = k \times VaR(Y_i), \forall k > 0$ —, the impact on the $VaR_{99.5\%}(Y_i)$ will also be proportional. Therefore, if we assume that the VaR of the policy's total claims (X_i) is greater than the premium which would be charged for reinsuring the whole policy — which is reasonable, because we are working with an extreme α —, we conclude that the effect of the $VaR_{99.5\%}(Y_i)$ is greater. If all policies follow this assumption, we determine that the contract which maximises the $VaR_{0.5\%}(W)$ is the trivial full reinsurance.

Therefore, again, we wish to strike a balance between maximising the expected value and the Value-at-Risk, which correlate negatively with one another. And, again, we must ensure that risk-aversion is respected at all points, which we can do by maximising the following function:

$$\begin{aligned}(E(W) - W_{riskless}) \times \left(VaR_{0.5\%}(W) - VaR_{0.5\%}(W_{maximum\,risk})\right) = \\ = \left(E(W) - P(c-d)\right) \times \left(VaR_{0.5\%}(W) - VaR_{0.5\%}(W_{no\,reinsurance})\right)\ ,\end{aligned} \tag{18}$$

where $W_{maximum\ risk} = W_{no\ reinsurance}$ is the wealth if nothing is reinsured, the scenario which maximises risk and, therefore, minimises the Value-at-Risk on the left tail. With these subtractions, we ensure that both factors are always positive and that the function translates a risk-averse behaviour.

[11] Indeed, the authors cited earlier do not have this problem, because they work with reinsurance that is not necessarily proportional. Therefore, they can simply minimise the VaR or the TVaR.

4. DATA SET AND PROBLEM SETTING

4.1. Data set

For this work, we have built a portfolio of 998 policies, based on fire insurance data received from a Portuguese insurer. The original data contained many more policies and, for each one, its premium, its capital insured and its total amount in reported claims over one year. We sorted the portfolio by capital insured and divided it into classes, to then randomly select a few policies belonging to each class; the number of policies selected within each class was meant to approximate the real distribution of the whole portfolio, but, due to the fact that there were too few policies with higher risks (which would be more relevant to decide the reinsurance contract), we decided to overrepresent such policies slightly. Lastly, for each policy among the 998 selected, we used its class average to estimate its probability of producing claims over the course of one year and, assuming such claims occur, the mean and the standard deviation of their total value.

The summary statistics of the portfolio which was produced by using this method are shown in **Table 1**.

Statistics	Mean	Minimum	1st Q	Median	3rd Q	Maximum
Capital	353 308.55	2619.73	50 000	81 379.84	200 000	19 792 309.33
Probability	0.0736	0.03	0.03	0.06	0.09	0.75
Avg. sev.	2067.19	1000	1160	1360	2300	35 000
STD of sev.	4236.27	1900	1900	1900	7500	50 000
P: Given	517.08	39.13	76.75	140.04	294.59	70 257.17
P: EV	517.08	57.74	66.97	157.04	398.38	50 518.92
P: STD	517.08	91.75	98.77	175.97	597.54	33 892.61
P: Variance	517.08	34.51	39.64	92.14	387.46	95 359.31

Table 1 — Means, minima, maxima and quartiles of the characteristics of the 998 policies used. 'Probability' refers to the probability that any claim will occur within one year. 'P' stands for 'premium', 'Q' for 'quartile', 'avg.' for 'average', 'sev.' for 'severity', 'STD' for 'standard deviation', and 'EV' for 'expected value'.

A few of the policy limits are much higher than the other ones, as we can see from the first line of the table. This is not a mistake, as the capital distribution is indeed severely skewed to the left: in fact, although more than half of all sums insured are below 100 000, there are 47 policies with a capital insured of at least 1 000 000, six of which even above 10 000 000 (and, as stated above, even these few higher risks are somewhat overrepresented). The implication is that, under many of the optimal reinsurance contracts we will find, only a minority of policies will actually be reinsured.

As mentioned earlier, primary premia were provided along with each policy. We could not identify the premium principle which had been used to determine them (which can perhaps mean that the insurer was using information specific to each policy, which we do not have, in addition to an unknown principle). Therefore, for the purposes of our optimisation, we decided to consider three more sets of premia, calculated with the expected value principle, with the standard deviation principle and with the variance principle. In order to do so, for each principle, we computed the loading which would keep the same total received premium of 516 049.29. This loading was found to be of 92.4530% for the expected value principle, 16.6590% for the standard deviation principle, and 0.0033% for the variance principle.

4.2. Claim frequency and claim severity distributions

As seen in the previous chapter, in order to perform many of the calculations necessary to find the optimal reinsurance contract, we must select a distribution for each X_i, that is, for the total value in yearly claims of each policy i.

This can be done in two ways: either the number of claims N_i follows a Poisson distribution and each claim has a severity given by X_{ij} (where $j = 1, \ldots, n_i$), or we use a Bernoulli distribution to model whether there were any claims or not (so that N_i only takes the values 0 and 1) and the total severity of all claims is given by $X_i \mid N_i = 1$. Naturally, unlike the Bernoulli approach, the Poisson method requires us to have knowledge of the severities of each specific claim.

If the Poisson approach is followed, X_i will be given by:

$$f_{X_i}(x_i) = \begin{cases} f_{N_i}(0) & x_i = 0 \\ \sum_{n_i=1}^{+\infty} f_{X_i|N_i}(x_i) f_{N_i}(n_i) & x_i > 0 \end{cases} = \begin{cases} e^{-\mu_i} & x_i = 0 \\ \sum_{n_i=1}^{+\infty} f_{X_i|N_i}(x_i) \frac{e^{-\mu_i} {\mu_i}^{n_i}}{n_i!} & x_i > 0 \end{cases}, \tag{19}$$

where μ_i is the Poisson parameter of policy i, which is the average number of claims.

Meanwhile, if we follow the Bernoulli approach, X_i is given by:

$$f_{X_i}(x_i) = \begin{cases} f_{N_i}(0) & x_i = 0 \\ f_{X_i|N_i=1}(x_i) f_{N_i}(1) & x_i > 0 \end{cases} = \begin{cases} 1 - p_i & x_i = 0 \\ p_i f_{X_i|N_i=1}(x_i) & x_i > 0 \end{cases},$$

(20)

where p_i is the Bernoulli parameter of policy i, which is the probability that there is a claim.

We have chosen the Bernoulli model, essentially because we do not have information on the individual claim severity for each policy.

Finally, we have chosen to model the total severity of all claims incurred by policy i through a Lognormal distribution, mainly because it is generally considered to be a medium-tailed distribution (that is, with a tail which is neither too light nor too heavy).

4.3. Problem setting

By resorting to the distribution explained in the previous section and assuming a portfolio comprising the 998 policies whose summary statistics were presented in Section 4.1, we have sought to find the optimal surplus reinsurance contract, according to the definition of surplus reinsurance presented in Section 3.1 and following each of the three optimisation criteria explained in Section 3.2.

We recall that the three criteria are each based on:

1. The expected utility of the wealth, which is the minimisation of **(8)**;
2. The standard deviation of the wealth, which is the maximisation of **(9)**, where the expected value of W is given by **(14)** and its variance by **(15)**; and
3. The Value-at-Risk of the wealth, which is the maximisation of **(18)**.

For the first two criteria, we have derived formulae which we can optimise numerically using adequate software. For the expected utility, we had to decide a value for β (which represents the coefficient of risk-aversion): we used $\beta = 0.3$ and $\beta = 0.6$.

For the Value-at-Risk criterion, since we were unable to obtain a formula for the VaR, we instead decided to simulate the expected value and the VaR for each possible reinsurance contract, at intervals of 1 000 000 for each of the variables M and L, such that $0 \leq M \leq 19\,000\,000$ and $M + 1\,000\,000 \leq L \leq 20\,000\,000$, plus the scenario of no reinsurance; and we found the optimal contract by identifying the highest value for **(18)** among the scenarios simulated. Such simulations were performed with 499 999 elements.[12]

Lastly, after finding the optimal contract under each criterion, we conducted simulations to determine the final risk measures — expected value, standard deviation and VaR — under that reinsurance contract, in order to compare the results produced by the different methods. These simulations were also performed with 499 999 elements.

These procedures, under all three optimisation criteria, were carried out for each of the four possible sets of premia mentioned in section 4.1, assuming a commission c of 10% and of 20%, and expenses d of 24%. The idea was to contrast the effect of a commission which is far from sufficient to reasonably cover the expenses against a commission which covers them more adequately.

All optimisations and simulations were computed using the R software.

[12] The odd number makes it easier to determine the Value-at-Risk at 99.5% of the claims. In fact, if the number of elements is one less than a multiple of 200, the $VaR_{99.5\%}$ can be identified by picking only one of the simulated elements. This presumes that, given a statistical sample with n elements, a quantile q is calculated from the element of order $(n+1)q$ — which is not the only formula that can be used, but it is the one we have used in this work. For more on quantile calculation, we recommend Kaas *et al*. (2008).

5. RESULTS

5.1. Presentation

All results obtained are presented in **Table 2**.

The left side of the table shows the eight scenarios tested (four possible sets of premia and two possible values for the commission), under each of the three criteria used. As stated earlier, in the case of the expected utility criterion, optimisations featured two different values for β.

At the top, the expected results under no reinsurance (in yellow) and the sure results under full reinsurance (in green) are shown, for comparison. These are the same for any premium principle, because the total premium amount of 516 049.29 has been kept. Under full reinsurance, there is, evidently, no risk, which is why the standard deviation under these scenarios is zero. The no reinsurance scenario provides the highest possible expected value and standard deviation, and the worst possible VaR; the full reinsurance scenario gives the lowest possible expected value and the best possible VaR (which, in this case, are equal, because no risk is taken).

In the central columns, the optimal reinsurance contract — defined by its M and its L — is presented. On the right side, the simulated risk measures of the wealth, under each optimal reinsurance contract, are shown.

We recall that, as explained earlier, the optimal reinsurance contracts for the Value-at-Risk criterion had to be obtained in a different way (through simulation), which is why their values for M and L are always rounded to the million, and which implies that better contracts with values of M and L in-between may have been missed.

Finally, we note that we did not allow the expenses (d) to vary, because we determined that it would cause no impact in the selection of the optimal reinsurance contract, as we will explain next.

Data				Optimum		Risk measures		
Premium	Commission	Beta	Criterion	M	L	E(W)	std(W)	VaR(W)
No reinsurance						125 122	83 897	-217 339
Full reinsurance, under a 10% commission						-72 247	0	-72 247
Full reinsurance, under a 20% commission						-20 642	0	-20 642
GIVEN	0.1	0.3	Utility	90 467	19 209 718	-30 756	24 502	-125 407
GIVEN	0.1	0.6	Utility	38 720	19 494 493	-47 352	14 331	-99 457
GIVEN	0.1	-	STD	2 939 263	7 950 670	84 783	63 336	-155 935
GIVEN	0.1	-	VaR	3 000 000	9 000 000	79 277	61 358	-153 404
GIVEN	0.2	0.3	Utility	65 769	19 194 770	3 433	20 453	-72 974
GIVEN	0.2	0.6	Utility	30 825	19 491 486	-6 349	11 864	-49 072
GIVEN	0.2	-	STD	2 982 069	6 923 361	99 714	66 557	-154 419
GIVEN	0.2	-	VaR	3 000 000	10 000 000	87 480	60 214	-138 916
EV	0.1	0.3	Utility	110 453	19 236 686	-7 596	27 216	-114 985
EV	0.1	0.6	Utility	47 420	19 506 689	-32 150	16 763	-93 112
EV	0.1	-	STD	2 147 191	10 664 714	77 434	55 115	-132 675
EV	0.1	-	VaR	3 000 000	10 000 000	89 171	60 323	-138 177
EV	0.2	0.3	Utility	93 774	19 231 179	23 650	24 936	-72 753
EV	0.2	0.6	Utility	40 353	19 505 217	5 656	14 778	-47 667
EV	0.2	-	STD	2 138 498	10 686 927	89 783	55 022	-118 637
EV	0.2	-	VaR	2 000 000	11 000 000	88 075	54 296	-117 760
STD	0.1	0.3	Utility	136 027	19 297 903	47 720	30 093	-73 431
STD	0.1	0.6	Utility	59 880	19 536 986	8 333	19 337	-63 457
STD	0.1	-	STD	55 839	19 198 075	5 072	18 588	-63 259
STD	0.1	-	VaR	1 000 000	16 000 000	98 756	48 197	-92 035
STD	0.2	0.3	Utility	120 206	19 311 957	70 951	28 388	-42 112
STD	0.2	0.6	Utility	55 333	19 547 216	41 222	18 473	-26 695
STD	0.2	-	STD	55 441	19 446 854	41 314	18 455	-26 504
STD	0.2	-	VaR	1 000 000	16 000 000	113 003	47 949	-74 723
VAR	0.1	0.3	Utility	93 277	19 206 831	-38 082	24 857	-133 887
VAR	0.1	0.6	Utility	32 921	19 485 848	-57 336	12 600	-102 961
VAR	0.1	-	STD	536 574	884 835	115 779	80 219	-211 968
VAR	0.1	-	VaR	2 000 000	7 000 000	62 339	60 220	-167 482
VAR	0.2	0.3	Utility	69 450	19 191 807	-3 332	21 120	-82 387
VAR	0.2	0.6	Utility	22 279	19 479 607	-14 572	9 082	-46 902
VAR	0.2	-	STD	383 847	795 700	116 204	77 801	-199 610
VAR	0.2	-	VaR	2 000 000	7 000 000	75 227	60 217	-155 134

Table 2 — Optimisation results obtained. 'EV' stands for 'expected value', 'STD' for 'standard deviation', 'VAR' for 'variance' and 'VaR' for 'Value-at-Risk'.

5.2. Discussion

As stated earlier, we did not include any presentation of results with different values for the expenses (d). This is because we have concluded that, when the expenses vary, under any of the criteria used, the optimal reinsurance contract does not change. Indeed, preliminary optimisations (not reported here) with different values for the expenses had led us to observe this fact, which can be explained from the expressions for each criterion:

1. For the expected utility, we see that, in **(6)**, the only term where *d* appears can be factored as $e^{-\beta P_i(1-d)} > 0$, which shows that changing the *d* is simply equivalent to multiplying $E\big(T(W_i)\big)$ by a positive constant, which does not change its optimum, and the same thing happens for the joint $E\big(T(W)\big)$, in **(7)**, where the factored term is $\prod_{V_i} e^{-\beta P_i(1-d)} = e^{-\beta P(1-d)} > 0$;
2. For the standard deviation criterion, we see, in **(14)**, that any change in *d* (say, Δd) produces a change of $\sum_{V_i}(-P_i\Delta d) = -P\Delta d$ in $E(W)$, which is then cancelled out in **(9)** by adding $P\Delta d$ to the numerator;
3. Similarly, for the Value-at-Risk criterion, any change Δd also produces in $E(W)$ a change of $-P\Delta d$ which is then cancelled in **(18)** and, through **(17)**, the same kind of cancelled change also happens in $VaR_{0.5\%}(W)$.

We note that varying the expenses, although it does not change the optimal point, will change the expected value and the Value-at-Risk of the wealth, improving them if the expenses are reduced and worsening them if they are increased. This implies, for example, that any optimal result producing a negative expected value should not be disregarded solely for that reason, as it can be turned into a positive one by reducing sufficiently the expenses.

The effect of changes in the commission is noticeable: as we would expect, since a higher commission makes reinsurance more attractive, results with a 20% commission tend to point to reinsuring more (that is, using a smaller *M* and perhaps a higher *L*) than under a 10% commission, although this is not always the case. Indeed, when the optimisation criterion is not the expected utility, the effect of changing the commission seems to become more negligeable. More relevant, though, is the fact that a commission increase always has a positive effect in the expected value and in the VaR of the wealth, which is to be expected: *ceteris paribus*, increasing the commission always increases the wealth, unless no reinsurance is in force. Notably, in scenarios of a 10% commission, some optimal contracts even yield undesirably negative expected values, which almost never happens with a 20% commission. We recall that the commission is aimed at partially, but not completely, covering the expenses: a 20% commission can perhaps be said to cover expenses of 24% satisfactorily, but a 10% commission falls short of having a similar effect.

Changing the premium principle also has an interesting impact on the optimal reinsurance contract. The expected value premium principle points to reinsuring less than the original real premia when using the expected utility criterion, but more when using the other two; the effect in the expected value is not constant but, interestingly, all Values-at-Risk are better under this premium principle than under the original premia. The standard deviation principle improves all VaRs even further; in this case, the expected utility criterion also requires reinsuring less, but, for the other two criteria, the optimal contracts are actually closer to those produced by the expected utility than under any other principle. Remarkably, it seems that usage of this premium principle produces overall better results than any other (which can perhaps be attributed to the fact that the premia follow the claim distributions more closely): not only are the VaRs the highest of all (as stated earlier), but also this is the only principle under which we can, for example, find optimal contracts with an expected value that is more than twice the standard deviation. Finally, the variance principle is the one producing the strangest outcomes (which can perhaps be attributed to the fact that the variance is not measured in the same scale as the claim): almost all VaRs are worse than those given by all the other premia sets, but, intriguingly, the standard deviation criterion suggests reinsuring very little of the high risks, by providing a low L.

The usage of different optimisation criteria is also interesting to analyse: the expected utility criterion almost always requires reinsuring much more than the other two; in fact, the only exception happens, as mentioned earlier, when using the standard deviation premium principle, under which the standard deviation criterion now approximates more closely the results given by the expected utility. Elsewhere, though, the expected utility criterion may be regarded as excessively risk-averse, as it frequently provides contracts which yield negative expected values and, when not so, expected values smaller than the associated standard deviations. This could be fixed by reducing the risk-aversion coefficient (β). Indeed, as expected, the higher coefficient of 0.6 always pointed to reinsuring more than the lower one of 0.3, thus improving the VaRs and the standard deviations, but reducing the expected values.

Finally, we note that — as stated in the Introduction — in no scenario did the optimal L match or exceed the highest of all sums insured V_i (in this case, 19 792 309.33) — although sometimes, especially under the expected utility criterion, the L was close to said value, but still from below. Thus, we confirm that it is possible to build a portfolio with which, using at least some optimality criteria and under certain conditions, the optimal surplus reinsurance contract does not imply reinsuring in such a way as to cover the highest of all risks; in other words, it is quite possible that the optimal surplus reinsurance contract features a non-negligeable reinsurance policy limit. This can perhaps be explained by the fact that there are so few risks so high that, mathematically, ensuring that they are fully covered simply does not compensate.

6. Conclusion

In this work, we have discussed ways of optimising surplus reinsurance, beginning with a theoretical approach and then applying it to a portfolio which was constructed based on real-life data on fire insurance.

We developed three possible optimisation criteria, based on the expected utility, the standard deviation, and the Value-at-Risk of the insurer's wealth. For the first two, it was possible to derive explicit expressions to be optimised numerically; for the Value-at-Risk, since this was not possible, optimisation was carried out by simulating several different reinsurance contracts. We applied each of these three methods to a portfolio of policies, and we varied the primary premium principle and the reinsurance commission.

Our findings do not point to any one of these three methods being better than the other ones, although the expected utility generally provided more risk-averse results, in the sense that more risk was reinsured. One disadvantage of the expected utility criterion is the vagueness of the concept of 'utility' and the need to provide an arbitrary risk-aversion coefficient. Meanwhile, a disadvantage of the Value-at-Risk method is its difficulty in calculating it, which does not allow for too much precision in the results obtained.

Interestingly, our results suggest that using the standard deviation as the primary premium principle tends to provide better results, after applying surplus reinsurance, than the other premia considered (the expected value principle, the variance principle, or the original premium with no clear principle), keeping the total premium constant. This improvement is noticeable in the sense that the Values-at-Risk are always higher and that the expected values, especially by comparison to their associated standard deviations, also tend to be higher. We attribute this to the fact that this principle causes the premium to follow more closely the distribution of each risk: given two risks with the same expected value but different standard deviations, it makes a distinction between the riskiest one and the safest one (unlike the expected value principle); and this distinction is measured in the same scale as the risk itself (unlike the variance principle).

Although we have also proven that varying the expenses does not change the optimal surplus reinsurance contract under any of the criteria we used, our findings highlight the need for the reinsurance commission to cover, as closely as possible, the insurer's expenses. Indeed, it was only when we optimised under a commission of 20% to cover expenses of 24% — as opposed to a commission of only 10% to cover the same amount in expenses — that we obtained reasonable results, in the sense that they could usually fulfil the basic desirability condition that the expected value of the wealth be positive. Otherwise, the optimal contract will usually require reinsuring unnecessarily less (that is, accepting unnecessarily more risk) to provide clearly worse results on average. The commission, of course, only has no impact if nothing is reinsured.

Finally, our results show that the reinsurance policy limit has its importance when optimising surplus reinsurance. As stated in the Introduction, this contrasts with the existing literature, which rarely considers the existence of a policy limit under surplus reinsurance — implicitly assuming that it can be ignored because, if it exists, it shall always be no smaller than the highest sum insured, at least for the optimal contract. In our work, doing away with this assumption has shown us that, very frequently — in fact, in all the scenarios we analysed —, the optimal reinsurance policy includes a limit which actually leaves some of the highest risks partially uncovered at the top.

Possible opportunities for future research on optimal surplus reinsurance, based on this work, could be, for example: modelling the severities with other distributions; determining what happens when the commission varies with the loss ratio, rather than remaining constant (including through mechanisms such as sliding scales and loss corridors); considering expenses that are not uniform across all policies; adding more premium principles to the analysis; exploring further the effects of the premium principle, notably the reason why the standard deviation premium principle yielded so much better results; identifying mathematically why a policy limit no smaller than the highest of all sums insured is not always optimal; introducing new optimisation criteria (some of which perhaps now depend on the expenses); or even adding a table of lines, rather than working with standard surplus reinsurance.

BIBLIOGRAPHY

1. Albrecher, H., Beirlant, J., Teugels, J.L. (2017). *Reinsurance: Actuarial and Statistical Aspects*. 1st edition. Wiley.
2. Borch, K. (1960). An Attempt to determine the Optimum Amount of Stop Loss Reinsurance. *Transactions of the XVI International Congress of Actuaries*, Vol. I, pp. 597–610.
3. Cai, J., Tan, K.S. (2007). Optimal Retention for a Stop-loss Reinsurance Under the VaR and CTE Risk Measures. *ASTIN Bulletin: The Journal of the IAA*, 37(1):93–112.
4. Cai, J., Tan, K.S., Weng, C., Zhang, Y. (2008). Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics*, 43(1):185–196.
5. Centeno, M.L. (1985). On Combining Quota-Share and Excess of Loss. *ASTIN Bulletin: The Journal of the IAA*, 15(1):49–63.
6. Centeno, M.L., Simões, O. (2009). Optimal reinsurance. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 103(2):387–404.
7. Clark, D.R. (2014). *Basics of Reinsurance Pricing: Actuarial Study Note*. Casualty Actuarial Society (CAS).
8. de Finetti, B. (1940). Il problema dei pieni. *Giornale dell'Istituto Italiano degli Attuari*, 11:1–88.
9. Denuit, M., Vermandele, C. (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics*, 22(3):229–233.
10. El Attar, A., El Hachloufi, M., Zine El Abidine, G. (2018). Optimization of Surplus Reinsurance Treaty using the Conditional Tail Expectation. *Pakistan Journal of Statistics and Operation Research*, 14(1):95–107.
11. El-Bolkiny, M.T., Wasif, J.A., Elmadawye, M.M., Spahr, R.W., Sunderman, M.A. (2018). Estimate the Optimal Proportional Reinsurance Method in Property Insurance. *Egyptian Journal of Business Studies*, 42(3):291–337.
12. Glineur, F., Walhin, J.F. (2006). de Finetti's Retention Problem for Proportional Reinsurance Revisited. *Blätter der DGVFM*, 27(3):451–462.

13. Guerra, M., Centeno, M.L. (2008). Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria. *Insurance: Mathematics and Economics*, 42(2):529–539.
14. Kaas, R., Goovaerts, M., Dhaene, J., Denuit, M. (2008). *Modern Actuarial Risk Theory: Using R*. 2nd edition. Springer.
15. Lampaert, I., Walhin, J.F. (2005). On the optimality of proportional reinsurance. *Scandinavian Actuarial Journal*, 2005(3):225–239.
16. Sharpe, W.F. (1994). The Sharpe Ratio. *Journal of Portfolio Management*, 21:48–58.
17. Veprauskaite, E., Sherris, M. (2012). An Analysis of Reinsurance Optimisation in Life Insurance. (Working Paper). ARC Centre of Excellence in Population Ageing Research (CEPAR), Australian School of Business, University of New South Wales.
18. Verbeek, H.G. (1966). On Optimal Reinsurance. *ASTIN Bulletin: The Journal of the IAA*, 4(1):29–38.
19. Verlaak, R., Beirlant, J. (2003). Optimal reinsurance programs: An optimal combination of several reinsurance protections on a heterogeneous insurance portfolio. *Insurance: Mathematics and Economics*, 33(2):381–403.