Received <day> <Month>, <year>; | Revised <day> <Month>, <year>; | Accepted <day> <Month>, <year>
DOI: xxx/xxxx

ARTICLE TYPE

A Sturm-Liouville Equation on the Crossroads of Continuous and Discrete Hypercomplex Analysis†

Isabel Cação[1] | M. Irene Falcão[2] | Helmuth R. Malonek*[1] | Graça Tomaz[1,3]

[1]University of Aveiro, Center for Research and Development in Mathematics and Applications, Aveiro, Portugal
[2]University of Minho, Centre of Mathematics, Braga, Portugal
[3]Polytechnic of Guarda, Research Unit for Inland Development, Guarda, Portugal

Correspondence
*Helmuth R. Malonek, Department of Mathematics, Campus Universitário de Santiago, 3810-193, Aveiro, Portugal.
Email: hrmalon@ua.pt

Summary

The paper studies discrete structural properties of polynomials that play an important role in the theory of spherical harmonics in any dimensions. These polynomials have their origin in the research on problems of Harmonic Analysis by means of generalized holomorphic (monogenic) functions of Hypercomplex Analysis. The Sturm-Liouville equation that occurs in this context supplements the knowledge about generalized Vietoris number sequences $\mathcal{V}_n$, first encountered as a special sequence (corresponding to $n = 2$) by L. Vietoris in 1958 in connection with positivity of trigonometric sums. Using methods of the calculus of holonomic differential equations we obtain a general recurrence relation for $\mathcal{V}_n$ and we derive an exponential generating function of $\mathcal{V}_n$ expressed by Kummer's confluent hypergeometric function.

KEYWORDS:
Hypercomplex Analysis, Clifford algebra, Vietoris' numbers, Sturm-Liouville equation.

1 | STARTING THE JOURNEY - SOME HISTORICAL REMARKS

> *The experience of past centuries shows that the development of mathematics was due not to technical progress (consuming most of the efforts of mathematicians at any given moment), but rather to discoveries of unexpected interrelations between different domains (which were made possible by these efforts).*
>
> Vladimir I. Arnold
>
> (in: *Polymathematics: Is Mathematics a Single Science or a Set of Arts*)

Vladimir I. Arnold, one of the most influential mathematicians of the recent past, is well known for having contributed with unconventional reasoning to an astounding number of different mathematical disciplines. This is marvelously visible in his contribution[1] to the volume of the International Mathematical Union celebrating the year 2000 as the Year of Mathematics. In the beginning of his paper he mentioned that

> *"According to J. J. Sylvester (1876) a mathematical idea should not be petrified in a formal axiomatic setting, but should be considered instead as flowing as a river."*

†Dedicated to Klaus Guerlebeck on the occasion of his 65th birthday.

His further explanations go far beyond our own subject but as confirmation of Sylvester's observation he called attention to general concepts such as *complexification, quanternionization, symplectization* like examples of *informal generalization of all mathematics* not having ready axioms for this purpose. Since the present paper has a lot to do with two *different* perspectives on *complexification* (or, more concrete, *hypercomplexification*) it seemed to us very appealing to follow Arnold's philosophy.

Most important in this context is the fact that the two aforementioned different *hypercomplexifications* have nothing to do with the usual *quaternionization* by Cayley-Dickson's doubling process as counterpart of two complex variables. In fact, in Section 2 we go along the usual road and refer to the ordinary way of hypercomplexification by paravectors as generalization of complex numbers. Section 3 leads us to a second road and a hypercomplexification by several hypercomplex variables, which for the first time was systematically used in 1990[2]. As we will show, both methods are closely related, but imply different insights into the structure of the hypercomplex polynomials that we are studying. In order to get an explanation for some historically investigated but also partially overlooked connections of real, complex and hypercomplex analysis, it seems opportune to mention some historical reasons for different approaches to quaternions or hypercomplex numbers during the second half of the last century. [1]

About 50 years ago, the influential paper of E. M. Stein and G. Weiss[6] directed the attention of physicists and mathematicians to

> "*... the correspondence of irreducible representations of several rotation groups to first order constant coefficient partial differential equations generalizing the Cauchy-Riemann equations.*"

They showed how certain aspects of complex one-dimensional function theory extend to solutions of those systems of PDE. The list of systems includes the generalized Riesz system, the Moisil-Theodoresco system, spinor systems as n-dimensional generalization of Dirac equations, Hodge - de Rham equations, etc. But their motivation for proving that correspondence between representation groups and PDE were merely of qualitative nature and deeply connected with properties of harmonic functions in several real variables. Around the same time the interest in quaternions and their extension to Clifford Algebras together with strong relations to symmetry groups were renewed. It provoked a fast-growing number of papers by physicists working in Quantum Mechanics and Quantum-Field Theory.[7]

Decades later, mathematicians successfully developed analytical tools for the treatment of all kinds of those generalized Cauchy-Riemann or Dirac equations. To a great part they renewed or actualized research from the 30ties, mainly done by R. Fueter.[8–11][2] The paper of A. Sudbery[14] and, particularly, the book of F. Brackx, R. Delanghe and F. Sommen[15] became very influential. Higher-dimensional analysis in Clifford algebras soon was called *Clifford Analysis*. Naturally, this type of generalized function theory heavily relied on representation theoretic and algebraic tools, functional analytic and topological principals, etc. Much less it relied on instruments or results from classical complex function theory. Their authors were also not motivated by applications to current problems of complex function theory such as, for instance, approximation or value distribution.

---

[1]In his Gibbs lecture,[3] F. Dyson explained the discrepancies between physicists and mathematicians after W. K. Hamilton's invention of quaternions as the result of his search for the best way to describe spatial rotations as simply as rotations in the plane through complex numbers (see also M. J. Crowe[4]). It is noteworthy that around 1970 there is no reference in these two works[3,4] to R. Fueter or other mathematicians who used quaternions to generalize complex function theory. This seems even stranger since F. Dyson used the words of the French function theorist J. Hadamard[5] as an epigraph: "*It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.*"

[2]R. Fueter started his work on the foundation of quaternionic analysis being almost 50 years old. He was motivated by the number theoretic problem of complex multiplication[12]. Generalizations of Fueter's mapping approach to quaternion valued holomorphic functions of a quaternion variable generated by holomorphic functions of a complex variable have been proposed by M. Sce in 1957. Later, in 1997, they have been extended by T. Qian applying operator methods. During the next 20 years a huge number of papers on this subject was published. The recent book[13] is a detailed account of almost all variants in different frameworks.

The frequently cited results of Sudbery [14] strongly supported those developments by suggesting that Riemann's approach via conjugate harmonic functions (starting point in the work of Stein and G. Weiss [6]) would be *the only one meaningful approach to quaternionic analysis and generalized holomorphic functions*. For many years, the search for a larger number of intrinsic similarities with classical function theory was noticeably restricted. Those negative results on the impossibility to develop quaternionic analysis following other suitably generalized concepts, for example in the sense of Weierstrass or Cauchy, did not stop further investigations, but naturally they continued to be concentrated on more qualitative function theoretic aspects. The book of K. Gürlebeck and W. Sprössig [16] at the end of the 80ties was an exception and therefore highly appreciated. Dedicated to boundary value problems for Dirac and generalized Cauchy-Riemann systems, the book also contains the basics of a discrete hypercomplex function theoretic approach to harmonic functions. The years after the 80ties saw a flourishing interest in Clifford Analysis, including the creation of a specialized journal and the foundation of a series of international conferences.

What concerns more recent developments in hypercomplex analysis, it seems important to notice that during the last one and a half decades a certain revival of interest in basic analytical tools grew up, for example, in polynomials and a Cauchy like formula. As a result of this tendency a new theory of hypercomplex functions *without being harmonic functions* was created by G. Gentili and D. Struppa. [17] We quote: [18]

"The fundamental question (...) is what function theory should be used to develop such a functional calculus if we are to obtain a calculus which shares the basic properties of the Riesz-Dunford functional calculus. In order to do so, one needs a function theory simple enough to include polynomials and yet developed enough to allow a Cauchy like formula. The theory of slice regular functions that we develop (...) satisfies both requirements."

In fact, the development of the new *theory of slice regular functions* for having at hand hypercomplex power functions did not come unexpected. Besides the restriction to Riemann's approach a second crucial drawback challenged for decades the research in hypercomplex analysis. This drawback substantially restricted the class of functions useful for a treatment in more analytically oriented research. It was a seemingly simple but basic fact: power functions of the considered hypercomplex variable (and corresponding polynomials or power series) do not belong to the set of generalized holomorphic (monogenic) functions in the sense of Fueter.

But two other attempts of the same inspiration [17,19] are older and have also been explicitly referred [17]: the concept of *"Modified Clifford Analysis"*, introduced by H. Leutwiler [20] in 1992 and some years later the introduction of *holomorphic Cliffordian functions* by G. Laville and I. Ramadanoff in 1998 [21]. Both attempts have pursued the same goal, namely the determination of function classes which include the power function x^k; $k \in \mathbb{N}$, in terms of the considered hypercomplex variable x and, as important consequence, the identity function $f(x) = x$. Indeed, in this work [21] we find as motivation the remark:

"(...) we think that the most important thing in the theory of one complex variable is the fact that the identity (i.e. z) and its powers (...) are holomorphic."

Seeming a rather simple reason for constructing new function classes or even theories, it was for all three theories a fundamental need for dealing with polynomials. Whereas H. Leutwiler changed from working with the Euclidean metric to the hyperbolic metric, G. Laville and I. Ramadanoff modified the order of the underlying generalized Cauchy-Riemann PDE's and correspondingly considered only an odd number of variables. Thereby they also succeeded the inclusion of power functions in the kernel of their differential operator of order $2m + 1$, $m \in \mathbb{N}_0$.

A different from all previously mentioned theories but overcoming the same problem was developed in a series of articles[22–24] by authors of the present paper. That time our approach was guided by the idea to stay inside of the class of Fueter's monogenic functions, without metric changes or increasing the order of Cauchy-Riemann PDE's. Therefore we constructed polynomials with the behavior of power-like functions under hypercomplex differentiation[25]. Our ideas, based on Appell's concept of power-like polynomials[26], found applications in quasi-conformal mappings[27], in the construction of generalized Hermite, Laguerre[28,29] as well as Bernoulli, Euler and others polynomials[30]. Appell polynomials also successfully have been applied to problems in 3-dimensional elasticity[31] as one of several other fields. We stop here our very short (and therefore incomplete) historical journey about some common background of different tendencies in the development of hypercomplex analysis. The analytic background for our own less conventional treatment of hypercomplex polynomials and powers series was the same, but the road we were driving and searching for new applications led us to combinatorics.

The present paper shows the application of analytical tools for detecting new properties of particular number sequences and vice versa. After some necessary technical preliminaries in Section 2, we explain in Section 3 how three different interpretations of the structure of the complex power z^k can almost trivially lead to different representations of those aforementioned Appell polynomials (confirming in some sense the second quote[21] of last section). Two of those interpretations are direct consequences of different perspectives on *hypercomplexification*, the third interpretation is based on a qualitative difference between power series expansion in complex and hypercomplex function theory.

After having shown those different viewpoints of hypercomplexification, we continue with some of their applications. They are results of a casual observation several years ago. Curiosity about the particular role of some coefficient sequences in the Appell polynomials called our attention to possibly hidden combinatorial relations between them. Finally, this curiosity led us to the observation that for the simplest non-complex case, the coefficients in those hypercomplex Appell polynomials coincide with a sequence of real numbers first encountered by L. Vietoris[32] in 1958 in connection with positivity of trigonometric sums[33]. The fact that all three interpretations called attention to special sequences of real numbers (in particular the generalized Vietoris sequence[34] $\mathcal{V}_n$) or sequences of vectors of real numbers in the third case, found its reflection in our papers[35,36]. Moreover, we could obtain a result on the divisibility of central binomial coefficients[35] whose proof is essentially based on an application of both types of coefficients. An independent from hypercomplex methods study of $\mathcal{V}_n$ via Jacobi polynomials (but different from those of Askey[33], where also applications in complex analysis are mentioned) can be found in a recent paper[34]. [3]

Some elementary tools for passing from continuous to discrete and back to continuous structures in the sense of D. Knuth's *concrete analysis*[37,38] were sufficient to detect a Sturm-Liouville type ODE on crossroads between continuous and discrete hypercomplex analysis. In connection with the holonomic calculus this fact allowed us to obtain an exponential generating function of $\mathcal{V}_n$ expressed by Kummer's confluent hypergeometric function depending on the dimension n of the real Euclidean vector space.

Section 4 closes our journey on different roads by analyzing the connection of the exponential generating function obtained in Section 3 with an exponential function introduced several years ago[22]. Fifteen years ago we did not have any knowledge about the relation of our research with Vietoris' numbers. The definition of that exponential function was a natural consequence of applying basic properties of the hypercomplex Appell polynomials[39] introduced one year before.

[3] Reading later the contribution of L Bourgain about connections between harmonic analysis and combinatorics as well as those of W. T. Gowers and R. P. Stanley on combinatorics and positivity in the same book where the epigraph of V. I . Arnold[1] is taken from, convinced us that at the end our casual observation was not so casual as we thought. It seems to be rooted in the deep relations of harmonic (hypercomplex) analysis to the other fields, but this time not expressed through representation theoretic facts, but simply through number sequences and monogenic Appell polynomials.

2 | ON THE USUAL ROAD TO GENERALIZED HOLOMORPHIC FUNCTIONS

For an independent and easy reading we remember some basics of hypercomplex analysis. Readers interested in more historical background of Clifford Analysis in general or in the consideration of hypercomplex analysis as function theory in co-dimension 1 may consult the references [15,40]. A short overview can also be found in a very recent survey [41].

The simplest example of *complexification* for working with the algebra of complex numbers in the sense mentioned by V. I. Arnold is the identification of $\mathbb{R}^2$ with $\mathbb{C}$, formally expressed by the two conjugate variables $z = x + iy$ and $\bar{z} = x - iy$. For a system of two real differentiable functions $u = u(x, y)$ and $v = v(x, y)$ and demanding that $f = f(z, \bar{z})$ is complex differentiable with respect to z, this leads to the usual system of Cauchy-Riemann equations . If we recall the complex partial derivatives (Wirtinger derivatives)

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big) \text{ and } \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$$

then the complex form of the Cauchy-Riemann equations is given as

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}} = 0. \tag{1}$$

If and only if (1) is fulfilled, then the complex derivative of f is the result of the action of the *conjugate Cauchy-Riemann operator*

$$\partial := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$$

on f and therefore given by $f' = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$. Due to (1) the complex differentiable or, equivalently, complex holomorphic function f does not depend on the conjugate variable $\bar{z}$ and we have that in fact

$$f' = \frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}. \tag{2}$$

The step from *complexification* to an analogue *hypercomplexification* now seems to be obvious. Instead of pairs of real variables (x, y) one has to consider $(n + 1)$ - dimensional vectors $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ treated in the manner of elements of a *Hypercomplex System* [42,43] which includes as simplest cases the complex numbers and also quaternions. For that an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ in form of n vectors $\{e_1, e_2, \cdots, e_n\}$ should be chosen, being subject to a non-commutative *binary* product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n., \text{ where } \delta_{kl} \text{ is the Kronecker symbol.} \tag{3}$$

Then *hypercomplexification* is realized by adding the real (scalar) part $x_0 = \mathrm{Sc}(x)$ and the purely imaginary (vector) part $\underline{x} = \mathrm{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$ for dealing in this form with only *one* hypercomplex variable x. [4]

This idea goes back to a number of papers published around 1880 by J. J. Sylvester about the correspondence between quaternions and their generalizations on the one hand and matrices on the other hand [44]. He also noticed that each hypercomplex number has an associated matrix based on the table of the multiplication rules (3). His discovery connected hypercomplex numbers to the nascent theory of transformation groups and thereby formed one of the fundamentals in the history of Representation Theory in the beginning of the last century.

Nowadays the (formal) sum

$$x = x_0 + \underline{x} \tag{4}$$

is called paravector and $\mathcal{A}_n \cong \mathbb{R}^{n+1}$ describes the embedding of $\mathbb{R}^{n+1}$ in a 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$ with $\mathcal{A}_n \subset C\ell_{0,n}$, generated by $e_k,\ k = 1, \dots, n$. A basis of $C\ell_{0,n}$ over $\mathbb{R}$ with unity $e_0 = 1$ is the set

[4] Speaking here of a *purely imaginary part* should not be confused with *the* imaginary part y of the complex number $z = x + yi$, being itself a real variable.

$$e_A = e_{h_1} e_{h_2} \dots e_{h_r}, \ 1 \le h_1 < \cdots < h_r \le n, \ e_\emptyset = e_0 = 1.$$

where $\{e_A : A \subseteq \{1, \dots, n\}\}$.

The reason for this usual type of hypercomplexification seems well justified by having in mind real analytic functions and their continuation into the complex plane through their series representation. Indeed, the easiest way to extend a real analytic function $f(x)$ automatically to a complex analytic (holomorphic) function is by substituting the real variable x in its series expansion by the complex variable z. The holomorphy of the obtained complex analytic function with real coefficients inside a corresponding circle of convergence is guaranteed by the well known fact that a holomorphic function has a series expansion that only depends on the (holomorphic) variable z, but not on its conjugate $\bar{z}$. The usual introduction of all elementary complex functions makes use of this fact. But does the same method of hypercomplexification (working analogously with a paravector $x = x_0 + \underline{x}$ instead of $z \in \mathbb{C}$) also automatically leads to meaningful *generalized holomorphic functions* as continuation from the one-dimensional real space $\mathbb{R}$ to the space $\mathbb{R}^{n+1}$? The answer to this question for $n \ge 2$ is negative.

The reason for this dilemma will become clear after having recalled all the other ingredients for a treatment of Clifford algebra valued functions of a paravector variable in analogy to complex holomorphic functions, i. e. functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. In concrete applications like, for example, pseudo-conformal mappings, 3D - elasticity problems and our particular aim of polynomials, the range of f coincides with $\mathbb{R}^{n+1}$ and f will be a mapping from $\mathcal{A}_n$ to $\mathcal{A}_n$.

Remark 1. In almost all problems we will deal with, it is our aim to guarantee that the considered general case formally includes also the real case and the complex case as particular cases of $\mathbb{R}^{n+1}$. If this is guaranteed, then the restrictions to one or to the other particular case have to be considered in different manner. For the complex case it is enough to choose $n = 1$, but the real case is the result of setting $\underline{x} = 0$.

Now it still remains to recall the usual generalized Cauchy-Riemann differential operator for the definition of generalized holomorphic (monogenic) functions in its kernel. The clarification about the possibility to use its hypercomplex conjugated operator as a meaningful derivation operator can be found in the work of Gürlebeck and Malonek[25]. Using a differential form calculus[2] and a corresponding Cauchy integral, this paper contains the proof that the existence of the hypercomplex derivative *is necessary and sufficient*[5] for defining monogenic functions, i.e. solutions of the generalized Cauchy-Riemann system[41]. The necessary topological basics are almost obvious.

For this purpose we recall that in analogy with Wirtinger's complex partial differential operator $\overline{\partial} := \frac{\partial}{\partial \bar{z}}$ the generalized Cauchy-Riemann differential operator $\overline{\partial}$ is a formal paravector of partial derivatives given by

$$\overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}\right) = \frac{1}{2}\left(\partial_0 + \partial_{\underline{x}}\right), \tag{5}$$

where

$$\partial_0 := \frac{\partial}{\partial x_0} \quad \text{and} \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

[5]Notice that this result corrected again the opinion suggested in Sudbery's paper from 1979[14] that only Riemann's approach via conjugated harmonic functions leads to a meaningful generalization of the system of Cauchy-Riemann equations. The two works[2,25] have shown that also a suitable generalization of Cauchy's approach via complex derivability leads to the same generalized system of Cauchy-Riemann equations and thereby to monogenic functions. Both papers clarified the fact that differentiability as property of local linear approximation and derivability (the existence of a hypercomplex derivative) are, contrary to the complex case $n = 1$, dual and have to be considered for $n > 1$ in hypercomplex dimension one resp. in co-dimension one of $\mathbb{R}^{n+1}$.

Since the *conjugate* of x is given by $\bar{x} = x_0 - \underline{x}$ analogously

$$\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}}) \tag{6}$$

is just the *conjugate generalized Cauchy-Riemann operator*. Then the hypercomplex derivative f' of a monogenic function [6] is obtained as

$$f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f. \tag{7}$$

Furthermore, the *norm* $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$. At the end of this preliminaries it remains to notice that in the case of $n \geq 1$ the vector part $\underline{x} = e_1x_1 + \cdots + e_nx_n$ is considered as a whole and has a non-positive square $\underline{x}^2 = -|(x_1, \cdots, x_n)|^2 = -\sum_1^n x_k^2$. This fact suggests the idea, already used by Fueter[8], to introduce for a *truly complexification* of the paravector $x = x_0 + \underline{x}$ a variable imaginary unit. If we define

$$\boldsymbol{\omega} = \omega(\underline{x}) = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|}, & \underline{x} \neq 0 \\ 0, & \underline{x} = 0 \end{cases} \tag{8}$$

then we notice that for $\underline{x} \neq 0$, $\boldsymbol{\omega}$ belongs to the unit sphere S^{n-1} in $\mathbb{R}^n$. Moreover, $\boldsymbol{\omega}$ behaves like an *imaginary unit*, since $\boldsymbol{\omega}^2 = -1$ without being for $n = 1$ a simple substitution of the constant imaginary unit i. In this case, $\boldsymbol{\omega} = \mathrm{sign}(x_1)e_1$ or $\boldsymbol{\omega} = \mathrm{sign}(y)i$ in the usual notation of the complex variable[7] $z = x + iy \in \mathbb{C}$. The usefulness of $\boldsymbol{\omega}$ lies in the fact that each paravector x has a second type of representation, which we simply call *complex-like form*

$$x = x_0 + |\underline{x}|\boldsymbol{\omega}, \tag{9}$$

because of its analogy to the complex variable representation. This very suggestive relationship shows *en passant* that x^k is a paravector, too. It plays a decisive role in some of the following considerations, also because of the interesting behavior under differentiation by $\partial_{\underline{x}}$ that in the past only rarely has been exploited[45,46]. Notice that for dealing with the representation (9) one would have to use

$$\partial_{\underline{x}}\omega = \frac{1-n}{|\underline{x}|} \quad \text{and} \quad \partial_{\underline{x}}|\underline{x}| = \omega. \tag{10}$$

We are now prepared to drive along different roads for an encounter with three different visions on monogenic hypercomplex Appell polynomials. They all have their natural origins in different interpretations of the power z^k of $z \in \mathbb{C}$.

3 | ON DIFFERENT ROADS: FROM COMPLEX POWERS TO HYPERCOMPLEX APPELL POLYNOMIALS

Everybody knows that sometimes you just can't see the forest for the trees. This happened also in some sense with a pretty obvious possibility to overcome the question of a suitable replacement of the holomorphic power z^k of $z \in \mathbb{C}$ by a monogenic function $f = f(x),\ x \in \mathcal{A}_n$, maintaining as most as possible its properties. Before arriving to the core of our *philosophy of*

---

[6]To be more exact, C^1-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex differentiable in Ω[25], i.e. it has a uniquely defined areolar derivative f' in the sense of Pompeiu in each point of Ω[40]. In fact, like in the complex case, hypercomplex differentiability implies also real differentiability.

[7]As R. Fueter[8] already noticed, for inverting this representation, i.e. for producing from iy the vector part of a paravector in a given complex holomorphic function, one should take the *sign* of $\sqrt{-(\underline{x})^2}$ according to the *sign* of y. Later on Fueter's remark has often been neglected and proposed that only complex holomorphic functions defined in the upper half plan should be used for that purpose. But Fueter himself discussed this question and included remarks about Schwarz' reflection principle in his paper.

hypercomplexification of $\mathbb{R}^{n+1}$ at the end of this section, we need a more intrinsic analysis of the structure of x^k under the action of the differential operators (5) and (6).

For the paravector x itself the action of the generalized Cauchy-Riemann operator (5) results in

$$\overline{\partial}x = x\overline{\partial} = \frac{1}{2}(\partial_0 x_0 + \partial_{\underline{x}}\underline{x}) = \frac{1}{2}(1-n). \tag{11}$$

This shows that for $n \geq 2$ the paravector x is not monogenic. Moreover, the application of the conjugate operator (6) leads to

$$\partial x = x\partial = \frac{1}{2}(\partial_0 x_0 - \partial_{\underline{x}}\underline{x}) = \frac{1}{2}(1+n). \tag{12}$$

Both formulae, (11) and (12) together, indicate that only the particular complex case $n = 1$ as specification of the general hypercomplex case gives the desired result, i.e. x itself is a monogenic function with derivative equal to 1. But where really lies the reason that already a simple change from z^k to x^k does not give the desired result for all $n > 1$ and $k \geq 1$? Of course, for a comfortable dealing with polynomials in the usual sense in terms of powers of a hypercomplex variable one could simply try to modify Fueter's theory. We saw in Section 1 that this was exactly the driving force for H. Leutwiler,[20] G. Laville and I. Ramadanoff[21] and also G. Gentili and D. Struppa.[17]

But we remember that a paravector power restricted to the real case $x^k|_{\underline{x}=0} = x_0^k$ resp. complex case $x^k|_{n=1} = z^k$, where $z = x_0 + x_1 e_1$ are both characterized by two common properties:

$$(x_0^k)' = kx_0^{k-1} \quad \text{resp.} \quad (z^k)' = kz^{k-1},\ k = 1, \dots,\ \text{and}\ \ x_0^0 = z^0 = 1. \tag{13}$$

Moreover, the real and complex derivatives can be considered as particular cases of the hypercomplex derivative (7). Why not ask for an easy to handle monogenic hypercomplex power function instead of x^k with exactly the same properties as indicated by the simplest cases in (13)?

Driving now along three different roads, we focus on three different interpretations of the structure of the prototype z^k, $z \in \mathbb{R}^2 \cong \mathcal{A}_1$ and some of their specific consequences and applications.

3.1 | The binomial expansion of x^k and a monogenic hypercomplex power-like polynomial

At the first glance, the use of complex numbers $z \in \mathbb{C} \cong \mathbb{R}^2 \cong \mathcal{A}_1 \cong C\ell_{0,1}$ as special representatives of $\mathbb{R}^{n+1} \cong \mathcal{A}_n$ for ideas about extensions to higher dimensions could not be very helpful, because of being an exceptional case: the only one commutative sub-algebra. But we consider the paravector x^k for $n \geq 2$ now as a binomial with two commuting elements and analyze more detailed its binomial expansion

$$x^k = (x_0 + \underline{x})^k = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}\underline{x}^s = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}|\underline{x}|^s \boldsymbol{\omega}^s \tag{14}$$

having in mind the almost identical expansion of the complex power

$$z^k = (x_0 + e_1 x_1)^k = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} x_1^s e_1^s.$$

Of course, the essential difference between the variable imaginary unit $\boldsymbol{\omega} \in \mathbb{R}^n$ and the constant imaginary unit $e_1 \cong i \in \mathbb{C}$ consists in its behavior under derivation (10). Since it is depending from the dimension n we have to count with non-desired influences on the binomial expansion after the action of the generalized Cauchy-Riemann (Wirtinger) operators. This can be shown by a simple calculation which opens also the way to a general *remediation* in form of a different binomial expansion.

If we consider a reduced binomial expansion of $x^k = (x_0 + \underline{x})^k$ for $k = 2, \ldots$

$$x^k = (x_0 + \underline{x})^k = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \underline{x}^s = x_0^k + k x_0^{k-1} \underline{x} + M(x_0, \underline{x}),$$

where $M(x_0, \underline{x})$ is the remaining sum of terms with lower powers of x_0, we see that after the formal application of the hypercomplex derivation operator the leading term of order $k-1$ (in x_0) is equal to

$$\partial x^k = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})(x_0^k + k x_0^{k-1} \underline{x} + M(x_0, \underline{x})) = \frac{1}{2}(k - k\partial_{\underline{x}} \underline{x}) x_0^{k-1} + N(x_0, \underline{x}), \tag{15}$$

where $N(x_0, \underline{x})$ is a sum of terms with lower powers of x_0. Taking into account that $\partial_{\underline{x}} \underline{x} = -n$ we notice that a constant correction factor $d_1 = \frac{1}{n}$, i. e. the use of $d_1\, \underline{x}$ could guarantee that at least the constant factor of the first element in the expansion of ∂x^k would be the desired value k. An analogue correction of the remaining higher order powers of $\underline{x}^s = |\underline{x}|^s \boldsymbol{\omega}^s,\ s = 2, \ldots$ in $M(x_0, \underline{x})$ also seems reasonable. But unfortunately, the derivatives $\partial_{\underline{x}}\, \underline{x}^s$ are depending from the parity of s, because

$$\partial_{\underline{x}}\, \underline{x}^s = \begin{cases} -s\underline{x}^{s-1}, & \text{if } s \text{ is even} \\ -(n+s-1)\underline{x}^{s-1}, & \text{if } s \text{ is odd} \end{cases}$$

which can be verified by straightforward computations. It implies that different correction factors d_s for every $s = 1, \ldots,$ of $\underline{x}^s$ have to be used. Suppose such process is viable and produces in $M(x_0, \underline{x})$ the same common factor k of all of its terms in an appropriate way, we would be close to our objective, i.e. to obtain a homogeneous polynomial with the analogous with (13) properties, but now with respect to the hypercomplex derivative. Only one additional property should still be taken into account, namely that the unknown so far correction coefficients d_s must ensure that a *monogenic homogeneous polynomial* is obtained. Only in this case the until now formally applied operator (6) as hypercomplex derivative (7) leads really to a monogenic hypercomplex function as generalization of x^k. The disadvantage is, of course, that the binomial expansion (14) of x^k will not anymore be possible and we have to pass to a homogeneous polynomial $P_k(x_0, \underline{x}; n,)$ given in the form

$$P_k(x_0, \underline{x}; n) = \sum_{s=0}^{k} \binom{k}{s} d_s(n)\, x_0^{k-s} \underline{x}^s = \sum_{s=0}^{k} \binom{k}{s} d_s(n) x_0^{k-s}\, |\underline{x}|^s \boldsymbol{\omega}^s. \tag{16}$$

That at the end the correction coefficients are also depending from the dimension of the vector part $\underline{x}$ of x is obvious and has been used in this representation (16). The particular choice of $d_0(n) = 1$ for all $n \in \mathbb{N}$ is justified by the second property in (13) and guaranties that $P_k(1, 0; n) = 1$.

Remark 2. We stress the (trivial) fact that $d_s(1) = 1$, identically for all values of s, characterizes the particular case $n = 1$ and consequently

$$P_k(x_0, x_1 e_1; 1) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \underline{x}^s = (x_0 + e_1 x_1)^k$$

are the usual powers of the holomorphic variable $z = x_0 + e_1 x_1$. The expected inclusion of powers of the real variable x_0 in our consideration by choosing $\underline{x} = 0$ in (16) also doesn't need any further explanation.

Remark 3. It is easy to see that for a fixed n a binomial expansion of $P_k(x_0, \underline{x}; n)$ itself is obtained in the form

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} P_{k-s}(x_0, 0) P_s(0, \underline{x}). \tag{17}$$

Nevertheless, since the nature of the polynomial $P_k(x_0, 0) = x_0^k$ is different from that of the dependent of n polynomial $P_k(0, \underline{x}) = d_k(n)\underline{x}^k$, the formula (17) is hiding the fact that the elements of the corresponding *Pascal triangles* with respect to x_0 and $\underline{x}$ in (16) are not identical with the ordinary Pascal triangle as the Table 1 shows.

d_k											
$d_0 = 1$						1					
$d_1 = \frac{1}{2}$					1		$\frac{1}{2}$				
$d_2 = \frac{1}{2}$				1		$2 \cdot \frac{1}{2}$		$\frac{1}{2}$			
$d_3 = \frac{3}{8}$			1		$3 \cdot \frac{1}{2}$		$3 \cdot \frac{1}{2}$		$\frac{3}{8}$		
$d_4 = \frac{3}{8}$		1		$4 \cdot \frac{1}{2}$		$6 \cdot \frac{1}{2}$		$4 \cdot \frac{3}{8}$		$\frac{3}{8}$	
$d_5 = \frac{5}{16}$	$\dots$		$\dots$		$\dots$		$\dots$		$\dots$		$\dots$

TABLE 1 Real coefficients in $P_k(x_0, \underline{x}; 2);\ k = 0, \dots, 5$

3.2 | Paravectors versus several hypercomplex variables

Now our aim is to see in the complex variable z a second and completely different source for interpretation as particular case $n = 1$ of some generalized to $\mathbb{R}^{n+1} \cong \mathcal{A}_n$ hypercomplex object. This *interpretation* leaves the usual road to monogenic functions of a paravector valued variable. It is, in some sense, an example of driving on a road without *having ready axioms for this purpose* in the spirit of the words of V. I. Arnold[8].

For this purpose we have to look with new eyes to $z = x + iy$. We saw in the previous subsection that (under some general restrictions on the considered function) a simple way of coming *from an interval on the real axis to a bi-dimensional domain in the complex plane* can be succeeded as linear extension of the real variable x by the pure imaginary variable iy. In a more advanced context, we find this idea behind the roots of Schwarz' reflection principle. Of course, this is a very simplified argumentation, but sufficient for the interpretation of the usual approach which uses the same method by extension *from an interval on the real axis to a n-dimensional domain in* $\mathbb{R}^n$. In this case the real variable $x = x_0$ figured as an exceptional variable compared with all the others considered as a whole. What happened if we consider $y = x_1$ as the result of specifying $\underline{x} \in \mathbb{R}^n$, but without being connected to $i = e_1$, simply as the first component of the real vector $(x_1, \dots, x_n)$? Vice versa, this perspective opens the eyes for another extension from the real to the hypercomplex, i.e. another type of *hypercomplexification of the variable* which now should be understand as extending $y = x_1 \in \mathbb{R}$ to $(x_1, \dots, x_n) \in \mathbb{R}^n$ and leaving x_0 unchanged as exceptional variable. As consequence immediately the question arises about the role of the generators $e_k,\ k = 1, \dots, n,$ of $C\ell_{0,n}$. The answer seems now obvious, because complex numbers as model for such an *algebraisation* include the imaginary unit i. Wouldn't it be the easiest way, by analogy, to include all generators together into a vector of *n imaginary units*? Yes, the way to do this is open after rotation of the complex plane whereby x and y change there places:

$$y - xi = -iz = z(-i) \quad \text{and correspondingly} \quad y + xi = i\bar{z} = \bar{z}i. \tag{18}$$

Remark 4. On pages 46 and 68 of the book of Brackx *et al.*[15] the authors postulate "*Notice however that for a complete analogy* (with the usual complex notation $z = x + yi$) *one has to think of x_1 as the real part and of x_0 as the imaginary part of the variable and to identify e_1 with $-i$*". All this happens here naturally by rotations and without artificial changes.

Introducing now the vectors

$$\mathbf{x} = (x_1, \dots, x_n)^T \quad \text{and analogously} \quad \mathbf{i} = (-e_1, \dots, -e_n)^T \tag{19}$$

[8] We call attention to the fact, that this subsection is nothing more than an unusual interpretation of a simple formula as basis for an easy understanding of the approach to monogenic functions realized in Malonek[2].

and writing as always $x_0 = x$, the *hypercomplex extension* of (18) to a *vector of n hypercomplex variables* follows in a self-explaining way. With $\mathbf{z} = (z_1, \ldots, z_n)^T$ we get

$$\mathbf{z} := \mathbf{x} + x_0\mathbf{i}. \tag{20}$$

In this way our road led us to a second hypercomplex structure of $\mathbb{R}^{n+1}$, simply by a different interpretation of $z \in \mathbb{C}$, namely

$$\mathbb{R}^{n+1} \cong \mathcal{H}^n := \{\mathbf{z} : z_k = x_k - x_0 e_k; x_0, x_k \in \mathbb{R},\ k = 1, \ldots, n\}.$$

The main advantage of this approach lies in the fact that all components of $\mathbf{z}$ are monogenic variables since $\bar{\partial} z_k = 0$.

The $\mathcal{H}^n$-approach to monogenic functions leads in a very natural and direct way to power series in several hypercomplex variables (generalizing Weierstrass' approach to complex holomorphic functions). In general, the non-commutative multiplication in Clifford algebras causes many difficulties. But the systematical use of n hypercomplex variables simplifies essentially notations and calculations, and allows to work with monogenic polynomials in almost the same way as with multivariate polynomials in several real or complex variables. Indeed, (20) opens even the eyes for some kind of duality with the treatment of holomorphic functions in several complex variables. The classical extension from one complex variable to several variables joints two n-dimensional real vectors $\mathbf{x}$ and $\mathbf{y}$ by *complexification* in the form of

$$\mathbf{z} := \mathbf{x} + i\,\mathbf{y}. \tag{21}$$

Now formula (20) shows that one real vector $\mathbf{x}$ together with a vector $\mathbf{i}$ of n imaginary units multiplied by the exceptional variable x_0 constitutes also a meaningful set of variables. Literally speaking, our road crosses the bifurcation of complex function theory of one variable into two different holomorphic function theories, one of several complex variables and the other one of several hypercomplex variables. Needless to say, that compared with the hypercomplex one variable $x \in \mathcal{A}_n$ we see that $\mathbf{z} \in \mathcal{H}^n$ is the basis for a hypercomplex function theory in co-dimension 1 of $\mathbb{R}^{n+1}$. This explains once again the particular role of $\mathbb{C}$ for $n = 1$, where both concepts coincide[41].

We finish this subsection with two of the most important basic consequences for the work with the components of $\mathbf{z} \in \mathcal{H}^n$ in the context of hypercomplex monogenic polynomials. Proofs and further details, including series or corresponding alternating differential forms, can be found in Malonek *et al.*.[40,41,47]

(i) If $\nu = (\nu_1, \ldots, \nu_n)$ is a multi-index, all homogeneous monogenic polynomials of degree $|\nu| = k$ can be obtained as linear combinations (from the left or from the right)

$$P(\mathbf{z}) = \sum_{|\nu|=k} \mathbf{z}^{\nu} c_\nu \text{ resp. } \sum_{|\nu|=k} c_\nu \mathbf{z}^{\nu},\quad c_\nu \in \mathcal{C}\ell_{0,n}$$

of generalized powers $\mathbf{z}^{\nu}$ defined in the form of an n-nary symmetric product

$$\mathbf{z}^{\nu} := z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} = \underbrace{z_1 \times \cdots \times z_1}_{\nu_1} \times \cdots \times \underbrace{z_n \times \cdots \times z_n}_{\nu_n} = \frac{1}{k!} \sum_{\pi(i_1,\ldots,i_k)} z_{i_1} \cdots z_{i_k}, \tag{22}$$

where the sum is taken over **all** permutations[9] of $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $z_j = x_j - x_0 e_j,\quad j = 1, \ldots, n$. Moreover, all functions of the form $f(\mathbf{z}) = \mathbf{z}^{\nu} \in \mathcal{A}_n$, are left and right monogenic and $\mathcal{C}\ell_{0,n}$ - linear independent. Therefore they can be used as basis for generalized monogenic power series. Of course, the definition of the n-nary symmetric product is not restricted to the components of the hypercomplex vector, but can be understood as applicable to any $n-$ tuple of elements of commutative or non-commutative rings. It is here not our aim to develop a vector calculus in $\mathcal{H}^n$, for instance with

[9] In page 68 of Brackx *et al.* book[15] one can read *"where the sum runs over all* **distinguishable** *permutations"*. This caused problems with heavy notations and the loss of important algebraic relations .

admission of $\mathcal{A}_n$ as scalar field and the symmetric product as multiplication. As a curious example we only mention that the product $z = x \times \mathbf{i}$ with $x \in \mathcal{A}_n$ realizes the projection of $x \in \mathcal{A}_n$ into $\mathbf{z} \in \mathcal{H}^n$ since $z_k = x_k - x_0 e_k = -\frac{1}{2}(e_k x + x e_k)$. Several relations, showing other deep connections of $x \in \mathcal{A}_n$ and $\mathcal{H}^n$ can be found in Malonek.[49]

(ii) We recall that (20) was the result of interchanging the real and the imaginary part of the complex variable. Extending from the real to the complex and, vice versa, restricting the complex to the real, means now extending $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ resp. restricting $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$.

Remark 5. In section III.2.3 of Delanghe *et al.*[48] on *Generalized power functions* one can read: "*As z^α may be defined as the holomorphic extension of $x^\alpha, x \in \,]0, \infty[$, it seems natural to replace x^α on $]0, \infty[$, by some kind of power function $\underline{x}^\alpha, \underline{x} \in \mathbb{R}^m \setminus \{0\}$*". This leads to the concept of *Cauchy-Kovalevskaya continuation*[48], but for $n \geq 2$ and $\alpha > 1$ this does not allow to include the extension from the real to the complex and, vice versa, the restriction of the complex to the real as a reversible process in the usual way.

Since $\mathbf{z}^\mu|_{x_0=0} = \mathbf{x}^\mu$, the partial derivatives with respect to x_k of the generalized powers $\mathbf{z}^\mu$ are straightforward and the key stone for an easy handling in analogy with real powers. This is, for instance, confirmed by

$$\frac{\partial \mathbf{z}^\mu}{\partial x_k} = \mu_k \mathbf{z}^{\mu - \tau_k}, \quad \text{together with Euler's formula} \quad \sum_{|\mu|=k} z_k \frac{\partial \mathbf{z}^\mu}{\partial x_k} = k\, \mathbf{z}^\mu, \tag{23}$$

where τ_k is the multi-index with 1 at place k and zero otherwise. Euler's formula is also true for any polynomial of homogeneous degree k. From (23) we get like in the real and complex calculus

$$\frac{\partial^{|\nu|}}{\partial \mathbf{x}^\nu} \mathbf{z}^\mu = \frac{\partial^{|\nu|}}{\partial \mathbf{x}^\nu} \mathbf{x}^\mu = \begin{cases} \mu!, & \text{if } \nu = \mu \\ 0, & \text{if } \nu \neq \mu \end{cases}$$

and this implies for arbitrary $n \in \mathbb{N}$:

$$P(\mathbf{z}) = \sum_{|\mu|=0}^{k} \frac{1}{\mu!} \mathbf{z}^\mu \frac{\partial^{|\mu|} P(\mathbf{0})}{\partial \mathbf{x}^\mu} \quad \text{or} \quad P(\mathbf{z}) = \sum_{|\mu|=0}^{k} \frac{1}{\mu!} \frac{\partial^{|\mu|} P(\mathbf{0})}{\partial \mathbf{x}^\mu} \mathbf{z}^\mu. \tag{24}$$

Remark 6. The expansion of $P(\mathbf{z})$ in its Taylor series identical with the expansion of $P(\mathbf{x})$ is expression of the uniqueness theorem for generalized hypercomplex Taylor series (an inheritance from the same relation between Taylor series in one or several complex variables). Of course, Taylor series expansions or convergence questions are irrelevant for polynomials, but extending the theory to power series it becomes one of the main concerns. R. Krausshar's book[50] uses the $\mathcal{H}^n$- approach for dealing with hypercomplex generalized Eisenstein series.

A useful interpretation of (24) is that an arbitrary $\mathcal{C}\ell_{0,n}$- valued multivariate polynomial in $\mathbb{R}^n$ can be extended to a left resp. right monogenic polynomial in $\mathbb{R}^{n+1}$ through its Taylor expansion simply by substituting $\mathbf{x}^\mu$ in (24) by $\mathbf{z}^\mu$ and changing at the same time the ordinary product to the symmetric product (22).[10]

We are now prepared for the final goal of this subsection, i.e. to find in terms of hypercomplex variables a homogeneous monogenic polynomial $P_k(\mathbf{z})$ of order k as substitution for $x^k = (x_0 + \underline{x})^k$ with the same properties as (16) for simulating z^k with respect to (13).

For this purpose we consider the paravector x^k which, as we have seen before, is not monogenic in $\mathbb{R}^{n+1}$. Consider now its restriction to the hyperplane $x_0 = 0$ which results in the pure vector part $\underline{x}^k \in \mathbb{R}^n$. Then we have, with the monogenic extension

[10] This is just the essence of the so-called Cauchy-Kovalevskaya extension of a real-analytic $\mathcal{C}\ell_{0,n}$-valued function in $\mathbb{R}^n$ to a monogenic function in $\mathbb{R}^{n+1}$.

described by (24), a method on hand to see that its Taylor expansion

$$\underline{x}^k = \sum_{|\nu|=k} \binom{k}{\nu} x_1^{\nu_1} \cdots x_n^{\nu_n} \cdot e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}, \quad \text{where like usual} \quad \binom{k}{\nu} := \frac{k!}{\nu!} = \frac{k!}{\nu_1! \cdots \nu_n!}, \tag{25}$$

indicates directly a (left, or analogously also a right) monogenic homogeneous polynomial $P_k(\mathbf{z};n)$ of the form

$$P_k(\mathbf{z};n) = \sum_{|\nu|=k} \binom{k}{\nu} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \cdot e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}. \tag{26}$$

Even though we constructed already directly an appropriate monogenic function by using $P_k(\mathbf{z};n)|_{x_0=0} = \underline{x}^k$, equations (13) demand also to guarantee that its hypercomplex derivative is $P_k' = kP_{k-1}$ and the initial value $P_k(1) := P_k(\mathbf{z};n)|_{x_0=1,\underline{x}=0}$, $k = 0, 1, \ldots$, should be 1. Since the hypercomplex derivative is obtained by applying ∂ to a monogenic function f, these conditions together demand that the coefficient of x_0^k, the highest order of x_0 in the k- homogeneous polynomial (26), is equal to 1. Direct inspections confirm that for $n \geq 2$ this is not the case, since $\underline{x} = 0$ in (26) implies $z_k = -x_0 e_k$ and

$$P_k(\mathbf{z};n)|_{\underline{x}=0} = (-1)^k \sum_{|\nu|=k} \binom{k}{\nu} x_0^k \cdot e_1^{\nu_1} \times \cdots \times e_n^{\nu_n} \cdot e_1^{\nu_1} \times \cdots \times e_n^{\nu_n} = (-1)^k x_0^k \sum_{|\nu|=k} \binom{k}{\nu} [e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}]^2. \tag{27}$$

From (27) it follows immediately how to define a correction factor $c_k(n)$ to $P_k(\mathbf{z};n)$ for this purpose. The seemingly complicated expression[11] of a real number $c_k(n)$ in form of

$$c_k(n) := (-1)^k \Big(\sum_{|\nu|=k} \binom{k}{\nu} [e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}]^2 \Big)^{-1} \tag{28}$$

turns out to be as factor of $P_k(\mathbf{z};n)$ the last step for satisfying the necessary condition for obtaining the desired correction of x^k in terms of $\mathbf{z}$. Distinguishing the resulting polynomial by a different notation from that of Subsection 3.1 we get its representation

$$\mathcal{P}_k(\mathbf{z};n) := c_k(n) P_k(\mathbf{z};n) = c_k(n) \sum_{|\nu|=k} \binom{k}{\nu} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \cdot e_1^{\nu_1} \times \cdots \times e_n^{\nu_n} = x_0^k + M(x_0, \mathbf{x}) \tag{29}$$

as leading monic polynomial of order k in x_0 separated from the remaining sum of terms with lower powers of x_0 contained in $M(x_0,\mathbf{x})$. Doing the same for fixed n and all $k = 1, \ldots$, the sequence of $\big(\mathcal{P}_k(\mathbf{z};n)\big)_{k=1}^{\infty}$ contains homogeneous monogenic polynomials with leading monic polynomial in x_0 according to the order k. Its elements $\mathcal{P}_k^n$ satisfy the analogue with (13) properties

$$(\mathcal{P}_k^n)' = k\mathcal{P}_{k-1}^n \quad \text{and} \quad \mathcal{P}_k^n(1) = 1 \tag{30}$$

with respect to the hypercomplex derivative.

A similar situation was studied in Subsection 3.1 and resulted in (15). The difference between (29) and (15) is an essential one. In the ladder our aim was to find a monogenic polynomial on the basis of the formally applied hypercomplex derivation operator (6). In the former we knew already that the same was not a formal application, but used for obtaining the given monogenic polynomial with a monic leading polynomial x_0^k. This means that only one correction factor $c_k(n)$ for each homogeneous order was needed, not k factors as in (16). The use of $\partial = -\partial_{\underline{x}}$ instead of the equivalent $\partial = \partial_0$ would lead to exactly the same result, but needs a slightly more complicated reasoning by applying, for example, Euler's formula (23). We stop here without further studying the structure of the remaining terms in $M(x_0,\mathbf{x})$. Table 2 serves as example for the case $n = 2$.

Remark 7. Table 2 shows interesting relations to the ordinary binomial Pascal triangle. More than the triangle in Table 1 it permits to infer properties characterizing its hypercomplex origins. For instance:

(i) It is in some sense symmetric with respect to the middle vertical but with alternating generators in the odd rows. Reading from the left side started with a multiple of e_1 and ended with the same multiple of e_2. This is characteristic for the odd

[11] Cação *et al.*[51] study examples of combinatorial identities in terms of generators of $C\ell_{0,n}$ and symmetric products of them inspired by expressions like in (28).

c_k										
$c_0 = 1$					1					
$c_1 = \frac{1}{2}$				$\frac{1}{2}e_1$		$\frac{1}{2}e_2$				
$c_2 = \frac{1}{2}$			$-\frac{1}{2}$		0		$-\frac{1}{2}$			
$c_3 = \frac{3}{8}$		$-\frac{3}{8}e_1$		$-\frac{3}{8}e_2$		$-\frac{3}{8}e_1$		$-\frac{3}{8}e_2$		
$c_4 = \frac{3}{8}$	$\frac{3}{8}$		0		$2\frac{3}{8}$		0		$\frac{3}{8}$	
$c_5 = \frac{5}{16}$	$\cdots$	$\cdots$		$\cdots$		$\cdots$		$\cdots$		$\cdots$

TABLE 2 Hypercomplex coefficients in $\mathcal{P}^2_k(\mathbf{z})$; $k = 0, \ldots, 5$

rows with an even number of elements for allowing this pairwise symmetry. Neglecting the odd rows would reduce to symmetry by reflection as usual.

(ii) Every even row contains only real coefficients with interchanging from row to row signs. This changes of signs can be explained by the influence of $x \in \mathcal{A}_n$ in its complex-like form (9) and reflects the periodicity of $\boldsymbol{\omega}^k$ modulo 4. By the same reasons the minus signs in the rows reflect just the hidden even powers of e_1 resp. e_2.

(iii) The appearance of *zeros* in the even line is a consequence of non-commutativity of the generators and their annihilating effects in the corresponding symmetric products. Looking only to the even rows without taking care of the zero-gaps one recognizes the ordinary Pascal triangle multiplied by the rational correction coefficients.

(iv) Important seems also the effect of using several hypercomplex variables and passing to hypercomplex coefficients instead of real coefficients like in the non-symmetric Pascal triangle of Subsection 3.1. The symmetry relations between these coefficients are guarantee for not leaving the result of forming a paravector valued polynomial instead of a polynomial in other sub-modules of $\mathcal{C}\ell_{0,n}$.

We notice that the use of several hypercomplex variables allows to work almost in the same way as in the calculus of multivariate real or complex polynomials. This is the reason for that in the case of $n \geq 3$, i.e. dealing with more than two hypercomplex variables, one can not anymore expect to obtain a generalized bi-dimensional Pascal triangle like in Table 2 . The higher dimensional case with $n \geq 3$ needs to be handled with polyhedral geometry. As a first step, the case $n = 3$ and the corresponding hypercomplex Pascal polyhedron has been studied.[52] A remark on the expression of complicated formulae in representation theory and hypergeometric functions theory by polyhedral geometry can be found in V. I. Arnolds paper.[1]

Due to the fact that the inner structure of all $\mathcal{P}^n_k$ is in a certain sense independent from n (only the dimension of $\mathbb{R}^n$, i.e. the number of hypercomplex variables matters, whereas the form of their appearance is fixed) we recognize the particular role of the sequence of coefficients depending from n. In other words, the properties of the sequence of $c_k(n)$ for fixed n and variable k are determining the properties of all of the hypercomplex monogenic power-like function sequences $\left(\mathcal{P}^n_k(\mathbf{z})\right)_{k=1}^{\infty}$ of the form

$$\mathcal{P}^n_k(\mathbf{z}) = c_k(n) \sum_{|\nu|=k} \binom{k}{\nu} \mathbf{z}^\nu \cdot \mathbf{i}^\nu. \tag{31}$$

The following table of sequences of $c_k(n)$ contains for $n = 1$ the constant coefficients of the complex geometric series and for $n = 2$ Vietoris' numbers.[32,33] They are followed by its generalization introduced by the authors[35] together with a related sequence of integers (A283208).[53]

	0	1	2	3	4	5	6	7	8	9	10
$c_k(1)$	1	1	1	1	1	1	1	1	1	1	1
$c_k(2)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{35}{128}$	$\frac{35}{128}$	$\frac{63}{256}$	$\frac{63}{256}$
$c_k(3)$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{11}$
$c_k(4)$	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{5}{64}$	$\frac{5}{64}$	$\frac{7}{128}$	$\frac{7}{128}$	$\frac{21}{512}$	$\frac{21}{512}$
$c_k(5)$	1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{35}$	$\frac{3}{35}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{33}$	$\frac{1}{33}$	$\frac{3}{143}$	$\frac{3}{143}$
$c_k(6)$	1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{7}{384}$	$\frac{7}{384}$	$\frac{3}{256}$	$\frac{3}{256}$
$c_k(7)$	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{5}{231}$	$\frac{5}{231}$	$\frac{5}{429}$	$\frac{5}{429}$	$\frac{1}{143}$	$\frac{1}{143}$

TABLE 3 Generalized Vietoris numbers $c_k(n)$ for $n = 1, 2, \dots 7;\ k = 0, 1, \dots 10$.

3.3 | Monogenic homogeneous polynomials in terms of x and $\overline{x}$

Here we refer to a third interpretation of $z = x+iy$ as starting point for the simulation of z^k, but now in a way that does not consist in an interpretation of the form as it was the case in the previous subsections. This third interpretation is based on a qualitative difference between power series expansion in complex and hypercomplex function theories. Formally it seems to be nothing more than a formal rewriting of (16) by substituting the real and vector part of $x = x_0 + \underline{x}$ in the polynomial representation (16) in Subsection 3.1 through

$$x_0 = \frac{x+\bar{x}}{2} \quad \text{and} \quad \underline{x} = \frac{x-\bar{x}}{2}. \tag{32}$$

However, in fact it is the confirmation of a principal difference between holomorphic functions in the complex plane whose series expansion depends only on $z \in \mathbb{C}$ and not on its conjugate $\bar{z}$. The series expansion of hypercomplex holomorphic function for $n \geq 2$ in $x \in \mathcal{A}_n$ as well as in $\bar{x} \in \mathcal{A}_n$ relies on the different behavior of x and $\bar{x}$ under the action of the generalized Cauchy-Riemann operators. From (12) it follows that $\bar{\partial}\bar{x} = \overline{x\partial} = \frac{1}{2}(1+n)$, i.e. $\bar{x}$ is a non-monogenic variable. Formula (11) together with (12) and the corresponding values of the Wirtinger operators applied to $\overline{x}$ allow that an easy linear combination of x and $\bar{x}$ is holomorphic with hypercomplex derivative equal to 1. Indeed, for $f(x,\bar{x}) = \alpha x + \beta\bar{x}$ we get easily as solutions of the system

$$\bar{\partial} f = \frac{\alpha}{2}(1-n) + \frac{\beta}{2}(1+n) = 0, \quad \text{and} \quad \partial f = \frac{\alpha}{2}(1+n) + \frac{\beta}{2}(1-n) = 1, \tag{33}$$

the values $\alpha = \frac{1}{2}(1+\frac{1}{n});\ \beta = \frac{1}{2}(1-\frac{1}{n})$. Written in x_0 and $\underline{x}$ this gives $f = x_0 + \frac{1}{n}\underline{x}$ and, as has to been expected, coincides with $P_1(x_0, \underline{x}; n)$ in (16).

As result of the substitutions (32) in (16) an identical homogeneous monogenic polynomial is obtained, but now in terms of x and its conjugate $\bar{x}$:

$$P_k(x,\bar{x};n) = P_k(x_0,\underline{x};n) = \sum_{s=0}^{k} T_s^k(n) x^{k-s}\bar{x}^s, \tag{34}$$

where

$$T_s^k(n) := \binom{k}{s} \frac{(\frac{n+1}{2})_{k-s}(\frac{n-1}{2})_s}{(n)_k}, \tag{35}$$

and $(.)_k$ stands for the Pochhammer symbol defined by $(a)_s = a(a+1)(a+2)\dots(a+s-1)$, $(a)_0 = 1, \quad s \geq 0$.

The coincidence of both polynomials generalizing $z^k \in \mathbb{C}$ as well as $x^k \in \mathbb{R}$ with different representations for $n \geq 2$ is trivially guaranteed, but how both representations are related to the result of our approach by several hypercomplex variables in

Subsection 3.2? All three have by construction the same initial value 1 for $x_0 = 1$ on their restrictions to the hyperplane $\underline{x} = 0$, but what happens if we chose in all $x_0 = 0$?

By considering $x_0 = 0$ in (26) from Subsection 3.2, we first notice that in this case (26) reduces to (25). Thereby we end up with

$$\mathcal{P}_k^n(\mathbf{z})|_{x_0=0} = c_k(n) P_k^n(\mathbf{x}; n) = c_k(n)\underline{x}^k. \tag{36}$$

Doing the same in (16) we get

$$P_k(0, \underline{x}; n) = d_k(n)\underline{x}^k. \tag{37}$$

The uniqueness theorem of the Taylor expansion of hypercomplex holomorphic functions and its coincidence with the expansion of $\underline{x}^k$ the direct comparison of (37) with (36) reveals that $d_k(n) = c_k(n)$.

But what about the expression of $c_k(n)$ in terms of the coefficients $T_s^k(n)$ in the third representation (34)?

Using the fact that $\underline{\bar{x}} = -\underline{x}$, one obtains from (34) on both sides

$$c_k(n)\underline{x}^k = \mathcal{P}_k^n(0, \underline{x}) = \sum_{s=0}^{k}(-1)^s T_s^k(n)\underline{x}^k = \left(\sum_{s=0}^{k}(-1)^s T_s^k(n)\right)\underline{x}^k,$$

which means that the coefficients $c_k(n)$ are given as the following alternating sum

$$c_k(n) = \sum_{s=0}^{k}(-1)^s T_s^k(n), \; k = 0, 1, \dots . \tag{38}$$

The aforementioned polynomials were introduced initially[22,23] as functions of a paravector variable x and its conjugate $\bar{x}$. More details on the connection between both types of coefficients $c_k(n)$ and $T_s^k(n)$ has been studied in Cação *et al.*.[36]

Remark 8. In connection with this different representations of $\mathcal{P}_k^n$ we observe the following facts.

(i) The case of a real variable can be formally included in the above definitions as the case $\underline{x} = 0$ or equivalently by requiring that $T_0^0(0) = 1$ and $T_s^k(0) = 0$, for $0 < s \leq k$.

(ii) For the first time and by using only elementary combinatorial relations the explicit values of the $c_k(n)$ have been determined in Falcão and Malonek[54]

$$c_s(n) = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{s}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{s}{2} \rfloor}} = \begin{cases} \frac{s!!(n-2)!!}{(n+s-1)!!}, & \text{if } s \text{ is odd} \\ c_{s-1}(n), & \text{if } s \text{ is even,} \end{cases} \tag{39}$$

constituting the elements of the n-parameter generalized Vietoris sequence.[35]

(iii) This representation highlights the fact that these polynomials are *special* monogenic polynomials in the sense of Abul-Ez and Constales[55]. In that work, a monogenic polynomial P is said to be special if there exist constants $a_{ij} \in \mathcal{A}_n$ for which

$$P(x) = \sum_{i,j}{}' \bar{x}^i x^j a_{i,j},$$

where the primed sigma stands for a finite sum. This paper is concerned with the extension of the Whittaker - Cannon theory without having that time at its disposal any concept of hypercomplex differentiability which allows - as we have just shown - an explicit determination of those coefficients $a_{i,j}$.

We close Subsection 3.3 with Table 4 where the first rows in the non-symmetric real Pascal triangle for $n = 2$ are presented and their relations to the elements m_r of the aforementioned sequence (A283208) is underlined.

$\frac{1}{2^{m_r}}$											
$\frac{1}{2^{m_0}} = 1$						1					
$\frac{1}{2^{m_1}} = \frac{1}{4}$					$\frac{3}{4}$		$\frac{1}{4}$				
$\frac{1}{2^{m_2}} = \frac{1}{8}$				$\frac{5}{8}$		$\frac{2}{8}$		$\frac{1}{8}$			
$\frac{1}{2^{m_3}} = \frac{1}{64}$			$\frac{35}{64}$		$\frac{15}{64}$		$\frac{9}{64}$		$\frac{5}{64}$		
$\frac{1}{2^{m_4}} = \frac{1}{128}$		$\frac{63}{128}$		$\frac{28}{128}$		$\frac{18}{128}$		$\frac{12}{128}$		$\frac{7}{128}$	
$\frac{1}{2^{m_5}} = \frac{1}{512}$	$\frac{231}{512}$		$\frac{105}{512}$		$\frac{70}{512}$		$\frac{50}{512}$		$\frac{35}{512}$		$\frac{21}{512}$

TABLE 4 Hypercomplex coefficients in $\mathcal{P}_k^n(x, \bar{x})$; $k = 0, 1, \ldots, 5$

3.4 | Examples of applications of hypercomplex sequences of homogeneous Appell polynomials

In the first part of this section we saw that different interpretations of the form of z^k were sufficient for a hypercomplex generalization of Appell's historical concept of power-like polynomial sequences, since $x^k, x \in \mathbb{R}$, and its complex pendant $z^k, z \in \mathbb{C}$, are their prototypes. Starting point for Appell's intentions were nothing more than the use of the relations (13) as result of differentiation, identical for real and complex variables. In 1880 Appell[26] considered for that purpose a sequence of polynomials $P_0(x),\ P_1(x), \ldots,\ P_{n-1}(x), P_n(x), \ldots$ such that $P_k(x)$ is of exact degree k, for each $k = 0, 1, \ldots$, and moreover, two consecutive terms are linked by the relation

$$P_k'(x) = kP_{k-1}(x),\ k = 1, 2, \ldots. \tag{40}$$

Finally Appell proved that all sequences of polynomials satisfying (40) have the general form

$$P_k(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} x^{k-s},\ k = 0, 1, \ldots, \tag{41}$$

with $\alpha_s, s = 0, \ldots, k$ real arbitrary coefficients ($\alpha_0 \neq 0$).

The classical Bernoulli polynomials, Euler polynomials, Hermite polynomials, and many others are well known examples of Appell sequences. Nowadays such sequences are simply known as Appell sequences.

Since the 80ties , the search for an appropriate definition of generalized power functions in the framework of Clifford analysis was imminent in several works like Brackx *et al.*[15], or later Abul-Ez and Constales[55], and in a more detailed way in Delanghe *et al.*[48], but without having available in that time the concept of a well defined hypercomplex derivative. Of course, well defined generalized power functions would also have been the key to an appropriate definition of generalized power series, particularly, the definition of exponential functions. The book of Gürlebeck *et al.*[63] contains in Chapter 11 on *Elementary Functions* more details about these problems.

Having the hypercomplex derivative (7) at disposal, Appell sequences or Appell sets in the hypercomplex context were first considered in Falcão et al.[39,22,23] in order to construct the homogeneous monogenic polynomials $\mathcal{P}_k^n$ introduced in the previous sections. [12]

[12] In these papers the approach was not done by interpretations of the structure of the complex power function as it has been done in the previous subsections. The initial idea was suggested by the analysis of a quasi-conformal mapping from a sphere in $\mathbb{R}^3$ into a cube in $\mathbb{R}^3$ in terms of special paravector valued monogenic functions.

Naturally, and having in mind (40) and (30), a sequence of homogeneous $\mathcal{A}_n$-valued monogenic polynomials $\left(\mathcal{F}_k\right)_{k\geq 0}$ is called *hypercomplex Appell sequence* if $\mathcal{F}_k$ is of exact degree k and

$$\mathcal{F}_k' := \partial \mathcal{F}_k = k\,\mathcal{F}_{k-1},\ k = 1, 2, \dots,\ \text{ where }\ \mathcal{F}_0 = 1. \tag{42}$$

Using the results of the previous subsections, we recognize now that the explicitly constructed homogeneous polynomials $\mathcal{P}_k^n$ are exactly this type of Appell polynomials. Moreover, in the general hypercomplex case with $n \geq 1$ they can serve as replacement of the powers in Appell's classical approach.

For the transition to general Appell sequences in terms of $\mathcal{P}_k^n$ of different homogeneous degrees we consider now

$$\mathbb{P}_k^n := \operatorname{span}_{\mathbb{R}}\{\mathcal{P}_0^n, \mathcal{P}_1^n, \dots, \mathcal{P}_k^n\}$$

and obtain the same as in the classical case (41)

Theorem 1. A sequence of polynomials $\mathcal{F}_k$ in $\mathbb{P}_k^n$ is an Appell sequence if and only if

$$\mathcal{F}_k(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} \mathcal{P}_{k-s}^n,\ k = 0, 1, \dots, \tag{43}$$

where $\alpha_k = \mathcal{F}_k(0)$ and $\alpha_0 \neq 0$.

A proof can be found in Cação *et al.*[64]. Important to notice that this paper contains two more possible representations based on the complex-like structure of $\mathcal{P}_k^n$, also mentioned in (16), namely a *matrix* as well as a *determinant representation*.

As applications of the hypercomplex Appell sequences to other hypercomplex polynomials we can refer the monogenic Hermite and Laguerre polynomials constructed in[28,56]. Monogenic Hermite polynomials containing the classical ones as particular cases are given by

$$H_k^n(x) = \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{r!} \frac{k!}{(k-2r)!} \frac{(-1)^r}{4^r} \gamma_r(n) \mathcal{P}_{k-2r}^n(x),\ k = 0, 1, \dots$$

where $\gamma_r(n) = \sum_{s=0}^{r} \binom{r}{s} c_s(n)$ denotes the binomial transform of the sequence $(c_s(n))_{s\geq 0}$ given by (39). The sequence $(H_k^n)_{k\geq 0}$ is clearly an Appell sequence and therefore its elements are of the form

$$H_k^n(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} \mathcal{P}_{k-s}^n,\ k = 0, 1, \dots,$$

where, in this case,

$$\alpha_{2s} = \frac{(-1)^s (2s)!}{4^s s!} \gamma_s \quad \text{ and } \quad \alpha_{2s+1} = 0,\ s = 0, \dots, \lfloor \tfrac{k}{2} \rfloor.$$

The monogenic Laguerre polynomials $L_k^n(x)$ presented in Cação *et al.*[28] are not an Appell sequence. However, using the associated Laguerre polynomials also constructed in Cação *et al.*[28] given by

$$L_k^{n,(\alpha)}(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{1}{2^r\, k!}\ \gamma_r(n)\ \tfrac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-r+1)} \mathcal{P}_{k-r}^n(x),$$

it is clear that the sequence of polynomials $\mathcal{L}_k^{(j)} = (-1)^k k! L_k^{n,(j-k)}$, with $j \in \mathbb{N}$, is an Appell sequence whose elements can be written as

$$\mathcal{L}_k^{(j)}(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} \mathcal{P}_{k-s}^n(x),\ k = 0, 1, \dots,$$

with

$$\alpha_s = \frac{(-1)^s j!}{2^s (j-s)!} \gamma_s,\ s = 0, 1, \dots, k.$$

The aim of the following section is a journey on combinatorial roads which directly brings us to the crossroads with a Sturm-Liouville equation and hypergeomentric functions.

4 | THE ROAD TO A GENERAL RECURRENCE FOR GENERALIZED VIETORIS NUMBERS

4.1 | Recurrences for $n = 2$ and $n = 4$ in analogy with Catalan numbers

In Cação *et al.* [34] the authors proved that Vietoris' numbers $c_n(n)$ can be generated via the Gauss' hypergeometric function.

Theorem 2. Let $G(.;n)$ be the following real-valued function depending on a parameter $n \in \mathbb{N}$:

$$G(t;n) = \begin{cases} \frac{1}{t}\left[(1+t)\,{}_2F_1\left(\frac{1}{2},1;\frac{n}{2};t^2\right) - 1\right], & \text{if } t \in (-1,0) \cup (0,1) \\ 1, & \text{if } t = 0. \end{cases} \tag{44}$$

Then, for any fixed $n \in \mathbb{N}$, $G(.;n)$ is a one-parameter generating function of the sequence $\mathcal{V}_n$.

Examples of closed formulae for $G(.;n)$ can be easily obtained: [34]

- $n = 1$,

$$G(t;1) = \frac{1}{t}\left[(1+t)\,{}_2F_1\left(\frac{1}{2},1;\frac{1}{2};t^2\right) - 1\right] = \frac{1}{1-t}.$$

- $n = 2$,

$$G(t;2) = \frac{1}{t}\left[(1+t)\,{}_2F_1\left(\frac{1}{2},1;1;t^2\right) - 1\right] = \frac{\sqrt{1+t} - \sqrt{1-t}}{t\sqrt{1-t}}. \tag{45}$$

- $n = 3$,

$$G(t;3) = \frac{1}{t}\left[(1+t)\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};t^2\right) - 1\right] = \frac{1}{t}\left(\frac{t+1}{t}\ln\sqrt{\frac{1+t}{1-t}} - 1\right). \tag{46}$$

- $n = 4$,

$$G(t;4) = \frac{1}{t}\left[(1+t)\,{}_2F_1\left(\frac{1}{2},1;2;t^2\right) - 1\right] = \frac{2t + 1 - \sqrt{1-t^2}}{t(1+\sqrt{1-t^2})}. \tag{47}$$

We underline the formal similarity of Vietoris' numbers

$$c_{2k}(2) = c_{2k-1}(2) = \frac{1}{2^{2k}}\binom{2k}{k}$$

and the Catalan numbers

$$C_k = \frac{1}{k+1}\binom{2k}{k},$$

both represented by weighted central binomial coefficients of the $2k-$th line in the Pascal triangle. In the case of Vietoris' numbers the weight is equal to the *sum* of binomial coefficients in the $2k-$th line, whereas in the case of Catalan numbers the weight is equal to the *number* of binomial coefficients in the $k-th$ line of the Pascal triangle.

Also the case $n = 4$ is linked to the Catalan numbers, because

$$c_{2k+1}(4) = c_{2k+2}(4) = \frac{1}{4^{k+1}}C_{k+1}.$$

It is well known that the Catalan numbers, whose generating function is

$$f(t) = \frac{1-\sqrt{1-4t}}{2t}, \tag{48}$$

satisfy in form of a convolution the recurrence

$$C_{k+1} = \sum_{s=0}^{k} C_{k-s} C_s, \; k = 0, 1, \ldots, \quad \text{with} \;\; C_0 = 1. \tag{49}$$

It is then reliable to expect some similarity on recurrence relations between Catalan numbers and Vietoris numbers for both cases $n = 2$ and $n = 4$.

Indeed, for the Vietoris numbers $c_k(2)$ the use of the generating function (45) leads to the following result.[34]

Theorem 3. Vietoris' numbers $c_k(2)$ satisfy the recurrence relation

$$\begin{aligned} c_{k+1}(2) &= c_k(2) - \frac{1}{2}\Big(\sum_{s=0}^{k} c_s(2) c_{k-s}(2) - \sum_{s=0}^{k-1} c_{s-1}(2) c_{k-s}(2) \Big), \; k = 0, 1, \ldots, \\ c_0(2) &= 1 \end{aligned} \tag{50}$$

where, by convention, $c_{-1}(2) = 0$.

Similarly, starting from the generating function (47), the following recurrence relation is obtained for the case $n = 4$.

Theorem 4. Vietoris' numbers $c_k(4)$ satisfy the recurrence relation

$$\begin{aligned} c_{k+3}(4) &= -c_{k+2}(4) + \tfrac{1}{2} c_{k+1}(4) + \tfrac{1}{4} \sum_{s=0}^{k} c_s(4) c_{k-s}(4), \; k = 0, 1, \ldots \\ c_0(4) &= 1, \; c_1(4) = c_2(4) = \frac{1}{4} \end{aligned}$$

where, by convention, $c_{-1}(4) = 0$.

Proof. For simplicity we consider $c_m := c_m(4)$, $m = -1, 0, 1, 2, \ldots$. Our aim is to find a recurrence for the coefficients of the ordinary generating function $H := G(t; 4)$. Applying now (47) we easily obtain

$$(Ht + 1)\sqrt{1 - t^2} = 2t + 1 - Ht.$$

Squaring both sides and multiplying by t, we get

$$(Ht^2)^2 = (4 + 4t - 2t^2) Ht - 5t^2 - 4t. \tag{51}$$

Explicitly writing the relevant terms in (51) as

$$Ht = \sum_{k=0}^{+\infty} c_k t^{k+1}, \; Ht^2 = \sum_{k=0}^{+\infty} c_k t^{k+2}, \text{and } (Ht^2)^2 = \sum_{k=0}^{+\infty} \Big(\sum_{s=0}^{k} c_s c_{k-s} \Big) t^{k+4}.$$

and replacing in (51) $(Ht^2)^2$ and Ht by the corresponding series, the left hand side and the right hand side can be written, respectively, as

$$\sum_{k=0}^{+\infty} \left(\sum_{s=0}^{k} c_s c_{k-s} \right) t^{k+4},$$

and

$$(4c_0 - 4)t + (4c_1 + 4c_0 - 5)t^2 + (4c_2 + 4c_1 - 2c_0)t^3 + 4 \sum_{k=0}^{+\infty} (c_{k+3} + c_{k+2} - \tfrac{1}{2} c_{k+1}) t^{k+4}.$$

The comparison of the two sides leads to

$$c_0 = 1, \; c_1 = c_2 = \tfrac{1}{4}, \; \sum_{s=0}^{k} c_s c_{k-s} = 4(c_{k+3} + c_{k+2} - \tfrac{1}{2} c_{k+1}),$$

and the theorem is proved. □

4.2 | The case $n = 3$ via a holonomic differential equation and a conjecture

In the case $n = 3$ our search for a recurrence relation of the generalized Vietoris' numbers $c_k := c_k(3)$ follows a different strategy compared with the cases $n = 2$ resp. $n = 4$. [13]

The expression of $G(t; 3)$ in the form (46) compared with the explicit expressions (45) and (47) of $G(t; 2)$ and $G(t; 4)$, respectively, does not suggest to obtain easily an algebraic equation for $G(t; 3)$ by suitable manipulations from (46). The obvious reason is the presence of the transcendental logarithmic function in (46) instead of square roots as in (45) and (47).

Our attempts were guided by the fact that also ordinary differential equations can be used for the detection of recurrence relations from their solutions. This led us to the application of methods from the theory of holonomic functions.[14]

The first step is to separate in the expression (46) of $G := G(t; 3)$ the logarithm in different ways, i.e.,

$$1 + Gt = \frac{t+1}{2t} \ln\left(\frac{1+t}{1-t}\right), \tag{52}$$

$$\frac{2 + 2Gt}{1+t} = \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right). \tag{53}$$

Differentiating (52) leads to

$$tG' + G = -\frac{1}{2t^2} \ln\left(\frac{1+t}{1-t}\right) + \frac{1}{t(1-t)} = \frac{1}{1-t}\left[-\frac{1+t}{2t^2} \ln\left(\frac{1+t}{1-t}\right) + \frac{1}{t} + \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)\right],$$

$$tG' + G = \frac{1}{1-t}\left[-G + \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)\right]. \tag{54}$$

Now the logarithmic part in (54) can be substituted by the rational function on the left side of (53) and we obtain

$$tG' + G = \frac{1}{1-t}\left(-G + \frac{2+2Gt}{1+t}\right)$$

But this is just the desired first order differential equation for G, namely

$$(t - t^3)G' - (t^2 + t - 2)G = 2. \tag{55}$$

Since G is an entire hypergeometric function with coefficients c_k in its Taylor series expansion we can compare the coefficients in (55) of the power t^k of t. The separate case of $k = 0$ follows immediately by inspection of (55) after choosing $t = 0$:

$$2G(0) = 2c_0 = 2, \text{ or } c_0 = 1, \tag{56}$$

as expected.

Consider now $k \geq 1$. For completing our task, the calculus of transforming holonomic differential equations in holonomic recurrences and vice versa is used. [15]

Therefore the correspondences of terms in G or G' of (55) are symbolically

$$\left[t^k\right](tG') = kc_k,\ \left[t^k\right](t^3G') = (k-2)c_{k-2},\ \left[t^k\right](t^2G) = c_{k-2}, \text{ and } \left[t^k\right](tG) = c_{k-1}. \tag{57}$$

[13] Recall that the method used for $n = 2$ and $n = 4$ was mainly suggested by analogy to the generating function of Catalan numbers (48) and their recurrence formula (49).

[14] If a function depends on a discrete variable (typically named n, m, or k), then it is called *holonomic if it satisfies a recurrence*, and if it depends on a continuous variable (typically named x, t, or z), then it is called *holonomic if it satisfies a differential equation* with polynomial coefficients. For more details, particularly about more advanced computational methods than we have used we refer to Kauers and Paule [57] and also Zeilberger. [58]

[15] In Zeilberger [58] it can also be found an example for using this calculus as method for obtaining differential recurrence relations for the Legendre polynomials. In this context we recall three-term recurrence relations for building blocks of an orthogonal basis of monogenic functions in form of shifted Appell polynomials with coefficients from $\mathcal{V}_n$. [59]

Applying them to (55) we obtain the recurrence relation

$$kc_k - (k-2)c_{k-2} - c_{k-2} - c_{k-1} + 2c_k = (k+2)c_k - (k-1)c_{k-2} - c_{k-1} = 0. \tag{58}$$

This recurrence relation needs still an interpretation in the case $k = 1$. It is evident that the generating function G does not include any negative power of t, so that c_{-1} in (58) for $k = 1$ must be considered as equal to 0. This coefficient appeared formally as coefficient in the twice differentiated function G. It means that finally

$$c_1 - c_0 + 2c_1 = 0, \text{ or } c_1 = \frac{1}{3}c_0 = \frac{1}{3}.$$

Shifting now in (58) the index k by 1 we end up with

$$\begin{aligned}&(k+3)c_{k+1} = kc_{k-1} + c_k,\ k \geq 1\\ &c_0 = 1;\ c_1 = \frac{1}{3}.\end{aligned}$$

Resuming, we formulate the result as

Theorem 5. Let $c_k(3),\ k = 0, 1, \ldots$, be the coefficients of the generalized Vietoris sequence $\mathcal{V}_3$ with the generating function (46). Then $\mathcal{V}_3$ fulfills the recurrence relation

$$\begin{aligned}&(k+3)c_{k+1}(3) = kc_{k-1}(3) + c_k(3),\ k \geq 1\\ &c_0(3) = 1;\ c_1(3) = \frac{1}{3}.\end{aligned} \tag{59}$$

Remark 9. After having found more complicated than (59) recurrences for $\mathcal{V}_2$ and $\mathcal{V}_4$, the case $\mathcal{V}_3$ came as surprise and showed some potential for further attempts to obtain similar recurrence formulas also for $n \geq 5$ or even for arbitrary dimension n.

Some trials show quickly that the number 3 in (59) could probably be changed to an arbitrary n. Indeed, as we know (see Appendix) all $c_k(1),\ k = 0, 1, \ldots$ are identically equal to 1. From (59), replacing 3 by 1,we see that

$$(k+1)c_{k+1}(1) = kc_{k-1}(1) + c_k(1);\ c_0(1) = c_1(1) = 1,$$

i.e., $(k+1) \cdot 1 = k \cdot 1 + 1$ is true.

For $n = 2$, we have $c_0(2) = 1, c_1(2) = \frac{1}{2}, c_2(2) = \frac{1}{2}, \ldots$. Indeed

$$(1+2)c_2(2) = 1.c_0(2) + c_1(2) = \frac{3}{2}$$

leads to $c_2(2) = \frac{1}{2}$, and so on.

The same happened for other values of $n \geq 4$ and therefore we conjecture that the recurrence formula

$$(n+k)c_{k+1}(n) = kc_{k-1}(n) + c_k(n),\ k \geq 1 \tag{60}$$

$$c_0(n) = 1;\ c_1(n) = \frac{1}{n} \tag{61}$$

is a general recurrence for the generalized Vietoris numbers $\mathcal{V}_n$. But how to prove that $\mathcal{V}_n$ is the solution of this recurrence without involving more advanced tools than the holonomic calculus?

4.3 | The generalized Vietoris numbers as solution of a general recurrence

To answer the last question of the previous subsection we use induction to prove

Theorem 6. The solution of the recurrence (60)-(61), with non-constants coefficients, is given by the generalized Vietoris numbers $c_k(n)$ written in the form

$$c_k(n) = \begin{cases} \dfrac{k!!}{n(n+2)\dots(n+k-1)}, & \text{if } k \text{ is odd} \\ c_{k-1}(n), & \text{if } k \text{ is even} \end{cases}. \tag{62}$$

For simplicity we postpone the proof of the theorem after having proved a lemma. It shows that the characteristic property of $\mathcal{V}_n$ being a sequence with pairwise repeated elements is inherent in (60).

Lemma 1. The recurrence (60) implies that

$$c_{2m}(n) = c_{2m-1}(n),\ m = 1, 2, \dots . \tag{63}$$

Proof. The proof is straigtforward, using induction. Consider, for simplicity, $c_k := c_k(n)$. For $k = 1$, (60) together with (61) give

$$(n+1)c_2 = c_1 + c_0 = (n+1)c_1$$

and (63) is true for $m = 1$, since $c_2 = c_1$. The essential induction step is now to prove that $c_{2m} = c_{2m-1}$ implies also $c_{2m+2} = c_{2m+1}$. For $k = 2m$, (60) and the induction hypothesis imply

$$(2m+n)c_{2m+1} = c_{2m} + 2mc_{2m-1} = c_{2m} + 2mc_{2m} = (2m+1)c_{2m}$$

which in turn together with (60), for $k = 2m+1$, gives

$$(2m+1+n)c_{2m+2} = c_{2m+1} + (2m+1)c_{2m} = c_{2m+1} + (n+2m)c_{2m+1} = (n+2m+1)c_{2m+1}$$

resulting in $c_{2m+2} = c_{2m+1}$. This proves the assertion, i.e. the second relation in (62). □

Now the proof of the theorem itself is straightforward.

Proof. (of Theorem 6)

For k odd, $(k+1)$ is even and we can use Lemma 1 rewriting the left side of (60) in the form

$$(n+k)c_k(n) = kc_{k-1}(n) + c_k(n).$$

Reordering leads immediately to the relation between a coefficient of odd index k and its predecessor of even order, i.e.

$$(n-1+k)c_k(n) = kc_{k-1}(n). \tag{64}$$

Now we substitute successively $c_k := c_k(n)$ by their predecessors according to Lemma 1 and (64). This leads, for k odd, to

$$c_k = \frac{k}{n-1+k}c_{k-1} = \frac{k(k-2)}{(n-1+k)(n-3+k)}c_{k-3} = \cdots = \frac{k!!}{(n-1+k)(n-3+k)\cdots(n+2)n}c_0.$$

The decreasing sequence of indices k stopped when it reached the value 1 and together with $c_0 = 1$ we proved the first assertion of (62) completely. The second, as already previously mentioned, is the content of Lemma 1. □

5 | ON THE CROSSROADS - A STURM-LIOUVILLE EQUATION

5.1 | From the general recurrence to the exponential generating function of $\mathcal{V}_n$

The further study of recurrence (60) will show that it can be taken as starting point for the detection of a second generating function for $\mathcal{V}_n$, namely an exponential generating function. Therefore we apply the holonomic calculus, but now in inverse form compared with Subsection 4.2. This means that we transform (60) in a statement about a corresponding differential equation for an exponential generating function and find its solution $F(t,n)$.

Doing so we have found an approach for relating Gauss' rational generating function $G(t,n)$ for $c_k(n)$ to an exponential generating function $F(t,n)$ without using some general transformation method between one type and the other type. Instead of this, we came from one generating function (*the continuous object*) to the other by using the obtained recurrence (*the discrete object*) as vehicle for the determination of an exponential generating function (*another continuous object*) by "inverting the recurrence".

The result of this strategy is the following

Theorem 7. Let

$$F(t;n) = \sum_{k=0}^{+\infty} c_k(n)\,\frac{t^k}{k!} \tag{65}$$

be an unknown so far exponential generating function of the sequence $\mathcal{V}_n = \left(c_k(n)\right)_0^\infty$. Then $F(t,n)$ is the solution of the second order holonomic differential equation

$$tF''(t) + nF'(t) - (1+t)F(t) = 0 \tag{66}$$

with the initial conditions

$$F(0) = 1, F'(0) = \frac{1}{n} \tag{67}$$

Proof. As previously mentioned, the proof consists in transforming the holonomic recurrence (60)-(61) valid for the generalized Vietoris numbers $c_k := c_k(n)$, into a holonomic differential equation. To do so we remember the action of derivation or multiplication by the variable or a constant on the coefficients of an exponential generating function (65). The relevant correspondences are given by

$$\left[t^k\right] F = c_k,\ \left[t^k\right](tF) = kc_{k-1},\ \left[t^k\right](nF') = nc_{k+1},\ \text{ and }\ \left[t^k\right](tF'') = kc_{k+1}.$$

All together they automatically imply for $F = F(t;n)$ the holonomic differential equation

$$nF' + tF'' = (1+t)F,$$

which is the equation (66) we were looking for. It is evident, that the initial values of $\mathcal{V}_n$ imply the initial values of the generating function $F(t;n)$. We find them by using the first coefficient $F(0) = c_0 = 1$ and by comparing both sides of (66) for $t = 0$. From

$$nF'(0) = F(0) = 1$$

follows the second initial condition $F'(0) = \frac{1}{n}$ immediately. □

In turn, the initial values (67) will be needed to determine the corresponding special solution of the second order ODE (66) that we are interested in.

To identify the class of ODE's to which (66) belongs, we multiply this equation by t^{n-1}. Then we can join the first two terms and end up by recognizing an initial value problem for the Sturm-Liouville equation

$$(t^n F')' - t^{n-1}(1+t)F = 0, \quad F(0) = 1, F'(0) = \frac{1}{n}. \tag{68}$$

It is well known that it can be solved by using confluent hypergeometric functions[60]. More precisely we have

Corollary 1. With the notations introduced in Theorem 7, we have

$$F(t;n) = e^{-t} M\left(\tfrac{n+1}{2}, n, 2t\right), \tag{69}$$

where $M(a,b,z)$ is Kummer's confluent hypergeometric function ${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$.

5.2 | The exponential generating function $F(t;n)$ and its relations with paravector valued exponential functions

The approach used in the last subsection allow us to compare exponential functions in the hypercomplex framework that have been studied in the past, coming back to the question about the role of *hypercomplexification* as mentioned in the introduction.

In 2007, authors of this paper introduced an exponential function[22] as a first application of Appell polynomials in the framework of hypercomplex function theory, in the form

$$\mathrm{Exp}_n(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{k!} = e^{x_0}\Gamma(\tfrac{n}{2})\Big(\frac{2}{|\underline{x}|}\Big)^{\frac{n}{2}-1}\Big(J_{\frac{n}{2}-1}(|\underline{x}|) + \boldsymbol{\omega} J_{\frac{n}{2}}(|\underline{x}|)\Big), \tag{70}$$

where $\boldsymbol{\omega}$ is the unit paravector (8) and $J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}$ is the Bessel function of the first kind.

It follows immediately from the representation of $\mathcal{P}_k^n$ by binomial expansion that

$$\mathrm{Exp}_n(x) = e^{x_0}\,\mathrm{Exp}_n(\underline{x}) = e^{x_0} F(\underline{x};n), \tag{71}$$

which allows us to write

Corollary 2. With the notations introduced in Theorem 7, we have

$$F(\underline{x};n) = \Gamma\left(\tfrac{n}{2}\right)\left(\tfrac{|\underline{x}|}{2}\right)^{1-\frac{n}{2}}\left(J_{\frac{n}{2}-1}(|\underline{x}|) + \boldsymbol{\omega} J_{\frac{n}{2}}(|\underline{x}|)\right) \tag{72}$$

Moreover, due to the fact that Appell sequences imply automatically a direct link to a corresponding exponential function[61], we have the following result.

Theorem 8. The exponential generating function of the special monogenic polynomials $\mathcal{P}_k^n(x)$ is

$$\mathrm{Exp}_n(x\,t) = \widetilde{\mathrm{Sc}}\,[\mathrm{Exp}_n] + \boldsymbol{\omega}\widetilde{\mathrm{Vec}}\,[\mathrm{Exp}_n] = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)t^k}{k!},$$

where

$$\widetilde{\mathrm{Sc}}[\mathrm{Exp}_n] := e^{x_0\,t}\Gamma\left(\tfrac{n}{2}\right)\left(\tfrac{2}{|\underline{x}|t}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(|\underline{x}|t) \quad \text{and} \quad \widetilde{\mathrm{Vec}}\,[\mathrm{Exp}_n] := e^{x_0\,t}\Gamma\left(\tfrac{n}{2}\right)\left(\tfrac{2}{|\underline{x}|t}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}}(|\underline{x}|t).$$

It is worth noting that (69) can be written in terms of modified Bessel functions of the first kind. In fact, when $b-2a$ is a nonnegative integer, the Kummer function can be expressed as[62]

$$M\left(\nu+\tfrac{1}{2}, 2\nu+1-m, 2z\right) = \Gamma(\nu-m)e^z\left(\tfrac{z}{2}\right)^{m-\nu}\sum_{k=0}^{m} \frac{(-1)^k(2\nu-2m)_k(\nu-m+k)}{(2\nu+1-m)_k k!} I_{\nu+k-m}(z),$$

where $I_\alpha(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}$ is the modified Bessel function of the first kind. This allows to write

Corollary 3. With the notations introduced in Theorem 7, we have

$$F(t;n) = \Gamma\left(\tfrac{n}{2}\right)\left(\tfrac{t}{2}\right)^{1-\frac{n}{2}}\left(I_{\frac{n}{2}-1}(t) + I_{\frac{n}{2}}(t)\right) \tag{73}$$

The expression (73) is related to hyperbolic trigonometric functions, in the case of odd dimensions, as is illustrated in Table 6 .

All these results show a deep connection with the results of Laville and Ramadanoff[45]. Their approach to an exponential function via an integral transform is based on the relationship of the real and complex exponential function and the hyperbolic

	$F(\underline{x};n)$
n = 1	$\cos(\lvert\underline{x}\rvert)+\boldsymbol{\omega}\sin(\lvert\underline{x}\rvert)$
n = 2	$J_0(\lvert\underline{x}\rvert)+\boldsymbol{\omega}(\underline{x})J_1(\lvert\underline{x}\rvert)$
n = 3	$\dfrac{\sin(\lvert\underline{x}\rvert)}{\lvert\underline{x}\rvert}+\boldsymbol{\omega}\dfrac{\sin(\lvert\underline{x}\rvert)-\lvert\underline{x}\rvert\cos(\lvert\underline{x}\rvert)}{\lvert\underline{x}\rvert^2}$
n = 4	$\dfrac{2}{\lvert\underline{x}\rvert}\left(J_1(\lvert\underline{x}\rvert)+\boldsymbol{\omega}J_2(\lvert\underline{x}\rvert)\right)$
n = 5	$\dfrac{3\sin(\lvert\underline{x}\rvert)-3\lvert\underline{x}\rvert\cos(\lvert\underline{x}\rvert)}{\lvert\underline{x}\rvert^3}+\boldsymbol{\omega}\dfrac{9\sin\lvert\underline{x}\rvert-3\lvert\underline{x}\rvert^2\sin(\lvert\underline{x}\rvert)-9\lvert\underline{x}\rvert\cos(\lvert\underline{x}\rvert)}{\lvert\underline{x}\rvert^4}$
n = 6	$\dfrac{8}{\lvert\underline{x}\rvert^2}\left(J_1(\lvert\underline{x}\rvert)+\boldsymbol{\omega}J_2(\lvert\underline{x}\rvert)\right)$

TABLE 5 The generating function $F(\underline{x};n);\ n=1,\dots,6$, in terms of Bessel functions.

	$F(t;n)$
n = 1	e^t
n = 2	$I_0(t)+I_1(t)$
n = 3	$\dfrac{e^t t-\sinh(t)}{t^2}$
n = 4	$\dfrac{2}{t}\left(I_1(t)+I_2(t)\right)$
n = 5	$\dfrac{3(t-3)t\cosh(t)+3((t-1)t+3)\sinh(t)}{t^4}$
n = 6	$\dfrac{8}{t^2}\left(I_2(t)+I_3(t)\right)$

TABLE 6 The generating function $F(t;n);\ n=1,\dots,6$, in terms of modified Bessel functions.

functions sinh and cosh. Laville and Ramadanoff developed an interesting by itself transform calculus which led to exactly the same exponential function.

6 | FINAL REMARK

The different recurrent relations as well as the now detected role of a Sturm-Liouville equation on the crossroads of continuous and discrete confirm in some sense the opinion of V. I. Arnold in the epigraph that no axiomatic scheme should petrify the free flow of generalizations. Both approaches, by Appell polynomials as well as by integral transforms can enrich our knowledge about the role of hypergeometric functions in the setting of hypercomplex analysis bridging real, complex and hypercomplex analysis.

ACKNOWLEDGMENTS

This work was supported by Portuguese funds through the CMAT - Research Centre of Mathematics of University of Minho - and through the CIDMA-Center of Research and Development in Mathematics and Applications (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT - Fundação para a Ciência e Tecnologia"), within projects UIDB/00013/2020, UIDP/00013/2020, UIDB/04106/2020 and UIDP/04106/2020.

Conflict of interest

The authors declare no potential conflict of interests.

References

1. Arnold, VI. Polymathematics: Is Mathematics a Single Science or a Set of Arts ? In: Arnold VI , Atiyah M, Lax P, Mazur B, eds. *Mathematics: Frontiers and Perspectives* IMU-International Mathematical Union, AMS, 1999 : 403–416.

2. Malonek H. A new hypercomplex structure of the Euclidean space $\mathbb{R}^{m+1}$ and the concept of hypercomplex differentiability. *Complex Variables* 1990;14:25–33.

3. Dyson F. Missed Opportunities. *Bull. Am. Math. Soc.* 1972; 78: 635–652.

4. Crowe MJ. *A History of Vector Analysis: the Evolution of the Idea of a Vectorial System*. Dover;1967.

5. Hadamard J. *The psychology of invention in the mathematical field*. Princeton Univ. Press, Princeton, N.J.; 1945.

6. Stein EM, Weiss G. Generalization of the Cauchy-Riemann Equations and Representations of the Rotation Group. *American Journal of Mathematics* 1968; 90(1):163 - 196.

7. Edmonds JD. Quaternion quantum theory: new physics or number mysticism? *Amer. J. Phys.* 1974; 42:220 - 223.

8. Fueter R. Analytische Funktionen einer Quaternionenvariablen. *Comment. Math. Helv.* 1932; 4:9–20.

9. Fueter R. Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. *Comment. Math. Helv.* 1935, 7: 307–330.

10. Fueter R. Zur Theorie der regulären Funktionen einer Quaternionenvariablen. *Monatshefte f Math. und Physik.*1935,43:69–74.

11. Fueter R. Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen. *Comment. Math. Helv.* 1935-36;8(1): 371–378.

12. Malonek HR. Rudolf Fueter and his motivation for hypercomplex function theory. *Adv. Appl. Clifford Algebras* 2001;11(S2):219–229.

13. Colombo F, Sabadini I, Struppa. *Michele Sce's Works in Hypercomplex Analysis - A Translation with Commentaries*. Birkhäuser Basel 2020, Hardcover ISBN 978-3-030-50215-7.

14. Sudbery A. Quaternionic Analysis. *Math. Proc. Camb. Phil. Soc.* 1979; 85: 199-225.

15. Brackx F, Delanghe R, Sommen F. *Clifford analysis* . Pitman, Boston-London-Melbourne; 1982. ISBN 0 273 08535 2.

16. Gürlebeck K, Sprößig W.*Quaternionic analysis and elliptic boundary value problems.* International Series of Numerical Mathematics, Birkhäuser Verlag. Basel, 1990;89.

17. Gentili G, Struppa DC. A new approach to Cullen-regular functions of a quaternionic variable. *C. R. Acad. Sci. Paris* 2006; 342: 741–744.

18. Colombo F, Sabadini I, Struppa DC. *Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions*. Progress in Mathematics, 289, Birkhäuser Basel; 2011.

19. Gentili G, Struppa DC. A new theory of regular functions of a quaternionic variable. *Advances in Mathematics* 2007; 216(1): 279–301.

20. Leutwiler H. Modified Clifford analysis. *Complex Variables, Theory Appl.* 1992;17(3-4):153–171.

21. Laville G, Ramadanoff I. Holomorphic cliffordian functions, *Adv. in Appl. Clifford Algebras* 1998; 8: 323–340.

22. Falcão MI, Malonek HR. Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. In: Simos TE, Psihoyios G, Tsitouras C, eds. *AIP Conference Proceedings* . 2007; 936:738–741.

23. Malonek HR, Falcão MI. Special monogenic polynomials—properties and applications. In: Simos TE, Psihoyios G, Tsitouras C, eds. *AIP Conference Proceedings* .2007;936:764–767.

24. Cação I, Malonek HR. On complete sets of hypercomplex Appell polynomials. In: Simos TE, Psihoyios G, Tsitouras C, eds. *AIP Conference Proceedings* . 2008;1048: 647–650.

25. Gürlebeck K, Malonek HR. A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications. *Complex Variables Theory Appl.* 1999;39:199–228.

26. Appell, P. Sur une classe de polynômes, *Ann. Sci. École Norm. Sup.* 1880;9(2): 119–144.

27. Malonek HR, De Almeida R. A note on a generalized Joukowski transformation. *Appl. Math. Lett.* 2010;23(10):1174–1178.

28. Cação I, Falcão MI, Malonek HR. Laguerre derivative and monogenic Laguerre polynomials: An operational approach. *Math. Comput. Model.* 2011;53(5-6): 1084–1094.

29. Cação I, Falcão MI, Malonek HR. On generalized hypercomplex Laguerre-type exponentials and applications. In: Murgante D, Gervasi O, Iglesias A, Taniar, D, Apduhan B, eds. *Computational Science and Its Applications - ICCSA 2011.* Lecture Notes in Computer Science, Springer-Verlag Berlin Heidelberg 2011; 6784: 271–286.

30. Aceto L, Malonek H, Tomaz G. Matrix approach to hypercomplex Appell polynomials. *Appl. Numer. Math.* 2017;116: 2–9.

31. Bock S. On Monogenic Series Expansions with Applications to Linear Elasticity. *Adv. Appl. Clifford Algebras* 2014; 24: 931 - 943.

32. Vietoris L. Über das Vorzeichen gewisser trigonometrischer Summen. *Sitzungsber. Österr. Akad. Wiss.* 1958;167:125-135.

33. Askey R. Vietoris's inequalities and hypergeometric series. In Milovanović GV. (ed) *Recent progress in inequalities* Kluwer Acad. Publ., Dordrecht, 1998, pp. 63–76.

34. Cação I, Falcão MI, Malonek HR. On generalized Vietoris' number sequences. *Discrete Applied Mathematics* 2019;269:77-85.

35. Cação I, Falcão MI, Malonek HR. Hypercomplex Polynomials, Vietoris' Rational Numbers and a Related Integer Numbers Sequence. *Complex Analysis and Operator Theory.* 2017;11(5):1059-1076.

36. Cação I, Malonek HR, Falcão MI, Tomaz G. Combinatorial Identities Associated with a Multidimensional Polynomial Sequence. *J. Integer Seq.* 2018; 21, Article 18.7.4.

37. Graham RL, Knuth DE, Patashnik O. *Concrete Mathematics: A foundation A Foundation for Computer Science*. Reading, Mass: Addison-Wesley, 1994.

38. Wilf H. *Generatingfunctionology*. Peters AK, Wellesley, MA, 2006.

39. Falcão MI, Cruz J, Malonek HR. Remarks on the generation of monogenic functions. In: Gürlebeck K, Könke C. eds. 17th Int. Conf. on the Appl. of Computer Science and Mathematics in Architecture and Civil Engineering, Weimar; 2006.

40. Malonek H. Selected topics in hypercomplex function theory. In: Eriksson SL, ed. *Clifford algebras and potential theory* University of Joensuu, 2004;7:111–150.

41. Malonek HR, Cação I, Falcão MI, Tomaz G. Harmonic Analysis and Hypercomplex Function Theory in Co-dimension One. In: Karapetyants A, Kravchenko V, Liflyand E. eds. *Modern Methods in Operator Theory and Harmonic Analysis. OTHA 2018.* Springer, Cham. Springer Proceedings in Mathematics & Statistics. 2019; 291:93–115.

42. Bourbaki N. *Algèbre, Chapitres 1 à 3.* Springer-Verlag Berlin Heidelberg (Edition originale publiée par Masson, Paris, 1970) 2007; ISBN 978-3-540-33849-9.

43. van der Waerden BL. *A History of Algebra. From al-Khwārizmī to Emmy Noether.* Springer-Verlag Berlin Heidelberg 1985. ISBN 978-3-642-51601-6.

44. Hawkins, T. Hypercomplex numbers, lie groups, and the creation of group representation theory. *Arch. Hist. Exact Sci.* 1972, 8, 243–287

45. Laville G, Ramadanoff I. An integral transform generating elementary functions in Clifford analysis. *Math. Methods Appl. Sci.* 2006; 29:637–654.

46. Malonek HR, Falcão MI. On Special Functions in the Context of Clifford Analysis. In: Simos Theodore E., Psihoyios George, Tsitouras Ch., eds. *AIP Conference Proceedings* 2010, 1281, 1492.

47. Malonek H. Power series representation for monogenic functions in $\mathbb{R}^{n+1}$ based on a permutational product. *Complex Variables, Theory Appl.* 1990;15:181-191.

48. Delanghe R, Sommen F, Souček V. *Clifford algebra and spinor-valued functions. A function theory for the Dirac operator. Related REDUCE software by F. Brackx and D. Constales.* Mathematics and its Applications, Dordrecht: Kluwer Academic Publishers. 1992;53.

49. Malonek H. *Zum Holomorphiebegriff in höheren Dimensionen*, Habilitation Thesis, Halle, 1987

50. Kraußhar RS. Generalized Analytic Automorphic Forms in Hypercomplex Spaces, *Frontiers in Mathematics,* Birkhäuser Verlag, Basel, 2004.

51. Cação I, Falcão MI, Malonek HR. Combinatorial Identities in the Context of Hypercomplex. Function Theory. In: Simos TE, Psihoyios G, Tsitouras C, eds. *AIP Conference Proceedings* . 2018, 2800041-2800044.

52. Cruz C, Falcão MI, Malonek HR. About Pascal's tetrahedron with hypercomplex entries, In: Simos TE, Psihoyios G, Tsitouras C, eds. *AIP Conference Proceedings* 2013, 1558, 509-512.

53. Cação I, Falcão MI, Malonek HR. Minimal Exponent Integer Sequence A283208 associated with Vietoris sequence *The On Line Encyclopedia of Integer Sequences*, 2017, March 08, http://oeis.org/A283208

54. Falcão MI, Malonek HR. A note on a one-parameter family of non-symmetric number triangles. *Opuscula Mathematica.* 2012;32(4): 661-673.

55. Abul-Ez MA. Constales D. Basic sets of polynomials in Clifford analysis. *Complex Variables, Theory Appl.* 1990; 14(1): 177–185.

56. Cação I, Malonek HR. On an hypercomplex generalization of Gould-Hopper and related Chebyshev polynomials. In: Murgante D, Gervasi O, Iglesias A, Taniar, D, Apduhan B, eds. *Computational Science and Its Applications - ICCSA 2011.* Lecture Notes in Computer Science, Springer-Verlag Berlin Heidelberg, 2011, 6784:316–326.

57. Kauers M, Paule P. *The Concrete Tetrahedron: Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates.* Text and Monographs in Symbolic Computation, Springer Verlag, Wien, 2011.

58. Zeilberger, D.: A holonomic systems approach to special function identities. *Journal of Computational and Applied Mathematics* 1990;32:321–368.

59. Cação I, Falcão MI. ,Malonek HR. Three-Term Recurrence Relations for Systems of Clifford Algebra-Valued Orthogonal Polynomials. *Adv. Appl. Clifford Algebras* 27, 2017, 71–85.

60. Campos LMBC. On Some Solutions of the Extended Confluent Hypergeometric Differential Equation, *Journal of Computational and Applied Mathematics* 2001; 137:177 – 200.

61. Boas RP, Buck RC. *Polynomial Expansions of Analytic Functions*, Springer, Berlin, 1958.

62. Prudnikov AP, Brychkov AY, Marichev OI. *Integrals and Series: More Special Functions*, Vol. 3, Gordon and Breach Science Publishers, New York,1990.

63. Gürlebeck, K, Habetha K, Sprößig W. *Holomorphic Functions in the Plane and n-Dimensional Space*, Translated from the 2006 German original. Birkhäuser Verlag, Basel, 2008.

64. Cação I, Falcão MI, Malonek HR. Matrix Representations of a Special Polynomial Sequence in Arbitrary Dimension *Computational Methods and Function Theory* 2012; 12:371–391.

How to cite this article: