One-Loop Analytic Results for the Higgs Boson Plus Four Partons and Searches for Supersymmetry

BUDGE, LUCY,ELIZABETH

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Lucy Budge

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Institute for Particle Physics Phenomenology
Department of Physics
Durham University
United Kingdom

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Abstract: In this thesis we present compact analytic expressions for the production of the Higgs boson plus two jets at one-loop mediated by both a scalar and a fermion. The results are derived using generalised unitarity methods and retain the full mass dependence of the mediating particle. We use the relationship between the fermion and scalar theories to simplify the algebra in the fermion theory; many of the required integral coefficients are identical and for those that differ, the difference is of a lower rank than the scalar result.

We use these calculations to study the production of the Higgs boson plus two jets in the Minimal Supersymmetric Standard Model, assuming stop squarks are the dominant mediator. This is a potential channel for an indirect search for stop squarks, in particular we focus on the region where the lightest stop squark mass is similar to that of the top quark. However, although the 1-jet process shows improved discrimination over the inclusive process, we find there is no benefit gained from the 2-jet process.

Contents

Declaration

The work in this thesis is based on research carried out in the Department of Physics at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification. Chapters 1 and 2 are reviews of the literature, chapter 4 is based on ref. [1] and chapters 3 and 5 are based on ref. [2].

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Outside of Physics, I would never have made it to the end of a PhD without the support of my Mum, Dad and sister Alice; the Norwich High girls who are still always there for advice (numquam moribimur); Melissa and Erin. You've all helped me through school, undergrad and postgrad so much.

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While I'm still confused and uncertain, it's on a much higher plane, d'you see, and at least I know I'm bewildered about the really fundamental and important facts of the universe.

— from *Equal Rites* by Terry Pratchett

Chapter 1

The Standard Model

The Standard Model (SM) of particle physics is the theory describing the fundamental particles in our universe as well as three of the four fundamental forces. In order to mathematically describe the Standard Model we need to build a quantum field theory (QFT) that combines quantum mechanics and special relativity.

To begin with, in section 1.1 we will introduce quantum field theories, moving on to study the Standard Model Lagrangian in section 1.2. We will then briefly discuss the known limits of the Standard Model in section 1.3, introducing supersymmetry as a potential extension of the SM.

1.1 Quantum Field Theory

Just as for classical field theories, the dynamics of the theory are calculated by starting with the Lagrangian, $L = T - V$, where T is the kinetic energy and V the potential energy, and building the action, S,

$$S = \int d^4x \mathcal{L}(t, \vec{x}) = \int dt L(t), \tag{1.1.1}$$

where $\mathcal{L}$ is the Lagrangian density, but is commonly referred to as the Lagrangian. The dependency on spacetime within the Lagrangian is through the field(s) it describes: $\phi_i(x)$. From the action we can derive the Euler-Lagrange equations of motion by using the principle of least action, $\delta S = 0$,

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right), \tag{1.1.2}$$

which tell us about the dynamics of our field ϕ in the classical limit.

The Lagrangian can be built by considering the symmetries of our theory, since the Lagrangian itself must obey these symmetries. It is then a case of writing down every possible term with mass dimension less than or equal to four. This ensures that our theory is renormalisable, a concept that will be discussed in section 1.1.2. Symmetries play another important role in our understanding of the theory: Noether's theorem tells us that for every continuous symmetry there is a conserved current j^μ,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \delta\phi, \tag{1.1.3}$$

where $\delta\phi$ is an infinitesimal change in the field ϕ.

The field ϕ can be either a boson (integer spin) or fermion (half-integer spin). These two types of particles behave very differently. Fermions obey Fermi-Dirac statistics: no two fermions will be in the same state. On the other hand, an unlimited number of bosons can be in the same state, known as Bose-Einstein statistics.

Particle theory aims to describe not only the basic properties of particles but how they interact with particles of both the same type and different types. To do this we study the process of moving from an initial state i to a final state f,

$$\left\langle t_f, \vec{x}_f \mid t_i, \vec{x}_i \right\rangle = \left\langle \vec{x}_f \left| e^{-\frac{i}{\hbar} H\left(t_i - t_f\right)} \right| \vec{x}_i \right\rangle \propto \int \mathcal{D}x \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x, \dot{x}) \right). \tag{1.1.4}$$

This equation can be understood as an integration over all possible paths from one state to the other. By analogy with statistical physics this is known as the partition function $\mathcal{Z}$. Promoting the partition function to a generating functional with a source term $J(x)$,

$$\mathcal{Z}[J] \equiv \int \mathcal{D}\phi e^{iS[\phi] + i \int d^4x \phi(x) J(x)}, \tag{1.1.5}$$

we can calculate the matrix element of any process by differentiating the generating

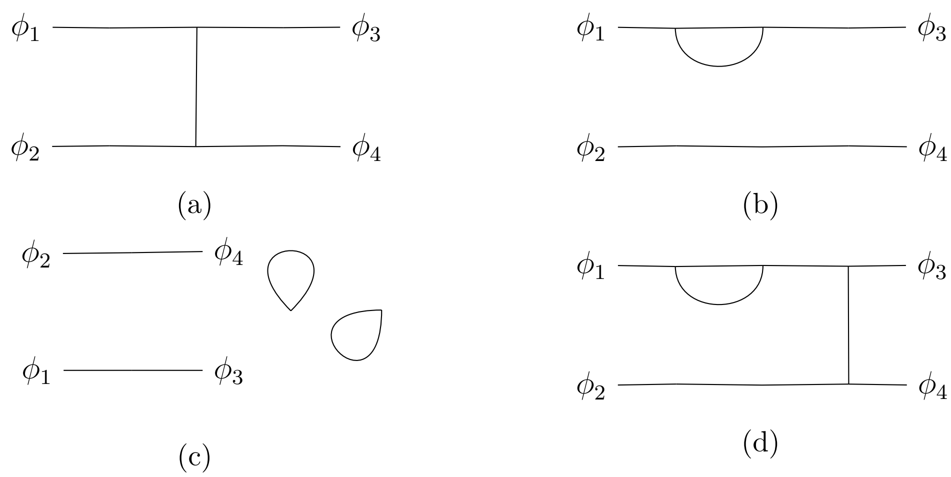


Figure 1.1: The four types of diagrams produced from Wick's theorem: fully connected (a), partially connected (b), disconnected (c) and unamputated (d).

functional with respect to the source term and setting the source to zero,

$$\langle 0 \left| \mathcal{T} \left(\phi \left(x_1 \right) \ldots \phi \left(x_n \right) \right) \right| 0 \rangle = \left. \frac{i^{-n} \delta^n \ln \mathcal{Z}[J]}{\delta J \left(x_1 \right) \ldots \delta J \left(x_n \right)} \right|_{J=0} . \tag{1.1.6}$$

The factors of $\phi(x)$ which are brought down by the functional derivatives act on the vacuum states to create or annihilate particles, leading to an expression that describes the scattering of particles of the field ϕ. The notation $\mathcal{T} \left(\phi \left(x_1 \right) \ldots \phi \left(x_n \right) \right)$ means "time-ordered", so that the operator ϕ_i with the largest value of t is on the left, and sequentially to the smallest on the right. We can then use Wick's theorem to convert from time ordered to normal ordered (all creation operators to the left of all annihilation operators),

$$\mathcal{T} \left\{ \phi_1 \phi_2 \ldots \phi_n \right\} =: \phi_1 \phi_2 \ldots \phi_n : + : \text{all possible contractions} :, \tag{1.1.7}$$

where $: \phi_1 \phi_2 \ldots \phi_n :$ represents a normal ordered expression, and a contraction of a pair of fields is

$$\ldots \overbrace{\phi(x_i) \ldots \phi}(x_n) = \ldots D_F(x_i - x_n) \tag{1.1.8}$$

with

$$D_F(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip \cdot x} . \tag{1.1.9}$$

Applying Wick's theorem to eq. (1.1.6) produces a sum of terms that can be considered diagramatically, where a contraction between two fields is an internal propagator. These diagrams can be divided into four types: fully connected, partially connected, disconnected and unamputated. These types are illustrated in fig. (1.1) for a $2 \to 2$ scattering process. Only fully connected (every part of the diagram is connected to all external lines) and amputated (once loops on external lines have been removed from unamputated diagrams) contribute to the scattering matrix. Partially connected diagrams are however important for renormalisation.

Once the disconnected and partially connected diagrams have been discarded, we can derive the Lehmann-Symanzik-Zimmermann (LSZ) formula for calculating the scattering matrix, $\mathcal{S}$, for an N particle process,

$$\mathcal{S} = \frac{1}{Z[0]} \int \mathcal{D}\phi e^{iS_0[\phi]} \left(\frac{\delta}{\delta\phi(p)} \right)^N e^{iS_{int}[\phi]} . \tag{1.1.10}$$

The $\mathcal{S}$-matrix can be decomposed into the identity (i.e. nothing happens) plus a term proportional to the matrix element, or amplitude, $\mathcal{A}$,

$$\mathcal{S} = \mathbb{1} + iT = \mathbb{1} - i(2\pi)^4 \delta^{(4)}(p_I - p_F)\mathcal{A}. \tag{1.1.11}$$

It is imperative that a theory can be experimentally tested, and to test a quantum field theory's predictions we use this invariant matrix element $\mathcal{A}$ to calculate the cross section, σ, for the $2 \to n$ particle scattering process,

$$\sigma = \frac{1}{|\vec{v}_a - \vec{v}_b| 2E_{\vec{p}_a} 2E_{\vec{p}_b}} \int \left(\prod_{i=1}^{n} \frac{d^3\vec{p}_i}{2E_{\vec{p}_i}(2\pi)^3} \right) (2\pi)^4 \delta^{(4)} \left(p_a + p_b - \sum_{i=1}^{n} p_i \right) |\mathcal{A}|^2 , \tag{1.1.12}$$

where the subscripts a and b refer to the initial particles, $\vec{v} = \vec{p}/E_{\vec{p}}$, and the index i sums over the n final particles.

1.1.1 Feynman Diagrams

Feynman diagrams are a common tool for the calculation of amplitudes so that we do not have to go through Wick's theorem and the LSZ formula for every process. Feynman diagrams provide a "recipe book" that gives us a much simpler way to

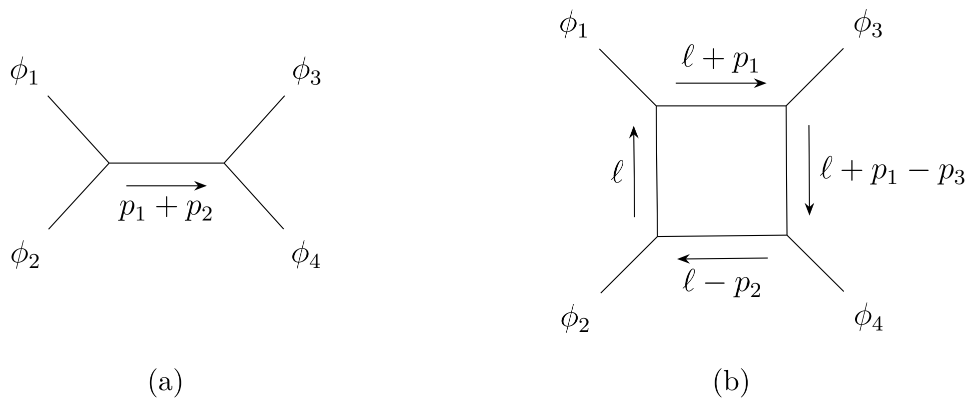


Figure 1.2: Two possible diagrams for the scattering process $\phi_1\phi_2 \to \phi_3\phi_4$ where the field ϕ has a triple self-coupling. (a) shows a tree level diagram while (b) is a one-loop diagram.

calculate the matrix element. The scattering process is drawn as a series of diagrams. Two possible diagrams for a $2 \to 2$ scattering process are shown in fig. (1.2).

Each line represents the propagation of a particle and each allowed propagator and vertex is defined by the terms of the Lagrangian. Feynman rules are derived from the Lagrangian and tell us what factor should be assigned to each external particle, vertex and internal propagator. All diagrams allowed by the Lagrangian must be included in the sum, as they have a 1:1 correspondence with terms in the expansion of eq. (1.1.10). Each vertex in a Feynman diagram contributes one or more factors of the coupling constant. Comparing the two diagrams in fig. (1.2), it is easy to see that these will have different orders of the coupling constant. For a coupling constant that is much less than 1, the expansion of eq. (1.1.10) can be done to a fixed order of the coupling, depending on the desired precision.

1.1.2 Loop Level

The diagram in fig. (1.2b) contains a closed loop of particles. In this case momentum conservation is not enough to fix the momentum of the internal lines; each loop has one unconstrained value of momentum, labelled as ℓ in fig. (1.2b). When calculating loop diagrams it is therefore necessary to integrate over all possible values of the momentum,

$$\int_{-\infty}^{\infty} \frac{d^4\ell}{(2\pi)^4} \mathcal{A} \,. \tag{1.1.13}$$

These integrations, however, are often non-trivial and give rise to infinities: as $\ell \to 0$ (infrared) or $\ell \to \infty$ (ultraviolet).

Infrared divergences also arise when the momentum of an external particle goes to zero, or when it becomes collinear with another of the external particles. In both of these cases, an n-particle process becomes indistinguishable from an $n-1$-particle process. The KLN theorem [3, 4] tells us that at any fixed order in the coupling constant the infrared divergences will cancel when all components are included. This means including diagrams with real corrections (the soft and collinear emissions) as well as the virtual (loop level) corrections.

In the case of hadronic collisions this situation is slightly more complicated. We describe the initial state using parton distribution functions (PDFs) [5–7], which give the probability of a parton having a specific fraction of the hadron momentum. To remove initial state collinear singularities we must therefore ensure to use a renormalised PDF [8].

To remove the ultraviolet divergences we begin by assuming that all quantities (fields, couplings, masses) are bare and therefore we are unable to measure them. We can define their renormalised counterparts as, for example, $\phi_B = Z_\phi^{1/2}\phi$ and $m_B = Z_m m$, where the subscript B denotes the bare quantity. The constants Z_i are the renormalisation constants and must be included in the Feynman rules. This allows us to remove UV divergences, however the couplings and masses are now quantities that must be measured rather than predictions of the theory. As mentioned in section 1.1, for a theory to be renormalisable we require every term in the Lagrangian to have a mass dimension of four or less.

After renormalisation we have a theory where the ultraviolet divergences are removed by cancelling them with equivalent divergences. This process can be better defined by regularising the theory. This is commonly done by dimensional regularisation, where the theory is analytically continued to $D = 4 - 2\epsilon$ dimensions, and the limit $\epsilon \to 0$ is then taken. Care is required in particular for any Dirac algebra as this must also be done in D dimensions. Regularisation introduces an arbitrary parameter, μ, which physical results do not depend on.

There are various techniques for evaluating these integrals: for example Passarino-Veltman reduction [9] and Feynman parameters among others. Beyond one-loop the integrals become more complicated and several methods exist specifically for these: differential equations, Mellin-Barnes and HyperInt for example. For reviews of these techniques see refs [10, 11]).

An additional effect of the process of renormalisation is that coupling parameters, including masses, become dependent on the energy scale they are measured at. This dependence is described by the β-function,

$$\beta(g) = \mu^2 \frac{\partial g}{\partial \mu^2}\,, \tag{1.1.14}$$

where g is the coupling parameter.

1.2 The Standard Model Lagrangian

The easiest way to describe the Standard Model is in terms of particle content: the six quarks (up, down, charm, strange, top and bottom), six leptons (electron, muon, tau and the three associated neutrinos), four types of gauge bosons (gluons G^a, photon A, Z and $W^{\pm}$ bosons) and one scalar boson (Higgs boson). It is also important to consider how these particles interact, for example photons interact with any particle charged under electromagnetism while gluons interact only with particles charged under colour: quarks and other gluons.

For the Standard Model to have any predictive power it is necessary for these qualitative descriptions to be understood mathematically, in the language of quantum field theory introduced in section 1.1. The Standard Model Lagrangian can be concisely written, at the classical level, as

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
& + i\bar{\Psi}\not{D}\Psi \\
& + D_\mu\Phi^\dagger D^\mu\Phi - V(\Phi) \\
& + \bar{\Psi}_i\hat{Y}_{ij}\Psi_j\Phi + \text{h.c.}\,.
\end{aligned} \tag{1.2.1}
$$

The aim of this section is to describe this famous Lagrangian. The gauge group of the Standard Model, before spontaneous symmetry breaking, is $SU(3)_c\times SU(2)_L\times U(1)_Y$, where the subscripts refer to colour, weak isospin and hypercharge respectively.

The Standard model contains three types of field: spin-0 scalar bosons Φ, spin-1/2 fermions Ψ and spin-1 vector bosons W^i_μ and B_μ. The first line of eq. (1.2.1) describes the kinetic and interaction dynamics of the vector (gauge) bosons, the second line concerns the fermions, Ψ, and their propagation as well as interactions with the gauge bosons. The third line contains the terms describing the Higgs boson's, Φ, propagation and interactions with the gauge fields and itself, and the fourth line displays the Yukawa terms: interactions between the Higgs field and fermions leading to quarks and charged leptons becoming massive after electroweak symmetry breaking (EWSB), which will be discussed in section 1.2.1.

If we were to write down a Lagrangian that contains only matter fields it would not be invariant under spacetime dependent transformations, a property that we require of the Standard Model Lagrangian. This is fixed with the addition of gauge bosons, including promoting partial derivatives to covariant derivatives which, for the Standard Model, are

$$
D_\mu \equiv \partial_\mu + i\frac{g_s}{\sqrt{2}}G^a_\mu t_a + i\frac{g}{2}W^I_\mu\sigma_I + ig'Y\mathbb{1} \tag{1.2.2}
$$

where G^a_μ and t^a are the $SU(3)_c$ gauge bosons and generators respectively. We use the normalisation $\mathrm{tr}(t^at^b) = \delta^{ab}$, $t^a = 1/\sqrt{2}\lambda^a$ where λ^a are the Gell-Mann matrices. W^I_μ and σ_I are the gauge bosons and generators of $SU(2)_L$, σ_I are the Pauli matrices, and $g' Q_Y \mathbb{1}$ is the $U(1)_Y$ hypercharge. The number of generators of an $SU(N)$ gauge group, in the adjoint representation, is equal to $N^2 - 1$ and each generator has an associated gauge boson. The $SU(N)$ generators, T^a, satisfy,

$$[T^a, T^b] = i f^{abc} T^c \,, \tag{1.2.3}$$

where f_{abc} are the structure constants of the group.

The field strength tensor, $F^{\mu\nu}$, contains terms describing the kinematics and self-interactions of the gauge bosons,

$$\begin{aligned} F^{\mu\nu} &= -\frac{1}{ig}[D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig \left[A_\mu, A_\nu\right] \end{aligned} \tag{1.2.4}$$

where $A^\mu = A^\mu_a T^a$ is a generic gauge field and A^μ_a are individual bosons. The presence of the commutator leads to gauge boson self-interactions in the non-Abelian SU(N) groups. For Abelian groups, such as $U(1)$, this commutator disappears, and there are no vector boson self-interactions.

The matter fields, fermions, denoted by Ψ in eq. (1.2.1), each transform differently under the different gauge groups. To understand how fermions transform under $SU(2)_L$ we must understand chirality: known as the left and right handed fields,

$$\Psi_L = P_L \Psi = \frac{1-\gamma_5}{2}\Psi \qquad \Psi_R = P_R \Psi = \frac{1+\gamma_5}{2}\Psi \tag{1.2.5}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $P_{L,R}$ are the left and right handed projectors. $\Psi_{L,R}$ are irreducible representations of the Lorentz group. The left-handed doublets, $q_L = (u_L, d_L)$ and $\ell_L = (\nu_\ell, \ell^-)$, transform under the fundamental representation of $SU(2)_L$, while the right-handed singlet fermions are uncharged under this group. The quarks (left and right handed) are charged under colour, while the leptons are uncharged. These representations are shown in table (1.1).

The last particle to discuss is the Higgs boson. The Higgs is a complex scalar field that transforms as a doublet under $SU(2)_L$, has a hypercharge of 1/2 and is

Matter Field	$SU(3)_c$	$SU(2)_L$	Y	Q
(ν_L, ℓ_L^-)	**1**	**2**	-1	(1/2,-1/2)
ℓ_R^-	**1**	**1**	-2	-1
(u_L, d_L)	**3**	**2**	1/3	(2/3,-1/3)
u_R	**3**	**1**	4/3	2/3
d_R	**3**	**1**	-2/3	-1/3

Table 1.1: The representation of each matter field in the Standard Model gauge groups, as well as the electromagnetic charge $Q = \frac{1}{2}Y + I_3$, where I_3 is the third component of the weak isospin.

uncharged under $SU(3)_c$,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} . \tag{1.2.6}$$

1.2.1 Electroweak Sector

Our description of the Standard Model gauge bosons, specifically the photon, $W^\pm$ and Z bosons, only exists after spontaneous symmetry breaking (SSB) has occurred. While the Standard Model Lagrangian is invariant under the full $SU(2)_L \times U(1)_Y$ gauge group, the ground state is not. This leads to SSB of the unified $SU(2)_L \times U(1)_Y$ to $U(1)_{\text{EM}}$,

$$SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{\text{EM}} . \tag{1.2.7}$$

Specifically, it is the presence of the Higgs boson that causes SSB. The Higgs potential, $V(\Phi)$, is explicitly

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2 . \tag{1.2.8}$$

The minimum is at

$$\Phi^\dagger \Phi = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} , \tag{1.2.9}$$

where $v = 1/\sqrt{\sqrt{2}G_F} = 246$ GeV is the vacuum expectation value (VEV) of the Higgs boson. Since $\Phi^\dagger \Phi \propto \phi_1 + \phi_2 + \phi_3 + \phi_4$, both $V(\Phi)$ and the minimum of $V(\Phi)$ are invariant under four-dimensional rotations, and therefore the solutions to eq. (1.2.9) exist on the surface of a 4D sphere,

$$\Phi = \frac{v}{\sqrt{2}} e^{i\theta}\,, \tag{1.2.10}$$

assuming $\mu^2 > 0$. It is this non-zero minimum value that causes spontaneous symmetry breaking, since there are infinite possible values for ground state configuration. When the system falls into one of these values the symmetry is broken.

To preserve the $U(1)$ invariance we must have $\phi^+ = 0$, however we choose $\theta = 0$ without loss of generality. We can then expand the Lagrangian in eq. (1.2.1) around the vacuum with

$$\Phi = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}\,, \tag{1.2.11}$$

which gives us the Higgs self-interactions

$$\mathcal{L}_h = \frac{1}{2}\partial_\mu h \partial^\mu h - \mu^2 h^2 - \lambda v h^3 - \frac{1}{4}\lambda h^4\,. \tag{1.2.12}$$

The mass of the Higgs boson is therefore $m_h \equiv \sqrt{2\lambda} v$.

Defining

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}\,, \tag{1.2.13}$$

we can study the terms in $D_\mu \Phi^\dagger D^\mu \Phi$ that are dependent on v,

$$\begin{aligned} D_\mu \langle\Phi\rangle^\dagger D^\mu \langle\Phi\rangle = \frac{v^2}{2}\Bigg(&\frac{g^2}{4}\left(W^1_\mu - iW^2_\mu\right)\left(W^{1\mu} + iW^{2\mu}\right) \\ &+ \left(\frac{g'}{2}B_\mu - \frac{g}{2}W^3_\mu\right)\left(\frac{g'}{2}B^\mu - \frac{g}{2}W^{3\mu}\right)\Bigg)\,. \end{aligned} \tag{1.2.14}$$

Redefining our gauge bosons to align with the mass basis,

$$D_\mu \langle\Phi\rangle^\dagger D^\mu \langle\Phi\rangle = \frac{v^2}{2}\left(\frac{g^2}{2}W^+_\mu W^{-\mu} + \frac{g^2}{4\cos^2\theta_w}Z_\mu Z^\mu\right)\,, \tag{1.2.15}$$

where the complex $W^\pm_\mu$ bosons are defined by

$$W^\pm_\mu \equiv \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu)\,, \tag{1.2.16}$$

and the Z_μ boson is defined by a combination of the W^3_μ, from $SU(2)_L$, and the

$U(1)_Y$ boson B_μ,

$$\frac{\sqrt{g^2+g'^2}}{2}Z_\mu \equiv \frac{g}{2}W^3_\mu - \frac{g'}{2}B_\mu \tag{1.2.17}$$

The one remaining degree of freedom is the photon, A_μ, which is defined by the combination orthogonal to the Z boson,

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}, \tag{1.2.18}$$

where θ_W is the weak angle, $\tan\theta_W = g'/g$. These terms give us masses for the $W^\pm$ and Z bosons [12]:

$$\begin{aligned} M_W &= \frac{gv}{2} = 80.370 \pm 0.012 \text{ GeV}\,, \\ M_Z &= \frac{gv}{2\cos\theta_W} = 91.1876 \pm 0.0021 \text{ GeV}. \end{aligned} \tag{1.2.19}$$

The photon remains massless, since it did not appear in eq. 1.2.15. Examining the Yukawa terms in eq. (1.2.1), for the first generation fermions,

$$\mathcal{L}_Y = -Y_{u^i}\bar{q}^i_L\tilde{\Phi}u^i_R - Y_{d^i}\bar{q}^i_L\Phi d^i_R - Y_{e^i}\bar{\ell}^i_L\Phi e^i_R + \text{h.c.} \tag{1.2.20}$$

where $\tilde{\Phi} = i\sigma_2\Phi^*$. After performing the expansion in eq. (1.2.11) we have terms that couple the fermions to the Higgs boson, and terms that give the fermions masses,

$$m_\Psi = \frac{vY_\Psi}{\sqrt{2}}. \tag{1.2.21}$$

There is no mass term in eq. (1.2.20) for the neutrino, since in the current formulation of the Standard Model there are no right-handed neutrinos.

1.2.2 Quantum Chromodynamics

While quarks are fundamental constituents of the Standard Model, they are only found in bound states known as hadrons, held together by the strong force. The theory describing the strong force is known as quantum chromodynamics (QCD). Every quark has a colour charge denoting its charge under the $SU(3)_c$ group: red, green or blue. The eight gluons, since they are in the adjoint representation, carry a dual colour charge.

Two very interesting properties of QCD are asymptotic freedom and colour confinement. Due to colour confinement, we will only ever detect what are known as "colourless states", or more specifically colour singlets. The most common of these hadrons are mesons (quark antiquark pair) and baryons (three quarks with different charges under $SU(3)$).

Asymptotic freedom can be seen by studying the QCD β-function. We can rewrite the definition of the β-function, eq. (1.1.14), in terms of $\alpha_s = g_s^2/4\pi$,

$$\beta(\alpha_s(\mu^2)) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = -\alpha_s \sum_{n=0}^{\infty} \beta_n (\frac{\alpha_s}{4\pi})^{n+2}\,, \tag{1.2.22}$$

and working to first order,

$$\beta_0^{\mathrm{QCD}} = 11 - \frac{2}{3} N_q. \tag{1.2.23}$$

Since $N_q = 6$, the QCD β-function is less than zero. The dependence of α_s on the energy scale μ^2 is, to lowest order,

$$\alpha_s(\mu^2) = \alpha_s(Q^2) \left(\frac{1}{1 + \beta_0 \frac{\alpha_s(Q^2)}{12\pi} \ln \frac{\mu^2}{Q^2}} \right)\,, \tag{1.2.24}$$

where $\alpha_s(Q^2)$ is a known value. Conventionally, α_s is measured at $Q^2 = M_Z^2$ [12],

$$\alpha_s(m_Z^2) \approx 0.1179. \tag{1.2.25}$$

The significance of $\beta_0^{QCD} > 0$ now becomes apparent. At low energies, below around 200 MeV, the size of α_s increases dramatically. Perturbation theory quickly becomes inapplicable in this region. As the energy scale increases, however, the QCD coupling strength decreases: known as asymptotic freedom. This allows us to use perturbation theory for the processes we study at particle colliders.

In a hadron collider such as the LHC, QCD is a huge source of radiation since at that scale it is the strongest of the four fundamental forces, and the initial hadronic state supplies plenty of particles with colour charges. Even at the energy of the Z

boson mass, α_s is large enough to make higher order corrections significant when studying the processes at current hadron colliders. The composite structure of hadrons causes the results from hadron colliders to be much “messier” than those from lepton colliders. However, due to the large amounts of energy lost by the light electrons through Bremsstrahlung radiation, it is easier to achieve higher centre of mass energies with protons.

1.3 Beyond the Standard Model

The Standard Model is an incredibly successful theory and has been extensively tested. However we know that it is not complete. As previously mentioned there are no mass terms for neutrinos, but the experimentally proven phenomenon of neutrino oscillations tells us that they must in fact have masses (see, for example, ref. [13]). Additionally, the Standard Model is currently inconsistent with general relativity and we do not yet have a description of dark matter, generally thought to be an as yet undiscovered particle or particles [14].

1.3.1 Supersymmetry

One popular extension to the Standard Model is based on the concept of supersymmetry (SUSY): a spacetime symmetry that maps bosons onto fermions and vice-versa, known as superpartners [15].

The popularity of SUSY stems from its potential ability to solve multiple issues within high energy physics alongside it being the most general interacting QFT for a theory containing a finite number of particles [16]. The simplest SUSY theories predict a supersymmetric partner for each Standard Model particle at the same mass, and with the same charges under the SM gauge group.

The Minimal Supersymmetric Standard Model (MSSM) is the supersymmetric extension of the Standard Model with the least number of additional particles and interactions. Experimental evidence tells us that the new superpartners must be much heavier than their Standard Model counterparts, and therefore the supersymmetry must be broken. In the MSSM this happens through soft supersymmetry breaking [17].

The MSSM requires an additional Higgs field, as well as the Higgs superpartners, to avoid gauge anomalies. This is described by the two Higgs doublet model:

$$\phi_1 = \begin{pmatrix} \phi_1^+ \\ \frac{h_1+v_1+a_1}{\sqrt{2}} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \phi_2 \\ \frac{h_2+v_2+a_2}{\sqrt{2}} \end{pmatrix}, \qquad \tan\beta = \frac{v_1}{v_2} \tag{1.3.1}$$

After soft SUSY breaking there are five physical particles: h^0, H^0, $A^0 = -a_1 \sin\beta + a_2 \cos\beta$ and two charged Higgs bosons $H^\pm$. Either h^0 or H^0 is identified as the previously discovered Higgs boson, and the Standard Model VEV is $v^2 = v_1^2 + v_2^2$.

1.4 Structure of this Thesis

In chapter 2 we will study methods for calculating amplitudes within QCD. These methods are applied to the processes $0 \to ggggh$, $0 \to q\bar{q}ggh$ and $0 \to q'\bar{q}'q\bar{q}$ to obtain compact analytic expressions. In chapter 3 we give results for the case where these processes are mediated by a coloured scalar and in chapter 4 mediated by a fermion. In chapter 5 we will use these analytic expressions to study the production of a Higgs boson and two jets at hadron colliders, focusing on distinguishing between the Standard Model and MSSM results.

Chapter 2

Amplitude Techniques for QCD

Scattering amplitudes are a basic building block of cross sections. While Feynman diagrams are an incredibly useful tool for calculating amplitudes, the number of diagrams can quickly become unmanageable. For tree-level processes with n external gluons, the number of diagrams scales as $n!$. This complexity can hide the simplicity of the final result. Computer programs can be used to evaluate each diagram, however these codes may be slow to run or be susceptible to numerical instabilities, for example due to spurious poles.

Possible solutions are to find methods of simplifying or reducing the diagrams that must be calculated, or to avoid Feynman diagrams completely.

Looking specifically at QCD, one of the most effective simplification techniques is colour decomposition. This involves separating the kinematic degrees of freedom of the amplitude from the colour factors,

$$\mathcal{A} = \sum_i C_i A_i \tag{2.0.1}$$

where $\mathcal{A}$ is the full amplitude, C_i the colour factors in terms of the $SU(3)_c$ generators t^a in the fundamental representation and A_i are the kinematic factors, known as the colour ordered sub-amplitudes. For the matrix element it is necessary to sum over the colours,

$$\sum_{\text{colours}} |\mathcal{A}|^2 = \sum_{\text{colours}} \sum_{i,j} A_i^\dagger C_i^\dagger C_j A_j = \sum_{i,j} A_i^\dagger \mathcal{C}_{ij} A_j \tag{2.0.2}$$

where $\mathcal{C}_{ij} = \sum_{\text{colours}} C_i^\dagger C_j$. There are common relations used to simplify these colour factors such as the Fierz identity:

$$t_{ij}^a t_{kl}^a = \left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl} \right) , \tag{2.0.3}$$

with the normalisation chosen such that $\text{tr}\left(t^a t^b\right) = \delta^{ab}$. The number of sub-amplitudes that need to be independently calculated can be reduced due to cyclic symmetry and by using various identities and relations, for example the photon decoupling identity, the Kleiss-Kuijf relations [18] and the BCJ identity [19].

We will now study some specific techniques in detail. Starting with spinor helicity notation in section 2.1 and then introducing BCFW recursion and generalised unitarity in section 2.2. In section 2.4 we will discuss momentum twistors and high-precision floating-point reconstruction as two methods for simplifying amplitude expressions.

2.1 Spinor Helicity

The helicity of a particle is defined as the projection of its spin onto its momentum,

$$h \equiv \frac{\vec{s} \cdot \vec{p}}{|\vec{s}|\,|\vec{p}|} \, . \tag{2.1.1}$$

For a massless particle the helicity states coincide with the chiral left and right handed particles defined in eq. (1.2.5). The Yukawa mass terms in the Standard Model, eq. (1.2.21), mix the left and right handed chiral states for massive particles, and so the equivalence of helicity and chirality no longer holds.

Kinematic factors can often be simplified by exchanging the 4-momentum p_i^μ for a smaller representation of the Lorentz group: helicity spinors. We will use the conventional notation for these helicity spinors, reflecting that in ref. [20], and introduced below.

2.1.1 Massless Particles

For an on-shell massless momentum,

$$p^{\dot{\alpha}\alpha} = p_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \tag{2.1.2}$$

with $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. The indices α and $\dot{\alpha}$ each transform in a fundamental representation of $SU(2)$ and are raised and lowered by the anti-symmetric Levi-Civita tensor, $\epsilon_{\alpha\beta}\lambda^\beta = \lambda_\alpha$. Since $\det(p^{\dot{\alpha}\alpha}) = p^2 = 0$, the matrix can be decomposed into bispinor form:

$$p^{\dot{\alpha}\alpha} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \, , \tag{2.1.3}$$

where λ and $\tilde{\lambda}$ are both two-component Weyl spinors. For real momentum p^μ, $\tilde{\lambda} = \pm\lambda^*$, however if p^μ is complex then λ and $\tilde{\lambda}$ are independent. One example of an explicit expression for λ and $\tilde{\lambda}$ is,

$$\lambda_{p\alpha} = \sqrt{2E} \begin{pmatrix} -e^{-i\phi} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}_p^{\dot{\alpha}} = \sqrt{2E} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix} \tag{2.1.4}$$

for $p^\mu = E\,(1, \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$.

The spinor bra-ket notation is a convenient and common way of representing the λ and $\tilde{\lambda}$ spinors:

$$\begin{aligned} &\lambda_p^\alpha \equiv |p\rangle_\alpha \, , \quad \tilde{\lambda}_p^\alpha \equiv |p]^{\dot{\alpha}} \\ \Rightarrow \quad & p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \equiv |p\rangle\, [p| \quad \text{and} \quad p^\mu \equiv \frac{1}{2}\langle p|\sigma^\mu|p] \, . \end{aligned} \tag{2.1.5}$$

This notation is particularly convenient for spinor products:

$$\langle ij \rangle = \lambda_i^\alpha \lambda_{j\alpha} \quad \text{and} \quad [ij] = \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_{j\dot{\alpha}} \, , \tag{2.1.6}$$

$$\Rightarrow \left(p_i + p_j\right)^2 = 2\, p_i \cdot p_j = -\langle ij \rangle\, [ij] \, , \tag{2.1.7}$$

where i and j label particles with momenta p_i and p_j respectively. It is worth noting that $[ii] = \langle ii \rangle = 0$. Additionally we define spinor sandwiches as,

$$\langle i|j|\ell] = \langle ij\rangle[j\ell] \text{ and } [i|j|\ell\rangle = [ij]\langle j\ell\rangle$$
$$\langle i|(j+k)|\ell] = \langle ij\rangle[j\ell] + \langle ik\rangle[k\ell] \text{ and } [i|(j+k)|\ell\rangle = [ij]\langle j\ell\rangle + [ik]\langle k\ell\rangle\,, \tag{2.1.8}$$

where j and k label particles with massless momenta. A useful relation for the simplification of amplitudes is the Schouten identity,

$$\langle ij\rangle\langle kl\rangle = \langle jk\rangle\langle li\rangle + \langle lj\rangle\langle ki\rangle\,, \tag{2.1.9}$$

for four massless 4-momenta $p_i, \ldots, p_l$. An equivalent expression holds for square brackets.

We are able to build polarisation vectors of gauge bosons from these spinors by solving the massless equation of motion, $p_\mu p_\nu A^\nu - p^2 A_\mu = 0$,

$$\varepsilon^\mu_+(p) = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle} \quad \text{and} \quad \varepsilon^\mu_-(p) = -\frac{1}{\sqrt{2}}\frac{[q|\bar{\sigma}^\mu|p\rangle}{[qp]}\,, \tag{2.1.10}$$

where q is a null reference vector that is not proportional to p. The final answer will be independent of q,

$$\begin{aligned}\varepsilon^\mu_+(p,q') - \varepsilon^\mu_+(p,q) &= \frac{1}{\sqrt{2}}\frac{\langle q'|\sigma^\mu|p]}{\langle q'p\rangle} - \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle} \\ &= \frac{1}{\sqrt{2}}\frac{\langle qp\rangle[p|\sigma^\mu|q'\rangle + \langle q|\sigma^\mu|p]\langle pq'\rangle}{\langle qp\rangle\,\langle q'p\rangle} \\ &= \frac{\sqrt{2}\,\langle qq'\rangle}{\langle qp\rangle\,\langle q'p\rangle}p^\mu\,,\end{aligned} \tag{2.1.11}$$

where we have used the anti-commutation property of Dirac gamma matrices, $\{\sigma^\mu, \sigma^\nu\} = 2g^{\mu\nu}$, or equivalently the Schouten identity defined in eq. (2.1.9).

2.1.2 Massive Particles

If the particle in question has a non-zero mass the matrix $p_{\alpha\dot{\alpha}}$ is no longer degenerate, since $\det(p_{\alpha\dot{\alpha}}) = m^2$, and so cannot be written in the form $\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$. It is, however, possible to expand it in terms of two degenerate matrices, following the notation of ref. [21],

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha{}^a \tilde{\lambda}_{\dot{\alpha}a}\,, \tag{2.1.12}$$

where the little group indices $a = 1, 2$ are raised and lowered by the antisymmetric ϵ_{ab}. This momentum decomposition leads to the two-dimensional version of the Dirac equation:

$$p^{\dot{\alpha}\alpha}\lambda^a_{p\alpha} = m\tilde{\lambda}^{\dot{\alpha}a}_p, \qquad p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}a}_p = m\lambda^a_{p\alpha}\,. \tag{2.1.13}$$

Switching to spinor bra-ket notation,

$$p_{\alpha\dot{\alpha}} = |p^a\rangle_\alpha\,[p_a|_{\dot{\alpha}}, \;\; \langle p^a p^b\rangle = -m\epsilon^{ab}, \;\; [p^a p^b] = m\epsilon^{ab}\,. \tag{2.1.14}$$

An explicit spinor realisation is,

$$\begin{aligned} \lambda^a_{p\alpha} &= \begin{pmatrix} \sqrt{E-P}\cos\frac{\theta}{2} & -\sqrt{E+p}e^{-i\varphi}\sin\frac{\theta}{2} \\ \sqrt{E-P}e^{i\varphi}\sin\frac{\theta}{2} & \sqrt{E+p}\cos\frac{\theta}{2} \end{pmatrix}, \\ \tilde{\lambda}^a_{p\dot{\alpha}} &= \begin{pmatrix} -\sqrt{E+P}e^{i\varphi}\sin\frac{\theta}{2} & -\sqrt{E-p}\cos\frac{\theta}{2} \\ \sqrt{E+P}\cos\frac{\theta}{2} & -\sqrt{E-p}e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}, \end{aligned} \tag{2.1.15}$$

for $p^\mu = (E, P\cos\varphi\sin\theta, P\sin\varphi\sin\theta, P\cos\theta)$. Spinor helicity notation is more natural for massless fermions, since they are readily described in terms of two dimensional Weyl spinors. Massive fermions, on the other hand, are generally described by Dirac spinors. It is therefore useful to be able to switch between the two notations:

$$u^a_p = \begin{pmatrix} \lambda^a_{p\alpha} \\ \tilde{\lambda}^{\dot{\alpha}a}_p \end{pmatrix} = \begin{pmatrix} |p^a\rangle_\alpha \\ |p^a]^{\dot{\alpha}} \end{pmatrix}, \tag{2.1.16}$$

$$\overline{u}^a_p = \begin{pmatrix} -\lambda^{\alpha a}_p \\ \tilde{\lambda}^a_{p\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} -\langle p^a|^\alpha \\ [p^a|_{\dot{\alpha}} \end{pmatrix}. \tag{2.1.17}$$

2.2 Constructing QCD Amplitudes

The simplest QCD amplitude is the tree-level three-point gluon amplitude. However, for three massless, on-shell particles all kinematic invariants vanish. There is therefore no way to construct a massless, non-trivial three-point amplitude with real momenta. If the momenta are complex, λ_i and $\tilde{\lambda}_i$ are independent, allowing us to construct non-physical but mathematically useful three-point gluon amplitudes.

Studying the kinematics of three point amplitudes with complex momenta, it is easy to show that if any $[ij] \neq 0$, then all $\langle ij \rangle = 0$. The amplitude is therefore composed of products of only λ_i (holomorphic) or $\tilde{\lambda}_i$ (anti-holomorphic). Additionally, any tree-level amplitude where all n gluons have the same helicity, or, for $n > 3$, only one is different, is equal to zero.

The simplest physical, non-trivial amplitudes are therefore ones where two gluons have a different helicity to the rest. Amplitudes where exactly two particles have negative helicity are known as maximally helicity violating (MHV) amplitudes, and those with exactly two positive helicity particles are anti-maximally helicity violating ($\overline{\text{MHV}}$).

These MHV and $\overline{\text{MHV}}$ amplitudes can be calculated for $n \geq 3$ gluons from the Parke-Taylor formulae [22],

$$A\left(1^+, ..., i^-, ..., j^-, ..., n^+\right) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle (n-1)\, n \rangle \langle n1 \rangle}\,, \tag{2.2.1}$$

$$A\left(1^-, ..., i^+, ..., j^+, ..., n^-\right) = \frac{[ij]^4}{[12]\,[23]\,...\,[(n-1)\, n]\,[n1]}\,. \tag{2.2.2}$$

The relation $p_{\alpha\dot{\alpha}} = -|p]_\alpha \langle p|_{\dot{\alpha}}$ is invariant under the scaling

$$|p\rangle \to t|p\rangle\,, \quad |p] \to t^{-1}|p]\,, \tag{2.2.3}$$

called little group scaling. When an amplitude contains massless particles only it can be completely described by angle and square brackets, and we can therefore determine that,

$$A_n(\{|1\rangle, |1], h_1\}, \ldots, \{t_i|i\rangle, t_i^{-1}|i], h_i\}, \ldots)$$
$$= t_i^{-2h_i} A_n(\{|1\rangle, |1], h_1\}, \ldots, \{|i\rangle, |i], h_i\}, \ldots)\,, \tag{2.2.4}$$

where h_i is the helicity of particle i. Three point amplitudes are completely fixed by this property. As already discussed, these amplitudes are either holomorphic or anti-holomorphic. Assuming an amplitude is holomorphic we have,

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = \langle 12\rangle^x \langle 23\rangle^y \langle 13\rangle^z\,, \tag{2.2.5}$$

and little group scaling fixes

$$x + z = -2h_1\,, \;\; x + y = -2h_2 \;\text{ and }\; y + z = -2h_3\,. \tag{2.2.6}$$

Studying the 3-gluon amplitude $A_3(1^+, 2^-, 3^-)$ as an example,

$$x = -1\,, \;\; y = 3 \;\text{ and }\; z = -1$$
$$\Rightarrow A_3(1^+, 2^-, 3^-) = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle}\,. \tag{2.2.7}$$

Had we begun by assuming that $A_3(1^+, 2^-, 3^-)$ was anti-holomorphic we would have found a solution with an incorrect mass dimension, which must therefore be unphysical. However, the same reasoning can be applied to amplitudes with one negative helicity and two positive helicity gluons to obtain the anti-holomorphic relations.

Moving to 4-particle amplitudes, the presence of an internal propagator requires us to consider locality in order to determine the structure of the amplitude. If an intermediate particle goes on-shell, the singularity structure of the amplitude is $1/p^2$, and in this limit the amplitude factorises into a product of two on-shell processes occurring at distinct points. This allows us to build higher point amplitudes from the lower point amplitudes we already know.

2.2.1 BCFW Recursion

Britto-Cachazo-Feng-Witten (BCFW) recursion uses lower point amplitudes to construct higher point ones [23, 24]. This method introduces fewer terms than results from Feynman diagrams, as it uses only on-shell particles, while Feynman diagram expansions contain off-shell propagators. BCFW recursion is additionally a useful tool in proving formulae for all n, such as the Parke-Taylor formulae, eqs. (2.2.1, 2.2.2).

If we want to calculate an n-point QCD amplitude A_n using BCFW recursion we first deform the momentum of two external particles: $i < n$ and n,

$$\begin{aligned} p_i &= \lambda_i\tilde{\lambda}_i \to \hat{p}_i(z) = \lambda_i\left(\tilde{\lambda}_i - z\tilde{\lambda}_n\right) \quad \text{and} \\ p_n &= \lambda_n\tilde{\lambda}_n \to \hat{p}_n(z) = \left(\lambda_n + z\lambda_i\right)\tilde{\lambda}_n, \end{aligned} \tag{2.2.8}$$

where z is a complex parameter, which allows us to construct non-trivial three point amplitudes. We denote this deformed amplitude as $\hat{A}_n(z)$. Choosing our deformations such that

$$\lim_{z\to\infty}\hat{A}_n(z) = 0 \tag{2.2.9}$$

then, according to Cauchy's theorem,

$$\oint_{z=\infty}\frac{\mathrm{d}z}{2\pi i}\frac{\hat{A}_n(z)}{z} = \sum_{\mathrm{Res}(z=z_0)}\frac{\hat{A}(z)}{z}dz + \hat{A}(z=0) = 0 \tag{2.2.10}$$

and the residues are, with $q = \lambda_i\tilde{\lambda}_n$,

$$\begin{aligned} \sum_{\mathrm{Res}(z=z_0)}\frac{\hat{A}(z)}{z}dz &= \sum_{i=3}^{n-1}\frac{1}{z_i}\frac{-1}{2p_i\cdot q}\sum_{s=\pm1}\hat{A}_L^{-s}(z_i)\hat{A}_R^{s}(z_i) \\ \Rightarrow\ A_n &= \sum_{i=3}^{n-1}\sum_{s=\pm1}\hat{A}_L^{-s}(z_i)\frac{1}{p_i^2}\hat{A}_R^{s}(z_i)\,, \end{aligned} \tag{2.2.11}$$

where s is the helicity of the internal particle linking the two amplitudes. The full amplitude can therefore be computed by summing over products of deformed, lower-point, on-shell amplitudes multiplied by undeformed propagators, i.e. where $z = 0$.

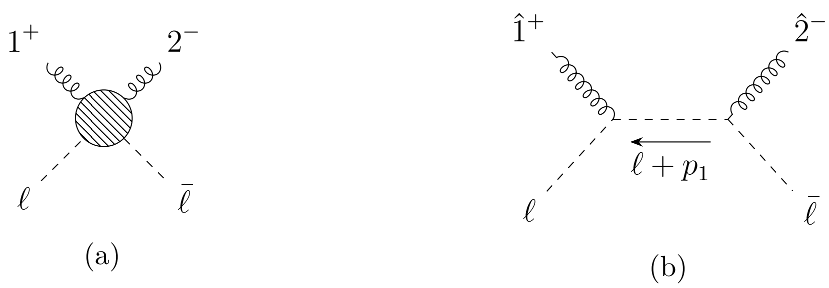


Figure 2.1: Diagrams for the BCFW calculation of two scalars (momenta ℓ and $\bar{\ell}$) plus one positive helicity gluon (1^+) and one negative helicity gluon (2^-). (a) illustrates the outgoing particles, while (b) shows the one diagram that contributes to the BCFW calculation. All momenta are outgoing.

2.2.1.1 BCFW calculation of two scalar, two gluon scattering

As an illustration of BCFW recursion we can calculate the colour-ordered subamplitude for the scattering of two gluons, one with positive helicity and one negative, and two scalars. The scalar-gluon coupling is defined in the Lagrangian in eq. (3.2.1) and the external particles are illustrated in fig. (2.1a). The standard Feynman diagram expansion contains three diagrams, however using BCFW recursion requires us to calculate only one diagram, shown in fig. (2.1b). We define the shifted momenta as,

$$\begin{aligned} |1\rangle &\to |\hat{1}\rangle = |1\rangle - z|2\rangle\,, \\ |2] &\to |\hat{2}] = |2] + z|1]\,, \end{aligned} \tag{2.2.12}$$

while $|\hat{1}] = |1]$ and $|\hat{2}\rangle = |2\rangle$. Applying eq. (2.2.11) to this example,

$$A_2^{\text{tree}}(\ell; 1^+, 2^-; \bar{\ell}) = \frac{\langle b_1|\ell|\hat{1}]}{\langle b_1 \hat{1}\rangle} \frac{1}{d(\ell_1)} \frac{\langle \hat{2}|\ell_1|b_2]}{[\hat{2} b_2]}\,, \tag{2.2.13}$$

where $d(\ell_1) = (\ell + p_1)^2 - m^2$ and m is the mass of the scalar particle. b_1 and b_2 are the polarisation vectors for gluons 1 and 2 respectively. We have used the three particle amplitudes given in eq. (3.4.1). Setting $b_1 = p_2$ and $b_2 = p_1$,

$$A_2^{\text{tree}}(\ell; 1^+, 2^-; \bar{\ell}) = \frac{\langle 2|\ell|1]}{\langle 2\hat{1}\rangle} \frac{1}{d(\ell_1)} \frac{\langle 2|\ell_1|1]}{[\hat{2}1]}\,. \tag{2.2.14}$$

The choice of polarisation vectors leads to a particularly quick simplification, since $\langle ii\rangle = [ii] = 0$,

$$A_2^{\rm tree}(\ell; 1^+, 2^-; \bar{\ell}) = -\frac{\langle 2|\ell|1]^2}{\langle 1|2|1]d(\ell_1)}\,. \qquad (2.2.15)$$

2.3 Generalised Unitarity

2.3.1 Scalar Integral Basis

An n-point, one loop amplitude can be written as a sum of scalar[1] basis integrals I_N^D with coefficients $C_N(D)$,

$$\mathcal{A}_{1-\rm loop} = \sum_{N=1}^{n}\sum_{K_N} C_{N;K_N}(D) I_N^D = \sum_{N=1}^{n}\sum_{K_N} C_{N;K_N}(4) I_N^D + R\,, \qquad (2.3.1)$$

where D is the number of dimensions and R is the rational part. The K_N are the sets of ordered external momenta grouping; defined by dividing the n external particles into N groups. The rational terms arise from non-cut constructible integrals, and contain no branch cuts in four dimensions [25]. An n-point integral is defined as cut constructible in four dimensions if

$$r < \max(n-1, 2)\,, \qquad (2.3.2)$$

where r is the rank of the integral [26]. Therefore, up to rank-3 pentagons, rank-2 boxes, rank-1 triangles and bubbles are cut-constructible.

We will use a common notation for these basis integrals, where pentagon integrals (I_5^D) are denoted by E_0, box integrals (I_4^D) by D_0, continuing for the triangle and bubble integrals. Additionally, we will use the lowercase letters to define the coefficients. We define the denominators of the integrals as follows,

$$d(\ell) = \ell^2 - m^2 + i\varepsilon\,. \qquad (2.3.3)$$

[1] the term “scalar” is due to there being no dependence in the numerator of the integrand on the loop momentum.

The notation for the momenta running through the propagators we use is,

$$\begin{aligned}\ell_i &= \ell + p_i = \ell + q_1 \\ \ell_{ij} &= \ell + p_i + p_j = \ell + q_2 \\ \ell_{ijk} &= \ell + p_i + p_j + p_k = \ell + q_3\end{aligned} \tag{2.3.4}$$

and so on for any number of subscripts. The q_i are the off-set momenta and the p_i are the external momenta. In terms of these denominators the scalar integrals are,

$$\begin{aligned}B_0(p_1;m) &= \frac{\bar{\mu}^{4-D}}{r_\Gamma}\frac{1}{i\pi^{D/2}}\int \mathrm{d}^D\ell\,\frac{1}{d(\ell)\,d(\ell_1)}\,,\\ C_0(p_1,p_2;m) &= \frac{1}{i\pi^2}\int \mathrm{d}^4\ell\,\frac{1}{d(\ell)\,d(\ell_1)\,d(\ell_{12})}\,,\\ D_0(p_1,p_2,p_3;m) &= \frac{1}{i\pi^2}\int \mathrm{d}^4\ell\,\frac{1}{d(\ell)\,d(\ell_1)\,d(\ell_{12})\,d(\ell_{123})}\,,\\ E_0(p_1,p_2,p_3,p_4;m) &= \frac{1}{i\pi^2}\int \mathrm{d}^4\ell\,\frac{1}{d(\ell)d(\ell_1)\,d(\ell_{12})\,d(\ell_{123})\,d(\ell_{1234})}\,.\end{aligned} \tag{2.3.5}$$

where $r_\Gamma = 1/\Gamma(1-\epsilon) + O(\epsilon^3)$, $\bar{\mu}$ is an arbitrary mass scale and ϵ is the dimensional regularisation parameter, $D = 4 - 2\epsilon$. These factors appear in the B_0 integral since it contains divergences and must therefore be regularised.

Though in general an n-point amplitude will contain integrals up to I_n^D, in four dimensions all one loop amplitudes can be written as a linear combination of scalar basis integrals with four or fewer external legs [27],

$$\mathcal{A}_{1-\text{loop}} = \sum_{N=1}^{4} C_N(4) I_N^D + R\,. \tag{2.3.6}$$

By reducing an amplitude to this basis we do not need to do complicated integrations, since these integrals are well known and can be evaluated using existing libraries [28–30]. Instead we must find only the coefficients and rational parts. These are all rational polynomials in terms of the kinematic variables (e.g. Mandlestam variables, masses and spinor products).

The reduction of an n-point amplitude to a sum of $N \leq 4$-point integrals arises due to the external momenta becoming linearly dependent in 4 dimensions for $n \geq 6$, and for $n = 5$ one of the external momenta can be eliminated by momentum conservation. N-point integrals can be recursively expressed in terms of $N-1$-point integrals down to $N = 5$. These $N = 5$ pentagon integrals can then be expressed as a sum of five $N = 4$ box integrals [31–33].

To demonstrate this reduction, we write the general integral I_N^D in the form,

$$I_N^D = I_{\text{red}} + I_{\text{fin}} = \int d\bar{\ell} \frac{\sum_{l=1}^N b_l \left(k_l^2 - m_l^2\right)}{\prod_{l=1}^N \left(k_l^2 - m_l^2 + i\delta\right)} + \int d\bar{\ell} \frac{\left[1 - \sum_{l=1}^N b_l \left(k_l^2 - m_l^2\right)\right]}{\prod_{l=1}^N \left(k_l^2 - m_l^2 + i\delta\right)}, \quad (2.3.7)$$

where we have made an ansatz that we can cancel denominators by writing propagators into the numerator with coefficients b_k. $k_l^\mu = \ell^\mu + r_l^\mu$ are the propagator momenta, ℓ^μ is the loop momentum and $d\bar{\ell} = d^D\ell / i\pi^{\frac{D}{2}}$. Performing Feynman parametrisation and the momentum shift $\tilde{\ell}^\mu = \ell^\mu + \sum_{i=1}^N z_i r_i$, where z_i are the Feynman parameters, on I_{fin},

$$I_{\text{fin}} = \Gamma(N) \int_0^\infty d^N z\, \delta \left(1 - \sum_{l=1}^N z_l\right) \int d\bar{\tilde{\ell}} \frac{\left[1 - \sum_{l=1}^N b_l \left(\tilde{q}_l^2 - m_l^2\right)\right]}{\left(\tilde{\ell}^2 - R^2\right)^N}, \quad (2.3.8)$$

with

$$R^2 = -\frac{1}{2} z \cdot S \cdot z - i\delta, \quad \tilde{q}_j^\mu = \tilde{\ell}^\mu + \sum_{i=1}^N (\delta_{ij} - z_i) r_i^\mu, \quad (2.3.9)$$

and

$$S_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2. \quad (2.3.10)$$

If $(S \cdot b)_j = 1$ is fulfilled for all $j = 1, \ldots, N$ then eq. (2.3.8) reduces to,

$$I_{\text{fin}} = -\Gamma(N) \left(\sum_{l=1}^N b_l\right) \int_0^\infty d^N z\, \delta \left(1 - \sum_{l=1}^N z_l\right) \int d\bar{\tilde{l}} \frac{l^2 + R^2}{\left(\tilde{l}^2 - R^2\right)^N}. \quad (2.3.11)$$

After loop integration

$$I_{\text{fin}} = -\left(\sum_{l=1}^N b_l\right) (N - D - 1) I_N^{D+2}. \quad (2.3.12)$$

We now have that $b_j = \sum_{i=1}^N S_{ij}^{-1}$ and therefore,

$$I_N^D(S) = \sum_{j=1}^N b_j I_{N-1}^D[j] - \left(\sum_{l=1}^N b_l\right) (N - D - 1) I_N^{D+2}, \quad (2.3.13)$$

where

$$I_{N-1}^{D}[j] = \int d\bar{\ell}\frac{\left(k_j^2 - m_j^2\right)}{\prod_{l=1}^{N}\left(k_l^2 - m_l^2 + i\delta\right)} \,. \tag{2.3.14}$$

We have required that $\det(S) \neq 0$, which holds for $N < 7$. In this thesis we will be studying 5-point amplitudes and can safely assume that S is invertible. We will use the notation $|S_{1\times2\times3\times4}|$ for the determinant of S, which is related to the Gram determinant, G,

$$\begin{aligned}16\;|S_{1\times2\times3\times4}| &= s_{12}\,s_{23}\,s_{34}\left(s_{14}\,s_{23} - (s_{12}+s_{13})\,(s_{24}+s_{34})\right) + m^2\,G\,,\\ G &= (s_{12}\,s_{34} - s_{13}\,s_{24} - s_{14}\,s_{23})^2 - 4\,s_{13}\,s_{14}\,s_{23}\,s_{24}\,.\end{aligned} \tag{2.3.15}$$

For pentagon integrals in $D = 4-2\epsilon$ dimensions the factor of $N-D-1$ in the second term of eq. (2.3.13) is $\mathcal{O}(\epsilon)$ and the scalar pentagon integrals are finite in $D = 6-2\epsilon$. When the limit $\epsilon \to 0$ is taken this term vanishes, and we are left with the pentagon integral being reduced to a sum over box integrals.

Returning to $D = 4-2\epsilon$ dimensions we can identify the terms which contribute to the rational coefficients [34]. Writing a general n-point one-loop amplitude in terms of scalar integrals with $N \leq 5$,

$$\mathcal{A}_{1-\text{loop}} = \sum_{N=1}^{n}\sum_{K_N} C_{N;K_N}(D) I_N^D \,, \tag{2.3.16}$$

and decomposing the loop momenta, ℓ^ν, into a four-dimensional component $\bar{\ell}^\nu$ plus the -2ϵ component,

$$\begin{aligned}\ell^\nu &= \bar{\ell}^\nu + \ell_\epsilon^\nu\,,\\ \Rightarrow \ell^2 &= \bar{\ell}^2 + \ell_\epsilon^\nu \equiv \bar{\ell}^2 - \mu^2\,,\end{aligned} \tag{2.3.17}$$

we can determine the dimensional dependence of the coefficients in terms of μ^2. A renormalisable theory requires n to be the maximum rank in an n-point tensor integral, so we have a maximum power of μ^4 in box integrals and μ^2 in both triangle and bubble integrals.

Separating the integrals into 4 and $D-4$ dimensions, and writing the $D-4$ dimensional integrals in terms of higher dimensional scalar integrals using [35, 36],

$$I_n^D\left[\mu^{2r}\right] = \frac{1}{2^r} I_n^{D+2r}[1] \prod_{k=0}^{r-1}(D-4+2k)\,, \tag{2.3.18}$$

we have a new basis of integrals with explicit dependence on the space-time dimension D,

$$\begin{aligned}\mathcal{A}_{1-\text{loop}} = & \frac{D-4}{2}\sum_{K_5}\hat{C}_{5;K_5} I_{5;K_5}^{D+2} + \sum_{K_4}\hat{C}_{4;K_4} I_{4;K_4}^{D} + \frac{D-4}{2}\sum_{K_4} C_{4;K_4}^{[2]} I_{4;K_4}^{D+2} \\ & + \frac{(D-4)(D-2)}{4}\sum_{K_4} C_{4;K_4}^{[4]} I_{4;K_4}^{D+4} + \sum_{K_3} C_{3;K_3} I_{3;K_3}^{D} \\ & + \frac{D-4}{2}\sum_{K_3} C_{3;K_3}^{[2]} I_{3;K_3}^{D+2} + \sum_{K_2} C_{2;K_2} I_{2;K_2}^{D} + \frac{D-4}{2}\sum_{K_2} C_{2;K_2}^{[2]} I_{2;K_2}^{D+2} + C_1 I_1^D\,,\end{aligned} \tag{2.3.19}$$

where

$$\begin{aligned}\hat{C}_{4;K_4} &= C_{4;K_4}^{[0]} + \sum_{i=1}^{5}\sum_{j} S_{ij}^{-1} C_{5;K_5^{(i)}}\,,\\ \hat{C}_{5;K_5} &= C_{5;K_5}\sum_{i,j} S_{ij}^{-1}\,,\end{aligned} \tag{2.3.20}$$

and $K_5^{(i)}$ represents the sets of momentum grouping obtained by "pinching" (see fig. (2.2)) one of the pentagon propagators.

The scalar pentagon and box integrals are finite in $6-2\epsilon$ dimensions, therefore, when we take $D \to 4-2\epsilon$ limit, the rational terms are given by the remaining integrals over μ,

$$I_4^{4-2\epsilon}\left[\mu^4\right] \overset{\epsilon\to 0}{\to} -\frac{1}{6}\,, \tag{2.3.21}$$

$$I_3^{4-2\epsilon}\left[\mu^2\right] \overset{\epsilon\to 0}{\to} -\frac{1}{2}\,, \tag{2.3.22}$$

$$I_2^{4-2\epsilon}\left[\mu^2\right] \overset{\epsilon\to 0}{\to} -\frac{1}{6}\left(s-3\left(m_1^2+m_2^2\right)\right)\,, \tag{2.3.23}$$

and the rational terms are [37–39],

$$R_n = -\frac{1}{6}\sum_{K_4} C_{4;K_4}^{[4]} - \frac{1}{2}\sum_{K_3} C_{3;K_3}^{[2]} - \frac{1}{6}\sum_{K_2}\left(K_2^2 - 3\left(m_1^2+m_2^2\right)\right) C_{2;K_2}^{[2]}\,. \tag{2.3.24}$$

2.3.2 Unitarity Cuts

Generalised unitarity is a method used to calculate the coefficients $C_N(D)$ of the scalar integrals in eq. (2.3.1) [26,40–44]. The name of this technique comes from the unitarity of the S-matrix, defined in eq. (1.1.11), which leads to

$$\begin{aligned} \mathbb{1} = S^\dagger S = (\mathbb{1} + iT^\dagger)(\mathbb{1} - iT) &= \mathbb{1} + T^\dagger T + i(T - T^\dagger) \\ \Rightarrow 2\,\mathrm{Im}(T) &= T^\dagger T\,. \end{aligned} \tag{2.3.25}$$

Sandwiching $T^\dagger T$ between initial and final states,

$$\langle f|T^\dagger T|i\rangle = \sum_{b_i} \langle f|T^\dagger|b_i\rangle \langle b_i|T|i\rangle\,, \tag{2.3.26}$$

where $\sum_{b_i} |b_i\rangle \langle b_i|$ is a sum over all possible on-shell states b_i. This is therefore a sum over all possible cuts of a loop amplitude, where a cut is a replacement of the off-shell propagator with an on-shell one,

$$\frac{i}{p^2 + i\varepsilon} \to 2\pi\delta^{(+)}(p^2). \tag{2.3.27}$$

Unitarity turns this process around and fuses on-shell tree amplitudes together to build the loop amplitude.

A quadruple cut will single out that specific box coefficient (see ref. [40] for details on quadruple cuts), however a triple cut will contain both the specified triangle coefficient and any box coefficients that share the propagators [45]. Double cuts then contain the bubble coefficient along with both triangle and box coefficients [46–48].

2.3.2.1 Box Diagrams

Box coefficients are straightforward to calculate. To begin with we express the loop momentum, without loss of generality as,

$$\ell^\nu = \alpha p_i^\nu + \beta p_j^\nu + \frac{\gamma}{2}\langle i|\sigma^\nu|j] + \frac{\delta}{2}\langle j|\sigma^\nu|i] + \ell_\epsilon^\nu\,, \tag{2.3.28}$$

where i and j are two external momenta.

Putting the four propagators on shell determines α, β, γ, δ and μ^2. On-shell cut conditions are quadratic in nature, and therefore there are two possible solutions for the loop momentum.

The box coefficient can then be calculated by taking the average of the results from these two solutions for ℓ^μ,

$$d = \frac{1}{2} \sum_{\sigma,s} A_1^{(0)} A_2^{(0)} A_3^{(0)} A_4^{(0)} \,, \tag{2.3.29}$$

where the $A_i^{(0)}$ are the tree amplitudes at each corner and the sum is over the two discrete solutions of the loop momenta $\ell_i^2 = m^2$, where m is the mass of the propagator, and all possible helicities, s, within the loop.

2.3.2.2 Triangle Diagrams

Triangle coefficients can be calculated using a method developed by Forde [45]. This uses a parametrisation of the loop momentum of the form,

$$\ell^\mu = t\, a_0^\mu + \frac{1}{t} a_1^\mu + a_2^\mu \,, \tag{2.3.30}$$

where t is the only free parameter in the cut integral after evaluating the three delta functions and a_i^μ are orthogonal null vectors. Compared to a box coefficient that contains the same three cuts, the fourth propagator will be inside the integrand, corresponding to a pole of the form $(\ell - q)^2$. In order to remove the contributions due to box coefficients we will isolate the terms containing these poles, and discard them. Here we will describe the method for massless internal propagators, for simplicity.

Taking the external outgoing momenta at each corner to be K_1, K_2 and K_3, we can expand the cut propagators as,

$$\ell_a^\mu = x_a K_1^{\flat,\mu} + y_a K_2^{\flat,\mu} + \frac{t}{2} \langle K_1^\flat | \sigma^\mu | K_2^\flat] + \frac{x_a y_a}{2t} \langle K_2^\flat | \sigma^\mu | K_1^\flat] \,, \tag{2.3.31}$$

where the “flat” momenta, $K_i^\flat$, are massless momenta that the external momenta can be expanded in terms of,

$$
\begin{aligned}
K_1 &= K_1^\flat + \frac{K_1^2}{\gamma} K_2^\flat \,, \\
K_2 &= K_2^\flat + \frac{K_2^2}{\gamma} K_1^\flat \,, \\
K_3 &= -\left(1 + \frac{K_2^2}{\gamma}\right) K_1^\flat - \left(1 + \frac{K_1^2}{\gamma}\right) K_2^\flat \,,
\end{aligned}
\tag{2.3.32}
$$

and where $\gamma = \langle K_1^\flat | K_2^\flat | K_1^\flat]$.

The on-shell conditions for the three propagators, $l_a^2 = 0$, give the constraints necessary to derive the x_a and y_a coefficients,

$$
\begin{aligned}
x_0 &= -\frac{K_2^2(\gamma + K_1^2)}{\gamma^2 - K_1^2 K_2^2} \,, \quad y_0 = \frac{K_1^2(\gamma + K_2^2)}{\gamma^2 - K_1^2 K_2^2} \,, \\
x_1 &= x_0 - 1 \,, \quad y_1 = y_0 - \frac{K_1^2}{\gamma} \,, \\
x_2 &= x_0 + \frac{K_2^2}{\gamma} \,, \quad y_2 = y_0 + 1 \,.
\end{aligned}
\tag{2.3.33}
$$

The pole due to the uncut propagator, when compared to a box coefficient, would have two solutions if it were to go on shell. This gives us the following expression for the triangle coefficient, after partial fractioning,

$$
\begin{aligned}
\int d^4 l \prod_{a=0}^{2} \delta\left(l_a^2\right) A_1 A_2 A_3 = \int dt J_t \left(\sum_{i=0}^{m} f_i t^i\right) \\
+ \int d^4 l \prod_{a=0}^{2} \delta\left(l_a^2\right) \left(\sum_{\text{poles } t_j} \frac{\text{Res}_{t=t_j} A_1 A_2 A_3}{t - t_j}\right) \,,
\end{aligned}
\tag{2.3.34}
$$

where J_t contains the Jacobian due to transforming from an integral over ℓ^μ to t and factors obtained from the integrals over the three delta functions. The poles in eq. (2.3.34) correspond exactly to the contributions from a triple-cut scalar box, and therefore can be discarded.

We are left with a polynomial in t. Using that $\langle K_1^{\flat,\pm} | \not{K}_1 | K_2^{\flat,\pm} \rangle$, $\langle K_1^{\flat,\pm} | \not{K}_2 | K_2^{\flat,\pm} \rangle$ and $\langle K_1^{\flat,\pm} | \gamma^\mu | K_2^{\flat,\pm} \rangle \langle K_1^{\flat,\pm} | \gamma_\mu | K_2^{\flat,\pm} \rangle$ are all equal to zero and an argument similar to that of Ossola, Papadopoulos and Pittau [49], we have that

$$
\int d^4 \ell \frac{\langle K_1^\flat | \not{\ell} | K_2^\flat]^n}{\ell_0^2 \ell_1^2 \ell_2^2} = 0 \,,
$$

$$\int d^4\ell \frac{\langle K_2^\flat|\not{\ell}|K_1^\flat]^n}{\ell_0^2\ell_1^2\ell_2^2} = 0\,. \tag{2.3.35}$$

Our parametrisation of the loop momentum therefore implies that all positive powers of t and all negative powers must integrate to 0. Therefore, only terms independent of t will contribute to the triangle coefficient.

2.3.2.3 Bubble Diagrams

These are more complicated than the box and triangle coefficients due to the dual cut not being sufficient to fully localise the cut integral, leaving a phase space integral that must be evaluated. In addition, we must remove the contributions from boxes and, in amplitudes with non-zero triple cuts, triangles that share the same cuts.

To calculate many of the bubble coefficients in this thesis we use a method developed by Mastrolia [46], based on Stokes' theorem. This method changes the phase space integration to an integration over the conjugate variables z and $\bar{z}$,

$$\frac{1}{2\pi i}\int\int_D dz d\bar{z}\frac{1}{(1+z\bar{z})^2}\,, \tag{2.3.36}$$

where the integration is over the whole complex plane D. We can then expand the loop momentum ℓ^μ as,

$$\ell^\mu = \frac{1}{1+z\bar{z}}(p^\mu + q^\mu + \frac{1}{2}z\langle q|\sigma^\mu|p] + \frac{1}{2}\bar{z}\langle p|\sigma^\mu|q])\,. \tag{2.3.37}$$

To perform the integration we can use the Generalised Cauchy formula,

$$\frac{1}{2\pi i}\int_L dz\frac{f(z)}{z-z_0} - \frac{1}{2\pi i}\int\int_D f_{\bar{z}}\frac{dzd\bar{z}}{z-z_0} = f(z_0), \quad z_0 \in D\,, \tag{2.3.38}$$

where L is the boundary of D and

$$f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}\,. \tag{2.3.39}$$

Taking a function f that vanishes on the boundary L, the first term in eq. (2.3.38) disappears and we are left with one term, which we can integrate by first doing an indefinite integral over $\bar{z}$ and then finding the residues in z. In general, we will be left with an expression that contains rational terms and terms containing logarithms. The double-cut of a two-point scalar function is rational, and since we want to isolate this contribution we can discard the logarithmic terms.

2.3.3 Higgs Plus Four Quarks in the Heavy Quark Limit

As an example of the unitarity method we will calculate some coefficients for the Higgs plus four parton processes. The Higgs boson does not couple directly to gluons, the process is mediated by a quark loop. The mass of the top quark is much greater than that of the other quarks and so will dominate this process, since the Higgs coupling is proportional to the quark mass. Taking the limit $m_t \to \infty$, valid when the quark mass is much heavier than other relevant scales, the loop can be replaced with an effective $gg \to h$ vertex,

$$\mathcal{L}_{\text{eff}} = \frac{g_s^2}{48\pi^2 v}\, h\, tr(G_{\mu\nu}G^{\mu\nu}) \tag{2.3.40}$$

where we define $G_{\mu\nu} \equiv F_{\mu\nu}$ as the field strength tensor for the gluon field. As an example of generalised unitarity we will compute some integral coefficients with massless propagators for $0 \to h\bar{q}q\bar{q}'q'$ in this effective field theory (EFT). The matching coefficients for this Lagrangian are known up to α_s^4 [50–57].

The EFT in eq. (2.3.40) has been used extensively in the study of Higgs boson production via gluon fusion. It has been used to calculate $gg \to h$ at N^{3}LO [58–64].

Higgs plus 1-jet is known at NNLO in the heavy top limit [65, 65–68] while Higgs plus 2-jet results are known at NLO [69, 70].

In order to improve the accuracy of these calculations as the transverse momentum of the final state particles increases beyond the mass of the top quark, terms of $\mathcal{O}(1/m_t)$ have been calculated [71–74].

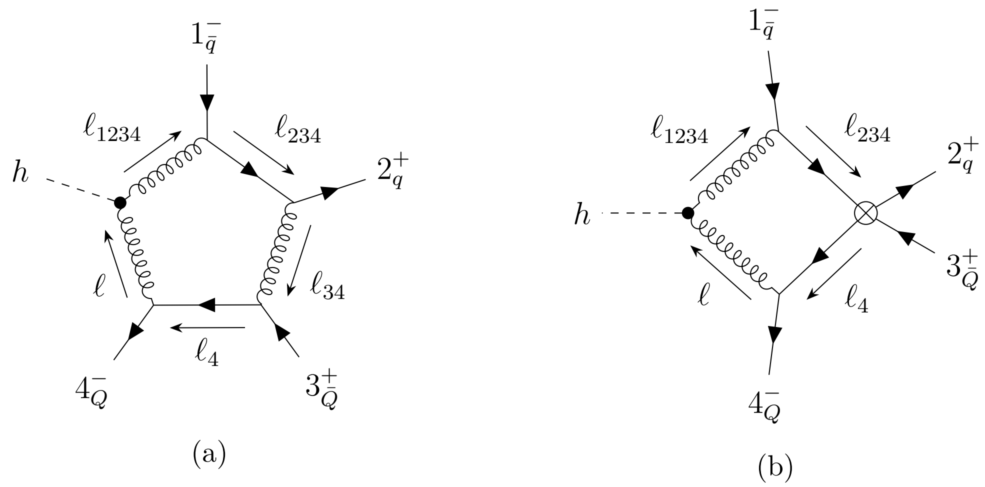


Figure 2.2: Diagrams for the calculation of $d^{\mathrm{EFT}}_{1\times23\times4}$ in $A_4\left(h, 1_{\bar{q}}^-, 2_q^+, 3_{\bar{Q}}^+, 4_{\bar{Q}}^-\right)$. (a) shows the full "uncut" diagram, where the hgg effective vertex is denoted by a solid circle. In (b) the pinched propagator has been reduced to an effective vertex. All external momenta are outgoing, and the notation is defined in eq. (2.3.4).

2.3.3.1 Two mass box coefficient in $A_4\left(h, 1_{\bar{q}}^-, 2_q^+, 3_{\bar{Q}}^+, 4_{\bar{Q}}^-\right)$

We will calculate the box coefficient with external legs $(1_{\bar{q}}^-, 2_q^+ + 3_{\bar{Q}}^+, 4_{\bar{Q}}^-, h)$, labelled as $d^{\mathrm{EFT}}_{1\times23\times4}$, for the amplitude $A_4\left(h, 1_{\bar{q}}^-, 2_q^+, 3_{\bar{Q}}^+, 4_{\bar{Q}}^-\right)$, following the results in ref. [75]. The diagram for this coefficient is shown in figure (2.2b). It has two massless external quark legs p_1^2, p_4^2 and two diagonally opposite massive legs m_h^2, s_{23}. There is only one helicity configuration in this case that is non-vanishing, since the amplitude with the Higgs and two gluons vanishes unless both gluons have the same helicity [76]. Additionally, the vertices containing legs 1 and 4 vanish for one of the loop momentum solutions, so eq. (2.3.29) reduces to

$$\begin{aligned} d^{\mathrm{EFT}}_{1\times23\times4} =& \frac{1}{2} A_2^{(0)}\left(h, \ell_{1234}^-, -\ell^-\right) A_3^{(0)}\left(-\ell_{1234}^+, 1_{\bar{q}}^-, \ell_{234}^+\right) \\ &\times A_4^{(0)}\left(-\ell_{234}^-, 2_q^+, 3_{\bar{Q}}^+, \ell_4^-\right) A_3^{(0)}\left(-\ell_4^+, 4_{\bar{Q}}^-, \ell^+\right) . \end{aligned} \tag{2.3.41}$$

Inserting expressions for these tree amplitudes,

$$d^{\mathrm{EFT}}_{1\times23\times4} = \frac{i^4}{2} \times \left(-\langle \ell_{1234}\,(-\ell)\rangle^2\right) \times \frac{[(-\ell_{1234})\,\ell_{234}]^2}{[1\ell_2]}$$

$$\times \left(-\frac{[23]^2}{[(-\ell_{234})\,2]\,[3\ell_4]} \right) \times \frac{[\ell\,(-\ell_4)]^2}{[(-\ell_4)\,4]}. \qquad (2.3.42)$$

The structure of 3-point kinematics requires that the positive chirality spinors appearing in the vertices containing legs 1 and 4 are proportional,

$$\lambda_{\ell_{1234}} \propto \lambda_{\ell_{234}} \propto \lambda_1, \quad \lambda_{\ell_4} \propto \lambda_\ell \propto \lambda_4 \,. \qquad (2.3.43)$$

These relations, along with momentum conservation, allow us to remove all explicit loop momenta,

$$d^{\mathrm{EFT}}_{1\times 23\times 4} = \frac{1}{2}\langle 4|(2+3)|1]\langle 1|(2+3)|4]\frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle}\,. \qquad (2.3.44)$$

2.3.3.2 Triangle Coefficients in $A_4\left(h, 1^-_{\bar{q}}, 2^+_q, 3^+_{\bar{Q}}, 4^-_Q\right)$

There are no scalar three-mass triangle coefficients in the heavy quark limit for the process $0 \to h\bar{q}q\bar{q}'q'$. This is due to the specific helicity configurations required for the tree level amplitudes to be non-vanishing. Each tree level amplitude must have two negative helicity particles, however with two external negative helicity particles it is not possible to fulfil this condition.

The two- and one-mass triangles contain single log terms at order $1/\epsilon$, and therefore the coefficients are completely determined by the known infra-red properties of the amplitude, and can be found in ref. [75].

2.3.3.3 Two Mass Bubble Coefficient in $A_4\left(h, 1^-_{\bar{q}}, 2^+_q, 3^+_{\bar{Q}}, 4^-_Q\right)$

There are several methods for the calculation of bubble coefficients, here we will write the cut loop momentum integral as an integral over the spinor variables $\ell \equiv |\ell\rangle$ and $\tilde{\ell} \equiv |\ell]$, a method proposed in refs. [41,42]. By transforming the integrand into a total derivative in $\tilde{\ell}$ we are left with a single integral over ℓ, which can then be solved using the residue theorem. This method enables us to neatly separate the box and triangle coefficients by using Schouten identities, eq. (2.1.9), to partial fraction the

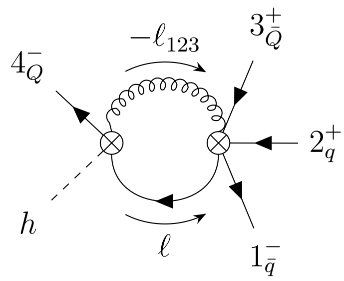


Figure 2.3: The two mass bubble coefficient, $b^{\rm EFT}_{123}$, in the $A_4\left(h, 1^-_{\bar{q}}, 2^+_q, 3^+_{\bar{Q}}, 4^-_Q\right)$ amplitude. The notation for the momenta is defined in eq. (2.3.4).

integrand. These terms must be discarded from the result for the bubble coefficient, but can serve as an independent check of the box coefficients.

In general the coefficient will be given by an integral of the form

$$\begin{aligned}
&\int d^4\ell_1 \delta^{(+)}(\ell_1^2)\delta^{(+)}(\ell_2^2) A_1^{(0)} A_2^{(0)} \\
&= \int_0^\infty dt\; t \int \langle \ell d\ell\rangle [\ell d\ell] \delta\left(P^2 - t\langle \ell|P|\ell]\right) f(\ell, \tilde{\ell}) \,,
\end{aligned} \tag{2.3.45}$$

where ℓ is whichever of the cut momenta ℓ_1, ℓ_2 is more convenient, P is the momentum flowing out of one of the bubble vertices and $f(\ell, \tilde{\ell})$ is a product of the tree amplitudes.

We can use this process to calculate $b^{\rm EFT}_{123}$ in the $A_4\left(h, 1^-_{\bar{q}}, 2^+_q, 3^+_{\bar{Q}}, 4^-_Q\right)$ amplitude, fig. (2.3), again following ref. [75],

$$\begin{aligned}
b^{\rm EFT}_{123} &= i \int {\rm dLIPS}\, A_3^{(0)}\left(h, -\ell^-_{123}, \ell^+, 4^-_Q\right) \times A_5^{(0)}\left(\ell^+_{123}, 1^-_{\bar{q}}, 2^+_q, 3^+_{\bar{Q}}, -\ell^-\right) \\
&= i^3 \int {\rm dLIPS} \left(-\frac{\langle(-\ell_{123})4\rangle^2}{\langle \ell 4\rangle}\right) \frac{\langle 1\,(-\ell)\rangle^3 \langle 23\rangle}{\langle 12\rangle\langle 23\rangle \langle 3\,(-\ell)\rangle \langle(-\ell)\,\ell_{123}\rangle \langle \ell_{123}1\rangle} \\
&= \int {\rm dLIPS} \frac{\langle \ell 1\rangle^3 \langle \ell_{123}4\rangle^2}{\langle 12\rangle \langle \ell 3\rangle \langle \ell\ell_{123}\rangle \langle \ell_{123}1\rangle \langle \ell 4\rangle}\,,
\end{aligned} \tag{2.3.46}$$

where dLIPS is the Lorentz invariant phase space. We can eliminate ℓ_{123} by multiplying the integrand by factors of $[\ell\,\ell_{123}]/[\ell\,\ell_{123}]$. Replacing dLIPS using eq. (2.3.45) and performing the integral over t,

$$\begin{aligned}
b^{\rm EFT}_{123} &= -\int \langle \ell {\rm d}\ell\rangle [\ell {\rm d}\ell] \frac{P^2 \langle \ell 1\rangle^3 \langle 4|P|\ell]^2}{\langle \ell|P|\ell]^3 \langle 12\rangle \langle \ell 3\rangle \langle 1|P|\ell] \langle \ell 4\rangle} \\
&= -\int \langle \ell d\ell\rangle [\ell d\ell] \left\{ \frac{P^2 \langle \ell 1\rangle \langle 1|P|\ell] \langle \ell 4\rangle}{\langle \ell|P|\ell]^3 \langle 12\rangle \langle \ell 3\rangle} - \frac{2P^2 \langle \ell 1\rangle \langle 14\rangle}{\langle \ell|P|\ell]^2 \langle 12\rangle \langle \ell 3\rangle}\right.
\end{aligned}$$

$$+\frac{P^2\langle\ell 1\rangle\langle 14\rangle^2}{\langle\ell|P|\ell]\langle 12\rangle\langle\ell 3\rangle\langle 1|P|\ell]\langle\ell 4\rangle}\Bigg\}\,, \tag{2.3.47}$$

where the second step involved the use of the Schouten identities, eq. (2.1.9). The term proportional to $1/\langle\ell|P|\ell]$ is the box coefficient to be discarded. The integral over $|\ell]$ can be transformed into a total derivative using the relation

$$[\ell d\ell]\left(\frac{[\eta\ell]^n}{\langle\ell|P|\ell]^{n+2}}\right)=[d\ell\partial_\ell]\left(\frac{1}{n+1}\frac{1}{\langle\ell|P|\eta]}\frac{[\eta\ell]^{n+1}}{\langle\ell|P|\ell]^{n+1}}\right) \tag{2.3.48}$$

and setting $|\eta]=|3]$ in the $n=0$ term,

$$b_{123}^{\mathrm{EFT}}=\int\langle\ell d\ell\rangle\,[d\ell\partial_\ell]\left\{\frac{1}{2}\frac{\langle 1|P|\ell]^2\langle\ell 4\rangle}{\langle\ell|P|\ell]^2\langle 12\rangle\langle\ell 3\rangle}+\frac{2P^2\langle\ell 1\rangle\langle 14\rangle[3\ell]}{\langle\ell|P|\ell]\langle 12\rangle\langle\ell 3\rangle\langle\ell|P|3]}\right\}. \tag{2.3.49}$$

This integral has simple poles for $|\ell\rangle=|3\rangle$ in the first term and $|\ell\rangle=P|3]$ in the second term, therefore,

$$b_{123}^{\mathrm{EFT}}=\left[\frac{1}{2}\frac{\langle 12\rangle[23]^2\langle 34\rangle}{(s_{123}-s_{12})^2}-2\frac{[23]\langle 14\rangle}{s_{123}-s_{12}}\right], \tag{2.3.50}$$

where we have used the notation

$$s_{ij\ldots n}=(p_i+p_j+\ldots+p_n)^2\,. \tag{2.3.51}$$

2.3.4 Unitarity with Massive Propagators

In order to calculate the full amplitude for any Higgs boson plus n-parton process we must have massive propagators. Techniques for generalised unitarity with massive internal propagators were pioneered in ref. [35] and comprehensive reviews can be found in refs. [20, 77]. Below we give an example calculation for a box coefficient in the Higgs plus four gluon amplitude with a massive scalar mediator.

2.3.4.1 Two Mass Box Coefficient in $A_4^{1234}(g^+,g^+,g^+,g^+;h)$

Here we will calculate the box coefficient $\tilde{d}_{1\times 2\times 34}$ with external legs $(1_g^+,2_g^+,3_g^++4_g^+,h)$ and a coloured triplet scalar particle propagating in the loop, described by the Lagrangian in eq. 3.2.1. As before, we begin by sewing together the tree amplitudes

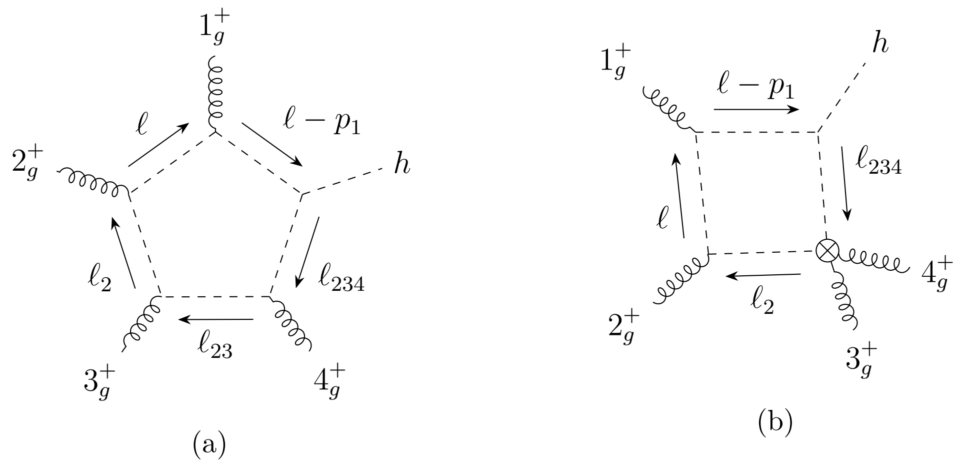


Figure 2.4: Diagrams for the calculation of $\tilde{d}_{1\times2\times34}$ in $A_4^{1234}(g^+, g^+, g^+, g^+; h)$. (a) shows the full "unpinched" diagram and in (b) the pinched propagator has been reduced to an effective vertex. The notation for the loop momenta is defined in eq. (2.3.4).

at each corner of the box,

$$\begin{aligned}\tilde{d}_{1\times2\times34} &= \frac{1}{2}\sum_{\sigma} -4\, A_3(-\ell, 1^+, \ell - p_1) A_3(-\ell_2, 2^+, \ell) A_4(-\ell_{234}, 4^+, 3^+, \ell_2) \\ &= \frac{1}{2}\sum_{\sigma} -4\, \frac{\langle 1|\ell|2]}{\langle 12\rangle} \frac{\langle 2|\ell|1]}{\langle 12\rangle} \frac{m^2[34]}{\langle 34\rangle d(\ell_{23})}\,, \end{aligned} \tag{2.3.52}$$

where the tree amplitudes are given in section 3.4 and the sum over σ is the sum over the two solutions for the loop momentum. We are working with colour stripped amplitudes, and have pulled out the factor of $-i\lambda/4$ due to the Higgs-scalar-scalar coupling. This is to be consistent with the results in chapter 3. We have set the reference vectors in the three-point amplitudes to p_2 for the amplitude containing 1^+ and p_1 for the 2^+ amplitude.

In order to remove the dependence on the loop momentum, ℓ, we can expand it as (c.f. eq. (2.3.28)),

$$\ell^\nu = \alpha p_1^\nu + \beta p_2^\nu + \frac{\gamma}{2}\langle 2|\sigma^\nu|1] + \frac{\delta}{2}\langle 1|\sigma^\nu|2]\ . \tag{2.3.53}$$

Using the requirement that the cut propagators are on-shell,

$$(\ell - p_1)^2 = m^2 \Rightarrow \alpha = 0,$$

$$(\ell + p_2)^2 = m^2 \Rightarrow \beta = 0,$$
$$\ell^2 = m^2 \Rightarrow \delta\gamma = -\frac{m^2}{s_{12}}, \tag{2.3.54}$$

and from the final cut propagator $\ell_{234} = m^0$,

$$\gamma^2 \langle 1|(3+4)|2] + \gamma\, s_{234} - \frac{m^2}{s_{12}} \langle 2|(3+4)|1] = 0\,. \tag{2.3.55}$$

This gives us our two solutions for the loop momentum, from the two solutions for γ. The simplest approach, avoiding square roots, is to consider products and sums of the two solutions,

$$\prod_{i=1,2} \gamma_i = -\frac{m^2 \langle 2|(3+4)|1]}{s_{12} \langle 1|(3+4)|2]}\,, \tag{2.3.56}$$
$$\sum_{i=1,2} \gamma_i = \frac{s_{234}}{\langle 1|(3+4)|2]}\,. \tag{2.3.57}$$

After expanding the loop momenta in the numerator of eq. (2.3.52),

$$\tilde{d}_{1\times 2\times 34} = \frac{1}{2} \sum_{\gamma_i} -4m^4 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \frac{1}{d_i(\ell_{23})}\,, \tag{2.3.58}$$

we can identify the effective pentagon coefficient as,

$$\hat{e}_{\{1\times 2\times 3\times 4\}} = -4m^4 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,, \tag{2.3.59}$$

which can also be obtained by calculating the pentagon coefficient in d dimensions and taking the $\mu^2 \to 0$ limit. The denominator factor $d_i(\ell_{23})$ depends on γ_i,

$$d_i(\ell_{23}) = \ell_{i,23} - m^2 = \langle 1|3|2]\,\gamma_i + s_{23} - \frac{m^2 \langle 2|3|1]}{s_{12}\gamma_i}\,, \tag{2.3.60}$$

and expanding the sum over σ leaves us with the expression,

$$\frac{d_1(\ell_{23}) + d_2(\ell_{23})}{d_1(\ell_{23}) d_2(\ell_{23})}\,, \tag{2.3.61}$$

which can be simplified using the relations,

$$[32]\,\langle 2|(3+4)|1]\,\langle 1\,3\rangle = s_{23}s_{13} + [3\,2]\,\langle 2\,4\rangle\,[4\,1]\,\langle 1\,3\rangle\;, \tag{2.3.62}$$

and,

$$[3\,2]\,\langle 2\,4\rangle\,[4\,1]\,\langle 1\,3\rangle = \not{p}_3 P_L\,\not{p}_2\not{p}_4\not{p}_1\,, \tag{2.3.63}$$

where P_L is the left-handed projector defined in eq. (1.2.5). We can then use the relation,

$$\mathrm{tr}\{P_L\,\not{p}_1\not{p}_2\not{p}_3\not{p}_4\} = \langle 1\,2\rangle\,[2\,3]\,\langle 3\,4\rangle\,[4\,1]\ , \tag{2.3.64}$$

to arrive at the final result,

$$\tilde{d}_{1\times 2\times 34}(1^+,2^+,3^+,4^+) = \mathcal{C}^{(4)}_{1\times 2\times 3\times 4}\,\tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^+\}} \tag{2.3.65}$$

where $\mathcal{C}^{(4)}_{1\times 2\times 3\times 4}$ is the relevant coefficient obtained from the reduction of the pentagon integral to box integrals,

$$\begin{aligned}\mathcal{C}^{(4)}_{1\times 2\times 3\times 4} = -\frac{1}{2}\,\frac{s_{12}}{16\,|S_{1\times 2\times 3\times 4}|}\,&[s_{1234}\,s_{23}\,(s_{234}-2\,s_{34}) + s_{234}\,(s_{12}\,(s_{234}-s_{23})\\ &+ s_{34}\,(s_{123}+s_{23}) - s_{234}\,s_{123})]\,,\end{aligned} \tag{2.3.66}$$

where $|S_{1\times 2\times 3\times 4}|$ is defined in eq. (2.3.15).

2.3.4.2 Two Mass Bubble Coefficient in $A_4^{1234}(g^+,g^+,g^+,g^-;h)$

In this example we will calculate the two mass bubble coefficient $\tilde{b}_{34}$ with external legs $(3_g^+ + 4_g^+, 1_g^+ + 2_g^+ + h)$, illustrated in fig. (2.5). We will calculate this with a massless coloured triplet scalar propagating in the loop, though the final result is identical to the case with a massive scalar, or with a massive fermion, as will be explained in section 3.3.

Here we give an example of the method developed by Mastrolia, described above, for the bubble in fig. (2.5),

$$\begin{aligned}\tilde{b}_{34} &= \int \frac{d\bar{z}}{(1+z\bar{z})^2} A(1_g^+,2_g^+,h)A(3_g^+,4_g^+)\,,\\ &= \frac{4}{s_{34}\,\langle 1\,2\rangle^2}\int \frac{d\bar{z}}{(1+z\bar{z})^2}\,\frac{\langle 4\,\ell\rangle\,[3\,\ell]^2}{[4\,\ell]}\left\{1 - \frac{\langle 1(\ell-3-4)\rangle\,\langle 2\,\ell\rangle\rangle}{\langle 1\,\ell\rangle\,\langle 2(\ell-3-4)\rangle}\right\}\,,\end{aligned} \tag{2.3.67}$$

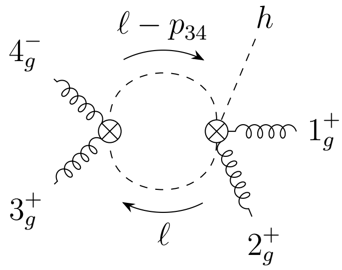


Figure 2.5: The two mass bubble coefficient, $\tilde{b}_{34}$, in the $A_4^{1234}(g^+, g^+, g^+, g^-; h)$ amplitude.

where the tree diagrams are given in section 3.4 with $m = 0$, and we have again pulled out a factor of $-\lambda/4$. We expand ℓ^μ as,

$$\ell^\mu = \frac{1}{1 + z\bar{z}}(p_3^\mu + p_4^\mu + \frac{1}{2}z \langle 4|\sigma^\mu|3] + \frac{1}{2}\bar{z} \langle 3|\sigma^\mu|4]) \,, \tag{2.3.68}$$

and perform some scaling to remove an overall factor,

$$\langle \ell | = \sqrt{t}\langle \lambda | \,, \quad |\ell] = \sqrt{t}|\lambda] \,, \tag{2.3.69}$$

where $t = 1/(1 + z\bar{z})$. We then have,

$$\tilde{b}_{34} = 4 \int \frac{d\bar{z}}{(1 + z\bar{z})^3} \frac{\bar{z}^2}{\langle 1\,2\rangle^2} \Bigg\{ -1 + \frac{1}{(\bar{z} - \langle 2\,4\rangle / \langle 2\,3\rangle)(z + \langle 1\,3\rangle / \langle 1\,4\rangle)} \left(-1 + z\bar{z}\frac{\langle 1\,3\rangle \langle 2\,4\rangle}{\langle 1\,4\rangle \langle 2\,3\rangle} + \bar{z}\frac{\langle 1\,3\rangle}{\langle 1\,4\rangle} - z\frac{\langle 2\,4\rangle}{\langle 2\,3\rangle} \right) \Bigg\} . \tag{2.3.70}$$

We then need to perform the integration over $\bar{z}$. By using algebraic manipulation such as,

$$\frac{\bar{z}}{(1 + z\bar{z})^3} = \frac{(1 + z\bar{z}) - 1}{z(1 + z\bar{z})^3} \tag{2.3.71}$$

and

$$\bar{z}^n = \left(\bar{z} - \langle 2\,4\rangle \langle 2\,3\rangle + \frac{\langle 2\,4\rangle}{\langle 2\,3\rangle} \right) \bar{z}^{n-1} \,, \tag{2.3.72}$$

where $n \geq 1$, the integration over $\bar{z}$ is then fairly straightforward. The next step is to find the terms which come from the bubble coefficient only, that is those that do not contain logarithms. To do this we calculate the residues with respect to z, for example our solution so far contains terms such as,

$$\left(z + \frac{\langle 1\,3\rangle}{\langle 1\,4\rangle}\right)^{-1} \left(\frac{\langle 1\,3\rangle}{\langle 1\,4\rangle}\bar{z} - 1\right)^{-2}, \tag{2.3.73}$$

and therefore the pole is at,

$$z = -\frac{\langle 1\,3\rangle}{\langle 1\,4\rangle}, \quad \bar{z} = -\frac{[1\,3]}{[1\,4]}. \tag{2.3.74}$$

The result for eq. (2.3.73) is therefore,

$$\frac{s_{14}^2}{\langle 1|(3+4)|1]^2}. \tag{2.3.75}$$

After finding the residues of the full calculation, and applying the Schouten identity (eq. (2.1.9)) to terms containing $\langle 1\,3\rangle\,\langle 2\,4\rangle$, we arrive at the final result,

$$\tilde{b}_{34}(1^+,2^+,3^+,4^-) = \frac{4}{\langle 1\,2\rangle^2\,\langle 1\,3\rangle\,\langle 2\,3\rangle}\left(\frac{\langle 2\,4\rangle^2\,\langle 1\,3\rangle\,[2\,3]}{(s_{23}+s_{24})} - \frac{\langle 1\,4\rangle^2\,\langle 2\,3\rangle\,[1\,3]}{(s_{13}+s_{14})}\right) \tag{2.3.76}$$

2.4 Further Simplification Techniques

There are two additional techniques used in obtaining the results presented in chapters 3 and 4: momentum twistors [78–81] and high-precision floating-point arithmetic [82] to simplify the analytic expressions for scalar integral coefficients.

2.4.1 Momentum Twistors

Momentum twistors can simplify kinematic results by intrinsically satisfying momentum conservation and Schouten identities. These are built in part from dual momentum coordinates [78],

$$x_i^\mu = x_0^\mu + \sum_{k=i}^{i} p_k^\mu, \tag{2.4.1}$$

where x_0^μ is a fixed point and $\sum_{i=1}^{N} p_i = 0$. Each particle is described by a four-component momentum twistor $Z_{iA}(\lambda_{i\alpha}, \mu_i^{\dot{\alpha}})$, where $\lambda_{i\alpha}$ is the two component holomorphic Weyl spinor (eq. (2.1.3)) and $\mu_i^{\dot\alpha}$ is defined as

$$\mu_i^{\dot\alpha} = \lambda_{i\alpha}(\sigma \cdot x_i)^{\alpha\dot\alpha} \,, \tag{2.4.2}$$

so we use $\mu_i^{\dot\alpha}$ instead of $\tilde\lambda_i^{\dot\alpha}$ in our definitions of the kinematics. The dual twistor is

$$W_i^A = \left(\tilde\mu_{i\alpha}, \tilde\lambda_i^{\dot\alpha}\right) = \frac{\varepsilon^{ABCD} Z_{(i-1)B} Z_{iC} Z_{(i+1)D}}{\langle i-1 i\rangle\langle i i+1\rangle} \,, \tag{2.4.3}$$

which leads to a definition for the anti-holomorphic spinor that satisfies momentum conservation,

$$\tilde\lambda_i = \frac{\langle i\,(i+1)\rangle\,\mu_{i-1} + \langle (i+1)\,(i-1)\rangle\,\mu_i + \langle (i-1)\,i\rangle\,\mu_{i+1}}{\langle i\,(i+1)\rangle\,\langle (i-1)\,i\rangle} \,. \tag{2.4.4}$$

An n particle scattering amplitude therefore has $4n$ momentum twistor components, however only $3n-10$ are independent. This is due to an overall Poincaré symmetry and a $U(1)$ symmetry for each particle.

For chapters 3 and 4 we will be using 6-point kinematics, so we require 8 momentum-twistor variables $(x_1, \ldots, x_8)$. We have chosen to parameterise these variables as,

$$\begin{aligned} Z &= \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_5 x_6 & x_6 & 1 \\ 0 & 0 & 1 & 1 & 1-\left(\frac{1-x_7}{x_2 x_5}\right) & -\left(\frac{1+x_8}{x_2 x_5 x_6}\right) \end{pmatrix} \,, \end{aligned} \tag{2.4.5}$$

where $y_i = \sum_{j=1}^{i} \Pi_{k=1}^{j} 1/x_k$. The spinors involving particles 1 to 4 become,

$$\begin{aligned} &\langle 1\,2\rangle = -1,\ \langle 1\,3\rangle = -1,\ \langle 1\,4\rangle = -1, \\ &\langle 2\,3\rangle = 1/x_1,\ \langle 2\,4\rangle = \frac{1+x_2}{x_1 x_2},\ \langle 3\,4\rangle = \frac{1}{x_1 x_2} \,, \end{aligned} \tag{2.4.6}$$

for the λ_i and

$$[1\,2] = x_1,\ \ [1\,3] = x_1 x_8,\ \ [1\,4] = -\frac{x_1(x_7+x_8)}{x_5},$$

$$[2\,3] = -\,x_1^2 x_2 x_5 x_6,\ \ [2\,4] = x_1^2 x_2 x_6,\ \ [3\,4] = -x_1^2 x_2 x_6 x_7\,, \tag{2.4.7}$$

for the $\tilde{\lambda}_i$. We can invert the above equations to obtain

$$\begin{aligned}
&x_1 = -\,\langle 1\,2\rangle\,[1\,2]\,,\ x_2 = +\frac{\langle 2\,3\rangle\,\langle 1\,4\rangle}{\langle 1\,2\rangle\,\langle 3\,4\rangle},\ x_3 = +\frac{\langle 3\,4\rangle\,\langle 1\,5\rangle}{\langle 1\,3\rangle\,\langle 4\,5\rangle},\\
&x_4 = -\frac{\langle 4\,5\rangle\,\langle 1\,6\rangle}{\langle 1\,4\rangle\,\langle 5\,6\rangle},\ x_5 = -\frac{\langle 1\,3\rangle\,[2\,3]}{\langle 1\,4\rangle\,[2\,4]},\ x_6 = -\frac{\langle 3\,4\rangle\,[2\,4]}{\langle 1\,3\rangle\,[1\,2]},\\
&x_7 = -\frac{\langle 1\,3\rangle\,[3\,4]}{\langle 1\,2\rangle\,[2\,4]},\ x_8 = +\frac{\langle 1\,3\rangle\,[1\,3]}{\langle 1\,2\rangle\,[1\,2]}\,.
\end{aligned} \tag{2.4.8}$$

The parametrisation of momentum-twistor variables is not unique; we have chosen a parametrisation where the momenta p_5 and p_6 appear only in the formulae for x_3 and x_4, and so these effectively decouple, similar to the method used in ref. [81].

The amplitude in question can now be simplified using simple algebra without having to account for momentum conservation or Schouten identities. This especially helps with identifying overall factors and denominator structures.

2.4.2 High-Precision Floating-Point Reconstruction

When lengthy expressions are hard to treat using momentum twistors variables, high-precision floating-point arithmetic can be used to study the singularity structure of integral coefficients, explained in detail in ref. [82]. After studying the singular and doubly singular limits in complex phase space, the integral coefficients can be reconstructed by solving linear systems for the rational coefficients of generic spinor trial functions. It is a useful method of removing square roots and massless projections of non-lightlike momenta that can occur as remnants of loop momentum parametrisation [45, 49]. In chapters 3 and 4 this method is particularly useful for some triangle and bubble coefficients.

Chapter 3

Higgs Boson Plus Two Jets with a Scalar Mediator

3.1 Introduction

Since the discovery of the Higgs boson at the LHC [83, 84], one of the main goals has been to study its properties in detail. On the theoretical side, this involves doing higher order calculations to keep up with the ever improving experimental precision. The dominant production mechanism for the Higgs at the LHC is gluon fusion, $gg \to h$. Key components of next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) calculations involve the processes $gg \to hg$ and $gg \to hgg$. These gluon fusion processes do not occur at tree level in the full theory, they are all mediated by a loop of massive, coloured particles. In the Standard Model these particles are the quarks, and since the coupling between quarks and the Higgs is proportional to the mass of the quark, this process is dominated by the top quark.

The $m_t \to \infty$ effective field theory defined in eq. (2.3.40) holds when the p_T of the final state gluons is less than the mass of the top quark, since the wavelength is too long to resolve the loop. This EFT has been successfully used to calculate $gg \to h$ up to NNNLO [58,61] and $gg \to hg$ to NLO [69,70]. As the energy and luminosity of the

LHC increases we are beginning to probe regions where the EFT breaks down [85]. This offers us the opportunity to learn about the loop propagators, but requires a calculation retaining the full dependence on the propagator mass.

These processes can be calculated numerically by several automatic procedures [70, 86–88], however a numerical calculation may be unstable in certain areas of the parameter space. Finding compact analytic expressions is more challenging, however they are generally more stable and quicker to compute.

Though obtaining compact analytic expressions is not always straightforward, we can exploit the relationship between the Higgs plus n parton processes with a fermion propagating in the loop and those with a scalar mediator. Scalar processes do not require the Dirac algebra present in fermion processes, therefore where these two theories will give the same result we can instantly simplify our expressions by calculating through the scalar theory. When the results are not equivalent, the difference is simpler than the full fermion result, as will be shown in section 3.3.

In addition, a full calculation of the Higgs plus four parton processes with a scalar mediator is an important tool in the search for evidence of new physics at hadron colliders. Many beyond the Standard Model theories include colored-triplet scalars, for example the quark superpartners in SUSY theories.

In this chapter we will give compact analytic expressions for the Higgs plus four parton processes mediated by a coloured triplet scalar in the full theory. These results were published in ref. [2], and are given assuming all particles are outgoing. Section 3.2 outlines the structure of the calculation and in section 3.3 we study the relationship between the scalar- and fermion-mediated theories. Section 3.4 contains the tree-level amplitudes used in our calculations and sections 3.5-3.7 contain the integral coefficients for the Higgs plus four parton processes. We conclude in section 3.8. Appendix A.1 contains numerical values for each coefficient, along with the full amplitudes at a given phase-space point. This is intended as an aid for an independent implementation of the formulae given here.

3.2 Structure of the Calculation

3.2.1 Coloured Triplet Scalar Theory

The production of a Higgs boson from gluon fusion mediated by a massive complex scalar field ϕ requires that scalar to transform as a triplet under $SU(3)_c$, i.e. be charged under colour. The Lagrangian for this field then reads,

$$\mathcal{L} = (D^\mu \phi_i^\dagger)(D_\mu \phi_i) - \lambda_{h\phi\phi} \phi_i^\dagger \phi_i h, \quad D_\mu \phi_i = \partial_\mu \phi_i + i \frac{g_s}{\sqrt{2}} (G_\mu^a t_a)_{ij} \phi_j \,, \tag{3.2.1}$$

where the coupling of the Higgs boson to the scalar field is denoted by $\lambda_{h\phi\phi}$. The $SU(3)$ generators $t^a = \lambda^a/\sqrt{2}$ are normalised such that

$$\mathrm{tr}(t^a t^b) \;=\; \delta^{ab}. \tag{3.2.2}$$

3.2.2 Decomposition to Scalar Integrals

The $0 \to hgggg$, $0 \to hgg\bar{q}q$ and $0 \to h\bar{q}q\bar{q}'q'$ amplitudes can all be expressed as a sum of colour ordered sub-amplitudes $A_n^{\{c_i\}}$, which can in turn be decomposed into scalar integrals, as discussed in section 2.3,

$$\begin{aligned}\mathcal{A}_n^{gggg}(\{p_i, h_i, c_i\}) =& \, i \frac{g_s^n}{16\pi^2} \frac{m^2}{v} \sum_{\{1,2,\ldots,n\}'} \mathrm{tr}\,(t^{c_1} t^{c_2} \ldots t^{c_n}) \\ & \times A_n^{\{c_i\}}(1^{h_1}, 2^{h_2}, \ldots n^{h_n}; h)\,,\end{aligned} \tag{3.2.3}$$

where m is the mass of the particle circulating in the loop. The sum, $\sum_{\{1,2,\ldots,n\}'}$, is over all $(n-1)!$ non-cyclic permutations of $1, 2, \ldots, n$.

Explicitly for the $0 \to hgggg$ case, eq. (3.2.3) becomes,

$$\begin{aligned}\mathcal{A}_4^{gggg}(\{p_i, h_i, c_i\}) =& \, i \frac{g_s^4}{16\pi^2} \left(\frac{m^2}{v}\right) \Bigg\{ \\ & \Big[\mathrm{tr}\,(t^{c_1} t^{c_2} t^{c_3} t^{c_4}) + \mathrm{tr}\,(t^{c_1} t^{c_4} t^{c_3} t^{c_2})\Big] A_4^{1234}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}; h) \\ & + \Big[\mathrm{tr}\,(t^{c_1} t^{c_3} t^{c_4} t^{c_2}) + \mathrm{tr}\,(t^{c_1} t^{c_2} t^{c_4} t^{c_3})\Big] A_4^{1342}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}; h)\end{aligned}$$

$$+ \left[\mathrm{tr}\,(t^{c_1}t^{c_4}t^{c_2}t^{c_3}) + \mathrm{tr}\,(t^{c_1}t^{c_3}t^{c_2}t^{c_4})\right] A_4^{1423}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}; h) \Bigg\} . \tag{3.2.4}$$

Squaring the amplitude eq. (3.2.4) and performing the sum over colours, for any one specific helicity configuration,

$$\sum_{\text{colours}} |\mathcal{A}_4^{gggg}|^2 = \left[\frac{g_s^4}{16\pi^2}\left(\frac{m^2}{v}\right)\right]^2 (N^2-1) \Bigg\{ 2N^2 \Big(\left|A_4^{1234}\right|^2 + \left|A_4^{1342}\right|^2 + \left|A_4^{1423}\right|^2 \Big) - 4\frac{(N^2-3)}{N^2} \left|A_4^{1234} + A_4^{1342} + A_4^{1423}\right|^2 \Bigg\} , \tag{3.2.5}$$

where N is the degree of the $SU(N)$ colour group, i.e. $N = 3$.

Similarly for the two quark, two gluon amplitude $0 \to hq\bar{q}gg$,

$$\mathcal{A}_4^{\bar{q}qgg}(\{p_i, h_i, c_i, j_i\}) = i\,\frac{g_s^4}{16\pi^2}\left(\frac{m^2}{v}\right)\Big[(t^{c_3}\, t^{c_4})_{j_2\, j_1} A_4^{34}(1^{h_1}, 2^{-h_1}, 3^{h_3}, 4^{h_4}; h) + (t^{c_4}\, t^{c_3})_{j_2\, j_1} A_4^{43}(1^{h_1}, 2^{-h_1}, 3^{h_3}, 4^{h_4}; h) \Big] . \tag{3.2.6}$$

where we have dropped the colour structure $\delta^{c_3\, c_4}\, \delta_{j_2\, j_1}/N$ as it makes no net contribution to the one-loop amplitude. This is likely due to some form of Furry's theorem [89], though we have not examined this in detail. In addition, A_4^{43} can be obtained from A_4^{34} through complex conjugation and permutation of momentum labels, so it is only necessary to calculate one of these amplitudes. Squaring and summing over colours yields

$$\sum |\mathcal{A}_4^{\bar{q}qgg}|^2 = \left(\frac{g_s^4}{16\pi^2}\right)^2 \left(\frac{m^2}{v}\right)^2 (N^2-1) \left[N\left(|A_4^{34}|^2 + |A_4^{43}|^2\right) - \frac{1}{N}|A_4^{34} + A_4^{43}|^2 \right] . \tag{3.2.7}$$

The four-quark amplitude, for quarks of different flavours q and q', takes the form,

$$\mathcal{A}_4^{4q}(\{p_i, h_i, j_i\}) = i\,\frac{g_s^4}{16\pi^2}\left(\frac{m^2}{v}\right) (t^{c_1})_{j_2\, j_1}\, (t^{c_1})_{j_4\, j_3} A_4^{4q}(1_{\bar{q}}^{h_1}, 2_q^{-h_1}, 3_{\bar{q}'}^{h_3}, 4_{q'}^{-h_3}) , \tag{3.2.8}$$

where the helicities of the quarks are fixed by those of the antiquarks. Performing the sum over colours we then have,

$$\sum |\mathcal{A}_4^{4q}(h_1, h_3)|^2 = \left(\frac{g_s^4}{16\pi^2}\right)^2 \left(\frac{m^2}{v}\right)^2 (N^2 - 1)\, |A_4^{4q}(h_1, h_3)|^2 \,. \tag{3.2.9}$$

For the case of identical quarks, $q = q'$, we first introduce the notation,

$$A_4^{4q'}(h_1, h_3) = A_4^{4q}(1_{\bar{q}}^{h_1}, 4_q^{-h_1}, 3_{\bar{q}}^{h_3}, 2_q^{-h_3})\,; \tag{3.2.10}$$

the sum over the colours for the identical case is then

$$\sum |\mathcal{A}_4^{4q}|^2 = \left(\frac{g_s^4}{16\pi^2}\right)^2 \left(\frac{m^2}{v}\right)^2 (N^2 - 1) \Bigg(|A_4^{4q}(h_1, h_3)|^2 + |A_4^{4q'}(h_1, h_3)|^2 \\ + \frac{\delta_{h_1 h_3}}{N} \left(A_4^{4q}(h_1, h_3) A_4^{4q'}(h_1, h_3)^* + A_4^{4q}(h_1, h_3)^* A_4^{4q'}(h_1, h_3) \right) \Bigg) . \tag{3.2.11}$$

Our results will be expressed in terms of the scalar integrals defined in eq. (2.3.5). For example, for the $0 \to ggggh$ sub-amplitude we have,

$$\begin{aligned} A_4^{1234}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}; h) =& \frac{\bar{\mu}^{4-D}}{r_\Gamma} \frac{1}{i\pi^{D/2}} \int \mathrm{d}^D \ell \, \frac{\mathrm{Num}(\ell)}{\prod_i d_i(\ell)} \\ =& \sum_{i,j,k,l} \tilde{e}_{i\times j\times k\times l}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})\, E_0(p_i, p_j, p_k, p_l; m) \\ &+ \sum_{i,j,k} \tilde{d}_{i\times j\times k}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})\, D_0(p_i, p_j, p_k; m) \\ &+ \sum_{i,j} \tilde{c}_{i\times j}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})\, C_0(p_i, p_j; m) \\ &+ \sum_{i} \tilde{b}_{i}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})\, B_0(p_i; m) + r(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})\,, \end{aligned} \tag{3.2.12}$$

where $\mathrm{Num}(\ell)$ is the numerator of the integrand, which in general depends on the loop momentum ℓ. We use a $\sim$ to denote a coefficient from the scalar theory. The sums run over groupings of external particles, excluding the Higgs. This expression has been written in D dimensions, as dimensional regularisation is used to regulate the UV singularities within the bubble integrals, though the end result is finite.

We choose to give our results in terms of the box, triangle and bubble integrals only, and will therefore reduce all pentagon integrals to box integrals, following the procedure in section 2.3.1. Using eq. (2.3.13) to reduce the pentagon ($N = 5$) in $D = 4$ dimensions,

$$
\begin{aligned}
E_0(p_1,p_2,p_3,p_4;m) = &\sum_{j=1}^{N} b_j I_4^4[j] \\
= &\,\mathcal{C}^{(1)}_{1\times2\times3\times4}\, D_0(p_2,p_3,p_4;m) + \mathcal{C}^{(2)}_{1\times2\times3\times4}\, D_0(p_{12},p_3,p_4;m) \\
&+ \mathcal{C}^{(3)}_{1\times2\times3\times4}\, D_0(p_1,p_{23},p_4;m) + \mathcal{C}^{(4)}_{1\times2\times3\times4}\, D_0(p_1,p_2,p_{34};m) \\
&+ \mathcal{C}^{(5)}_{1\times2\times3\times4}\, D_0(p_1,p_2,p_3;m)\,. \qquad (3.2.13)
\end{aligned}
$$

The $\mathcal{C}^{(i)}_{1\times2\times3\times4}$ are known as the pentagon reduction coefficients,

$$
\begin{aligned}
\mathcal{C}^{(1)}_{1\times2\times3\times4} &= -\frac{1}{2}\frac{s_{23}\,s_{34}\,[2\,s_{12}\,s_{24} + s_{13}\,s_{24} + s_{34}\,s_{12} - s_{23}\,s_{14}]}{16\,|S_{1\times2\times3\times4}|} \\
\mathcal{C}^{(2)}_{1\times2\times3\times4} &= -\frac{1}{2}\frac{s_{34}}{16\,|S_{1\times2\times3\times4}|}\,[s_{1234}\,s_{23}\,(s_{123} - 2\,s_{12}) + s_{123}\,(s_{34}\,(s_{123} - s_{23}) \\
&\quad + s_{12}\,(s_{234} + s_{23}) - s_{234}\,s_{123}))] \\
\mathcal{C}^{(3)}_{1\times2\times3\times4} &= -\frac{1}{2}\frac{[s_{14}\,s_{23} - (s_{12} + s_{13})\,(s_{24} + s_{34})]\,[s_{34}\,s_{12} + s_{23}\,s_{14} - s_{13}\,s_{24}]}{16\,|S_{1\times2\times3\times4}|} \\
\mathcal{C}^{(4)}_{1\times2\times3\times4} &= -\frac{1}{2}\frac{s_{12}}{16\,|S_{1\times2\times3\times4}|}\,[s_{1234}\,s_{23}\,(s_{234} - 2\,s_{34}) + s_{234}\,(s_{12}\,(s_{234} - s_{23}) \\
&\quad + s_{34}\,(s_{123} + s_{23}) - s_{234}\,s_{123}))] \\
\mathcal{C}^{(5)}_{1\times2\times3\times4} &= -\frac{1}{2}\frac{s_{12}\,s_{23}\,[2\,s_{34}\,s_{13} + s_{13}\,s_{24} + s_{34}\,s_{12} - s_{23}\,s_{14}]}{16\,|S_{1\times2\times3\times4}|}\,, \qquad (3.2.14)
\end{aligned}
$$

where $|S_{1\times2\times3\times4}|$ is defined in eq. (2.3.15).

The rational terms, $\tilde{r}$, are given by eq. (2.3.24), though since we have no box coefficients with m^4 dependence and no bubbles with m^2, this simplifies to

$$
\tilde{r}(1^{h_1},2^{h_2},3^{h_3},4^{h_4}) = \frac{1}{2}\sum_{i,j}\tilde{c}^{(2)}_{i\times j}(1^{h_1},2^{h_2},3^{h_3},4^{h_4})\,. \qquad (3.2.15)
$$

Some of the results presented here and in the following chapter use the simplification techniques introduced in section 2.4. For both of these we can decompose the momentum of the Higgs boson into two light-like momenta, p_5 and p_6, which could be considered as two massless decay products. This allows us to treat the processes as having 6-point massless kinematics.

This is where the usefulness of the decoupling of the x_3 and x_4 variables in eq. (2.4.8) becomes apparent, since our amplitudes will not contain the p_5 and p_6 momenta. In addition, in order to use the momentum twistor parametrisation in eq. (2.4.5) we must remove the overall phase of the coefficient that corresponds to the helicities of the external gluons. For example, for the all-plus case we must multiply by $\langle 1\,2\rangle^2 \langle 3\,4\rangle^2$.

3.3 Relationship Between Fermion and Scalar Theories

For simplicity it is useful to study the inclusive cross section to understand where the scalar and fermion mediated loop processes are similar. The Standard Model amplitude for inclusive Higgs boson production via gluon fusion, for a quark of mass m in the loop, is

$$\mathcal{H}_2^{gg} = i\,\frac{g_s^2}{16\,\pi^2}\,\delta^{AB}\left(\frac{m^2}{v}\right)\left[g^{\mu\nu} - \frac{p_1^\nu\, p_2^\mu}{p_1 \cdot p_2}\right] \\ \left[(2\,m_h^2 - 8m^2)\,C_0(p_1,p_2;m) - 4\right]\varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,, \tag{3.3.1}$$

where the gluons have colour labels A and B and ε represents a polarisation vector. C_0 is the scalar triangle integral defined in eq. (2.3.5). The same process mediated by a colour triplet scalar, mass m, has an amplitude of [90]

$$\mathcal{A}_2^{gg} = i\,\frac{g_s^2}{16\,\pi^2}\,\delta^{AB}\left(\frac{-\lambda}{4}\right)\left[g^{\mu\nu} - \frac{p_1^\nu\, p_2^\mu}{p_1 \cdot p_2}\right] \\ \left[-8m^2\,C_0(p_1,p_2;m) - 4\right]\varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,. \tag{3.3.2}$$

It is easy to see the similarities between these two amplitudes, especially when setting $(-\lambda/4) = m^2/v$. The origin of this m^2 factor in eq. (3.3.1) is due to one factor of m arising from the quark-quark-Higgs coupling and the other from the form of the fermion propagator.

The coefficient of the triangle integral proportional to m^2 and the rational terms are identical. This correspondence extends to the 1- and 2-jet cases for specific coefficients, which will be given explicitly at the end of this section.

Rewriting both of these amplitudes by extracting an overall factor,

$$\begin{aligned}
\mathcal{H}_2^{gg} &= i\,\frac{g_s^2}{16\,\pi^2}\,\delta^{AB}\left[\frac{1}{2}\Big(g^{\mu\nu}-\frac{p_1^\nu\,p_2^\mu}{p_1{\cdot}p_2}\Big)\right]\left(\frac{m_h^2}{v}\right)\varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,F_{1/2}(\tau)\,,\\
\mathcal{A}_2^{gg} &= i\,\frac{g_s^2}{16\,\pi^2}\,\delta^{AB}\left[\frac{1}{2}\Big(g^{\mu\nu}-\frac{p_1^\nu\,p_2^\mu}{p_1{\cdot}p_2}\Big)\right]\left(\frac{\lambda}{2}\frac{m_h^2}{m^2}\right)\varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\,F_0(\tau)\,,
\end{aligned} \tag{3.3.3}$$

where the scalar, $F_0(\tau)$, and fermion, $F_{1/2}(\tau)$, functions are defined as

$$F_0(\tau) = \tau\Big[1-\tau f(\tau)\Big]\,, \tag{3.3.4}$$

$$F_{1/2}(\tau) = -2\tau\Big[1+(1-\tau)f(\tau)\Big] = -2F_0(\tau)-2\tau f(\tau)\,, \tag{3.3.5}$$

and $\tau = 4m^2/m_h^2$. Additionally we have defined the triangle function,

$$\begin{aligned}
f(\tau) &= -\frac{m_h^2}{2}\,C_0(p_1,p_2;m)\\
&= -\frac{1}{4}\theta(1-\tau)\left[\ln\left(\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}}\right)-i\pi\right]^2+\theta(\tau-1)\left[\sin^{-1}(1/\sqrt{\tau})\right]^2\,.
\end{aligned} \tag{3.3.6}$$

In the Standard Model, considering the top quark only, $\tau \gg 1$, both F_0 and $F_{1/2}$ are in their asymptotic regions, illustrated in Figure 3.1. This validates the use of the effective field theory in eq. (2.3.40). In this limit, $m^2 \gg m_h^2$, a comparison of the asymptotic values and the overall factors in eq. (3.3.3) shows that the two theories are described by the same effective Lagrangian when $\lambda/8 = m^2/v$.

This asymptotic correspondence can also be seen by starting with the large mass expansion for the triangle integral,

$$C_0(p_1,p_2;m) \to -\frac{1}{2\,m^2}-\frac{s}{24m^4}+O\left(\frac{1}{m^6}\right),\qquad s = m_h^2 = 2p_1\cdot p_2\,. \tag{3.3.7}$$

and noticing that in both theories the terms within the square brackets in eqs. (3.3.1,3.3.2) scale as m_h^2/m^2.

More precisely, extracting the effective interactions for the fermion and scalar theor-

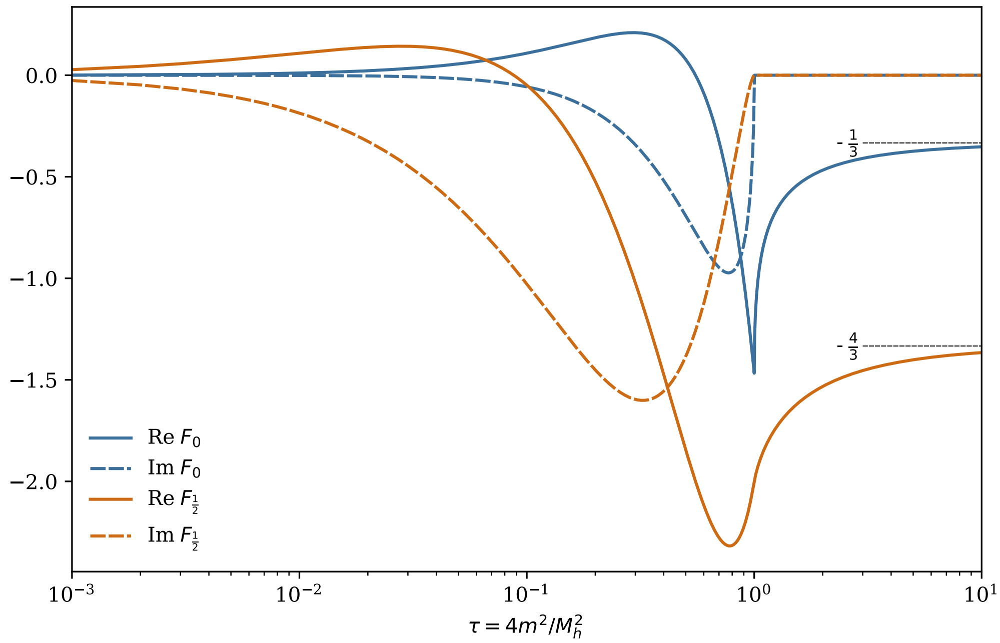


Figure 3.1: The functions $F_{1/2}$ and F_0, given in Eqs. (3.3.4) and (3.3.5) respectively, plotted as functions of $\tau = 4m^2/M_h^2$. The dashed lines on the right hand side show their asymptotes at large τ.

ies:

$$\mathcal{L}_{hgg} = -\frac{1}{4} C_f G_a^{\mu\nu}\, G_{\mu\nu\, a}\, h, \quad C_f = -\frac{g_s^2}{12\pi^2 v}, \tag{3.3.8}$$

for the fermion loop and

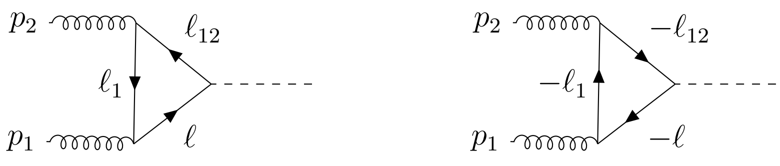
$$\tilde{\mathcal{L}}_{hgg} = -\frac{1}{4} C_s G_a^{\mu\nu}\, G_{\mu\nu\, a}\, h, \quad C_s = \frac{g_s^2}{24\pi^2 m^2}\left(-\frac{\lambda}{4}\right) \tag{3.3.9}$$

for the scalar one. Therefore, with $(-\lambda/4) = m^2/v$, the amplitudes are related by a factor of $-1/2$ when the EFT is valid,

$$m^2 A(\ldots; h) \to -\frac{1}{2} m^2 H(\ldots; h)\,. \tag{3.3.10}$$

3.3.1 Second Order Formalism

To study this correspondence in more detail we can use the second order formalism [91]. It is a description of fermions that is similar to that of scalars, which



(a) (b)

Figure 3.2: Triangle diagrams for Higgs boson production via gluon fusion. The momenta are defined as $\ell_1 = \ell + p_1$ and $\ell_{12} = \ell + p_1 + p_2$ and flow in the direction of the fermion arrows.

makes it an excellent tool to study the relationship between these two theories. The starting point for the second order formalism is an internal fermion propagator line interacting with a gauge boson, in our case a gluon. We will look at the numerator, $\mathbf{A}^\mu$, part of this expression,

$$\begin{aligned}\mathbf{A}^\mu(\ell, p_1) = (\not{\ell} + m)\gamma^\mu &= (\not{\ell} + \frac{1}{2}\not{p}_1 - \frac{1}{2}\not{p}_1 + m)\gamma^\mu \\ &= (2\ell^\mu + p_1^\mu)\mathbf{1} - \frac{1}{2}[\not{p}_1, \gamma^\mu] - \gamma^\mu(\not{\ell} + \not{p}_1 - m)\,, \end{aligned} \tag{3.3.11}$$

where ℓ is the momentum of the fermion propagator and p_1 is the momentum flowing out of the vertex along the gluon line. The first term on the second line of eq. (3.3.11) is similar to the vertex for a gluon interacting with a scalar field. Defining

$$\mathbf{B}^\mu(\ell, \ell_1) \equiv (\ell^\mu + \ell_1^\mu)\,\mathbf{1} - \frac{1}{2}\left[\not{p}_1, \gamma^\mu\right] \quad \text{and} \quad \mathbf{C}^\mu(\not{\ell}_1) \equiv \gamma^\mu(\not{\ell} + \not{p}_1 - m)\,, \tag{3.3.12}$$

we have that

$$\mathbf{A}^\mu(\ell, p_1) = \mathbf{B}^\mu(\ell, \ell_1) - \mathbf{C}^\mu(\not{\ell}_1)\,. \tag{3.3.13}$$

We can generate a propagator-like factor by following $\mathbf{C}^\mu(\not{\ell}_1)$ by the factor $\mathbf{A}^\nu(\ell_1 + q, -q)$,

$$\mathbf{C}^\mu(\ell_1)\mathbf{A}^\mu(\ell_1 + q, -q) = -d(\ell_1) \times \gamma^\mu\gamma^\nu\,, \tag{3.3.14}$$

where the denominator $d(\ell_1)$ follows the same notation as eq. (2.3.4). General fermion loops will contain chains of $\mathbf{A}^\mu/d(\ell_i)$. We can use eq. (3.3.14) to cancel denominators and to decompose the expression into terms equivalent to a scalar theory plus a correction of lower rank.

As an example, we can consider the relatively simple case of Higgs production via gluon fusion, fig. (3.2). To begin with, the integrand for the diagram in fig. (3.2a) is,

$$(-1) \times t^{c_1} t^{c_2} \times \frac{1}{m} \frac{\mathrm{tr}\{(\not{\ell} + m)\gamma^{\mu_1}(\not{\ell}_1 + m)\gamma^{\mu_2}(\not{\ell}_{12} + m)\}}{d(\ell)\, d(\ell_1)\, d(\ell_{12})}, \tag{3.3.15}$$

where the minus sign is due to the fermion loop and the factor of $1/m$ is for convenience. After applying the decomposition in eq. (3.3.13) twice we have,

$$(-1) \times t^{c_1} t^{c_2} \times \left[\frac{\mathrm{tr}\{\mathbf{B}^{\mu_1}(\ell, \ell_1)\mathbf{B}^{\mu_2}(\ell_1, \ell_{12})\}}{d(\ell)\, d(\ell_1)\, d(\ell_{12})} - \frac{\mathrm{tr}\{\gamma^{\mu_1}\gamma^{\mu_2}\}}{d(\ell)\, d(\ell_{12})} \right]. \tag{3.3.16}$$

Similarly, the result for the companion triangle in fig. (3.2b) is,

$$\begin{aligned}
&(-1) \times t^{c_2} t^{c_1} \times \frac{1}{m} \frac{\mathrm{tr}\{(-\not{\ell}_{12} + m)\gamma^{\mu_2}(-\not{\ell}_1 + m)\gamma^{\mu_1}(-\not{\ell} + m)\}}{d(\ell)\, d(\ell_1)\, d(\ell_{12})} \\
&= (-1) \times t^{c_2} t^{c_1} \times \left[\frac{\mathrm{tr}\{\mathbf{B}^{\mu_2}(-\ell_{12}, -\ell_1)\mathbf{B}^{\mu_1}(-\ell_1, -\ell)\}}{d(\ell)\, d(\ell_1)\, d(\ell_{12})} - \frac{\mathrm{tr}\{\gamma^{\mu_2}\gamma^{\mu_1}\}}{d(\ell)\, d(\ell_{12})} \right].
\end{aligned} \tag{3.3.17}$$

Once the diagrams have been added together the full amplitude is,

$$\begin{aligned}
(-1)\Bigg\{ & t^{c_1} t^{c_2} \times \left[\frac{(\ell^{\mu_1} + \ell_1^{\mu_1})(\ell_1^{\mu_2} + \ell_{12}^{\mu_2})\mathrm{tr}\{\mathbf{1}\} + \frac{1}{4}\mathrm{tr}\{[\not{p}_1, \gamma^{\mu_1}][\not{p}_2, \gamma^{\mu_2}]\}}{d(\ell)\, d(\ell_1)\, d(\ell_{12})} \right] \\
& + t^{c_2} t^{c_1} \times \left[\frac{(\ell^{\mu_1} + \ell_1^{\mu_1})(\ell_1^{\mu_2} + \ell_{12}^{\mu_2})\mathrm{tr}\{\mathbf{1}\} + \frac{1}{4}\mathrm{tr}\{[\not{p}_1, \gamma^{\mu_1}][\not{p}_2, \gamma^{\mu_2}]\}}{d(\ell)\, d(\ell_1)\, d(\ell_{12})} \right] \\
& - (t^{c_1} t^{c_2} + t^{c_2} t^{c_1}) g^{\mu_1 \mu_2} \frac{\mathrm{tr}\{\mathbf{1}\}}{d(\ell)\, d(\ell_{12})} \Bigg\}.
\end{aligned} \tag{3.3.18}$$

The fermion loop can therefore be considered as a (suitably normalised) scalar triangle with convection terms, e.g. $\ell^\mu + \ell_1^\mu$, plus spin flip terms (the commutators of gamma matrices). These spin flip terms do not have any dependence on the loop momentum so they are of lower rank than the scalar triangle contribution. In addition, there is no explicit mass dependence in the expression in eq. (3.3.18). The dependence appears when reducing to scalar integrals.

The same process can be applied to amplitudes with more gluons; the rank of the spin flip terms is always at least two powers of the loop momentum lower than the terms from the scalar theory.

Therefore, the full fermion amplitude can be written as

$$\text{Fermion theory} = \text{Scalar theory} + \Delta F\,, \tag{3.3.19}$$

where the correction, ΔF, is of lower rank by at least two powers of ℓ. Since the fermion theory contains rank-4 pentagons, rank-3 boxes, rank-2 triangles and rank-1 bubbles, ΔF must contain at most rank-2 pentagons, rank-1 boxes, rank-0 triangles and no bubbles; therefore ΔF is cut-constructible. This means that the bubble coefficients and rational terms can be fully calculated in the scalar theory. A further simplification is that the bubble coefficients have no dependence on the mass, and can therefore be calculated in the massless scalar theory.

Separating out the mass dependence in the triangle coefficients, we have

$$\begin{aligned} c_{i\times j} &= c^{(0)}_{i\times j} + m^2\, c^{(2)}_{i\times j} \\ \tilde{c}_{i\times j} &= \tilde{c}^{(0)}_{i\times j} + m^2\, \tilde{c}^{(2)}_{i\times j} \end{aligned} \tag{3.3.20}$$

for the fermion and scalar theories respectively. ΔF does not contain any m^2 triangle coefficients, so we have $c^{(2)}_{i\times j} = \tilde{c}^{(2)}_{i\times j}$. In addition, ΔF makes no contribution at all to certain triangle coefficients,

$$\begin{aligned} c_{3\times4}(1^+,2^+,3^+,4^-) &= \tilde{c}_{3\times4}(1^+,2^+,3^+,4^-)\,, \\ c_{2\times34}(1^+,2^+,3^+,4^-) &= \tilde{c}_{2\times34}(1^+,2^+,3^+,4^-)\,, \\ c_{1\times43}(1^+,2^+,3^+,4^-) &= \tilde{c}_{1\times43}(1^+,2^+,3^+,4^-)\,, \end{aligned} \tag{3.3.21}$$

$$\begin{aligned} c_{3\times4}(1^+,2^-,3^+,4^-) &= \tilde{c}_{3\times4}(1^+,2^-,3^+,4^-)\,, \\ c_{2\times34}(1^+,2^-,3^+,4^-) &= \tilde{c}_{2\times34}(1^+,2^-,3^+,4^-)\,, \end{aligned} \tag{3.3.22}$$

$$\begin{aligned} c_{2\times3}(1^+,2^+,3^-,4^-) &= \tilde{c}_{2\times3}(1^+,2^+,3^-,4^-)\,, \\ c_{1\times23}(1^+,2^+,3^-,4^-) &= \tilde{c}_{1\times23}(1^+,2^+,3^-,4^-)\,. \end{aligned} \tag{3.3.23}$$

This is all $c_{i\times j}$ that do not have a lone Higgs boson external leg.

Though we refer to ΔF as a correction, that does not mean its contribution to the amplitude is small. It is a useful technique to simplify the calculations in the fermion theory, however there is no constraint on the relative sizes of the two terms in eq. (3.3.19).

3.4 Tree-Level Amplitudes with Massive Scalars

It is necessary to calculate several tree level amplitudes for the unitarity construction of the loop amplitudes. We require amplitudes for a scalar—anti-scalar pair coupled to n partons, where $n \leq 3$, both with and without a Higgs boson. These partons consist of either n gluons or $n-2$ gluons and a quark—anti-quark pair. For the n gluon cases we will give the colour ordered sub-amplitudes (c.f. eq. (2.0.1)), $A_n^{\rm tree}\left(\ell;\sigma(1),\ldots,\sigma(n);\bar{\ell}\right)$ and $A_n^{\rm tree}\left(h,\ell;\sigma(1),\ldots,\sigma(n);\bar{\ell}\right)$ for those including the Higgs. ℓ ($\bar{\ell}$) is the momentum of the (anti-)scalar and $\sigma(i)$ is the momentum and helicity label of particle i. All momenta are taken to be outgoing, and the amplitudes are labelled by the particles not including the two external scalar particles.

One gluon

The amplitudes for two scalars and a gluon are,

$$A_1^{\rm tree}(\ell;1^+;\bar{\ell}) = \frac{\langle b|\ell|1]}{\langle b\,1\rangle}\,, \quad A_1^{\rm tree}(\ell;1^-;,\bar{\ell}) = \frac{\langle 1|\ell|b]}{[1\,b]}\,, \tag{3.4.1}$$

and the amplitude containing the Higgs is,

$$\begin{aligned} A_1^{\rm tree}(h,\ell;1^+;\bar{\ell}) &= -4\Big[\frac{\left\langle b|\bar{\ell}|1\right]}{\langle b\,1\rangle\,\left\langle 1|\bar{\ell}|1\right]} - \frac{\langle b|\ell|1]}{\langle b\,1\rangle\,\langle 1|\ell|1]}\Big] \\ &= -4\Big[\frac{\left\langle b|\bar{\ell}|1\right]\,\langle 1|\ell|1] - \langle b|\ell|1]\,\left\langle 1|\bar{\ell}|1\right]}{\langle b\,1\rangle\,\left\langle 1|\bar{\ell}|1\right]\,\langle 1|\ell|1]}\Big]\,. \end{aligned} \tag{3.4.2}$$

In eqs. (3.4.1,3.4.2), b is an arbitrary light-like momentum.

Two gluons

The amplitudes for two gluons and two scalars are given by,

$$A_2^{\text{tree}}(\ell; 1^+, 2^+; \bar{\ell}) = -\frac{m^2\ [1\,2]}{\langle 1\,2\rangle\ d(\ell_1)}\,, \tag{3.4.3}$$

$$A_2^{\text{tree}}(\ell; 1^+, 2^-; \bar{\ell}) = -\frac{\langle 2|\ell|1]^2}{\langle 1|2|1]\ d(\ell_1)}\,, \tag{3.4.4}$$

and with a Higgs boson,

$$\begin{aligned} A_2^{\text{tree}}(h, \ell; 1^+, 2^+; \bar{\ell}) = & -4\frac{[1\,2]}{\langle 1\,2\rangle}\Bigg\{\Bigg[\frac{m^2}{d(\ell_1)\,d(\ell_{12})} + \frac{m^2}{d(\bar{\ell}_2)\,d(\bar{\ell}_{12})}\Bigg] \\ & - \frac{1}{s_{12}}\Bigg[1 - \frac{\left\langle 1|\bar{\ell}|2\right]\ \langle 2|\ell|1]}{d(\ell_1)\,d(\bar{\ell}_2)}\Bigg]\Bigg\}, \end{aligned} \tag{3.4.5}$$

$$A_2^{\text{tree}}(h, \ell; 1^+, 2^-; \bar{\ell}) = -4\frac{1}{s_{12}}\Bigg\{\frac{\langle 2|\ell|1]^2}{d(\ell_{12})\,d(\ell_1)} + \frac{\left\langle 2|\bar{\ell}|1\right]^2}{d(\bar{\ell}_{12})\,d(\bar{\ell}_2)} - \frac{\left\langle 2|\bar{\ell}|1\right]\ \langle 2|\ell|1]}{d(\ell_1)\,d(\bar{\ell}_2)}\Bigg\}. \tag{3.4.6}$$

Three gluons

The calculation of this amplitude using BCFW recursion leaves spurious poles in the results [92], which can be removed by writing the results in the following form [93]

$$\begin{aligned} A_3^{\text{tree}}(\ell; 1^+, 2^+, 3^-; \bar{\ell}) = & \frac{\left\langle 3|\bar{\ell}(1+2)\ell|1\right]\ \left\langle 3|\bar{\ell}|2\right]}{\langle 1\,2\rangle\ s_{23}\,d(\ell_1)\,d(\ell_{12})} \\ & + \frac{[1\,2]\ \left\langle 3|(1+2)|\bar{\ell}|3\right\rangle}{\langle 1\,2\rangle\ s_{23}\,s_{123}}, \end{aligned} \tag{3.4.7}$$

$$\begin{aligned} A_3^{\text{tree}}(\ell; 1^+, 2^-, 3^+; \bar{\ell}) = & \frac{1}{s_{12}\,s_{23}}\Bigg[\frac{[2\,1]\ \langle 2|\ell|1]\ \left\langle 2|\bar{\ell}|3\right]^2}{d(\ell)d(\ell_{12})} + \frac{[3\,1]\ \langle 2|\ell|1]\ \left\langle 2|\bar{\ell}|3\right]}{d(\ell_{12})} \\ & - \frac{\left\langle 2|\bar{\ell}|(1+3)|2\right\rangle\ [1\,3]^2}{s_{123}}\Bigg]. \end{aligned} \tag{3.4.8}$$

We can then obtain $A_3^{\text{tree}}(\ell; 1^+, 2^-, 3^-; \bar{\ell})$ by using charge conjugation on Eq. (3.4.7),

$$\begin{aligned} A_3^{\text{tree}}(\ell; 1^+, 2^-, 3^-; \bar{\ell}) = & \frac{\left\langle 3|\bar{\ell}(2+3)\ell|1\right]\ \langle 2|\ell|1]}{[2\,3]\ s_{12}\,d(\ell_1)\,d(\ell_{12})} \\ & - \frac{\langle 2\,3\rangle\ [1|(2+3)|\ell|1]}{[2\,3]\ s_{12}\,s_{123}}\,. \end{aligned} \tag{3.4.9}$$

One gluon, two quarks

$$\begin{aligned}\mathcal{A}_3^{\bar{q}qg}(\ell; 1_{\bar{q}}^+, 2_q^-, 3_g^+; \bar{\ell}) =& i\sqrt{2}g_s^3 \Bigg\{ \Big[(t^{c_3})_{jj_1}\, \delta_{j_2,\bar{j}} - (t^{c_3})_{j_2\bar{j}}\, \delta_{j,j_1}\Big] \frac{[1\,3]\ \langle 2|\ell|(1+2+3)|2\rangle}{\langle 2\,3\rangle\ s_{12}\, s_{123}} \\ &- \Big[(t^{c_3})_{jj_1}\, \delta_{j_2,\bar{j}} - \frac{1}{N}\,(t^{c_3})_{j_2j_1}\, \delta_{j,\bar{j}}\Big] \frac{\langle 2|\ell|(1+2+3)|2\rangle}{\langle 1\,3\rangle\ \langle 2\,3\rangle\ s_{123}} \\ &+ \Big[(t^{c_3})_{jj_1}\, \delta_{j_2,\bar{j}} - \frac{1}{N}\,(t^{c_3})_{j\bar{j}}\, \delta_{j_2,j_1}\Big] \frac{\langle 2|\ell_{123}|1]\ \langle 2|\ell|3]}{\langle 2\,3\rangle\ s_{12}\, d(\ell_3)} \\ &+ \Big[(t^{c_3})_{j_2\bar{j}}\, \delta_{j,j_1} - \frac{1}{N}\,(t^{c_3})_{j\bar{j}}\, \delta_{j_2,j_1}\Big] \frac{\langle 2|\ell|1]\ \langle 2|\ell_{123}|3]}{\langle 2\,3\rangle\ s_{12}\, d(\ell_{12})} \Bigg\} \end{aligned} \tag{3.4.10}$$

$$\begin{aligned}\mathcal{A}_3^{\bar{q}qg}(\ell; 1_{\bar{q}}^+, 2_q^-, 3_g^-; \bar{\ell}) =& i\sqrt{2}g_s^3 \\ & \Bigg\{ - \Big[(t^{c_3})_{jj_1}\, \delta_{j_2,\bar{j}} - (t^{c_3})_{j_2\bar{j}}\, \delta_{j,j_1}\Big] \frac{\langle 2\,3\rangle\ [1|\ell|(1+2+3)|1]}{[1\,3]\ s_{12}\, s_{123}} \\ &- \Big[(t^{c_3})_{j_2\bar{j}}\, \delta_{j,j_1} - \frac{1}{N}\,(t^{c_3})_{j_2j_1}\, \delta_{j,\bar{j}}\Big] \frac{[1|\ell|(1+2+3)|1]}{[1\,3]\ [2\,3]\ s_{123}} \\ &- \Big[(t^{c_3})_{jj_1}\, \delta_{j_2,\bar{j}} - \frac{1}{N}\,(t^{c_3})_{j\bar{j}}\, \delta_{j_2,j_1}\Big] \frac{\langle 2|\ell_{123}|1]\ [1|\ell|3\rangle}{[1\,3]\ s_{12}\, d(\ell_3)} \\ &- \Big[(t^{c_3})_{j_2\bar{j}}\, \delta_{j,j_1} - \frac{1}{N}\,(t^{c_3})_{j\bar{j}}\, \delta_{j_2,j_1}\Big] \frac{\langle 2|\ell|1]\ [1|\ell_{123}|3\rangle}{[1\,3]\ s_{12}\, d(\ell_{12})} \Bigg\} \end{aligned} \tag{3.4.11}$$

3.5 Amplitude for $0 \to ggggh$ with a Scalar Mediator

3.5.1 Coefficients for $A_4^{1234}(g^+, g^+, g^+, g^+; h)$

The Higgs plus four gluon amplitude where all gluons have positive helicity is particularly simple, especially when the pentagon integral (E_0) is retained, rather than being reduced to box integrals,

$$\begin{aligned} A_4(h; 1_g^+, 2_g^+, 3_g^+, 4_g^+) =& \Bigg[\Bigg\{ \frac{4\,m^2}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3\,4\rangle\ \langle 4\,1\rangle} \Big[- \mathrm{tr}_+\{1\,2\,3\,4\} m^2 E_0(p_1, p_2, p_3, p_4; m) \\ &+ \frac{1}{2}((s_{12}+s_{13})(s_{24}+s_{34}) - s_{14}s_{23}) D_0(p_1, p_{23}, p_4; m) \end{aligned}$$

$$+\frac{1}{2}s_{12}s_{23}D_0(p_1,p_2,p_3;m)$$
$$\left.\left. + (s_{12}+s_{13}+s_{14})C_0(p_1,p_{234};m)\right] + 2\frac{s_{12}+s_{13}+s_{14}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\right\}$$
$$\left.+\left\{3 \text{ cyclic permutations}\right\}\right] . \tag{3.5.1}$$

where we have introduced the trace functions

$$\begin{aligned}\mathrm{tr}_+\{1\,2\,3\,4\} &= \mathrm{tr}\{\gamma_R\,\not{p}_1\,\not{p}_2,\not{p}_3,\not{p}_4\} = [1\,2]\ \langle 2\,3\rangle\ [3\,4]\,\langle 4\,1\rangle\ ,\\ \mathrm{tr}_-\{1\,2\,3\,4\} &= \mathrm{tr}\{\gamma_L\,\not{p}_1\,\not{p}_2,\not{p}_3,\not{p}_4\} = \langle 1\,2\rangle\ [2\,3]\ \langle 3\,4\rangle\,[4\,1]\ .\end{aligned} \tag{3.5.2}$$

The equalities on the far right hold only for lightlike momenta.

When presenting the results for the other helicity configurations we will, however, reduce the pentagon integrals to box integrals. For simplicity we therefore give the formulae once the pentagon integral has been reduced. This reduction of pentagon integrals to box integrals leaves us with an effective pentagon coefficient,

$$\begin{aligned}\tilde{e}_{1\times2\times3\times4}(1^+,2^+,3^+,4^+) &= -4\,m^4\frac{\mathrm{tr}_+\{1\,2\,3\,4\}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\\ &= -4\,m^4\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\,,\end{aligned} \tag{3.5.3}$$

which can be calculated by first calculating the coefficient in d dimensions and then taking the $\mu^2 \to 0$ limit.

It is not necessary to calculate every coefficient since many of these can be obtained from permutations of the momentum labels. The minimal set of coefficients we must calculate is shown in columns 1 and 3 of table 3.1.

3.5.1.1 Boxes

$\tilde{d}_{1\times2\times34}$

$$\tilde{d}_{1\times2\times34}(1^+,2^+,3^+,4^+) = \mathcal{C}^{(4)}_{1\times2\times3\times4}\,\tilde{e}_{\{1^+\times2^+\times3^+\times4^+\}} \tag{3.5.4}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$\tilde{d}_{1\times 2\times 34}$	$\tilde{d}_{2\times 3\times 41}, \tilde{d}_{3\times 4\times 12}, \tilde{d}_{4\times 1\times 23},$ $\tilde{d}_{1\times 4\times 32}, \tilde{d}_{2\times 1\times 43}, \tilde{d}_{3\times 2\times 14}, \tilde{d}_{4\times 3\times 21}$	$\tilde{c}_{1\times 234}$	$\tilde{c}_{2\times 341}, \tilde{c}_{3\times 412}, \tilde{c}_{4\times 123}$
$\tilde{d}_{1\times 23\times 4}$	$\tilde{d}_{2\times 34\times 1}, \tilde{d}_{3\times 41\times 2}, \tilde{d}_{4\times 12\times 3}$		
$\tilde{d}_{1\times 2\times 3}$	$\tilde{d}_{2\times 3\times 4}, \tilde{d}_{3\times 4\times 1}, \tilde{d}_{4\times 1\times 2}$		

Table 3.1: The minimal set of integral coefficients for $A_4^{1234}(g^+, g^+, g^+, g^+; h)$. The related coefficients can be obtained by simple permutations of momentum labels.

$\tilde{\boldsymbol{d}}_{\mathbf{1\times 23\times 4}}$

$$\tilde{d}_{1\times 23\times 4}(1^+, 2^+, 3^+, 4^+) = \mathcal{C}^{(3)}_{1\times 2\times 3\times 4}\, \tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^+\}} + \frac{2\,m^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\left[(s_{12}+s_{13})\,(s_{24}+s_{34}) - s_{14}\,s_{23}\right] \tag{3.5.5}$$

$\tilde{\boldsymbol{d}}_{\mathbf{1\times 2\times 3}}$

$$\tilde{d}_{1\times 2\times 3}(1^+, 2^+, 3^+, 4^+) = \mathcal{C}^{(1)}_{4\times 1\times 2\times 3}\, \tilde{e}_{\{4^+\times 1^+\times 2^+\times 3^+\}} + \mathcal{C}^{(5)}_{1\times 2\times 3\times 4}\, \tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^+\}} + \frac{2\,m^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\, s_{12}\, s_{23} \tag{3.5.6}$$

3.5.1.2 Triangle

$\tilde{\boldsymbol{c}}^{(\mathbf{0})}_{\mathbf{1\times 234}}$, $\tilde{\boldsymbol{c}}^{(\mathbf{2})}_{\mathbf{1\times 234}}$

$$\tilde{c}^{(0)}_{1\times 234}(1^+, 2^+, 3^+, 4^+) = 0 \tag{3.5.7}$$

$$\tilde{c}^{(2)}_{1\times 234}(1^+, 2^+, 3^+, 4^+) = 4(s_{12}+s_{13}+s_{14})\frac{1}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 3\,4\rangle\;\langle 4\,1\rangle} \tag{3.5.8}$$

3.5.1.3 Rational terms

$$\begin{aligned}\tilde{r}(1^+, 2^+, 3^+, 4^+) &= \frac{1}{2}\Big[\tilde{c}^{(2)}_{1\times 234}(1^+, 2^+, 3^+, 4^+) + \tilde{c}^{(2)}_{1\times 234}(2^+, 3^+, 4^+, 1^+) \\ &\quad + \tilde{c}^{(2)}_{1\times 234}(3^+, 4^+, 1^+, 2^+) + \tilde{c}^{(2)}_{1\times 234}(4^+, 1^+, 2^+, 3^+)\Big] \\ &= 4\,\frac{s_{1234}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\end{aligned} \tag{3.5.9}$$

3.5.2 Coefficients for $A_4^{1234}(g^+, g^+, g^+, g^-; h)$

The initial form of the effective pentagon coefficients for this helicity combination contained spurious poles due to factors of $1/\mathrm{tr}_5\{1\,2\,3\,4\}^2$ in the denominator, where

$$\begin{aligned}\mathrm{tr}_5\{1\,2\,3\,4\} &= \mathrm{tr}\{\gamma_5\,\not{p}_1\,\not{p}_2\,\not{p}_3\,\not{p}_4\} \\ &= [1\,2]\;\langle 2\,3\rangle\;[3\,4]\;\langle 4\,1\rangle - \langle 1\,2\rangle\;[2\,3]\;\langle 3\,4\rangle\;[4\,1]\;, \end{aligned} \tag{3.5.10}$$

where the second equality holds for lightlike momenta. These poles are unphysical since the factors of $\mathrm{tr}_5\{1\,2\,3\,4\}$ are cancelled in the full amplitude by corresponding factors in the box coefficients. However, leaving the pentagon coefficients in this form can lead to a loss of stability in calculations. To remove this unphysical pole we can write,

$$\mathrm{tr}_5\{1\,2\,3\,4\}^2 = (s_{12}\,s_{34} - s_{14}\,s_{23} - s_{13}\,s_{24})^2 - 4\,s_{14}\,s_{23}\,s_{24}\,s_{13} = G\,, \tag{3.5.11}$$

where G is as defined in eq. (2.3.15). We therefore have

$$s_{12}s_{23}s_{34}\,\langle 1|(2+3)|4]\,\langle 4|(2+3)|1] = m^2\,\mathrm{tr}_5\{1\,2\,3\,4\}^2 - 16\,|S_{1\times 2\times 3\times 4}|\;. \tag{3.5.12}$$

Using this equation to eliminate a factor of $\langle 4|(2+3)|1]$ in the numerator generates two terms: one which is free from the denominator $\mathrm{tr}_5\{1\,2\,3\,4\}^2$ and one containing a factor of $|S_{1\times 2\times 3\times 4}|$. The former we identify as the effective pentagon coefficient and the latter cancels the denominator factor when reducing the pentagon to boxes (eq. (3.2.14)), therefore explicitly absorbing the $1/\mathrm{tr}_5\{1\,2\,3\,4\}^2$ factors into the box coefficients and cancelling them.

We therefore arrive at the effective pentagon coefficients,

$$\tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}} = -4\,m^4\,\frac{[2\,3]\;\langle 4|(2+3)|1]}{\langle 2\,3\rangle\;\langle 1|(2+3)|4]}\,, \tag{3.5.13}$$

$$\tilde{e}_{\{4^-\times 1^+\times 2^+\times 3^+\}} = \tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}}\{1 \leftrightarrow 3\}\,, \tag{3.5.14}$$

$$\tilde{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}} = -4\,m^4\,\frac{[2\,3]^2\,\langle 3\,4\rangle\,\langle 2|(3+4)|1]}{\langle 2\,3\rangle^2\,[3\,4]\,\langle 1|(3+4)|2]}\,, \tag{3.5.15}$$

$$\tilde{e}_{\{3^+\times 4^-\times 1^+\times 2^+\}} = \tilde{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}}\{1 \leftrightarrow 3\}\,. \tag{3.5.16}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$\tilde{d}_{1\times2\times34}$	$\tilde{d}_{3\times2\times14}$	$\tilde{c}_{3\times4}$	$\tilde{c}_{4\times1}$
$\tilde{d}_{1\times4\times32}$	$\tilde{d}_{3\times4\times12}$	$\tilde{c}_{2\times34}$	$\tilde{c}_{2\times14}$
$\tilde{d}_{2\times1\times43}$	$\tilde{d}_{2\times3\times41}$	$\tilde{c}_{1\times43}$	$\tilde{c}_{3\times41}$
$\tilde{d}_{4\times3\times21}$	$\tilde{d}_{4\times1\times23}$	$\tilde{c}_{4\times123}$	
$\tilde{d}_{2\times34\times1}$	$\tilde{d}_{3\times41\times2}$	$\tilde{c}_{1\times234}$	$\tilde{c}_{3\times412}$
$\tilde{d}_{1\times23\times4}$	$\tilde{d}_{4\times12\times3}$	$\tilde{c}_{2\times341}$	
$\tilde{d}_{2\times3\times4}$	$\tilde{d}_{4\times1\times2}$	$\tilde{c}_{12\times34}$	$\tilde{c}_{23\times41}$
$\tilde{d}_{1\times2\times3}$		$\tilde{b}_{34}$	$\tilde{b}_{14}$
$\tilde{d}_{3\times4\times1}$		$\tilde{b}_{234}$	$\tilde{b}_{412}, \tilde{b}_{341}$
		$\tilde{b}_{1234}$	

Table 3.2: Minimal set of integral coefficients for $A_4^{1234}(g^+, g^+, g^+, g^-; h)$.

As before, not all coefficients need to be calculated separately, and the minimal set is shown in columns 1 and 3 of table 3.2. The related coefficients can mostly be obtained through symmetry properties and relabelling of momenta, except for the coefficient $\tilde{b}_{341}$,

$$\tilde{b}_{341}(1^+, 2^+, 3^+, 4^-) = -\tilde{b}_{234}(2^+, 3^+, 1^+, 4^-) - \tilde{b}_{234}(2^+, 1^+, 3^+, 4^-)\,. \tag{3.5.17}$$

Where a triangle coefficient has no mass dependence we will omit the superscripts in eq. (3.3.20).

3.5.2.1 Boxes

$\tilde{d}_{1\times2\times34}$

$$\begin{aligned}\tilde{d}_{1\times2\times34}(1^+, 2^+, 3^+, 4^-) = \mathcal{C}^{(4)}_{1\times2\times3\times4}\, &\tilde{e}_{\{1^+\times2^+\times3^+\times4^-\}} \\ &- 2\,m^2 \frac{[1\,2]}{\langle 1\,2\rangle}\left[\frac{\langle 2\,4\rangle^2\ \langle 4|(2+3)|1]}{\langle 2\,3\rangle\ \langle 3\,4\rangle\ \langle 2|(3+4)|1]}\right. \\ &\left. + \frac{[2\,3]\ \langle 1|(2+4)|3]^2}{[3\,4]\ \langle 1|(3+4)|2]\ \langle 1|(2+3)|4]}\right]\end{aligned} \tag{3.5.18}$$

$\tilde{d}_{1\times4\times32}$

$$\tilde{d}_{1\times4\times32}(1^+, 2^+, 3^+, 4^-) = \mathcal{C}^{(2)}_{2\times3\times4\times1}\, \tilde{e}_{\{2^+\times3^+\times4^-\times1^+\}}$$

$$+\,2\,m^2\frac{[2\,3]\;s_{14}s_{234}^2}{\langle 2\,3\rangle^2\,[3\,4]\,\langle 1|(3+4)|2]\,\langle 1|(2+3)|4]} \tag{3.5.19}$$

$\tilde{d}_{2\times 1\times 43}$

$$\begin{aligned}\tilde{d}_{2\times 1\times 43}(1^+,2^+,3^+,4^-) =\, & \mathcal{C}^{(2)}_{3\times 4\times 1\times 2}\,\tilde{e}_{\{3^+\times 4^-\times 1^+\times 2^+\}} \\ & + 2\,m^2\frac{[1\,2]}{\langle 1\,2\rangle}\Bigg[\frac{[1\,3]^2\;\langle 2|(1+4)|3]}{[1\,4]\;[3\,4]\;\langle 2|(3+4)|1]} \\ & + \frac{\langle 1\,4\rangle\;\langle 4|(1+3)|2]^2}{\langle 3\,4\rangle\;\langle 1|(3+4)|2]\;\langle 3|(1+4)|2]}\Bigg]\end{aligned} \tag{3.5.20}$$

$\tilde{d}_{4\times 3\times 21}$

$$\begin{aligned}\tilde{d}_{4\times 3\times 21}(1^+,2^+,3^+,4^-) =\, & \mathcal{C}^{(2)}_{1\times 2\times 3\times 4}\,\tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}} \\ & + 2\,m^2\frac{s_{34}\,s_{123}^2}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 1|(2+3)|4]\;\langle 3|(1+2)|4]}\end{aligned} \tag{3.5.21}$$

$\tilde{d}_{1\times 23\times 4}$

$$\tilde{d}_{1\times 23\times 4}(1^+,2^+,3^+,4^-) = \mathcal{C}^{(3)}_{1\times 2\times 3\times 4}\,\tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}} \tag{3.5.22}$$

$\tilde{d}_{2\times 34\times 1}$

$$\begin{aligned}\tilde{d}_{2\times 34\times 1}(1^+,2^+,3^+,4^-) =\, & \mathcal{C}^{(3)}_{2\times 3\times 4\times 1}\,\tilde{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}} \\ & + \frac{2\;\langle 2\,4\rangle}{\langle 1\,2\rangle\;\langle 2\,3\rangle}\Bigg[\frac{\langle 1\,4\rangle\;\langle 2\,4\rangle\;\langle 1|(3+4)|2]\;\langle 2|(3+4)|1]}{\langle 1\,2\rangle^2\;\langle 3\,4\rangle} \\ & - m^2\left(3\,\frac{\langle 1\,4\rangle\;\langle 2\,4\rangle\;[1\,2]}{\langle 1\,2\rangle\;\langle 3\,4\rangle} + 2\frac{[1\,3]\;[2\,3]}{[3\,4]} + \frac{\langle 2\,4\rangle\;[1\,4]\;[2\,3]}{\langle 2\,3\rangle\;[3\,4]}\right)\Bigg]\end{aligned} \tag{3.5.23}$$

$\tilde{d}_{2\times 3\times 4}$

$$\begin{aligned}\tilde{d}_{2\times 3\times 4}(1^+,2^+,3^+,4^-) =\, & \mathcal{C}^{(1)}_{1\times 2\times 3\times 4}\,\tilde{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}} + \mathcal{C}^{(5)}_{2\times 3\times 4\times 1}\,\tilde{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}} \\ & + 2\,m^2\frac{s_{234}\;\langle 3\,4\rangle\;[2\,3]^2}{\langle 2\,3\rangle\;\langle 1|(3+4)|2]\;\langle 1|(2+3)|4]}\end{aligned} \tag{3.5.24}$$

$\tilde{\boldsymbol{d}}_{\mathbf{1\times 2\times 3}}$

$$\begin{aligned}\tilde{d}_{1\times2\times3}(1^+,2^+,3^+,4^-) =&\, \mathcal{C}^{(5)}_{1\times2\times3\times4}\,\tilde{e}_{\{1^+\times2^+\times3^+\times4^-\}} + \mathcal{C}^{(1)}_{4\times1\times2\times3}\,\tilde{e}_{\{4^-\times1^+\times2^+\times3^+\}} \\ &+ 2\,m^2 \frac{s_{123}\,[1\,2]\,[2\,3]}{\langle 3|(1+2)|4]\,\langle 1|(2+3)|4]}\end{aligned} \tag{3.5.25}$$

$\tilde{\boldsymbol{d}}_{\mathbf{3\times 4\times 1}}$

$$\begin{aligned}\tilde{d}_{3\times4\times1}(1^+,2^+,3^+,4^-) =&\, \mathcal{C}^{(1)}_{2\times3\times4\times1}\,\tilde{e}_{\{2^+\times3^+\times4^-\times1^+\}} + \mathcal{C}^{(5)}_{3\times4\times1\times2}\,\tilde{e}_{\{3^+\times4^-\times1^+\times2^+\}} \\ &- 2m^2 \frac{1}{\langle 1\,3\rangle}\left[\frac{[2\,3]\,\langle 3\,4\rangle^2\,s_{14}}{\langle 2\,3\rangle^2\,\langle 1|(3+4)|2]} + \frac{[1\,2]\,\langle 1\,4\rangle^2\,s_{34}}{\langle 1\,2\rangle^2\,\langle 3|(1+4)|2]}\right] \\ &+ \frac{\langle 1\,4\rangle\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,\langle 1\,3\rangle^2\,\langle 2\,3\rangle}\left[2\,s_{14}\,s_{34} + 6\,m^2\,s_{13}\right]\end{aligned} \tag{3.5.26}$$

3.5.2.2 Triangles

$\tilde{\boldsymbol{c}}_{\mathbf{3\times 4}}$

$$\tilde{c}_{3\times4}(1^+,2^+,3^+,4^-) = 2s_{34}\,\frac{\langle 1\,4\rangle\,\langle 4\,3\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 1\,3\rangle^2} \tag{3.5.27}$$

$\tilde{\boldsymbol{c}}_{\mathbf{2\times 34}}$

$$\tilde{c}_{2\times34}(1^+,2^+,3^+,4^-) = -2(s_{23}+s_{24})\frac{\langle 1\,4\rangle\,\langle 2\,4\rangle^2}{\langle 1\,2\rangle^3\,\langle 2\,3\rangle\,\langle 3\,4\rangle} \tag{3.5.28}$$

$\tilde{\boldsymbol{c}}_{\mathbf{1\times 43}}$

$$\tilde{c}_{1\times43}(1^+,2^+,3^+,4^-) = -2(s_{13}+s_{14})\,\frac{(\langle 1\,2\rangle\,\langle 3\,4\rangle + \langle 1\,3\rangle\,\langle 2\,4\rangle)}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\left[\frac{\langle 1\,4\rangle}{\langle 1\,2\rangle\,\langle 1\,3\rangle}\right]^2 \tag{3.5.29}$$

$\tilde{\boldsymbol{c}}^{(\mathbf{0})}_{\mathbf{4\times 123}}$, $\tilde{\boldsymbol{c}}^{(\mathbf{2})}_{\mathbf{4\times 123}}$

$$\tilde{c}^{(0)}_{4\times123}(1^+,2^+,3^+,4^-) = 0 \tag{3.5.30}$$

$$\tilde{c}^{(2)}_{4\times123}(1^+,2^+,3^+,4^-) = 4\frac{s_{123}\,(s_{14}+s_{24}+s_{34})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3|(1+2)|4]\,\langle 1|(2+3)|4]} \tag{3.5.31}$$

$\tilde{c}^{(0)}_{1\times 234}$, $\tilde{c}^{(2)}_{1\times 234}$

$$\tilde{c}^{(0)}_{1\times 234}(1^+,2^+,3^+,4^-) = 2(s_{12}+s_{13}+s_{14})\frac{\langle 1\,4\rangle\ \langle 2\,4\rangle^2}{\langle 1\,2\rangle^3\ \langle 2\,3\rangle\ \langle 3\,4\rangle} \tag{3.5.32}$$

$$\begin{aligned}\tilde{c}^{(2)}_{1\times 234}(1^+,2^+,3^+,4^-) =& \frac{8\ \langle 4|(2+3)|1]^3}{s_{234}\,(s_{12}+s_{13}+s_{14})\ \langle 2\,3\rangle\ \langle 3\,4\rangle\ \langle 2|(3+4)|1]} \\ & - 4\,(s_{12}+s_{13}+s_{14})\Bigg\{\frac{\langle 1\,4\rangle^2\ [2\,3]}{\langle 1\,2\rangle^2\ \langle 3\,4\rangle\ \langle 1|(3+4)|2]\ \langle 1|(2+3)|4]} \\ & - \frac{[2\,3]}{\langle 1\,2\rangle\ \langle 1|(2+3)|4]\ \langle 1|(3+4)|2]\ \langle 2|(3+4)|1]} \\ & \times \Big[\frac{\langle 1\,2\rangle\,[1\,3]\,[2\,3]}{[3\,4]} + \frac{[1\,2]\,\langle 1\,4\rangle\,\langle 2\,4\rangle}{\langle 3\,4\rangle}\Big] \\ & - \Big[\frac{[1\,3]\ \langle 1\,4\rangle}{\langle 1\,2\rangle} + \frac{\langle 4|(2+3)|1]}{\langle 2\,3\rangle}\Big] \\ & \times \Big[\frac{\langle 2\,4\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle\ \langle 1|(2+3)|4]\ \langle 2|(3+4)|1]}\Big]\Bigg\}\end{aligned} \tag{3.5.33}$$

$\tilde{c}^{(0)}_{2\times 341}$, $\tilde{c}^{(2)}_{2\times 341}$

$$\tilde{c}^{(0)}_{2\times 341}(1^+,2^+,3^+,4^-) = 2(s_{12}+s_{23}+s_{24})\ \langle 2\,4\rangle^2\ \frac{\langle 1\,4\rangle^2\ \langle 2\,3\rangle^2 + \langle 1\,2\rangle^2\ \langle 3\,4\rangle^2}{\langle 1\,2\rangle^3\ \langle 2\,3\rangle^3\ \langle 1\,4\rangle\ \langle 3\,4\rangle} \tag{3.5.34}$$

$$\begin{aligned}\tilde{c}^{(2)}_{2\times 341}(1^+,2^+,3^+,4^-) =& \frac{4(s_{12}+s_{23}+s_{24})\,s_{134}\ [1\,2]\ [3\,2]}{\langle 1\,2\rangle\ \langle 3\,2\rangle\ [1\,4]\ [3\,4]\ \langle 1|(3+4)|2]\ \langle 3|(1+4)|2]} \\ & - \frac{8}{(s_{12}+s_{23}+s_{24})\,s_{134}}\frac{\langle 4|(1+3)|2]^4}{\langle 1\,4\rangle\ \langle 3\,4\rangle\ \langle 1|(3+4)|2]\ \langle 3|(1+4)|2]} \\ & + \Bigg\{\frac{4\,(s_{12}+s_{23}+s_{24})}{[(s_{13}+s_{14})\,(s_{23}+s_{24}) - s_{12}\,s_{34}]} \\ & \times \Big[\frac{[1\,3]^2\ [2\,3]}{\langle 1\,2\rangle\ [1\,4]\ [3\,4]} - \frac{[1\,2]\ \langle 1\,4\rangle\ \langle 2\,4\rangle^2}{\langle 1\,2\rangle^2\ \langle 2\,3\rangle\ \langle 3\,4\rangle}\Big]\Bigg\} \\ & + \Big\{1 \leftrightarrow 3\Big\}\end{aligned} \tag{3.5.35}$$

$\tilde{c}^{(0)}_{12\times34}$, $\tilde{c}^{(2)}_{12\times34}$

$$\tilde{c}^{(0)}_{12\times34}(1^+,2^+,3^+,4^-) = 0 \tag{3.5.36}$$

$$\begin{aligned}\tilde{c}^{(2)}_{12\times34}(1^+,2^+,3^+,4^-) &= -4\,\frac{[1\,2]}{\langle 1\,2\rangle\;\langle 3|(1+2)|4]\;\big[(s_{13}+s_{14})\,(s_{23}+s_{24}) - s_{12}\,s_{34}\big]}\\
&\times\Bigg[\big(\langle 1\,4\rangle\;[1\,3] - \langle 2\,4\rangle\;[2\,3]\big)\,(s_{13}+s_{23}-s_{14}-s_{24})\\
&+\langle 1\,2\rangle\;\langle 3\,4\rangle\;[1\,3]\;[2\,3]\left(2-\frac{(s_{13}+s_{23})\,(s_{13}+s_{14}+s_{23}+s_{24})}{s_{12}\,s_{34}}\right)\\
&+\langle 1\,4\rangle\;\langle 2\,4\rangle\;[1\,2]\;[3\,4]\left(2-\frac{(s_{14}+s_{24})\,(s_{13}+s_{14}+s_{23}+s_{24})}{s_{12}\,s_{34}}\right)\Bigg]\end{aligned} \tag{3.5.37}$$

3.5.2.3 Bubbles

$\tilde{b}_{34}$

$$\tilde{b}_{34}(1^+,2^+,3^+,4^-) = \frac{4}{\langle 1\,2\rangle^2\,\langle 1\,3\rangle\,\langle 2\,3\rangle}\left(\frac{\langle 2\,4\rangle^2\,\langle 1\,3\rangle\,[2\,3]}{(s_{23}+s_{24})} - \frac{\langle 1\,4\rangle^2\,\langle 2\,3\rangle\,[1\,3]}{(s_{13}+s_{14})}\right) \tag{3.5.38}$$

$\tilde{b}_{234}$

$$\begin{aligned}\tilde{b}_{234}(1^+,2^+,3^+,4^-) = \frac{4}{\langle 2\,3\rangle\;\langle 3\,4\rangle}\Bigg(&\frac{\langle 2\,4\rangle^2\;\langle 4|(2+3)|1]}{\langle 1\,2\rangle^2\;\langle 2|(3+4)|1]}\\
&-\frac{\langle 4|(2+3)|1]^3}{\langle 2|(3+4)|1]\;(s_{1234}-s_{234})^2}\\
&-\frac{\langle 2\,4\rangle^2\;[2\,3]\;\langle 3\,4\rangle}{\langle 1\,2\rangle^2\;(s_{23}+s_{24})}\Bigg)\end{aligned} \tag{3.5.39}$$

$\tilde{b}_{1234}$

The full amplitude must be finite in four dimensions; and since the bubble integrals are divergent, one bubble coefficient can be identified from the remainder of the others,

$$\tilde{b}_{1234}(1^+,2^+,3^+,4^-) = -\tilde{b}_{34} - \tilde{b}_{41} - \tilde{b}_{234} - \tilde{b}_{412} - \tilde{b}_{341}\,, \tag{3.5.40}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$\tilde{d}_{4\times3\times21}$	$\tilde{d}_{2\times1\times43}, \tilde{d}_{3\times2\times14}, \tilde{d}_{1\times4\times32}, \tilde{d}_{1\times2\times34}, \tilde{d}_{2\times3\times41}, \tilde{d}_{3\times4\times12}, \tilde{d}_{4\times1\times23}$	$\tilde{c}_{3\times4}$	$\tilde{c}_{4\times1}, \tilde{c}_{2\times3}, \tilde{c}_{1\times2}$
		$\tilde{c}_{2\times34}$	$\tilde{c}_{3\times41}, \tilde{c}_{4\times12}, \tilde{c}_{1\times23}$
			$\tilde{c}_{1\times43}, \tilde{c}_{2\times14}, \tilde{c}_{3\times21}, \tilde{c}_{4\times32}$
$\tilde{d}_{1\times23\times4}$	$\tilde{d}_{2\times34\times1}, \tilde{d}_{3\times41\times2}, \tilde{d}_{4\times12\times3}$	$\tilde{c}_{12\times34}$	$\tilde{c}_{23\times41}$
$\tilde{d}_{1\times2\times3}$	$\tilde{d}_{2\times3\times4}, \tilde{d}_{3\times4\times1}, \tilde{d}_{4\times1\times2}$	$\tilde{c}_{1\times234}$	$\tilde{c}_{2\times341}, \tilde{c}_{3\times412}, \tilde{c}_{4\times123}$
		$\tilde{b}_{34}$	$\tilde{b}_{12}, \tilde{b}_{23}, \tilde{b}_{41}$
		$\tilde{b}_{234}$	$\tilde{b}_{341}, \tilde{b}_{412}, \tilde{b}_{123}$
		$\tilde{b}_{1234}$	

Table 3.3: Minimal set of integral coefficients for $A_4^{1234}(g^+, g^-, g^+, g^-; h)$.

where for the sake of readability the momentum and helicity labels have been excluded on the right hand side.

3.5.2.4 Rational terms

$$\begin{aligned}\tilde{r}(1^+,2^+,3^+,4^-) = \frac{1}{2}\Big[&\tilde{c}^{(2)}_{12\times34}(1^+,2^+,3^+,4^-) + \tilde{c}^{(2)}_{12\times34}(3^+,2^+,1^+,4^-)\\ &+ \tilde{c}^{(2)}_{1\times234}(1^+,2^+,3^+,4^-) + \tilde{c}^{(2)}_{1\times234}(3^+,2^+,1^+,4^-)\\ &+ \tilde{c}^{(2)}_{4\times123}(1^+,2^+,3^+,4^-) + \tilde{c}^{(2)}_{2\times341}(1^+,2^+,3^+,4^-)\Big]\end{aligned} \tag{3.5.41}$$

3.5.3 Coefficients for $A_4^{1234}(g^+, g^-, g^+, g^-; h)$

The effective pentagon coefficients for this helicity combination are,

$$\tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}} = -4m^4 \frac{\langle 1\,2\rangle\,[3\,4]\,\langle 4|(2+3)|1]^2}{[1\,2]\,\langle 3\,4\rangle\,\langle 1|(2+3)|4]^2}\,, \tag{3.5.42}$$

$$\tilde{e}_{\{3^+\times4^-\times1^+\times2^-\}} = \tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}}\Big\{1 \leftrightarrow 3,\ 2 \leftrightarrow 4\Big\}, \tag{3.5.43}$$

$$\tilde{e}_{\{4^-\times1^+\times2^-\times3^+\}} = \tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}}\Big\{1 \to 4,\ 2 \to 1,\ 3 \to 2,\ 4 \to 3,\ \langle\rangle \leftrightarrow []\Big\}, \tag{3.5.44}$$

$$\tilde{e}_{\{2^-\times3^+\times4^-\times1^+\}} = \tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}}\Big\{1 \to 2,\ 2 \to 3,\ 3 \to 4,\ 4 \to 1,\ \langle\rangle \leftrightarrow []\Big\}. \tag{3.5.45}$$

As before, not all coefficients need to be calculated separately, and the minimal set is shown in columns 1 and 3 of table 3.3.

3.5.3.1 Boxes

$\tilde{d}_{4\times3\times21}$

$$\begin{aligned}\tilde{d}_{4\times3\times21}(1^+,2^-,3^+,4^-) =& \mathcal{C}^{(2)}_{1\times2\times3\times4}\,\tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}}\\ &- \frac{2\,\langle2|(1+3)|4]}{\langle1|(2+3)|4]\,\langle3|(1+2)|4]}\Bigg[\frac{\langle2\,3\rangle\,\langle2|(1+3)|4]\,s_{34}s_{123}^2}{\langle1\,2\rangle\,\langle3|(1+2)|4]^2}\\ &+ m^2\Bigg(2\frac{[1\,3]\,\langle4|(2+3)|1]}{[1\,2]} + \frac{[2\,3]\,\langle2|(1+3)|4]\,\langle4|(2+3)|1]}{[1\,2]\,\langle1|(2+3)|4]}\\ &+ 3\frac{\langle2\,3\rangle\,\langle2|(1+3)|4]\,\langle4|(1+2)|3]}{\langle1\,2\rangle\,\langle3|(1+2)|4]}\Bigg)\Bigg]\end{aligned} \tag{3.5.46}$$

$\tilde{d}_{1\times23\times4}$

$$\begin{aligned}\tilde{d}_{1\times23\times4}(1^+,2^-,3^+,4^-) =& \mathcal{C}^{(3)}_{1\times2\times3\times4}\,\tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}}\\ &- 2\,m^2\frac{\langle4|(2+3)|1]}{\langle1|(2+3)|4]}\left[\frac{\langle1\,2\rangle\,\langle2\,4\rangle^2}{\langle1\,4\rangle\,\langle2\,3\rangle\,\langle3\,4\rangle} + \frac{[1\,3]^2\,[3\,4]}{[1\,2]\,[1\,4]\,[2\,3]}\right]\end{aligned} \tag{3.5.47}$$

$\tilde{d}_{1\times2\times3}$

$$\begin{aligned}\tilde{d}_{1\times2\times3}(1^+,2^-,3^+,4^-) =& \mathcal{C}^{(5)}_{1\times2\times3\times4}\,\tilde{e}_{\{1^+\times2^-\times3^+\times4^-\}} + \mathcal{C}^{(1)}_{4\times1\times2\times3}\,\tilde{e}_{\{4^-\times1^+\times2^-\times3^+\}}\\ &+ \frac{\langle1\,2\rangle\,\langle2\,3\rangle}{\langle1|(2+3)|4]\,\langle3|(1+2)|4]}\Bigg[-2\frac{s_{12}s_{23}s_{123}}{\langle1\,3\rangle^2}\\ &+ 2m^2\Bigg(2\frac{[1\,3]\,s_{123}}{\langle1\,3\rangle}\\ &- [1\,3]^2 + \frac{[1\,2]\,[2\,3]\,\langle2\,4\rangle^2}{\langle1\,4\rangle\,\langle3\,4\rangle}\\ &- \frac{[2\,3]\,\langle2|(1+3)|4]\,\langle4|(2+3)|1]}{\langle3\,4\rangle\,\langle1|(2+3)|4]}\\ &+ \frac{[1\,2]\,\langle2|(1+3)|4]\,\langle4|(1+2)|3]}{\langle1\,4\rangle\,\langle3|(1+2)|4]}\Bigg)\Bigg]\end{aligned} \tag{3.5.48}$$

3.5.3.2 Triangles

$\tilde{c}_{3\times 4}$

$$\begin{aligned}\tilde{c}_{3\times4}(1^+,2^-,3^+,4^-) = {}&\frac{-2s_{34}}{s_{12}\,\langle 3|(1+2)|4]^3\,\langle 1\,3\rangle^2\,[2\,4]^2}\Big\{\,\langle 3|1|4]^3\,[s_{24}(s_{14}+s_{24})\\
&+s_{12}(s_{23}+s_{34})\\
&+\langle 3|1|4]^2\,\langle 3|2|4]\,\Big[s_{12}^2-s_{14}(s_{14}+s_{24})+s_{12}(3s_{234}-5s_{24})\Big]\\
&+\langle 3|2|4]^3\,[s_{13}(s_{13}+s_{23})+s_{12}(s_{14}+s_{34})]\\
&+\langle 3|1|4]\,\langle 3|2|4]^2\,\Big[s_{12}^2-s_{23}(s_{13}+s_{23})+s_{12}(3s_{134}-5s_{13})\Big]\,\Big\}\end{aligned}\tag{3.5.49}$$

$\tilde{c}_{2\times 34}$

$$\begin{aligned}\tilde{c}_{2\times34}(1^+,2^-,3^+,4^-) = {}&2(s_{23}+s_{24})\\
&\times\frac{s_{234}\;[2\,3]^2\Big(\,\langle 1|(2+4)|3]\;[2\,4]+\langle 1|(3+4)|2]\;[3\,4]\,\Big)}{\langle 1|(3+4)|2]^3\;[2\,4]^2\;[3\,4]}\end{aligned}\tag{3.5.50}$$

$\tilde{c}^{(0)}_{12\times 34}$, $\tilde{c}^{(2)}_{12\times 34}$

$$\begin{aligned}&\tilde{c}^{(0)}_{12\times34}(1^+,2^-,3^+,4^-)\\
&\quad=\Bigg\{2\,\frac{\langle 2\,3\rangle^3\;[3\,4]\;\langle 3|(1+2)|3]\;(\langle 3|(1+2)|3]\;[2\,3]-[1\,2]\;\langle 1\,4\rangle\;[3\,4])}{\langle 1\,2\rangle\;\langle 1|(3+4)|2]\;\langle 3|(1+2)|4]^3}\\
&\qquad+2\,\frac{\langle 2\,3\rangle^2\;[3\,4]\;\langle 4|(1+2)|3]\;(-2\,s_{23}-s_{24})}{\langle 1\,2\rangle\;\langle 1|(3+4)|2]\;\langle 3|(1+2)|4]^2}\\
&\qquad+2\,\frac{[1\,2]\,\langle 2\,3\rangle^2\,[3\,4]}{\langle 1|(3+4)|2]\,\langle 3|(1+2)|4]^3}(2s_{12}(s_{23}-s_{14}-s_{34})+2s_{13}s_{23}\\
&\qquad+2s_{23}^2+s_{14}s_{34}-s_{23}s_{34}+2\,[1\,2]\,\langle 1\,3\rangle\,\langle 2\,4\rangle\,[3\,4])\\
&\qquad+2\,\frac{[1\,4]^2}{\langle 1|(3+4)|2]\,\langle 3|(1+2)|4]^2}(\langle 1\,4\rangle\,\langle 2\,4\rangle\,(2(s_{13}-s_{24})-3(s_{34}+s_{14})\\
&\qquad-4(s_{12}+s_{23}))-2\,\langle 1\,3\rangle\,[2\,3]\,\langle 2\,4\rangle^2+3\,[1\,3]\,\langle 1\,4\rangle^2\,\langle 2\,3\rangle)\\
&\qquad+\frac{s_{14}^2\,s_{12}\,(6\,s_{13}-2\,s_{14}+2\,s_{23}+2\,s_{24})-s_{14}^4+s_{14}^2\,s_{23}^2}{\langle 1|(3+4)|2]^2\;\langle 3|(1+2)|4]^2}\end{aligned}$$

$$
\begin{aligned}
&- 8 \frac{s_{12}\, s_{13}\, s_{14}\, s_{23}}{\langle 1|(3+4)|2]^2\ \langle 3|(1+2)|4]^2} \\
&+ 4 \frac{s_{14}\, s_{1234}\ \langle 2|(3+4)|1]\ \langle 4|(1+2)|3]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)} \\
&+ 4 \frac{\langle 1\,2\rangle\ [1\,3]\ \langle 2|(3+4)|1]\ \langle 4|(1+2)|3]\ \langle 3|(1+4)|2]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)} \\
&+ \frac{\langle 1|(2+3)|4]\ \langle 2|(3+4)|1]\ \langle 3|(1+4)|2]\ \langle 4|(1+2)|3]}{\langle 1|(3+4)|2]^2\ \langle 3|(1+2)|4]^2\ \Delta_3(1,2,3,4)} \\
&\times \left(\Pi(4,3,2,1)\,\Pi(1,2,3,4) + \Delta_3(1,2,3,4)\right) \\
&- 3 \frac{s_{1234}\ \langle 2|(3+4)|1]\ \langle 4|(1+2)|3]}{2\ \langle 1|(3+4)|2]\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)^2} \Pi(4,3,2,1)\,\Pi(1,2,3,4) \\
&\times (s_{13} + s_{14} + s_{23} + s_{24}) \\
&+ 5 \frac{s_{1234}\ \langle 2|(3+4)|1]\ \langle 4|(1+2)|3]\ (s_{13} + s_{14} + s_{23} + s_{24})}{2\ \langle 1|(3+4)|2]\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)} \\
&\left. - 4 \frac{\langle 2|(3+4)|1]\ \langle 4|(1+2)|3]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]} \right\} \\
&+ \Big\{ 1 \leftrightarrow 3, 2 \leftrightarrow 4 \Big\} + \Big\{ 1 \leftrightarrow 2, 3 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,] \Big\} + \Big\{ 1 \leftrightarrow 4, 2 \leftrightarrow 3, \langle\,\rangle \leftrightarrow [\,] \Big\}
\end{aligned}
\tag{3.5.51}
$$

where

$$
\Pi(i,j,k,l) = s_{ik} + s_{jk} - s_{il} - s_{jl}\,. \tag{3.5.52}
$$

$$
\begin{aligned}
\tilde{c}^{(2)}_{12\times 34}(1^+, 2^-, 3^+, 4^-) = &\left\{ 4 \frac{\langle 2|(3+4)|1]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]} \left[\frac{[2\,3]^2\,(s_{23} - s_{14})}{[1\,2]\,[3\,4]\,\langle 1|(3+4)|2]} \right.\right. \\
&+ \frac{3}{2} \frac{[1\,3]\,[2\,3]}{[1\,2]\,[3\,4]} \\
&+ \langle 2\,4\rangle\ \frac{(\langle 3|(1+2)|3] - \langle 4|(1+2)|4])}{\Delta_3(1,2,3,4)} \\
&\left.\left. \times \left([2\,3] - \frac{\langle 1\,4\rangle\, p_{12} \cdot p_{34}}{\langle 1\,2\rangle\,\langle 3\,4\rangle} \right) \right] \right\} \\
&+ \Big\{ 1 \leftrightarrow 3, 2 \leftrightarrow 4 \Big\} + \Big\{ 1 \leftrightarrow 2, 3 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,] \Big\} \\
&+ \Big\{ 1 \leftrightarrow 4, 2 \leftrightarrow 3, \langle\,\rangle \leftrightarrow [\,] \Big\}
\end{aligned}
\tag{3.5.53}
$$

where Δ_3 is defined as

$$
\Delta_3(i,j,k,l) = (s_{ijkl} - s_{ij} - s_{kl})^2 - 4 s_{ij} s_{kl}\,. \tag{3.5.54}
$$

$\tilde{\boldsymbol{c}}^{(0)}_{\mathbf{1\times 234}}$, $\tilde{\boldsymbol{c}}^{(2)}_{\mathbf{1\times 234}}$

$$\begin{aligned}\tilde{c}^{(0)}_{1\times 234}(1^+,2^-,3^+,4^-) = &-2(s_{12}+s_{13}+s_{14})\frac{s_{234}\ \langle 1|(2+4)|3]^2}{[2\,3]\ [3\,4]\ \langle 1|(2+3)|4]^3\ \langle 1|(3+4)|2]^3}\\ &\times\left([2\,3]^2\ \langle 1|(2+3)|4]^2+[3\,4]^2\ \langle 1|(3+4)|2]^2\right)\end{aligned} \tag{3.5.55}$$

$$\begin{aligned}\tilde{c}^{(2)}_{1\times 234}(1^+,2^-,3^+,4^-) = 4\,&\frac{(s_{12}+s_{13}+s_{14})}{\langle 1|(3+4)|2]\ \langle 1|(2+3)|4]}\Bigg[\left(\frac{\langle 2\,4\rangle^2}{\langle 2\,3\rangle\ \langle 3\,4\rangle}-\frac{[1\,3]^2}{[1\,2]\ [1\,4]}\right)\\ &+\frac{\langle 1|(2+4)|3]}{\langle 1|(3+4)|2]}\left(\frac{\langle 1\,4\rangle\ \langle 2\,4\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle}+\frac{[1\,3]\ [2\,3]}{[1\,2]\ [3\,4]}\right)\\ &+\frac{\langle 1|(2+4)|3]}{\langle 1|(2+3)|4]}\left(\frac{\langle 1\,2\rangle\ \langle 2\,4\rangle}{\langle 1\,4\rangle\ \langle 2\,3\rangle}+\frac{[1\,3]\ [3\,4]}{[1\,4]\ [2\,3]}\right)\Bigg]\\ &-8\,\frac{[1\,3]^4}{[1\,2]\ [1\,4]\ [2\,3]\ [3\,4]\ (s_{12}+s_{13}+s_{14})}\end{aligned} \tag{3.5.56}$$

3.5.3.3 Bubbles

$\tilde{\boldsymbol{b}}_{\mathbf{34}}$

$$\begin{aligned}\tilde{b}_{34}(1^+,2^-,3^+,4^-) = \Bigg\{&4\,\frac{[1\,3]\ \langle 1\,4\rangle^2\ s_{134}}{\langle 1\,3\rangle\ (s_{13}+s_{14})\ \langle 1|(3+4)|2]^2}\\ &+4\,\frac{[1\,4]^2\ \langle 3\,4\rangle\ (s_{13}+s_{14})\ \langle 4|(1+2)|3]\ (2\,s_{123}+s_{124})}{[1\,2]\ \langle 1|(3+4)|2]\ \langle 3|(1+2)|4]^2\ \Delta_3(1,2,3,4)}\\ &+12\,\frac{\langle 1\,2\rangle\ [1\,4]^2\ \langle 3\,4\rangle\ \langle 4|(1+2)|3]\ s_{134}}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]^2\ \Delta_3(1,2,3,4)}\\ &+4\,\frac{\langle 4|(1+2)|3]\ \langle 4|(1+3)|4]}{\langle 1|(3+4)|2]^2\ \langle 3|(1+2)|4]}\\ &+8\,\frac{[1\,3]\ \langle 3\,4\rangle\ \langle 1|(2+3)|4]\ \langle 4|(1+2)|3]\ (s_{234}-s_{134})}{\langle 1|(3+4)|2]^2\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)}\\ &-2\,\frac{[1\,4]\ \langle 2\,3\rangle\ \langle 4|(1+2)|3]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]^2}\\ &+4\,\frac{[1\,4]\ \langle 2\,3\rangle\ (s_{13}+s_{14})\ \langle 4|(1+2)|3]\ (s_{123}-s_{124})}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]^2\ \Delta_3(1,2,3,4)}\\ &+3\,\frac{\langle 2|(3+4)|1]\ \langle 4|(1+2)|3]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)^2}\\ &\times(s_{123}-s_{124})\,(s_{234}-s_{134})\,(s_{134}+s_{234})\\ &+\frac{\langle 2|(3+4)|1]\ \langle 4|(1+2)|3]}{\langle 1|(3+4)|2]\ \langle 3|(1+2)|4]\ \Delta_3(1,2,3,4)}\end{aligned}$$

$$\times (s_{13} - 5\, s_{14} - 5\, s_{23} + s_{24} - 14\, s_{34}) \Bigg\}$$
$$+ \Big\{ 1 \leftrightarrow 2, 3 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,] \Big\} \tag{3.5.57}$$

where Δ_3 is given by eq. (3.5.54).

$\tilde{b}_{234}$

$$\begin{aligned}\tilde{b}_{234}(1^+, 2^-, 3^+, 4^-) = \Bigg\{ & \frac{4\, s_{234}\ [3\,4]}{[2\,4]} \Bigg(- \frac{\langle 2\,4\rangle\ [3\,4]}{(s_{24} + s_{34})\ \langle 1|(2+3)|4]^2} \\ & + \frac{[1\,3]\ \langle 1|(2+4)|3]}{(s_{1234} - s_{234})\ \langle 1|(2+3)|4]\ [2\,3]\ [3\,4]} \\ & \times \Bigg[\frac{[1\,3]}{(s_{1234} - s_{234})} - \frac{[3\,4]}{\langle 1|(2+3)|4]} \Bigg] \Bigg) \Bigg\} \\ & + \Big\{ 2 \leftrightarrow 4 \Big\}\end{aligned} \tag{3.5.58}$$

$\tilde{b}_{1234}$

Similarly to eq. (3.5.40) we have,

$$\tilde{b}_{1234}(1^+, 2^-, 3^+, 4^-) = -\tilde{b}_{34} - \tilde{b}_{12} - \tilde{b}_{23} - \tilde{b}_{41} - \tilde{b}_{234} - \tilde{b}_{341} - \tilde{b}_{412} - \tilde{b}_{123}\,, \tag{3.5.59}$$

where we have again suppressed the momentum and helicity labels on the right hand side for simplicity.

3.5.3.4 Rational terms

$$\begin{aligned}\tilde{r}(1^+, 2^-, 3^+, 4^-) = \frac{1}{2}\Big[& \tilde{c}^{(2)}_{12\times 34}(1^+, 2^-, 3^+, 4^-) + \tilde{c}^{(2)}_{12\times 34}(2^+, 3^-, 4^+, 1^-)|_{[]\leftrightarrow\langle\rangle} \\ & + \tilde{c}^{(2)}_{1\times 234}(1^+, 2^-, 3^+, 4^-) + \tilde{c}^{(2)}_{1\times 234}(2^+, 3^-, 4^+, 1^-)|_{[]\leftrightarrow\langle\rangle} \\ & + \tilde{c}^{(2)}_{1\times 234}(3^+, 4^-, 1^+, 2^-) + \tilde{c}^{(2)}_{1\times 234}(4^+, 1^-, 2^+, 3^-)|_{[]\leftrightarrow\langle\rangle} \Big]\end{aligned} \tag{3.5.60}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$\tilde{d}_{1\times2\times34}$	$\tilde{d}_{2\times1\times43}, \tilde{d}_{3\times4\times12}, \tilde{d}_{4\times3\times21}$	$\tilde{c}_{2\times3}$	$\tilde{c}_{4\times1}$
$\tilde{d}_{1\times4\times32}$	$\tilde{d}_{3\times2\times14}, \tilde{d}_{4\times1\times23}, \tilde{d}_{2\times3\times41}$	$\tilde{c}_{1\times23}$	$\tilde{c}_{2\times14}, \tilde{c}_{3\times41}, \tilde{c}_{4\times32}$
$\tilde{d}_{2\times34\times1}$	$\tilde{d}_{4\times12\times3}$	$\tilde{c}_{1\times234}$	$\tilde{c}_{2\times341}, \tilde{c}_{3\times412}, \tilde{c}_{4\times123}$
$\tilde{d}_{1\times23\times4}$	$\tilde{d}_{3\times41\times2}$	$\tilde{c}_{23\times41}$	
$\tilde{d}_{1\times2\times3}$	$\tilde{d}_{3\times4\times1}, \tilde{d}_{4\times1\times2}, \tilde{d}_{2\times3\times4}$	$\tilde{b}_{23}$	$\tilde{b}_{41}$
		$\tilde{b}_{234}$	$\tilde{b}_{341}, \tilde{b}_{412}, \tilde{b}_{123}$
		$\tilde{b}_{1234}$	

Table 3.4: Minimal set of integral coefficients for $A_4^{1234}(g^+, g^+, g^-, g^-; h)$.

3.5.4 Coefficients for $A_4^{1234}(g^+, g^+, g^-, g^-; h)$

The effective pentagon coefficients are given by,

$$\tilde{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}} = -\,4\,m^4 \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]}\,, \tag{3.5.61}$$

$$\tilde{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}} = -\,4\,m^4 \frac{[2\,3]\;\langle 3\,4\rangle^2\;[4\,1]}{\langle 2\,3\rangle\;[3\,4]^2\;\langle 4\,1\rangle}\,, \tag{3.5.62}$$

$$\tilde{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}} = -\,4\,m^4 \frac{[2\,3]\;\langle 3\,4\rangle^2\;[4\,1]}{\langle 2\,3\rangle\;[3\,4]^2\;\langle 4\,1\rangle}\,, \tag{3.5.63}$$

$$\tilde{e}_{\{4^-\times 1^+\times 2^+\times 3^-\}} = \tilde{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}}\{2 \leftrightarrow 4, 1 \leftrightarrow 3, \langle\,\rangle \leftrightarrow [\,]\}\,. \tag{3.5.64}$$

The minimal set of coefficients that needs to be calculated is given in Table 3.4.

3.5.4.1 Boxes

$\tilde{d}_{1\times2\times34}$

$$\tilde{d}_{1\times2\times34}(1^+, 2^+, 3^-, 4^-) = \mathcal{C}^{(4)}_{1\times2\times3\times4}\,\tilde{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}} \tag{3.5.65}$$

$\tilde{d}_{1\times4\times32}$

$$\begin{aligned}\tilde{d}_{1\times4\times32}(1^+, 2^+, 3^-, 4^-) = \mathcal{C}^{(2)}_{2\times3\times4\times1}\,\tilde{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}} \\ -\,2\,\frac{[2\,4]^2}{\langle 1|(2+3)|4]\;[3\,4]}\Bigg\{\frac{s_{14}\,s_{234}^2\,\langle 1|(3+4)|2]}{[2\,3]\;\langle 1|(2+3)|4]^2} \\ +\,m^2\Big[3\,\frac{\langle 1|(3+4)|2]\,\langle 4|(2+3)|1]}{[2\,3]\,\langle 1|(2+3)|4]}\end{aligned}$$

$$+\frac{[1\,4]\,\langle 3\,4\rangle\, s_{234}}{\langle 2\,3\rangle\,[2\,4]\,[3\,4]}+\frac{\langle 3\,4\rangle\;\langle 3|(2+4)|1]}{\langle 2\,3\rangle\,[2\,4]}\Bigg]\Bigg\} \tag{3.5.66}$$

$\tilde{\boldsymbol{d}}_{\mathbf{2\times 34\times 1}}$

$$\begin{aligned}\tilde{d}_{2\times 34\times 1}(1^+,2^+,3^-,4^-) = {} & \mathcal{C}^{(3)}_{2\times 3\times 4\times 1}\,\tilde{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}} \\ & - 2\,m^2\frac{\langle 3\,4\rangle\;\langle 1|(3+4)|2]\;\langle 2|(3+4)|1]}{\langle 1\,2\rangle\;\langle 1\,4\rangle\;\langle 2\,3\rangle\;[3\,4]^2}\end{aligned} \tag{3.5.67}$$

$\tilde{\boldsymbol{d}}_{\mathbf{1\times 23\times 4}}$

$$\begin{aligned}\tilde{d}_{1\times 23\times 4}(1^+,2^+,3^-,4^-) = {} & \mathcal{C}^{(3)}_{1\times 2\times 3\times 4}\,\tilde{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}} \\ & + 2\,m^2\,\frac{\langle 4|(2+3)|1]}{\langle 1\,2\rangle\;[3\,4]\;\langle 1|(2+3)|4]} \\ & \times\left[\frac{s_{12}\,[2\,4]^2}{[1\,4]\;[2\,3]}+\frac{s_{34}\,\langle 1\,3\rangle^2}{\langle 2\,3\rangle\;\langle 1\,4\rangle}\right]\end{aligned} \tag{3.5.68}$$

$\tilde{\boldsymbol{d}}_{\mathbf{1\times 2\times 3}}$

$$\begin{aligned}\tilde{d}_{1\times 2\times 3}(1^+,2^+,3^-,4^-) = {} & \mathcal{C}^{(5)}_{1\times 2\times 3\times 4}\,\tilde{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}} + \mathcal{C}^{(1)}_{4\times 1\times 2\times 3}\,\tilde{e}_{\{4^-\times 1^+\times 2^+\times 3^-\}} \\ & - 2\,m^2\frac{[1\,2]^2\;\langle 2\,3\rangle}{\langle 1\,2\rangle\;[1\,4]\;[3\,4]}\end{aligned} \tag{3.5.69}$$

3.5.4.2 Triangles

$\tilde{\boldsymbol{c}}_{\mathbf{2\times 3}}$

$$\begin{aligned}\tilde{c}_{2\times 3}(1^+,2^+,3^-,4^-) = {} & -2\,\frac{s_{23}}{\langle 2|(1+4)|3]^3}\left\{\frac{[1\,3]^2\;s_{134}\;\langle 2|(3+4)|1]}{[1\,4]\;[3\,4]}\right. \\ & \left. + \frac{\langle 2\,4\rangle^2\;s_{124}\;\langle 4|(1+2)|3]}{\langle 1\,4\rangle\;\langle 1\,2\rangle}\right\}\end{aligned} \tag{3.5.70}$$

$\tilde{\boldsymbol{c}}_{\mathbf{1\times 23}}$

$$\tilde{c}_{1\times 23}(1^+,2^+,3^-,4^-) = 2\,(s_{12}+s_{13})\frac{s_{123}\;\langle 1\,3\rangle^2\;\langle 3|(1+2)|4]}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 1|(2+3)|4]^3} \tag{3.5.71}$$

$\tilde{\boldsymbol{c}}^{(0)}_{\mathbf{23\times41}}$, $\tilde{\boldsymbol{c}}^{(2)}_{\mathbf{23\times41}}$

$$
\begin{aligned}
\tilde{c}^{(0)}_{23\times41}(1^+,2^+,3^-,4^-) = & -\tilde{c}^{(0)}_{12\times34}(2^+,3^-,1^+,4^-) \\
& - \left\{ 2\,\Delta_3(1,4,2,3)\left[\frac{(s_{13}-s_{24})}{\langle 2|(1+4)|3]\ \langle 1|(2+3)|4]}\right]^2 \right. \\
& \left. + 4\frac{\langle 3|(1+4)|2]\ \langle 4|(2+3)|1]}{\langle 2|(1+4)|3]\ \langle 1|(2+3)|4]} \right\}
\end{aligned}
\tag{3.5.72}
$$

$$
\begin{aligned}
\tilde{c}^{(2)}_{23\times41}(1^+,2^+,3^-,4^-) = & \left\{ 4\frac{\langle 2\,4\rangle\ \langle 3|(1+4)|2]}{\langle 2\,3\rangle\ \langle 1\,4\rangle\ \langle 2|(1+4)|3]} \right. \\
& \left[\frac{1}{\langle 1|(2+3)|4]}\left(\frac{\langle 2\,4\rangle\ (s_{13}-s_{24})}{\langle 2|(1+4)|3]} - 2\ \langle 3\,4\rangle\right) \right. \\
& \left.\left. - \frac{\langle 4|(2+3)|1]}{\Delta_3(1,4,2,3)}\left(\frac{\langle 1\,2\rangle\ \langle 3|(1+4)|2]}{\langle 1|(2+3)|4]} - \langle 3\,4\rangle\right)\right]\right\} \\
& + \{1\leftrightarrow 2, 3\leftrightarrow 4\} + \{1\leftrightarrow 3, 2\leftrightarrow 4, \langle\,\rangle\leftrightarrow[\,]\} \\
& + \{1\leftrightarrow 4, 2\leftrightarrow 3, \langle\,\rangle\leftrightarrow[\,]\}
\end{aligned}
\tag{3.5.73}
$$

Where Δ_3 is given by Eq. (3.5.54).

$\tilde{\boldsymbol{c}}^{(0)}_{\mathbf{1\times234}}$, $\tilde{\boldsymbol{c}}^{(2)}_{\mathbf{1\times234}}$

$$
\tilde{c}^{(0)}_{1\times234}(1^+,2^+,3^-,4^-) = -\,2\,(s_{12}+s_{13}+s_{14})\,s_{234}\ \frac{\langle 1|(3+4)|2]\,[2\,4]^2}{\langle 1|(2+3)|4]^3\,[2\,3]\,[3\,4]}
\tag{3.5.74}
$$

$$
\begin{aligned}
\tilde{c}^{(2)}_{1\times234}(1^+,2^+,3^-,4^-) = & \frac{-4}{s_{23}\,\langle 1|4|3]}\left\{ \frac{\langle 4|1|2]^2\,(\langle 3|4|1] - \langle 3|2|1])}{s_{14}(s_{12}+s_{13}+s_{14})} \right. \\
& + \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1|(2+3)|4]}\left(\langle 1|(3+4)|2] + \frac{(s_{12}+s_{13}+s_{14})}{s_{12}}\,\langle 1|3|2]\right) \\
& - 2\frac{\langle 1|(3+4)|2]\,\langle 4|(2+3)|1]\,\langle 3|1|2]}{(s_{12}+s_{13}+s_{14})\,\langle 1|(2+3)|4]} \\
& \left. + \frac{\langle 1|(3+4)|2]^2\,\langle 4|(2+3)|1]\,\langle 3|1|4]}{(s_{12}+s_{13}+s_{14})\,\langle 1|(2+3)|4]^2} \right\}
\end{aligned}
\tag{3.5.75}
$$

3.5.4.3 Bubbles

$\tilde{b}_{23}$

$$
\begin{aligned}
\tilde{b}_{23}(1^+,2^+,3^-,4^-) = \Bigg\{ & -4\,\frac{[2\,4]^2\;\langle 3\,4\rangle\; s_{234}}{[3\,4]\;\langle 4|(2+3)|4]\;\langle 1|(2+3)|4]^2} \\
& -4\,\frac{[1\,3]^2\;\langle 2\,3\rangle\;\langle 1|(2+3)|1]\;\langle 3|(4+1)|2]\,(2\,s_{124}+s_{134}-3\,s_{14})}{[1\,4]\;\langle 1|(2+3)|4]\;\langle 2|(1+4)|3]^2\;\Delta_3(1,4,2,3)} \\
& +12\frac{\langle 1\,4\rangle\;[1\,3]^2\;\langle 2\,3\rangle^2\;[2\,3]\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]\;\langle 2|(1+4)|3]^2\;\Delta_3(1,4,2,3)} \\
& +4\frac{\langle 2\,4\rangle\;[2\,4]\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]^2\;\langle 2|(1+4)|3]} \\
& -8\frac{[2\,4]\;\langle 2\,3\rangle\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]^2\;\langle 2|(1+4)|3]\;\Delta_3(1,4,2,3)} \\
& \times(\langle 1\,4\rangle\;[3\,4]\;\langle 4|(2+3)|1]+\langle 1\,4\rangle\;\langle 2\,3\rangle\;[1\,3]\;[2\,3] \\
& \quad +\langle 2\,4\rangle\;[2\,3]\;\langle 4|(2+3)|4]) \\
& -4\frac{\langle 2\,4\rangle\;[1\,3]\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]\;\langle 2|(1+4)|3]^2} \\
& +8\frac{\langle 1\,4\rangle\;[3\,4]\;[1\,3]\;\langle 2\,3\rangle\;\langle 4|(2+3)|1]\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]\;\langle 2|(1+4)|3]^2\;\Delta_3(1,4,2,3)} \\
& +3\frac{\langle 4|(2+3)|1]\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]\;\langle 2|(1+4)|3]\;\Delta_3(1,4,2,3)^2} \\
& \times(s_{124}-s_{134})\,(s_{123}-s_{234})\,(s_{234}+s_{123}) \\
& +\frac{\langle 4|(2+3)|1]\;\langle 3|(4+1)|2]}{\langle 1|(2+3)|4]\;\langle 2|(1+4)|3]\;\Delta_3(1,4,2,3)} \\
& \times(2s_{23}+5s_{24}+3s_{34}+3s_{12}+5s_{13})\Bigg\} \\
& +\Big\{1\leftrightarrow 4, 2\leftrightarrow 3, \langle\,\rangle\leftrightarrow[\,]\Big\}
\end{aligned}
\tag{3.5.76}
$$

where Δ_3 is given by eq. (3.5.54).

$\tilde{b}_{234}$

$$
\tilde{b}_{234}(1^+,2^+,3^-,4^-) = \frac{4\,s_{234}}{[3\,4]\;\langle 1|(2+3)|4]^2}\left(\frac{\langle 3\,4\rangle\;[2\,4]^2}{(s_{24}+s_{34})}\right.
$$

$$+\frac{[1\,2]\ \langle 1|(3+4)|2]}{[2\,3]}\left[\frac{\langle 1|(2+3)|4]\ [1\,2]}{(s_{12}+s_{13}+s_{14})^2}-\frac{[2\,4]}{(s_{12}+s_{13}+s_{14})}\right]\Bigg) \tag{3.5.77}$$

$\tilde{b}_{1234}$

Similarly to eq. (3.5.40) we have,

$$\tilde{b}_{1234}(1^+,2^+,3^-,4^-) = -\tilde{b}_{23}-\tilde{b}_{41}-\tilde{b}_{234}-\tilde{b}_{341}-\tilde{b}_{412}-\tilde{b}_{123}\,. \tag{3.5.78}$$

3.5.4.4 Rational terms

$$\begin{aligned}\tilde{r}(1^+,2^+,3^-,4^-) = \frac{1}{2}\Big[&\tilde{c}^{(2)}_{23\times 41}(1^+,2^+,3^-,4^-)\\ &+\tilde{c}^{(2)}_{1\times 234}(1^+,2^+,3^-,4^-)+\tilde{c}^{(2)}_{1\times 234}(2^+,1^+,4^-,3^-)\\ &+\tilde{c}^{(2)}_{1\times 234}(3^+,4^+,1^-,2^-)|_{[\,]\leftrightarrow\langle\,\rangle}+\tilde{c}^{(2)}_{1\times 234}(4^+,3^+,2^-,1^-)|_{[\,]\leftrightarrow\langle\,\rangle}\Big]\end{aligned} \tag{3.5.79}$$

3.6 Amplitude for $0 \to \bar{q}qggh$ with a Scalar Mediator

3.6.1 Coefficients for $A_4^{34}(\bar{q}^+, q^-, g^+, g^+; h)$

Moving to study amplitudes including two gluons, a quark and an antiquark alongside the Higgs boson, the coefficients contained in these amplitudes are shown in table 3.5.

3.6.1.1 Boxes

$\tilde{d}_{3\times 21\times 4}$

$$\begin{aligned}\tilde{d}_{3\times 21\times 4}(1^+_{\bar{q}},2^-_q,3^+_g,4^+_g) = &-2\,\frac{\langle 2\,4\rangle\ \langle 2\,3\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle^3}\Big[(s_{13}+s_{23})\,(s_{14}+s_{24})-s_{12}\,s_{34}\Big]\\ &+2\,m^2\left[\frac{[1\,3]\ [1\,4]}{[1\,2]\ \langle 3\,4\rangle}+3\,\frac{\langle 2\,3\rangle\ \langle 2\,4\rangle\ [3\,4]}{\langle 1\,2\rangle\ \langle 3\,4\rangle^2}\right]\end{aligned} \tag{3.6.1}$$

$1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g$		$1^+_{\bar{q}}, 2^-_q, 3^-_g, 4^+_g$		$1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g$
Coefficient	Related coefficient	Coefficient	Related coefficient	Coefficient
$\tilde{d}_{3\times21\times4}$		$\tilde{d}_{3\times21\times4}$		$\tilde{c}_{4\times123}$
$\tilde{d}_{4\times3\times21}$	$\tilde{d}_{3\times4\times12}$	$\tilde{d}_{4\times3\times21}$	$\tilde{d}_{3\times4\times12}$	$\tilde{b}_{123}$
$\tilde{c}_{3\times21}$	$\tilde{c}_{4\times12}$	$\tilde{c}_{3\times4}$		
$\tilde{c}_{12\times34}$		$\tilde{c}_{3\times21}$	$\tilde{c}_{4\times12}$	
$\tilde{c}_{4\times123}$		$\tilde{c}_{12\times34}$		
$\tilde{c}_{3\times412}$		$\tilde{c}_{4\times123}$	$\tilde{c}_{3\times412}$	
$\tilde{b}_{12}$		$\tilde{b}_{34}$		
$\tilde{b}_{123}$		$\tilde{b}_{12}$		
$\tilde{b}_{412}$		$\tilde{b}_{123}$	$\tilde{b}_{412}$	
$\tilde{b}_{1234}$		$\tilde{b}_{1234}$		

Table 3.5: The coefficients required for the $A_4^{34}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g)$, $A_4^{34}(1^+_{\bar{q}}, 2^-_q, 3^-_g, 4^+_g)$ and $A_4^{34}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g)$ amplitudes. The left-hand column shows the minimal sets that need to be calculated.

$\tilde{d}_{4\times3\times21}$

$$\tilde{d}_{4\times3\times21}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = 2\, m^2\, \frac{[3\,4]}{\langle 3\,4\rangle} \left[\frac{\langle 2\,3\rangle\ \langle 2|(1+3)|4]}{\langle 1\,2\rangle\ \langle 3|(1+2)|4]} - \frac{[1\,3]\ \langle 4|(2+3)|1]}{[1\,2]\ \langle 4|(1+2)|3]} \right] \tag{3.6.2}$$

3.6.1.2 Triangles

$\tilde{c}_{3\times21}$

$$\tilde{c}_{3\times21}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = 2\,(s_{13} + s_{23})\, \frac{\langle 2\,3\rangle\ \langle 2\,4\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle^3} \tag{3.6.3}$$

$\tilde{c}^{(0)}_{12\times34}$, $\tilde{c}^{(2)}_{12\times34}$

$$\tilde{c}^{(0)}_{12\times34}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = 0 \tag{3.6.4}$$

$$\begin{aligned}\tilde{c}^{(2)}_{12\times34}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = {}& \frac{4}{\langle 3\,4\rangle^2\ \langle 3|(1+2)|4]} \left[\frac{[1\,4]^2\ \langle 3\,4\rangle}{[1\,2]} - \frac{\langle 2\,3\rangle^2\ [3\,4]}{\langle 1\,2\rangle} \right] \\ & + \frac{4}{\langle 3\,4\rangle^2\ \langle 4|(1+2)|3]} \left[\frac{[1\,3]^2\ \langle 3\,4\rangle}{[1\,2]} - \frac{\langle 2\,4\rangle^2\ [3\,4]}{\langle 1\,2\rangle} \right]\end{aligned} \tag{3.6.5}$$

$\tilde{c}^{(0)}_{4\times123}$, $\tilde{c}^{(2)}_{4\times123}$

$$\tilde{c}^{(0)}_{4\times123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = -2\,(s_{14}+s_{24}+s_{34})\left[\frac{\langle 2\,3\rangle\,\langle 2\,4\rangle}{\langle 1\,2\rangle\,\langle 3\,4\rangle^3}\right] \tag{3.6.6}$$

$$\begin{aligned}\tilde{c}^{(2)}_{4\times123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = \frac{4}{\langle 3|(1+2)|4]}\Bigg\{&\frac{(s_{14}+s_{24}+s_{34})}{\langle 4|(1+2)|3]}\Bigg[\frac{\langle 2\,3\rangle\,\langle 2\,4\rangle\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle^2}\\ &+\frac{[1\,3]\,[1\,4]}{[1\,2]\,\langle 3\,4\rangle}\Bigg] - 2\,\frac{\langle 2|(1+3)|4]^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,(s_{14}+s_{24}+s_{34})\,s_{123}}\Bigg\}\end{aligned} \tag{3.6.7}$$

$\tilde{c}^{(0)}_{3\times412}$, $\tilde{c}^{(2)}_{3\times412}$

$$\tilde{c}^{(0)}_{3\times412}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = -2\,(s_{13}+s_{23}+s_{34})\left[\frac{\langle 2\,3\rangle\,\langle 2\,4\rangle}{\langle 1\,2\rangle\,\langle 3\,4\rangle^3}\right] \tag{3.6.8}$$

$$\begin{aligned}\tilde{c}^{(2)}_{3\times412}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = 4\,&\frac{(s_{13}+s_{23}+s_{34})}{\mathrm{tr}_-\{\not{p}_3\,\not{p}_{12}\,\not{p}_4\,\not{p}_{12}\}}\left[\frac{\langle 2\,4\rangle\,\langle 2\,3\rangle\,[4\,3]}{\langle 2\,1\rangle\,\langle 4\,3\rangle^2}+\frac{[1\,4]\,[1\,3]}{[2\,1]\,\langle 4\,3\rangle}\right]\\ &-8\,\frac{\langle 2|(1+4)|3]^2\,\langle 1|(2+4)|3]}{\langle 2\,1\rangle\,\langle 1\,4\rangle\,\langle 4|(1+2)|3]\,(s_{13}+s_{23}+s_{34})\,s_{124}}\end{aligned} \tag{3.6.9}$$

3.6.1.3 Bubbles

$\tilde{b}_{12}$

$$\tilde{b}_{12}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = \frac{4}{\langle 3\,4\rangle^2}\left[\frac{[1\,3]\,\langle 2\,3\rangle}{(s_{13}+s_{23})} - \frac{[1\,4]\,\langle 2\,4\rangle}{(s_{14}+s_{24})}\right] \tag{3.6.10}$$

$\tilde{b}_{123}$

$$\begin{aligned}\tilde{b}_{123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = \frac{4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle^2}\Bigg[&\frac{\langle 3\,4\rangle\,\langle 2\,4\rangle\,\langle 2|(1+3)|4]^2}{(s_{14}+s_{24}+s_{34})^2}\\ &-\frac{\langle 2\,3\rangle\,\langle 2\,4\rangle\,\langle 2|(1+3)|4]}{(s_{14}+s_{24}+s_{34})} - \frac{\langle 1\,2\rangle\,[1\,3]\,\langle 2\,3\rangle^2}{(s_{13}+s_{23})}\Bigg]\end{aligned} \tag{3.6.11}$$

$\tilde{b}_{412}$

$$\begin{aligned}\tilde{b}_{412}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^+_g) = 4\,\frac{1}{\langle 1\,2\rangle\,\langle 1\,4\rangle\,\langle 3\,4\rangle^2}\Bigg[&\frac{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 2|(1+4)|3]\,\langle 1|(2+4)|3]}{(s_{13}+s_{23}+s_{34})^2}\\ &+\frac{\langle 1\,3\rangle\,\langle 2\,4\rangle\,\langle 2|(1+4)|3]}{(s_{13}+s_{23}+s_{34})} - \frac{\langle 1\,2\rangle\,\langle 2\,4\rangle\,s_{14}}{(s_{14}+s_{24})}\Bigg]\end{aligned} \tag{3.6.12}$$

$\tilde{b}_{1234}$

As before, we can calculate the remaining bubble coefficient using the relation

$$\tilde{b}_{1234}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) = -\tilde{b}_{12} - \tilde{b}_{123} - \tilde{b}_{412}\,. \tag{3.6.13}$$

3.6.1.4 Rational terms

$$\tilde{r}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) = \frac{1}{2}\Big[\tilde{c}^{(2)}_{12\times 34}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) + \tilde{c}^{(2)}_{4\times 123}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) + \tilde{c}^{(2)}_{3\times 412}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+)\Big] \tag{3.6.14}$$

3.6.2 Coefficients for $A_4^{34}(\bar{q}^+, q^-, g^-, g^+; h)$

3.6.2.1 Boxes

$\tilde{d}_{3\times 21\times 4}$

$$\tilde{d}_{3\times 21\times 4}(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+) = 2\,m^2\,\frac{\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]}\left[\frac{\langle 2\,3\rangle\ \langle 2\,4\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle} - \frac{[1\,3]\ [1\,4]}{[1\,2]\ [3\,4]}\right] \tag{3.6.15}$$

$\tilde{d}_{4\times 3\times 21}$

$$\tilde{d}_{4\times 3\times 21}(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+) = \frac{-2}{\langle 4|(1+2)|3]}\left\{\frac{[1\,3]\ \langle 4|(2+3)|1]\ s_{34}\,s_{123}^2}{[1\,2]\ \langle 4|(1+2)|3]^2} + m^2\left[\frac{3\ [1\,3]\ \langle 3|(1+2)|4]\ \langle 4|(2+3)|1]}{[1\,2]\ \langle 4|(1+2)|3]} + \frac{\langle 2\,3\rangle\ \langle 2|(1+3)|4]}{\langle 1\,2\rangle}\right]\right\} \tag{3.6.16}$$

3.6.2.2 Triangles

$\tilde{c}_{3\times 4}$

$$\tilde{c}_{3\times 4}(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+) = \left\{\frac{2\,s_{34}\ [1\,3]\ \langle 2\,3\rangle}{\langle 4|(1+2)|3]^2} + \frac{2\,s_{34}\ [1\,3]^2\ \langle 3\,4\rangle\ (2\,s_{12} + s_{13} + s_{23})}{[1\,2]\ \langle 4|(1+2)|3]^3}\right\} - \left\{1 \leftrightarrow 2, 3 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,]\right\} \tag{3.6.17}$$

$\tilde{c}_{3\times 21}$

$$\tilde{c}_{3\times 21}(1_{\bar{q}}^{+}, 2_{q}^{-}, 3_{g}^{-}, 4_{g}^{+}) = 2\,(s_{13} + s_{23})\, s_{123}\, \frac{[1\,3]\ \langle 4|(2+3)|1]}{[1\,2]\ \langle 4|(1+2)|3]^{3}} \tag{3.6.18}$$

$\tilde{c}^{(0)}_{12\times 34}$, $\tilde{c}^{(2)}_{12\times 34}$

$$\begin{aligned}
\tilde{c}^{(0)}_{12\times 34}(1_{\bar{q}}^{+}, 2_{q}^{-}, 3_{g}^{-}, 4_{g}^{+}) =\, & 8\,(s_{124} - s_{123})\,(s_{12} + s_{34} + 2\,s_{13} + 2\,s_{23}) \\
& \frac{\langle 2\,4\rangle\ [1\,3]\ \langle 3|(1+2)|4]}{\langle 4|(1+2)|3]^{2}\,\Delta_3(1,2,3,4)} \\
& + \Big((9\,s_{13} - 7\,s_{23} - s_{14} - s_{24} + 4\,s_{34})\ \langle 2\,4\rangle\ [1\,4] \\
& - (9\,s_{14} - 7\,s_{24} - s_{13} - s_{23} + 4\,s_{34})\ \langle 2\,3\rangle\ [1\,3]\Big) \\
& \times \frac{1}{\langle 4|(1+2)|3]^{2}} \\
& + 12\,\frac{s_{1234}\,((s_{13} + s_{23})^{2} - (s_{14} + s_{24})^{2})}{\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)^{2}} \\
& \langle 2|(3+4)|1]\ \langle 3|(1+2)|4] \\
& + 4\,\Big(\Big\{3\,(s_{12} + s_{34}) + 4\,(s_{13} + s_{23} + s_{14})\Big\}\ [1\,3]\ \langle 2\,3\rangle \\
& - \Big\{3\,(s_{12} + s_{34}) + 4\,(s_{13} + s_{24} + s_{14})\Big\}\ [1\,4]\ \langle 2\,4\rangle\Big) \\
& \times \frac{\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]\ \Delta_3(1,2,3,4)} \\
& - 24\,\frac{[1\,3]\ \langle 2\,4\rangle\ \langle 3|(1+2)|4]^{2}}{\langle 4|(1+2)|3]\ \Delta_3(1,2,3,4)} - 8\,\frac{[1\,4]\ \langle 2\,3\rangle\ \langle 3|(1+2)|4]}{\Delta_3(1,2,3,4)} \\
& + 8\,\frac{[1\,4]\ \langle 2\,3\rangle}{\langle 4|(1+2)|3]} \\
& + \Bigg\{\frac{2\ \langle 2\,4\rangle^{2}\ [3\,4]\ (s_{14} + s_{24})^{2}}{\langle 1\,2\rangle\ \langle 4|(1+2)|3]^{3}} \\
& + \frac{[1\,3]\ \langle 2\,4\rangle\ (s_{14} + s_{24})\,(4\,s_{124} - 2\,s_{34})}{\langle 4|(1+2)|3]^{3}} \\
& + \frac{2\ \langle 2\,3\rangle\ \langle 2\,4\rangle\ [3\,4]\ (s_{14} + s_{24})}{\langle 1\,2\rangle\ \langle 4|(1+2)|3]^{2}} \\
& - \frac{\langle 2\,3\rangle\ [1\,3]\ (s_{14} + s_{24} - s_{13} - s_{23})}{\langle 4|(1+2)|3]^{2}}\Bigg\} \\
& - \Bigg\{1 \leftrightarrow 2, 3 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,]\Bigg\}
\end{aligned} \tag{3.6.19}$$

$$\tilde{c}^{(2)}_{12\times34}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = \Bigg\{\frac{4\,\langle 2\,3\rangle^2}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4|(1+2)|3]} - \frac{4\,\langle 2\,4\rangle^2\,\langle 3|(1+2)|4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4|(1+2)|3]^2}$$
$$- 8\,(s_{13}+s_{23}+s_{14}+s_{24})$$
$$\times \frac{\langle 2\,3\rangle\,\langle 2\,4\rangle\,\langle 3|(1+2)|4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)}$$
$$- 16\,\frac{[1\,3]\,\langle 2\,3\rangle\,\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)}\Bigg\}$$
$$- \Big\{1 \leftrightarrow 2, 3 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,]\Big\} \tag{3.6.20}$$

$\tilde{c}^{(0)}_{4\times123}, \tilde{c}^{(2)}_{4\times123}$

$$\tilde{c}^{(0)}_{4\times123}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = -\frac{2\,(s_{14}+s_{24}+s_{34})\,[1\,3]\,\langle 4|(2+3)|1]\,s_{123}}{[1\,2]\,\langle 4|(1+2)|3]^3} \tag{3.6.21}$$

$$\tilde{c}^{(2)}_{4\times123}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = \frac{4\,(s_{14}+s_{24}+s_{34})}{\langle 4|(1+2)|3]^2}\Big[\frac{\langle 2\,3\rangle\,\langle 2\,4\rangle}{\langle 1\,2\rangle\,\langle 3\,4\rangle} + \frac{[1\,3]\,[1\,4]}{[1\,2]\,[3\,4]}\Big]$$
$$- 8\,\frac{[1\,4]^2\,[2\,4]}{[1\,2]\,[2\,3]\,[3\,4]\,(s_{14}+s_{24}+s_{34})} \tag{3.6.22}$$

3.6.2.3 Bubbles

$\tilde{b}_{34}$

$$\tilde{b}_{34}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = 2\,\frac{\langle 3|(1+2)|4]}{s_{12}\,\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)}$$
$$\times \Big[3\,\frac{\langle 2|(3+4)|1]\,(s^2_{124}-s^2_{123})\,(s_{13}+s_{14}+s_{23}+s_{24})}{\Delta_3(1,2,3,4)}$$
$$+ 2\,\frac{(s_{124}-s_{123})}{\langle 4|(1+2)|3]}$$
$$\times (\langle 1\,2\rangle\,[1\,3]\,\langle 4|(2+3)|1] - \langle 2\,4\rangle\,[1\,2]\,\langle 2|(1+4)|3])$$
$$- 3\,(s_{123}+s_{124})\,(\langle 2\,3\rangle\,[1\,3] - \langle 2\,4\rangle\,[1\,4])\Big] \tag{3.6.23}$$

$\tilde{b}_{12}$

$$\tilde{b}_{12}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = \Bigg\{\frac{4}{\langle 4|(1+2)|3]^2}\Bigg[-\frac{\langle 1\,2\rangle\,[1\,3]^2\,\langle 3|(1+2)|4]}{[3\,4]\,(s_{13}+s_{23})}$$
$$+ \frac{\langle 3|(1+2)|4]}{\langle 3\,4\rangle\,\Delta_3(1,2,3,4)}\Big[s_{12}\,\langle 2\,4\rangle\,[1\,3]\,\langle 3\,4\rangle - 2\,\langle 1\,2\rangle\,[1\,3]^2\,\langle 3\,4\rangle^2$$

$$
\begin{aligned}
&- \langle 2\,3\rangle\ \langle 4|(1+2)|3]\ \langle 4|(2+3)|1] \\
&- \langle 2\,4\rangle^2\ [1\,2]\ (s_{13}+s_{23}+s_{14}+s_{24})\Big]\Big] \\
&- 12\,(s_{13}+s_{23})\,(2\,s_{12}+s_{13}+s_{23})\,\frac{\langle 2|(3+4)|1]\ \langle 3|(1+2)|4]}{\langle 4|(1+2)|3]\ \Delta_3(1,2,3,4)^2}\Bigg\} \\
&- \Bigg\{1\leftrightarrow 2, 3\leftrightarrow 4, \langle\,\rangle\leftrightarrow[\,]\Bigg\} \qquad (3.6.24)
\end{aligned}
$$

$\tilde{\boldsymbol{b}}_{\mathbf{123}}$

$$
\begin{aligned}
\tilde{b}_{123}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = 4\,s_{123}\Bigg[&\frac{[1\,4]^2\ \langle 4|(1+3)|2]}{[1\,2]\ [2\,3]\ \langle 4|(1+2)|3]\ (s_{14}+s_{24}+s_{34})^2} \\
&+\frac{[1\,4]\ \langle 4|(2+3)|1]}{[1\,2]\ \langle 4|(1+2)|3]^2\ (s_{14}+s_{24}+s_{34})} \\
&+\frac{\langle 2\,3\rangle\ [1\,3]}{(s_{13}+s_{23})\ \langle 4|(1+2)|3]^2}\Bigg] \qquad (3.6.25)
\end{aligned}
$$

$\tilde{\boldsymbol{b}}_{\mathbf{1234}}$

As before, the final bubble coefficient can be found through the relation

$$
\tilde{b}_{1234}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = -\tilde{b}_{12}-\tilde{b}_{34}-\tilde{b}_{123}-\tilde{b}_{412}\,. \qquad (3.6.26)
$$

3.6.2.4 Rational terms

$$
\begin{aligned}
\tilde{r}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = \frac{1}{2}\Big[&\tilde{c}^{(2)}_{12\times34}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g)+\tilde{c}^{(2)}_{4\times123}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) \\
&-\tilde{c}^{(2)}_{4\times123}(2^+_{\bar{q}},1^-_q,4^-_g,3^+_g)|_{[\,]\leftrightarrow\langle\,\rangle}\Big] \qquad (3.6.27)
\end{aligned}
$$

3.6.3 Coefficients for $A_4^{34}(\bar{q}^+,q^-,g^+,g^-;h)$

The majority of the coefficients for this amplitude can be calculated from those for $A_4^{34}(\bar{q}^+,q^-,g^-,g^+)$ by the simple operation $1\leftrightarrow 2\,,\langle\,\rangle\leftrightarrow[\,]$. The remaining coefficients are listed in the right-most column of table 3.5, and are given explicitly below.

3.6.3.1 Triangle

$\tilde{c}^{(0)}_{4\times 123}$, $\tilde{c}^{(2)}_{4\times 123}$

$$\tilde{c}^{(0)}_{4\times 123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g) = \frac{2\ \langle 2|(1+3)|4]\ s_{123}}{\langle 1\,2\rangle\ \langle 3|(1+2)|4]^2} \left\{ \langle 2\,4\rangle - \langle 3\,4\rangle \frac{\langle 2|(1+3)|4]}{\langle 3|(1+2)|4]} \right\} \tag{3.6.28}$$

$$\tilde{c}^{(2)}_{4\times 123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g) = -4\frac{(s_{14}+s_{24}+s_{34})}{\langle 3|(1+2)|4]^2}\left[\frac{\langle 2\,3\rangle\ \langle 2\,4\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle} + \frac{[1\,3]\ [1\,4]}{[1\,2]\ [3\,4]}\right] + 8\,\frac{\langle 2\,4\rangle^3}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3\,4\rangle\ (s_{14}+s_{24}+s_{34})} \tag{3.6.29}$$

3.6.3.2 Bubble

$\tilde{b}_{123}$

$$\tilde{b}_{123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g) = 4\,s_{123}\Bigg[-\frac{\langle 2\,4\rangle^2\ \langle 2|(1+3)|4]}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3|(1+2)|4]\ (s_{14}+s_{24}+s_{34})^2} - \frac{\langle 2\,4\rangle\ \langle 2|(1+3)|4]}{\langle 1\,2\rangle\ \langle 3|(1+2)|4]^2\ (s_{14}+s_{24}+s_{34})} + \frac{[1\,3]\ \langle 2\,3\rangle}{(s_{13}+s_{23})\ \langle 3|(1+2)|4]^2}\Bigg] \tag{3.6.30}$$

3.6.3.3 Rational terms

$$\tilde{r}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g) = \frac{1}{2}\Big[\tilde{c}^{(2)}_{12\times 34}(2^+_{\bar{q}}, 1^-_q, 3^-_g, 4^+_g)|_{[]\leftrightarrow\langle\rangle} + \tilde{c}^{(2)}_{4\times 123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g) - \tilde{c}^{(2)}_{4\times 123}(2^+_{\bar{q}}, 1^-_q, 4^+_g, 3^-_g)|_{[]\leftrightarrow\langle\rangle}\Big] \tag{3.6.31}$$

3.7 Amplitude for $0 \to \bar{q}q\bar{q}qh$

The calculation of this amplitude has previously been done for a fermion mediator in ref. [94], and can be done in a similar fashion for a scalar mediator. We begin with the tensor current for the process $0 \to ggh$, where the gluons have off-shell momenta

k_1 and k_2,

$$\mathcal{T}^{\mu_1\mu_2}(k_1,k_2) = -i\delta^{c_1c_2}\,\frac{g_s^2}{8\pi^2}\left(\frac{-\lambda}{4}\right)\left[\tilde{F}_T(k_1,k_2)\,T_T^{\mu_1\mu_2} + \tilde{F}_L(k_1,k_2)\,T_L^{\mu_1\mu_2}\right]. \quad (3.7.1)$$

The tensor structures are

$$T_T^{\mu_1\mu_2} = k_1\cdot k_2\,g^{\mu_1\mu_2} - k_1^{\mu_2}k_2^{\mu_1}\,, \quad (3.7.2)$$

$$T_L^{\mu_1\mu_2} = k_1^2k_2^2\,g^{\mu_1\mu_2} - k_1^2\,k_2^{\mu_1}k_2^{\mu_2} - k_2^2\,k_1^{\mu_1}k_1^{\mu_2} + k_1\cdot k_2\,k_1^{\mu_1}k_2^{\mu_2}\,, \quad (3.7.3)$$

and the form factors

$$\begin{aligned}\tilde{F}_T(k_1,k_2) = &-\frac{1}{\Delta(k_1,k_2)}\Big\{k_{12}^2\,(B_0(k_1;m)+B_0(k_2;m)-2B_0(k_{12};m))\\ &-2\,k_1\cdot k_{12}\,k_2\cdot k_{12}\,C_0(k_1,k_2;m)\\ &+(k_1^2-k_2^2)\,(B_0(k_1;m)-B_0(k_2;m))\Big\} - k_1\cdot k_2\,\tilde{F}_L(k_1,k_2)\,, \quad (3.7.4)\\ \tilde{F}_L(k_1,k_2) = &-\frac{1}{\Delta(k_1,k_2)}\Big\{\Big[2-\frac{3k_1^2\,k_2\cdot k_{12}}{\Delta(k_1,k_2)}\Big]\,(B_0(k_1;m)-B_0(k_{12};m))\\ &+\Big[2-\frac{3k_2^2\,k_1\cdot k_{12}}{\Delta(k_1,k_2)}\Big]\,(B_0(k_2;m)-B_0(k_{12};m))\\ &-\Big[4m^2+k_1^2+k_2^2+k_{12}^2-3\frac{k_1^2\,k_2^2\,k_{12}^2}{\Delta(k_1,k_2)}\Big]\,C_0(k_1,k_2;m)-2\Big\}\,, \quad (3.7.5)\end{aligned}$$

where $k_{12}=k_1+k_2$ and $\Delta(k_1,k_2)=k_1^2\,k_2^2-(k_1\cdot k_2)^2$. The amplitude can then be found by contracting eq. (3.7.1) with currents for the quark-antiquark lines,

$$\begin{aligned}A_4^{4q}(1_{\bar{q}}^+,2_q^-,3_{\bar{q}'}^+,4_{q'}^-;h) = &\,2\,\langle 2\,4\rangle\,[1\,3]\,\tilde{F}_L(p_{12},p_{34})\\ &+\left[\frac{\langle 2|(3+4)|1]\,\langle 4|(1+2)|3] + \langle 2\,4\rangle\,[1\,3]\,(2\,p_{12}.p_{34})}{s_{12}\,s_{34}}\right]\tilde{F}_T(p_{12},p_{34})\,. \quad (3.7.6)\end{aligned}$$

All the helicity combinations can be obtained from this single expression.

3.8 Conclusion

The results presented in this chapter are an important tool in calculations of Higgs boson plus jets at hadron colliders in extensions of the Standard Model containing coloured scalars. The compact analytic expressions retain the full dependence on the scalar mass and have been derived using unitarity techniques. We additionally used momentum twistors and high-precision floating-point reconstruction to simplify many of the resulting expressions. The relationship between fermion and scalar theories leads to these results being useful for the equivalent Standard Model calculation with a fermion mediator. The amplitudes given here are essential components of NLO Higgs plus 1-jet and NNLO inclusive cross sections for any theory involving colour-triplet scalars.

Chapter 4

Higgs Boson Plus Two Jets with Full Mass Dependence

4.1 Introduction

This chapter is based on the work in [1] and presents compact analytic amplitudes, retaining all mass effects, for all processes contributing to the Higgs plus four parton amplitudes $0 \to ggggh$, $0 \to \bar{q}qggh$ and $0 \to \bar{q}q\bar{q}'q'h$.

At hadron colliders there are four main production modes for a Higgs boson with a mass of around 125 GeV: gluon fusion, vector boson fusion, Higgs-strahlung (produced in association with a vector boson) and production in association with a top quark pair.

Gluon fusion is the dominant production mode at the LHC [95], necessitating detailed studies of the $gg \to h$ process. The NLO level calculation includes the Higgs plus 1-jet process, and at NNLO the Higgs plus 2-jet process. In addition to being an integral part of the $gg \to h$ at NNLO calculation, the one-loop $gg \to hgg$ process is an irreducible background to vector boson fusion [96].

Inclusive Higgs production, including mass effects, was recently calculated at NNLO [97], while Higgs plus 1-jet is known at NLO [98–100]. For the production of Higgs plus

2, or more, jets, results are only known to leading order [73, 94, 101, 102].

Compact analytic expressions are known for processes involving the Higgs and three partons [103, 104], however previous results for Higgs plus four parton amplitudes have contained expressions which were too long to report [73, 94]. Initially, simple analytic expressions were available for the $0 \to hgggg$ process with four positive helicity gluons only [105].

The calculation in this chapter is structured identically to the method given in section 3.2. The full amplitude is again calculated through the colour-ordered sub-amplitudes, for example $\mathcal{H}_n^{gggg}(\{p_i, h_i, c_i\})$,

$$\begin{aligned}\mathcal{H}_n^{gggg}(\{p_i, h_i, c_i\}) =& \, i\frac{g_s^n}{16\pi^2}\frac{m^2}{v}\sum_{\{1,2,\ldots,n\}'} \mathrm{tr}\,(t^{c_1}t^{c_2}\ldots t^{c_n}) \\ & \times H_n^{\{c_i\}}(1^{h_1}, 2^{h_2}, \ldots n^{h_n}; h)\,. \end{aligned} \tag{4.1.1}$$

The formulae for the squared amplitudes are reproduced below to ensure the difference in notation is clear.

The squared matrix element for the process with four gluons is,

$$\begin{aligned}\sum_{\text{colours}} |\mathcal{H}_4^{gggg}|^2 = & \left[\frac{g_s^4}{16\pi^2}\left(\frac{m^2}{v}\right)\right]^2 (N^2-1)\Bigg\{2N^2\Big(\left|H_4^{1234}\right|^2 + \left|H_4^{1342}\right|^2 + \left|H_4^{1423}\right|^2\Big) \\ & - 4\frac{(N^2-3)}{N^2}\left|H_4^{1234} + H_4^{1342} + H_4^{1423}\right|^2\Bigg\}\,, \end{aligned} \tag{4.1.2}$$

and the two gluon, two quark amplitude squared is

$$\sum |\mathcal{H}_4^{\bar{q}qgg}|^2 = \left(\frac{g_s^4}{16\pi^2}\right)^2\left(\frac{m^2}{v}\right)^2 (N^2-1)\,\left[N\left(|H_4^{34}|^2 + |H_4^{43}|^2\right) - \frac{1}{N}|H_4^{34} + H_4^{43}|^2\right]. \tag{4.1.3}$$

For the four quark amplitude with quarks of different flavours we have,

$$\sum |\mathcal{H}_4^{4q}(h_1, h_3)|^2 = \left(\frac{g_s^4}{16\pi^2}\right)^2\left(\frac{m^2}{v}\right)^2 (N^2-1)\,|H_4^{4q}(h_1, h_3)|^2\,, \tag{4.1.4}$$

while for identical quarks

$$\sum |\mathcal{H}_4^{4q}|^2 = \left(\frac{g_s^4}{16\pi^2}\right)^2\left(\frac{m^2}{v}\right)^2 (N^2-1)\,\Bigg(|H_4^{4q}(h_1, h_3)|^2 + |H_4^{4q'}(h_1, h_3)|^2$$

$$+\frac{\delta_{h_1h_3}}{N}\left(H_4^{4q}(h_1,h_3)H_4^{4q'}(h_1,h_3)^*+H_4^{4q}(h_1,h_3)^*H_4^{4q'}(h_1,h_3)\right)\Bigg). \tag{4.1.5}$$

We give expressions for the tree-level amplitudes that we require for this calculation in section 4.2 and in sections 4.3-4.5 we give the analytic results required to construct the Higgs plus four parton amplitudes. Section 4.6 is a discussion of the large mass limit and we conclude in section 4.7. In appendix A.2 we give the numerical results for all coefficients in this chapter at a given phase-space point, the same as for the scalar results in Appendix A.1. The full amplitudes for the fermion theory contain many coefficients that are identical to the scalar theory, as already discussed. Therefore in Appendix A.2 we only report results for the coefficients that differ.

4.2 Tree Amplitudes

The tree-level amplitudes containing 2 quarks and $n-2$ gluons required for this calculation can be calculated from the two formulae below [21],

$$\begin{aligned}&A\left(\underline{1}^a,3^+,4^+,\ldots,n^+,\overline{2}^b\right)=\\&\frac{im\left\langle 1^a2^b\right\rangle\left[3\left|\prod_{j=3}^{n-2}\left\{\not{p}_{13\ldots j}\not{p}_{j+1}+\left(s_{13\ldots j}-m^2\right)\right\}\right|n\right]}{\left(s_{13}-m^2\right)\left(s_{134}-m^2\right)\ldots\left(s_{13\ldots(n-1)}-m^2\right)\langle 34\rangle\langle 45\rangle\ldots\langle n-1\mid n\rangle},\end{aligned} \tag{4.2.1}$$

where all gluons have positive helicity, and

$$\begin{aligned}A\left(\underline{1}^a,3^-,4^+,\ldots,n^+,\overline{2}^b\right)=&-\frac{i\langle 3|1|2\mid 3\rangle\left(\langle 1^a3\rangle\left[2^b|1+2|3\right\rangle+\left\langle 2^b3\right\rangle\left[1^a|1+2|3\right\rangle\right)}{s_{12}\langle 34\rangle\ldots\langle n-1\mid n\rangle\langle 3|1|1+2\mid n\rangle}\\&+\sum_{k=4}^{n-1}\frac{im\left\langle 3\left|\not{p}_1\not{p}_{3\ldots k}\right|3\right\rangle}{s_{3\ldots k}\left(s_{13\ldots k}-m^2\right)\ldots\left(s_{13\ldots(n-1)}-m^2\right)}\\&\times\frac{\left(\left\langle 1^a2^b\right\rangle\left\langle 3\left|\not{p}_1\not{p}_{3\ldots k}\right|3\right\rangle+\langle 1^a3\rangle\left\langle 2^b3\right\rangle s_{3\ldots k}\right)}{\langle 34\rangle\ldots\langle k-1\mid k\rangle\left\langle 3\left|\not{p}_1\not{p}_{3\ldots k}\right|k\right\rangle}\\&\times\frac{\left\langle 3\left|\not{p}_{3\ldots k}\prod_{j=k}^{n-2}\left\{\not{p}_{13\ldots j}\not{p}_{j+1}+\left(s_{13\ldots j}-m^2\right)\right\}\right|n\right]}{\left\langle 3\left|\not{p}_1\not{p}_{3\ldots k}\right|k+1\right\rangle\langle k+1\mid k+2\rangle\ldots\langle n-1\mid n\rangle}\end{aligned} \tag{4.2.2}$$

for those where one gluon has negative helicity.

4.3 Amplitude for $0 \to ggggh$ with a Fermion Mediator

4.3.1 Coefficients for $H_4^{1234}(g^+, g^+, g^+, g^+; h)$

These results were originally published in ref. [105], and are included for completeness. The amplitude can be concisely written as

$$\begin{aligned} H_4^{1234}(1^+, 2^+, 3^+, 4^+; h) = \Bigg\{ & \frac{4m^2 - s_{1234}}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} \\ & \times \Big[- \mathrm{tr}_+\{1\,2\,3\,4\} m^2 E_0(p_1, p_2, p_3, p_4; m) \\ & + \frac{1}{2}((s_{12} + s_{13})(s_{24} + s_{34}) - s_{14}s_{23}) D_0(p_1, p_{23}, p_4; m) \\ & + \frac{1}{2} s_{12}s_{23} D_0(p_1, p_2, p_3; m) \\ & + (s_{12} + s_{13} + s_{14}) C_0(p_1, p_{234}; m) \Big] \\ & + 2\frac{s_{12} + s_{13} + s_{14}}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} \Bigg\} \\ & + \Bigg\{ 3 \text{ cyclic permutations} \Bigg\}. \end{aligned} \tag{4.3.1}$$

However, as with the results in chapter 3, we will focus on a basis containing box, triangle and bubble integrals only. The effective pentagon coefficient $\hat{e}$ for this amplitude is,

$$\hat{e}_{1\times2\times3\times4}(1^+, 2^+, 3^+, 4^+) = \frac{m^2\,(s_{1234} - 4m^2)\mathrm{tr}_+\{1\,2\,3\,4\}}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle}\,. \tag{4.3.2}$$

The remaining integral coefficients are shown in table 4.1.

Coefficient	Related coefficients	Coefficient	Related coefficients
$d_{1\times2\times34}$	$d_{2\times3\times41}, d_{3\times4\times12}, d_{4\times1\times23}$, $d_{1\times4\times32}, d_{2\times1\times43}, d_{3\times2\times14}, d_{4\times3\times21}$	$c_{1\times234}$	$c_{2\times341}, c_{3\times412}, c_{4\times123}$
$d_{1\times23\times4}$	$d_{2\times34\times1}, d_{3\times41\times2}, d_{4\times12\times3}$		
$d_{1\times2\times3}$	$d_{2\times3\times4}, d_{3\times4\times1}, d_{4\times1\times2}$		

Table 4.1: Minimal set of integral coefficients for $1_g^+ \, 2_g^+ \, 3_g^+ \, 4_g^+$. The rational terms are identical to the scalar case, given in eq. (3.5.9).

4.3.1.1 Boxes

$\boldsymbol{d_{1\times2\times34}}$

$$d_{1\times2\times34}(1^+,2^+,3^+,4^+) = \mathcal{C}^{(4)}_{1\times2\times3\times4}\,\hat{e}_{\{1^+\times2^+\times3^+\times4^+\}} \tag{4.3.3}$$

$\boldsymbol{d_{1\times23\times4}}$

$$\begin{aligned} d_{1\times23\times4}(1^+,2^+,3^+,4^+) = &\mathcal{C}^{(3)}_{1\times2\times3\times4}\,\hat{e}_{\{1^+\times2^+\times3^+\times4^+\}} \\ &+ \frac{1}{2}\frac{(4\,m^2 - s_{1234})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\left[(s_{12}+s_{13})\,(s_{24}+s_{34}) - s_{14}\,s_{23}\right] \end{aligned} \tag{4.3.4}$$

$\boldsymbol{d_{1\times2\times3}}$

$$\begin{aligned} d_{1\times2\times3}(1^+,2^+,3^+,4^+) = &\mathcal{C}^{(1)}_{4\times1\times2\times3}\,\hat{e}_{\{4^+\times1^+\times2^+\times3^+\}} + \mathcal{C}^{(5)}_{1\times2\times3\times4}\,\hat{e}_{\{1^+\times2^+\times3^+\times4^+\}} \\ &+ \frac{1}{2}\frac{(4\,m^2 - s_{1234})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,s_{12}\,s_{23} \end{aligned} \tag{4.3.5}$$

4.3.1.2 Triangles

$\boldsymbol{c^{(0)}_{1\times234}, c^{(2)}_{1\times234}}$

$$c^{(0)}_{1\times234}(1^+,2^+,3^+,4^+) = -(s_{12}+s_{13}+s_{14})\frac{s_{1234}}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 3\,4\rangle\;\langle 4\,1\rangle} \tag{4.3.6}$$

$$c^{(2)}_{1\times234}(1^+,2^+,3^+,4^+) = 4(s_{12}+s_{13}+s_{14})\frac{1}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 3\,4\rangle\;\langle 4\,1\rangle} \tag{4.3.7}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$d_{1\times 2\times 34}$	$d_{3\times 2\times 14}$	$c_{3\times 4}$ (3.5.27)	$c_{4\times 1}$
$d_{1\times 4\times 32}$	$d_{3\times 4\times 12}$	$c_{2\times 34}$ (3.5.28)	$c_{2\times 14}$
$d_{2\times 1\times 43}$	$d_{2\times 3\times 41}$	$c_{1\times 43}$ (3.5.29)	$c_{3\times 41}$
$d_{2\times 34\times 1}$	$d_{3\times 41\times 2}$	$c_{4\times 123}$	
$d_{4\times 3\times 21}$	$d_{4\times 1\times 23}$	$c_{1\times 234}$	$c_{3\times 412}$
$d_{1\times 23\times 4}$	$d_{4\times 12\times 3}$	$c_{2\times 341}$	
$d_{2\times 3\times 4}$	$d_{4\times 1\times 2}$	$c_{12\times 34}$	$c_{23\times 41}$
$d_{1\times 2\times 3}$		b_{34} (3.5.38)	b_{14}
$d_{3\times 4\times 1}$		b_{234} (3.5.39)	b_{412}, b_{341}
$d_{3\times 4\times 1}$		b_{1234} (3.5.40)	

Table 4.2: Minimal set of integral coefficients for $1_g^+\,2_g^+\,3_g^+\,4_g^-$. The brackets give the equation numbers for the coefficients that are identical to the scalar case, and hence have already been given in chapter 3. In addition, the rational terms are identical to the scalar case, eq. (3.5.41).

4.3.2 Coefficients for $H_4^{1234}(g^+,g^+,g^+,g^-;h)$

After removing spurious poles due to factors of $1/\mathrm{tr}_5\{1\,2\,3\,4\}^2$, as described in section 3.5.2, we have the effective pentagon coefficients,

$$\hat{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}} = (s_{123}-4m^2)\,m^2\left[\frac{[2\,3]\ \langle 4|(2+3)|1]}{\langle 2\,3\rangle\ \langle 1|(2+3)|4]}\right], \tag{4.3.8}$$

$$\hat{e}_{\{4^-\times 1^+\times 2^+\times 3^+\}} = \hat{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}}\{1\leftrightarrow 3\}, \tag{4.3.9}$$

$$\begin{aligned}\hat{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}} = &-m^2\frac{[2\,3]}{\langle 2\,3\rangle\,[3\,4]}\Bigg(\frac{[2\,3]\,\langle 2|(3+4)|1]\,\langle 4|(1+3)|2]}{\langle 1|(3+4)|2]}\\ &+[1\,3]\,\langle 4|(2+3)|1]+4m^2\frac{[2\,3]\,\langle 3\,4\rangle\,\langle 2|(3+4)|1]}{\langle 2\,3\rangle\,\langle 1|(3+4)|2]}\Bigg),\end{aligned} \tag{4.3.10}$$

$$\hat{e}_{\{3^+\times 4^-\times 1^+\times 2^+\}} = \hat{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}}\{1\leftrightarrow 3\}. \tag{4.3.11}$$

This creates very simple forms for the reduced pentagon contributions to the box coefficients. As before, it is not necessary to calculate every coefficient, the minimum set needed are given in columns one and three of table 4.2.

Many of these coefficients are identical to those with a scalar mediator, and these will not be repeated here.

4.3.2.1 Boxes

$\boldsymbol{d_{1\times 2\times 34}}$

$$\begin{aligned} d_{1\times 2\times 34}(1^+,2^+,3^+,4^-) =& \,\mathcal{C}^{(4)}_{1\times 2\times 3\times 4}\,\hat{e}_{\{1^+\times 2^+\times 3^+\times 4^-\}} \\ &+ \frac{1}{2}\frac{[1\,2]\,[2\,3]\,\langle 1|(2+4)|3]}{[3\,4]\,\langle 1|(3+4)|2]\,\langle 1|(2+3)|4]} \\ &\times \left([2\,3]\,s_{1234} - 4\frac{\langle 1|(2+4)|3]}{\langle 1\,2\rangle}m^2 \right) \\ &- \frac{1}{2}\frac{[1\,2]\,\langle 2\,4\rangle\,\langle 4|(2+3)|1]}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 2|(3+4)|1]}\left(\langle 4|(2+3)|1] + 4\frac{\langle 2\,4\rangle}{\langle 1\,2\rangle}m^2 \right) \end{aligned} \tag{4.3.12}$$

$\boldsymbol{d_{1\times 4\times 32}}$

$$\begin{aligned} d_{1\times 4\times 32}(1^+,2^+,3^+,4^-) =& \,\mathcal{C}^{(2)}_{2\times 3\times 4\times 1}\,\hat{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}} \\ &+ \frac{1}{2}\frac{[2\,3]\,s_{14}s_{234}(4m^2 s_{234} - s_{23}s_{1234})}{\langle 2\,3\rangle^2\,[3\,4]\,\langle 1|(3+4)|2]\,\langle 1|(2+3)|4]} \end{aligned} \tag{4.3.13}$$

$\boldsymbol{d_{2\times 1\times 43}}$

$$\begin{aligned} d_{2\times 1\times 43}(1^+,2^+,3^+,4^-) =& \,\mathcal{C}^{(2)}_{3\times 4\times 1\times 2}\,\hat{e}_{\{3^+\times 4^-\times 1^+\times 2^+\}} \\ &+ \frac{1}{2}\frac{[1\,2]\,\langle 4|(1+3)|2]^2\,(4m^2\,\langle 1\,4\rangle - \langle 1\,2\rangle\,\langle 4|(1+3)|2])}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 1|(3+4)|2]\,\langle 3|(1+4)|2]} \\ &+ \frac{1}{2}\frac{[1\,2]\,[1\,3]^2\,(4m^2\,\langle 2|(1+4)|3] - \langle 2\,1\rangle\,[1\,3]\,s_{1234})}{\langle 1\,2\rangle\,[1\,4]\,[3\,4]\,\langle 2|(3+4)|1]} \end{aligned} \tag{4.3.14}$$

$\boldsymbol{d_{2\times 34\times 1}}$

$$\begin{aligned} d_{2\times 34\times 1}(1^+,2^+,3^+,4^-) =& \,\mathcal{C}^{(3)}_{2\times 3\times 4\times 1}\,\hat{e}_{\{2^+\times 3^+\times 4^-\times 1^+\}} \\ &+ \frac{\langle 2\,4\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle}\Bigg[2\frac{\langle 1\,4\rangle\,\langle 2\,4\rangle\,\langle 1|(3+4)|2]\,\langle 2|(3+4)|1]}{\langle 1\,2\rangle^2\,\langle 3\,4\rangle} \\ &+ \frac{\langle 4|(1+3)|2]\,\langle 4|(2+3)|1]}{2\,\langle 3\,4\rangle} \\ &+ \frac{s_{1234}\,[1\,3]\,[2\,3]}{2\,[3\,4]} - 2m^2\Big(3\frac{\langle 1\,4\rangle\,\langle 2\,4\rangle\,[1\,2]}{\langle 1\,2\rangle\,\langle 3\,4\rangle} + 2\frac{[1\,3]\,[2\,3]}{[3\,4]} \end{aligned} \tag{4.3.15}$$

$$+\frac{\langle 2\,4\rangle\,[1\,4]\,[2\,3]}{\langle 2\,3\rangle\,[3\,4]}\Big)\Big]$$

$\boldsymbol{d_{4\times3\times21}}$

$$\begin{aligned} d_{4\times3\times21}(1^+,2^+,3^+,4^-) =& \mathcal{C}^{(2)}_{1\times2\times3\times4}\,\hat{e}_{\{1^+\times2^+\times3^+\times4^-\}} \\ &+\frac{(4\,m^2-s_{123})s_{34}\,s_{123}^2}{2\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 1|(2+3)|4]\,\langle 3|(1+2)|4]} \end{aligned} \tag{4.3.16}$$

$\boldsymbol{d_{1\times23\times4}}$

$$d_{1\times23\times4}(1^+,2^+,3^+,4^-) = \mathcal{C}^{(3)}_{1\times2\times3\times4}\,e_{\{1^+\times2^+\times3^+\times4^-\}} \tag{4.3.17}$$

$\boldsymbol{d_{2\times3\times4}}$

$$\begin{aligned} d_{2\times3\times4}(1^+,2^+,3^+,4^-) =& \mathcal{C}^{(1)}_{1\times2\times3\times4}\,\hat{e}_{\{1^+\times2^+\times3^+\times4^-\}} + \mathcal{C}^{(5)}_{2\times3\times4\times1}\,\hat{e}_{\{2^+\times3^+\times4^-\times1^+\}} \\ &+\frac{(4m^2 s_{234}-s_{23}s_{1234})\,\langle 3\,4\rangle\,[2\,3]^2}{2\,\langle 2\,3\rangle\,\langle 1|(3+4)|2]\,\langle 1|(2+3)|4]} \end{aligned} \tag{4.3.18}$$

$\boldsymbol{d_{1\times2\times3}}$

$$\begin{aligned} d_{1\times2\times3}(1^+,2^+,3^+,4^-) =& \mathcal{C}^{(5)}_{1\times2\times3\times4}\,e_{\{1^+\times2^+\times3^+\times4^-\}} + \mathcal{C}^{(1)}_{4\times1\times2\times3}\,e_{\{4^-\times1^+\times2^+\times3^+\}} \\ &+(4\,m^2-s_{123})\,\frac{s_{123}\,[1\,2]\,[2\,3]}{2\,\langle 3|(1+2)|4]\,\langle 1|(2+3)|4]} \end{aligned} \tag{4.3.19}$$

It is useful to note that this expression is symmetric under the exchange $1 \leftrightarrow 3$.

$\boldsymbol{d_{3\times4\times1}}$

$$\begin{aligned} d_{3\times4\times1}(1^+,2^+,3^+,4^-) =& \mathcal{C}^{(1)}_{2\times3\times4\times1}\,\hat{e}_{\{2^+\times3^+\times4^-\times1^+\}} + \mathcal{C}^{(5)}_{3\times4\times1\times2}\,\hat{e}_{\{3^+\times4^-\times1^+\times2^+\}} \\ &-\frac{1}{2}\frac{s_{14}\,[2\,3]\,\langle 3\,4\rangle\,(\langle 4|(1+3)|2]\,\langle 2\,3\rangle+4m^2\,\langle 3\,4\rangle)}{\langle 1\,3\rangle\,\langle 2\,3\rangle^2\,\langle 1|(3+4)|2]} \\ &-\frac{1}{2}\frac{[1\,2]\,\langle 1\,4\rangle\,s_{34}}{\langle 1\,2\rangle^2\,\langle 1\,3\rangle\,\langle 3|(1+4)|2]}(\langle 4|(1+3)|2]\,\langle 2\,1\rangle+4m^2\,\langle 1\,4\rangle) \\ &+\frac{1}{2}\frac{\langle 1\,4\rangle\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,\langle 1\,3\rangle\,\langle 2\,3\rangle}\left[4\frac{s_{14}s_{34}}{\langle 1\,3\rangle}+[1\,3]\left(s_{123}-12m^2\right)\right] \end{aligned} \tag{4.3.20}$$

Which is again symmetric under the exchange $1 \leftrightarrow 3$.

4.3.2.2 Triangles

$\boldsymbol{c^{(0)}_{4\times 123}, c^{(2)}_{4\times 123}}$

$$c^{(0)}_{4\times 123}(1^+,2^+,3^+,4^-) = -\frac{s_{123}^2\,(s_{14}+s_{24}+s_{34})}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3|(1+2)|4]\ \langle 1|(2+3)|4]} \tag{4.3.21}$$

$$c^{(2)}_{4\times 123}(1^+,2^+,3^+,4^-) = 4\frac{s_{123}\,(s_{14}+s_{24}+s_{34})}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3|(1+2)|4]\ \langle 1|(2+3)|4]} \tag{4.3.22}$$

$\boldsymbol{c^{(0)}_{1\times 234}, c^{(2)}_{1\times 234}}$

$$\begin{aligned} c^{(0)}_{1\times 234}(1^+,2^+,3^+,4^-) &= -2\frac{\langle 1|(2+3)|1]^2\,\langle 4|(2+3)|1]}{\langle 1|(2+3)|4]\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[1\,4]} \\ &+(s_{12}+s_{13}+s_{14})\left[\frac{1}{\langle 1|(2+3)|4]}\left(2\frac{\langle 4|(2+3)|1]\,[2\,3]^2}{s_{34}s_{234}}\right.\right. \\ &\left.-\frac{(s_{12}+s_{13})^2\,[3\,4]+s_{23}\,\langle 1\,2\rangle\,[2\,3]\,[1\,4]}{\langle 1\,2\rangle\,\langle 1\,3\rangle\,\langle 2\,3\rangle\,[1\,4]\,[3\,4]}\right) \\ &+\frac{\langle 3|(1+4)|2]\,\langle 1\,4\rangle\,[2\,3]^2}{\langle 1|(3+4)|2]\,\langle 1\,2\rangle\,\langle 1\,3\rangle\,[2\,4]\,s_{34}}-\frac{\langle 4|(2+3)|1]\,[1\,3]^2}{\langle 2|(3+4)|1]\,\langle 1\,2\rangle\,[1\,4]\,s_{34}} \\ &\left.-\frac{\langle 1\,4\rangle\,(s_{12}^2\,\langle 2\,3\rangle+2\,\langle 1\,3\rangle\,\langle 2\,4\rangle\,[1\,4]\,s_{24})}{\langle 1\,2\rangle^3\,\langle 2\,3\rangle\,\langle 1\,3\rangle\,\langle 3\,4\rangle\,[2\,4]\,[1\,4]}\right] \end{aligned} \tag{4.3.23}$$

$$\begin{aligned} c^{(2)}_{1\times 234}(1^+,2^+,3^+,4^-) &= \frac{8\ \langle 4|(2+3)|1]^3}{s_{234}\,(s_{12}+s_{13}+s_{14})\ \langle 2\,3\rangle\ \langle 3\,4\rangle\ \langle 2|(3+4)|1]} \\ &-4\,(s_{12}+s_{13}+s_{14})\left\{\frac{\langle 1\,4\rangle^2\ [2\,3]}{\langle 1\,2\rangle^2\ \langle 3\,4\rangle\ \langle 1|(3+4)|2]\ \langle 1|(2+3)|4]}\right. \\ &-\frac{[2\,3]}{\langle 1\,2\rangle\ \langle 1|(2+3)|4]\ \langle 1|(3+4)|2]\ \langle 2|(3+4)|1]} \\ &\times\left[\frac{\langle 1\,2\rangle\,[1\,3]\,[2\,3]}{[3\,4]}+\frac{[1\,2]\,\langle 1\,4\rangle\,\langle 2\,4\rangle}{\langle 3\,4\rangle}\right] \\ &-\left[\frac{[1\,3]\ \langle 1\,4\rangle}{\langle 1\,2\rangle}+\frac{\langle 4|(2+3)|1]}{\langle 2\,3\rangle}\right] \\ &\left.\times\left[\frac{\langle 2\,4\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle\ \langle 1|(2+3)|4]\ \langle 2|(3+4)|1]}\right]\right\} \end{aligned} \tag{4.3.24}$$

$c^{(0)}_{2\times 341}$, $c^{(2)}_{2\times 341}$

$$
\begin{aligned}
c^{(0)}_{2\times 341}(1^+,2^+,3^+,4^-) &= 2\frac{(s_{12}+s_{23}+s_{24})\,\langle 2\,4\rangle^2\left(\langle 1\,4\rangle^2\langle 2\,3\rangle^2+\langle 1\,2\rangle^2\langle 3\,4\rangle^2\right)}{\langle 1\,2\rangle^3\langle 2\,3\rangle^3\langle 1\,4\rangle\langle 3\,4\rangle}\\
&+\frac{\langle 4|(1+3)|2]^4}{s_{134}\langle 1\,4\rangle\langle 3\,4\rangle\langle 1|(3+4)|2]\,\langle 3|(1+4)|2]}+\frac{(s_{12}+s_{23})\,[1\,3]^2}{\langle 1\,2\rangle\,[1\,4]\,\langle 2\,3\rangle\,[3\,4]}\\
&+\left\{\frac{(s_{12}+s_{23}+s_{24})\,[1\,2]\,[1\,3]^3}{s_{134}\,[1\,4]\,[3\,4]\,\langle 2|(3+4)|1]}-\frac{[1\,2]\,\langle 2\,4\rangle^2\langle 4|(3+2)|1]}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 2|(3+4)|1]}\right.\\
&\left.-\frac{[1\,2]\,[1\,3]^2\langle 2\,4\rangle}{\langle 2\,3\rangle\,[3\,4]\,\langle 2|(3+4)|1]}\right\}\\
&+\{1\leftrightarrow 3\}
\end{aligned}
\tag{4.3.25}
$$

$$
\begin{aligned}
c^{(2)}_{2\times 341}(1^+,2^+,3^+,4^-) &= \frac{4(s_{12}+s_{23}+s_{24})\,s_{134}\,[1\,2]\,[3\,2]}{\langle 1\,2\rangle\,\langle 3\,2\rangle\,[1\,4]\,[3\,4]\,\langle 1|(3+4)|2]\,\langle 3|(1+4)|2]}\\
&-\frac{8}{(s_{12}+s_{23}+s_{24})\,s_{134}}\\
&\times\frac{\langle 4|(1+3)|2]^4}{\langle 1\,4\rangle\,\langle 3\,4\rangle\,\langle 1|(3+4)|2]\,\langle 3|(1+4)|2]}\\
&+\left\{\frac{4\,(s_{12}+s_{23}+s_{24})}{[(s_{13}+s_{14})\,(s_{23}+s_{24})-s_{12}\,s_{34}]}\right.\\
&\left.\times\left[\frac{[1\,3]^2\,[2\,3]}{\langle 1\,2\rangle\,[1\,4]\,[3\,4]}-\frac{[1\,2]\,\langle 1\,4\rangle\,\langle 2\,4\rangle^2}{\langle 1\,2\rangle^2\,\langle 2\,3\rangle\,\langle 3\,4\rangle}\right]\right\}\\
&+\left\{1\leftrightarrow 3\right\}
\end{aligned}
\tag{4.3.26}
$$

$c^{(0)}_{12\times 34}$, $c^{(2)}_{12\times 34}$

$$
\begin{aligned}
c^{(0)}_{12\times 34}(1^+,2^+,3^+,4^-) &= \frac{[1\,3]\,(s_{12}+s_{23})\,\langle 4|(2+3)|1]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[2\,4]\,\langle 2|(3+4)|1]}\\
&+\frac{[1\,2]\,\langle 2\,4\rangle^2\,\langle 4|(2+3)|1]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 2|(3+4)|1]}\\
&-\frac{[2\,3]\,\langle 4|(1+3)|2]^2}{\langle 1\,2\rangle\,[2\,4]\,\langle 3\,4\rangle\,\langle 1|(3+4)|2]}\\
&-\frac{[2\,3]^3\,s_{1234}}{\langle 1\,2\rangle\,[2\,4]\,[3\,4]\,\langle 1|(3+4)|2]}\\
&-\frac{[1\,3]^2\,[2\,3]\,s_{1234}}{\langle 1\,2\rangle\,[2\,4]\,[3\,4]\,\langle 2|(3+4)|1]}\\
&+\frac{[1\,2]\,s_{123}\,(s_{123}-s_{124})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[2\,4]\,\langle 3|(1+2)|4]}
\end{aligned}
$$

$$- \frac{[1\,2]\ \langle 1\,4\rangle\ \langle 4|(2+3)|1]}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ [2\,4]\ \langle 3\,4\rangle} \tag{4.3.27}$$

$$\begin{aligned}
c^{(2)}_{12\times 34}(1^+,2^+,3^+,4^-) = &-4\,\frac{[1\,2]}{\langle 1\,2\rangle\ \langle 3|(1+2)|4]\ \Big[(s_{13}+s_{14})\,(s_{23}+s_{24}) - s_{12}\,s_{34}\Big]} \\
&\times\Bigg[\left(\langle 1\,4\rangle\ [1\,3] - \langle 2\,4\rangle\ [2\,3]\right)(s_{13}+s_{23}-s_{14}-s_{24}) \\
&+\langle 1\,2\rangle\ \langle 3\,4\rangle\ [1\,3]\ [2\,3] \\
&\times\left(2 - \frac{(s_{13}+s_{23})\,(s_{13}+s_{14}+s_{23}+s_{24})}{s_{12}\,s_{34}}\right) \\
&+\langle 1\,4\rangle\ \langle 2\,4\rangle\ [1\,2]\ [3\,4] \\
&\times\left(2 - \frac{(s_{14}+s_{24})\,(s_{13}+s_{14}+s_{23}+s_{24})}{s_{12}\,s_{34}}\right)\Bigg]
\end{aligned} \tag{4.3.28}$$

4.3.3 Coefficients for $H_4^{1234}(g^+,g^-,g^+,g^-;h)$

The pentagon coefficients for gluons of alternating helicities also has spurious poles, in this case due to factors of $1/\mathrm{tr}_5\{1\,2\,3\,4\}^4$. We can use a similar method as in section 3.5.2 to remove them. After this modification we have just one necessary expression for the pentagon coefficients,

$$\begin{aligned}
\hat{e}_{\{1^+\times 2^-\times 3^+\times 4^-\}} = &-m^2\frac{\langle 1\,2\rangle\,[3\,4]\,\langle 4|(2+3)|1]}{[1\,2]\,\langle 3\,4\rangle\,\langle 1|(2+3)|4]} \\
&\times\left(\frac{[1\,3]^2\,\langle 3\,4\rangle}{[3\,4]} + \frac{\langle 2\,4\rangle^2\,[1\,2]}{\langle 1\,2\rangle} + 4m^2\frac{\langle 4|(2+3)|1]}{\langle 1|(2+3)|4]}\right).
\end{aligned} \tag{4.3.29}$$

The other coefficients are trivially related by symmetries:

$$\hat{e}_{\{3^+\times 4^-\times 1^+\times 2^-\}} = \hat{e}_{\{1^+\times 2^-\times 3^+\times 4^-\}}\Big\{1\leftrightarrow 3,\ 2\leftrightarrow 4\Big\} \tag{4.3.30}$$

$$\hat{e}_{\{4^-\times 1^+\times 2^-\times 3^+\}} = \hat{e}_{\{1^+\times 2^-\times 3^+\times 4^-\}}\Big\{1\to 4,\ 2\to 1,\ 3\to 2,\ 4\to 3,\ \langle\rangle\leftrightarrow[]\Big\} \tag{4.3.31}$$

$$\hat{e}_{\{2^-\times 3^+\times 4^-\times 1^+\}} = \hat{e}_{\{1^+\times 2^-\times 3^+\times 4^-\}}\Big\{1\to 2,\ 2\to 3,\ 3\to 4,\ 4\to 1,\ \langle\rangle\leftrightarrow[]\Big\}. \tag{4.3.32}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$d_{4\times3\times21}$	$d_{2\times1\times43}, d_{3\times2\times14}, d_{1\times4\times32}, d_{1\times2\times34}, d_{2\times3\times41}, d_{3\times4\times12}, d_{4\times1\times23}$	$c_{3\times4}$ (3.5.49)	$c_{4\times1}, c_{2\times3}, c_{1\times2}$
		$c_{2\times34}$ (3.5.50)	$c_{3\times41}, c_{4\times12}, c_{1\times23}$ $c_{1\times43}, c_{2\times14}, c_{3\times21}, c_{4\times32}$
$d_{1\times23\times4}$	$d_{2\times34\times1}, d_{3\times41\times2}, d_{4\times12\times3}$	$c_{12\times34}$	$c_{23\times41}$
$d_{1\times2\times3}$	$d_{2\times3\times4}, d_{3\times4\times1}, d_{4\times1\times2}$	$c_{1\times234}$	$c_{2\times341}, c_{3\times412}, c_{4\times123}$
		b_{34} (3.5.57)	b_{12}, b_{23}, b_{41}
		b_{234} (3.5.58)	$b_{341}, b_{412}, b_{123}$
		b_{1234} (3.5.59)	

Table 4.3: Minimal set of integral coefficients for $1_g^+\,2_g^-\,3_g^+\,4_g^-$. The brackets give the equation numbers for the coefficients that are identical to the scalar case, and hence have already been given in chapter 3. In addition, the rational terms are identical to the scalar case, eq. (3.5.60).

The minimal set of coefficients that must be calculated is shown in columns one and three of table 4.3, though when calculating coefficients for other colour orderings it is necessary to use functions given for $H_4^{1234}(g^+, g^+, g^-, g^-; h)$ in the next section.

4.3.3.1 Boxes

$d_{4\times3\times21}$

$$\begin{aligned}
d_{4\times3\times21}(1^+,2^-,3^+,4^-) =\, & \hat{e}_{\{1^+\times2^-\times3^+\times4^-\}}\,\mathcal{C}^{(2)}_{1\times2\times3\times4} \\
& + \frac{\langle 2|(1+3)|4]}{\langle 1|(2+3)|4]\,\langle 3|(1+2)|4]} \\
& \Bigg[-2\frac{\langle 2\,3\rangle\,\langle 2|(1+3)|4]\,s_{34}s_{123}^2}{\langle 1\,2\rangle\,\langle 3|(1+2)|4]^2} \\
& + \frac{1}{2}\frac{\langle 2\,4\rangle^2\,[3\,4]\,s_{123}}{\langle 1\,2\rangle} + \frac{1}{2}\frac{[1\,3]^2\,\langle 3\,4\rangle\,s_{123}}{[1\,2]} \\
& - 2m^2\Bigg(2\frac{[1\,3]\,\langle 4|(2+3)|1]}{[1\,2]} \\
& + \frac{[2\,3]\,\langle 2|(1+3)|4]\,\langle 4|(2+3)|1]}{[1\,2]\,\langle 1|(2+3)|4]} \\
& + 3\frac{\langle 2\,3\rangle\,\langle 2|(1+3)|4]\,\langle 4|(1+2)|3]}{\langle 1\,2\rangle\,\langle 3|(1+2)|4]}\Bigg)\Bigg]
\end{aligned} \tag{4.3.33}$$

$\boldsymbol{d_{1\times 23\times 4}}$

$$
\begin{aligned}
d_{1\times 23\times 4}(1^+,2^-,3^+,4^-) = {}& \hat{e}_{\{1^+\times 2^-\times 3^+\times 4^-\}}\,\mathcal{C}^{(3)}_{1\times 2\times 3\times 4} \\
&+ \left[\frac{1}{2}\frac{\langle 2\,4\rangle^3\,\langle 4|(2+3)|1]}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\left(1+4m^2\frac{\langle 1\,2\rangle}{\langle 2\,4\rangle\,\langle 1|(2+3)|4]}\right)\right] \\
&+ \Big[1\leftrightarrow 4,\ 2\leftrightarrow 3,\ \langle\rangle\leftrightarrow []\Big]
\end{aligned}
\tag{4.3.34}
$$

The symmetrisation only applies to the terms in square brackets.

$\boldsymbol{d_{1\times 2\times 3}}$

$$
\begin{aligned}
d_{1\times 2\times 3}(1^+,2^-,3^+,4^-) = {}& \mathcal{C}^{(5)}_{1\times 2\times 3\times 4}\,\hat{e}_{\{1^+\times 2^-\times 3^+\times 4^-\}} + \mathcal{C}^{(1)}_{4\times 1\times 2\times 3}\,\hat{e}_{\{4^-\times 1^+\times 2^-\times 3^+\}} \\
&+ \frac{\langle 1\,2\rangle\,\langle 2\,3\rangle}{\langle 1|(2+3)|4]\,\langle 3|(1+2)|4]}\Bigg[-2\frac{s_{12}s_{23}s_{123}}{\langle 1\,3\rangle^2} \\
&- \frac{1}{2}\frac{[1\,2]\,[2\,3]\,\langle 2\,4\rangle^2\,(s_{14}+s_{24}+s_{34})}{\langle 1\,4\rangle\,\langle 3\,4\rangle} \\
&+ \frac{1}{2}[1\,3]^2\,s_{123} + 2m^2\Bigg(2\frac{[1\,3]\,s_{123}}{\langle 1\,3\rangle} - [1\,3]^2 + \frac{[1\,2]\,[2\,3]\,\langle 2\,4\rangle^2}{\langle 1\,4\rangle\,\langle 3\,4\rangle} \\
&- \frac{[2\,3]\,\langle 2|(1+3)|4]\,\langle 4|(2+3)|1]}{\langle 3\,4\rangle\,\langle 1|(2+3)|4]} \\
&+ \frac{[1\,2]\,\langle 2|(1+3)|4]\,\langle 4|(1+2)|3]}{\langle 1\,4\rangle\,\langle 3|(1+2)|4]}\Bigg)\Bigg]
\end{aligned}
\tag{4.3.35}
$$

As expected, this is explicitly symmetric under the exchange $1\leftrightarrow 3$.

4.3.3.2 Triangles

$\boldsymbol{c^{(0)}_{12\times 34}, c^{(2)}_{12\times 34}}$

The simplest way to write this coefficient is in terms of the corresponding coefficient with a scalar loop, $\tilde{c}^{(0)}_{12\times 34}$ (eq. (3.5.51)),

$$
\begin{aligned}
c^{(0)}_{12\times 34}(1^+,2^-,3^+,4^-) = {}& \tilde{c}^{(0)}_{12\times 34}(1^+,2^-,3^+,4^-) \\
&+ \left\{\frac{\langle 2\,1\rangle^2\ [1\,3]^2\ \langle 3\,4\rangle^2 - \langle 2\,4\rangle^2\ \langle 1|(3+4)|1]\ \langle 4|(1+2)|4]}{\langle 1\,2\rangle\ \langle 3\,4\rangle\ \langle 1|(3+4)|2]\ \langle 3|(1+2)|4]}\right\} \\
&+ \Big\{1\leftrightarrow 3, 2\leftrightarrow 4\Big\} + \Big\{1\leftrightarrow 2, 3\leftrightarrow 4, \langle\,\rangle\leftrightarrow [\,]\Big\}
\end{aligned}
$$

$$+\Big\{1 \leftrightarrow 4, 2 \leftrightarrow 3, \langle\,\rangle \leftrightarrow [\,]\Big\} \tag{4.3.36}$$

$$\begin{aligned}
c^{(2)}_{12\times 34}(1^+,2^-,3^+,4^-) = &\Bigg\{4\frac{\langle 2|(3+4)|1]}{\langle 1|(3+4)|2]\,\langle 3|(1+2)|4]}\Bigg[\frac{[2\,3]^2\,(s_{23}-s_{14})}{[1\,2]\,[3\,4]\,\langle 1|(3+4)|2]} \\
&+\frac{3}{2}\frac{[1\,3]\,[2\,3]}{[1\,2]\,[3\,4]} \\
&+\langle 2\,4\rangle\,\frac{(\langle 3|(1+2)|3]-\langle 4|(1+2)|4])}{\Delta_3(1,2,3,4)} \\
&\times\left([2\,3]-\frac{\langle 1\,4\rangle\,p_{12}\cdot p_{34}}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\right)\Bigg]\Bigg\} \\
&+\Big\{1\leftrightarrow 3, 2\leftrightarrow 4\Big\}+\Big\{1\leftrightarrow 2, 3\leftrightarrow 4, \langle\,\rangle\leftrightarrow[\,]\Big\} \\
&+\Big\{1\leftrightarrow 4, 2\leftrightarrow 3, \langle\,\rangle\leftrightarrow[\,]\Big\}
\end{aligned} \tag{4.3.37}$$

where Δ_3 is given by eq. (3.5.54).

$c^{(0)}_{1\times 234}$, $c^{(2)}_{1\times 234}$

$$\begin{aligned}
&c^{(0)}_{1\times 234}(1^+,2^-,3^+,4^-) = -2\,\frac{[1\,3]^4}{[1\,2]\;[1\,4]\;[3\,2]\;[3\,4]} \\
&+(s_{12}+s_{13}+s_{14})\Bigg[\frac{[1\,3]^2\;[3\,4]}{\langle 1|(2+3)|4]\;[1\,4]\;[2\,3]\;[2\,4]}+\frac{[1\,3]^2\;[3\,2]}{\langle 1|(3+4)|2]\;[1\,2]\;[4\,3]\;[4\,2]} \\
&+\frac{\langle 2\,4\rangle^2\;\langle 1\,4\rangle}{\langle 1|(3+4)|2]\;\langle 1\,3\rangle\;\langle 1\,2\rangle\;\langle 3\,4\rangle}+\frac{\langle 2\,4\rangle^2\;\langle 1\,2\rangle}{\langle 1|(2+3)|4]\;\langle 1\,3\rangle\;\langle 1\,4\rangle\;\langle 3\,2\rangle} \\
&-2\,\frac{s_{234}\;\langle 1|(2+4)|3]^2\;\left([2\,3]^2\;\langle 1|(2+3)|4]^2+[3\,4]^2\;\langle 1|(3+4)|2]^2\right)}{\langle 1|(2+3)|4]^3\;\langle 1|(3+4)|2]^3\;[2\,3]\;[3\,4]}\Bigg]
\end{aligned} \tag{4.3.38}$$

$$\begin{aligned}
&c^{(2)}_{1\times 234}(1^+,2^-,3^+,4^-) = 4\,\frac{(s_{12}+s_{13}+s_{14})}{\langle 1|(3+4)|2]\;\langle 1|(2+3)|4]}\Bigg[\left(\frac{\langle 2\,4\rangle^2}{\langle 2\,3\rangle\;\langle 3\,4\rangle}-\frac{[1\,3]^2}{[1\,2]\;[1\,4]}\right) \\
&+\frac{\langle 1|(2+4)|3]}{\langle 1|(3+4)|2]}\left(\frac{\langle 1\,4\rangle\;\langle 2\,4\rangle}{\langle 1\,2\rangle\;\langle 3\,4\rangle}+\frac{[1\,3]\;[2\,3]}{[1\,2]\;[3\,4]}\right)+\frac{\langle 1|(2+4)|3]}{\langle 1|(2+3)|4]}\left(\frac{\langle 1\,2\rangle\;\langle 2\,4\rangle}{\langle 1\,4\rangle\;\langle 2\,3\rangle}+\frac{[1\,3]\;[3\,4]}{[1\,4]\;[2\,3]}\right)\Bigg] \\
&-8\,\frac{[1\,3]^4}{[1\,2]\;[1\,4]\;[2\,3]\;[3\,4]\;(s_{12}+s_{13}+s_{14})}
\end{aligned} \tag{4.3.39}$$

Coefficient	Related coefficients	Coefficient	Related coefficients
$d_{1\times2\times34}$	$d_{2\times1\times43}, d_{3\times4\times12}, d_{4\times3\times21}$	$c_{2\times3}$ (3.5.70)	$c_{4\times1}$
$d_{1\times4\times32}$	$d_{3\times2\times14}, d_{4\times1\times23}, d_{2\times3\times41}$	$c_{1\times23}$ (3.5.71)	$c_{2\times14}, c_{3\times41}, c_{4\times32}$
$d_{2\times34\times1}$	$d_{4\times12\times3}$	$c_{23\times41}$	
$d_{1\times23\times4}$	$d_{3\times41\times2}$	$c_{1\times234}$	$c_{2\times341}, c_{3\times412}, c_{4\times123}$
$d_{1\times2\times3}$	$d_{3\times4\times1}, d_{4\times1\times2}, d_{2\times3\times4}$	b_{23} (3.5.76)	b_{41}
		b_{234} (3.5.77)	$b_{341}, b_{412}, b_{123}$
		b_{1234} (3.5.78)	

Table 4.4: Minimal set of integral coefficients for $1_g^+\,2_g^+\,3_g^-\,4_g^-$. The brackets give the equation numbers for the coefficients that are identical to the scalar case, and hence have already been given in chapter 3. In addition, the rational terms are identical to the scalar case, eq. (3.5.79).

4.3.4 Coefficients for $H_4^{1234}(g^+, g^+, g^-, g^-; h)$

The effective pentagon coefficients for this amplitude do not contain any factors of $1/\mathrm{tr}_5\{1\,2\,3\,4\}^2$, and it is therefore straightforward to obtain,

$$\hat{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}} = m^2(s_{12}+s_{34}-4m^2)\frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]} \tag{4.3.40}$$

$$\hat{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}} = m^2\frac{\langle 3\,4\rangle}{[3\,4]}\left[(s_{34}-4\,m^2)\frac{[2\,3]\ \langle 3\,4\rangle\ [4\,1]}{\langle 2\,3\rangle\ [3\,4]\ \langle 4\,1\rangle} - [1\,2]^2\right] \tag{4.3.41}$$

$$\hat{e}_{\{4^-\times 1^+\times 2^+\times 3^-\}} = \hat{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}}\{2\leftrightarrow 4, 1\leftrightarrow 3, \langle\,\rangle\leftrightarrow[\,]\}\,. \tag{4.3.42}$$

The minimal set of coefficients that must be calculated is shown in columns one and three of table 4.4, though when calculating coefficients for other colour orderings it is necessary to use functions given for $H_4^{1234}(g^+, g^-, g^+, g^-; h)$ in the previous section.

4.3.4.1 Boxes

$d_{1\times2\times34}$

$$d_{1\times2\times34}(1^+,2^+,3^-,4^-) = \mathcal{C}^{(4)}_{1\times2\times3\times4}\,\hat{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}} \tag{4.3.43}$$

$d_{1\times4\times32}$

$$d_{1\times4\times32}(1^+,2^+,3^-,4^-) = \mathcal{C}^{(2)}_{2\times3\times4\times1}\,\hat{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}}$$

$$
\begin{aligned}
&-\frac{[2\,4]}{\langle 1|(2+3)|4]\;[3\,4]}\Bigg\{2\,\frac{[2\,4]\;s_{14}\,s_{234}^2\,\langle 1|(3+4)|2]}{[2\,3]\;\langle 1|(2+3)|4]^2}\\
&+\frac{1}{2}\,s_{234}\left[\frac{[1\,4]\;\langle 3\,4\rangle^2}{\langle 2\,3\rangle}+\frac{\langle 1\,4\rangle\;[1\,2]^2}{[2\,3]}\right]\\
&+2\,m^2\Bigg[3\,\frac{[2\,4]\,\langle 1|(3+4)|2]\,\langle 4|(2+3)|1]}{[2\,3]\,\langle 1|(2+3)|4]}+\frac{[1\,4]\,\langle 3\,4\rangle\,s_{234}}{\langle 2\,3\rangle\,[3\,4]}\\
&+\frac{\langle 3\,4\rangle\;\langle 3|(2+4)|1]}{\langle 2\,3\rangle}\Bigg]\Bigg\}
\end{aligned}
\tag{4.3.44}
$$

$\boldsymbol{d_{2\times 34\times 1}}$

$$
\begin{aligned}
d_{2\times 34\times 1}(1^+,2^+,3^-,4^-) =&\,\mathcal{C}^{(3)}_{2\times 3\times 4\times 1}\,\hat{e}_{\{2^+\times 3^-\times 4^-\times 1^+\}}\\
&+\frac{1}{2}\frac{\langle 3\,4\rangle\;\langle 1|(3+4)|2]\;\langle 2|(3+4)|1]\;(s_{34}-4m^2)}{\langle 1\,2\rangle\;\langle 1\,4\rangle\;\langle 2\,3\rangle\;[3\,4]^2}
\end{aligned}
\tag{4.3.45}
$$

$\boldsymbol{d_{1\times 23\times 4}}$

$$
\begin{aligned}
d_{1\times 23\times 4}(1^+,2^+,3^-,4^-) =&\,\mathcal{C}^{(3)}_{1\times 2\times 3\times 4}\,\hat{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}}\\
&-\frac{1}{2}\;\langle 4|(2+3)|1]\times\Bigg[\frac{[2\,1]\;[2\,4]}{[1\,4]\;[2\,3]\;[3\,4]}\left([2\,1]-\frac{4\,m^2\;[2\,4]}{\langle 1|(2+3)|4]}\right)\\
&+\frac{\langle 4\,3\rangle\;\langle 1\,3\rangle}{\langle 2\,3\rangle\;\langle 1\,4\rangle\;\langle 1\,2\rangle}\left(\langle 4\,3\rangle-\frac{4\,m^2\;\langle 1\,3\rangle}{\langle 1|(2+3)|4]}\right)\Bigg]
\end{aligned}
\tag{4.3.46}
$$

$\boldsymbol{d_{1\times 2\times 3}}$

$$
\begin{aligned}
d_{1\times 2\times 3}(1^+,2^+,3^-,4^-) =&\,\mathcal{C}^{(5)}_{1\times 2\times 3\times 4}\,\hat{e}_{\{1^+\times 2^+\times 3^-\times 4^-\}}+\mathcal{C}^{(1)}_{4\times 1\times 2\times 3}\,\hat{e}_{\{4^-\times 1^+\times 2^+\times 3^-\}}\\
&+\frac{[1\,2]^2\;\langle 2\,3\rangle\;(s_{12}-4\,m^2)}{2\;[3\,4]\;\langle 1\,2\rangle\;[1\,4]}
\end{aligned}
\tag{4.3.47}
$$

4.3.4.2 Triangles

$\boldsymbol{c^{(0)}_{23\times 41}, c^{(2)}_{23\times 41}}$

The full coefficient for this integral is defined in terms of the coefficient with a scalar running in the loop, $\tilde{c}^{(0)}_{23\times 41}$ (eq. (3.5.72)),

$$
c^{(0)}_{23\times 41}(1^+,2^+,3^-,4^-) = \tilde{c}^{(0)}_{23\times 41}(1^+,2^+,3^-,4^-)
$$

$$
\begin{aligned}
&+\left\{ - \frac{\langle 3\,2\rangle^2\ [2\,1]^2\ \langle 1\,4\rangle^2 - \langle 3\,4\rangle^2\ \langle 2|(1+4)|2]\ \langle 4|(2+3)|4]}{\langle 2\,3\rangle\ \langle 1\,4\rangle\ \langle 2|(1+4)|3]\ \langle 1|(2+3)|4]} \right\} \\
&+\left\{ 1 \leftrightarrow 2, 3 \leftrightarrow 4 \right\} + \left\{ 1 \leftrightarrow 3, 2 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,] \right\} \\
&+\left\{ 1 \leftrightarrow 4, 2 \leftrightarrow 3, \langle\,\rangle \leftrightarrow [\,] \right\}
\end{aligned} \tag{4.3.48}
$$

$$
\begin{aligned}
c^{(2)}_{23\times 41}(1^+,2^+,3^-,4^-) = &\left\{ 4 \frac{\langle 2\,4\rangle\ \langle 3|(1+4)|2]}{\langle 2\,3\rangle\ \langle 1\,4\rangle\ \langle 2|(1+4)|3]} \right. \\
&\left[\frac{1}{\langle 1|(2+3)|4]} \left(\frac{\langle 2\,4\rangle\ (s_{13}-s_{24})}{\langle 2|(1+4)|3]} - 2\ \langle 3\,4\rangle \right) \right. \\
&\left.\left. - \frac{\langle 4|(2+3)|1]}{\Delta_3(1,4,2,3)} \left(\frac{\langle 1\,2\rangle\ \langle 3|(1+4)|2]}{\langle 1|(2+3)|4]} - \langle 3\,4\rangle \right) \right] \right\} \\
&+\{1 \leftrightarrow 2, 3 \leftrightarrow 4\} + \{1 \leftrightarrow 3, 2 \leftrightarrow 4, \langle\,\rangle \leftrightarrow [\,]\} \\
&+\{1 \leftrightarrow 4, 2 \leftrightarrow 3, \langle\,\rangle \leftrightarrow [\,]\}
\end{aligned} \tag{4.3.49}
$$

Where Δ_3 is given by eq. (3.5.54).

$c^{(0)}_{1\times 234}, c^{(2)}_{1\times 234}$

$$
\begin{aligned}
c^{(0)}_{1\times 234}(1^+,2^+,3^-,4^-) = &-2(s_{12}+s_{13}+s_{14})\, s_{234} \frac{\langle 1|(3+4)|2]\ [2\,4]^2}{\langle 1|(2+3)|4]^3\ [2\,3]\ [3\,4]} \\
&+ \frac{2\ [1\,2]^3}{[1\,4]\ [2\,3]\ [3\,4]} - \frac{(s_{12}+s_{13}+s_{14})}{\langle 1\,2\rangle\ \langle 1|(2+3)|4]\ [3\,4]} \\
&\times \left(\frac{\langle 1\,3\rangle\ \langle 3\,4\rangle^2\ [3\,4]}{\langle 1\,4\rangle\ \langle 2\,3\rangle} - \frac{\langle 1\,2\rangle\ [1\,2]^2\ [2\,4]}{[1\,4]\ [2\,3]} \right)
\end{aligned} \tag{4.3.50}
$$

$$
\begin{aligned}
c^{(2)}_{1\times 234}(1^+,2^+,3^-,4^-) = &\frac{-4}{s_{23}\ \langle 1|4|3]} \left\{ \frac{\langle 4|1|2]^2\, (\langle 3|4|1] - \langle 3|2|1])}{s_{14}(s_{12}+s_{13}+s_{14})} \right. \\
&+ \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1|(2+3)|4]} \left(\langle 1|(3+4)|2] + \frac{(s_{12}+s_{13}+s_{14})}{s_{12}} \langle 1|3|2] \right) \\
&- 2 \frac{\langle 1|(3+4)|2]\, \langle 4|(2+3)|1]\, \langle 3|1|2]}{(s_{12}+s_{13}+s_{14})\, \langle 1|(2+3)|4]} \\
&\left. + \frac{\langle 1|(3+4)|2]^2\, \langle 4|(2+3)|1]\, \langle 3|1|4]}{(s_{12}+s_{13}+s_{14})\, \langle 1|(2+3)|4]^2} \right\}
\end{aligned} \tag{4.3.51}
$$

$1_{\bar q}^+, 2_q^-, 3_g^+, 4_g^+$		$1_{\bar q}^+, 2_q^-, 3_g^-, 4_g^+$		$1_{\bar q}^+, 2_q^-, 3_g^+, 4_g^-$
Coefficient	Related coeff.	Coefficient	Related coeff.	Coefficient
$d_{3\times21\times4}$		$d_{3\times21\times4}$		$c_{4\times123}$
$d_{4\times3\times21}$	$d_{3\times4\times12}$	$d_{4\times3\times21}$	$d_{3\times4\times12}$	b_{123} (3.6.30)
$c_{3\times21}$ (3.6.3)	$c_{4\times12}$	$c_{3\times21}$ (3.6.18)	$c_{4\times12}$	
$c_{12\times34}$		$c_{3\times4}$ (3.6.17)		
$c_{4\times123}$		$c_{12\times34}$		
$c_{3\times412}$		$c_{4\times123}$	$c_{3\times412}$	
b_{12} (3.6.10)		b_{34} (3.6.23)		
b_{123} (3.6.11)		b_{12} (3.6.24)		
b_{412} (3.6.12)		b_{123} (3.6.25)	b_{412}	
b_{1234} (3.6.13)		b_{1234} (3.6.26)		

Table 4.5: Minimal set of integral coefficients for $H_4^{34}(1_{\bar q}^+, 2_q^-, 3_g^+, 4_g^+)$, $H_4^{34}(1_{\bar q}^+, 2_q^-, 3_g^-, 4_g^+)$ and $H_4^{34}(1_{\bar q}^+, 2_q^-, 3_g^+, 4_g^-)$ together with the related coefficients that can be obtained from the base set. The brackets give the equation numbers for the coefficients that are identical to the scalar case, and hence have already been given in chapter 3. In addition, the rational terms are identical to the scalar cases (eqs. (3.6.14, 3.6.27, 3.6.31)).

4.4 Amplitude for $0 \to \bar{q}qggh$ with a Fermion Mediator

4.4.1 Coefficients for $H_4^{34}(\bar{q}^+, q^-, g^+, g^+; h)$

The coefficients that must be computed for this amplitude are shown in Table 4.5.

4.4.1.1 Boxes

$d_{3\times21\times4}$

$$\begin{aligned} d_{3\times21\times4}^{(0)}(1_{\bar q}^+, 2_q^-, 3_g^+, 4_g^+) = & -2\,\frac{\langle 2\,4\rangle\ \langle 2\,3\rangle\ \mathrm{tr}_-\{\not p_3\,\not p_{12}\,\not p_4\,\not p_{12}\}}{\langle 1\,2\rangle\ \langle 3\,4\rangle^3} \\ & -\frac{1}{2}\,\frac{[1\,3]\ [1\,4]\ s_{1234}}{[1\,2]\ \langle 3\,4\rangle} - \frac{1}{2}\,\frac{\langle 2|(1+3)|4]\ \langle 2|(1+4)|3]}{\langle 1\,2\rangle\ \langle 3\,4\rangle} \\ & + 2\,m^2\left[\frac{[1\,3]\ [1\,4]}{[1\,2]\ \langle 3\,4\rangle} + 3\,\frac{\langle 2\,3\rangle\ \langle 2\,4\rangle\ [3\,4]}{\langle 1\,2\rangle\ \langle 3\,4\rangle^2}\right] \end{aligned} \tag{4.4.1}$$

where

$$\mathrm{tr}_-\{\not{p}_3\,\not{p}_{12}\,\not{p}_4\,\not{p}_{12}\} = \langle 3|(1+2)|4]\;\langle 4|(1+2)|3] = (s_{13}+s_{23})\,(s_{14}+s_{24}) - s_{12}\,s_{34} \quad (4.4.2)$$

$\boldsymbol{d_{4\times 3\times 21}}$

$$\begin{aligned} d_{4\times3\times21}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) = {} & \frac{1}{2}\,\frac{[3\,4]\;\langle 2|(1+3)|4]^2}{\langle 1\,2\rangle\;\langle 3|(1+2)|4]} + \frac{1}{2}\,\frac{[1\,3]^2\;[3\,4]\;s_{1234}}{[1\,2]\;\langle 4|(1+2)|3]} \\ & + 2\,m^2 \left[\frac{\langle 2\,3\rangle\;[3\,4]\;\langle 2|(1+3)|4]}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\langle 3|(1+2)|4]} \right. \\ & \left. - \frac{[1\,3]\;[3\,4]\;\langle 4|(2+3)|1]}{[1\,2]\;\langle 3\,4\rangle\;\langle 4|(1+2)|3]}\right] \end{aligned} \quad (4.4.3)$$

4.4.1.2 Triangles

$\boldsymbol{c^{(0)}_{12\times 34}, c^{(2)}_{12\times 34}}$

$$\begin{aligned} c^{(0)}_{12\times34}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) = {} & \frac{[1\,4]^2\,(s_{123}-s_{124}-s_{34})}{[1\,2]\;\langle 3\,4\rangle\;\langle 3|(1+2)|4]} - \frac{\langle 2\,3\rangle^2\;[3\,4]^2}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\langle 3|(1+2)|4]} \\ & + \frac{[1\,3]^2\,(s_{124}-s_{123}-s_{34})}{[1\,2]\;\langle 3\,4\rangle\;\langle 4|(1+2)|3]} - \frac{\langle 2\,4\rangle^2\;[3\,4]^2}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\langle 4|(1+2)|3]} \\ & - \frac{4\;[1\,3]\;[1\,4]}{[1\,2]\;\langle 3\,4\rangle} \end{aligned} \quad (4.4.4)$$

$$\begin{aligned} c^{(2)}_{12\times34}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) = {} & \frac{4}{\langle 3\,4\rangle^2\;\langle 3|(1+2)|4]}\left[\frac{[1\,4]^2\;\langle 3\,4\rangle}{[1\,2]} - \frac{\langle 2\,3\rangle^2\;[3\,4]}{\langle 1\,2\rangle}\right] \\ & + \frac{4}{\langle 3\,4\rangle^2\;\langle 4|(1+2)|3]}\left[\frac{[1\,3]^2\;\langle 3\,4\rangle}{[1\,2]} - \frac{\langle 2\,4\rangle^2\;[3\,4]}{\langle 1\,2\rangle}\right] \end{aligned} \quad (4.4.5)$$

$\boldsymbol{c^{(0)}_{4\times 123}, c^{(2)}_{4\times 123}}$

$$\begin{aligned} c^{(0)}_{4\times123}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+) = {} & \frac{2\;\langle 2|(1+3)|4]^3}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 3|(1+2)|4]\,s_{123}} \\ & - (s_{14}+s_{24}+s_{34})\left[\frac{\langle 2|(1+4)|3]\;\langle 2|(1+3)|4]}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\mathrm{tr}_-\{\not{p}_3\,\not{p}_{12}\,\not{p}_4\,\not{p}_{12}\}} \right. \\ & \left. + \frac{[1\,3]\;[1\,4]\;s_{1234}}{[1\,2]\;\langle 3\,4\rangle\;\mathrm{tr}_-\{\not{p}_3\,\not{p}_{12}\,\not{p}_4\,\not{p}_{12}\}} + \frac{2\;\langle 2\,3\rangle\;\langle 2\,4\rangle}{\langle 1\,2\rangle\;\langle 3\,4\rangle^3}\right] \end{aligned} \quad (4.4.6)$$

$$c^{(2)}_{4\times 123}(1^+_{\bar q}, 2^-_q, 3^+_g, 4^+_g) = \frac{4}{\langle 3|(1+2)|4]}\Bigg\{ \frac{(s_{14}+s_{24}+s_{34})}{\langle 4|(1+2)|3]} \times \left[\frac{\langle 2\,3\rangle\ \langle 2\,4\rangle\ [3\,4]}{\langle 1\,2\rangle\ \langle 3\,4\rangle^2} + \frac{[1\,3]\ [1\,4]}{[1\,2]\ \langle 3\,4\rangle}\right] - 2\,\frac{\langle 2|(1+3)|4]^3}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ (s_{14}+s_{24}+s_{34})\, s_{123}}\Bigg\} \tag{4.4.7}$$

$\boldsymbol{c^{(0)}_{3\times 412}, c^{(2)}_{3\times 412}}$

$$\begin{aligned} c^{(0)}_{3\times 412}(1^+_{\bar q}, 2^-_q, 3^+_g, 4^+_g) = &+ 2\,\frac{\langle 2|(1+4)|3]^2\ \langle 1|(2+4)|3]}{\langle 2\,1\rangle\ \langle 1\,4\rangle\ \langle 4|(1+2)|3]\ s_{124}} \\ &+ (s_{13}+s_{23}+s_{34})\Bigg[- 2\,\frac{\langle 2\,4\rangle\ \langle 2\,3\rangle}{\langle 2\,1\rangle\ \langle 4\,3\rangle^3} \\ &+ \frac{[1\,4]\ [1\,3]\ [4\,3]\ \langle 4\,3\rangle}{[2\,1]\ \langle 4\,3\rangle\ \mathrm{tr}_-\{\not p_3 \not p_{12} \not p_4 \not p_{12}\}} \\ &+ \frac{\langle 2\,4\rangle\ \langle 2\,3\rangle\ [4\,3]^2}{\langle 2\,1\rangle\ \langle 4\,3\rangle\ \mathrm{tr}_-\{\not p_3 \not p_{12} \not p_4 \not p_{12}\}} \\ &- \frac{[1\,4]^2}{[2\,1]\ \langle 4\,3\rangle\ \langle 3|(1+2)|4]} - \frac{[1\,3]^2}{[2\,1]\ \langle 4\,3\rangle\ \langle 4|(1+2)|3]}\Bigg] \end{aligned} \tag{4.4.8}$$

$$\begin{aligned} c^{(2)}_{3\times 412}(1^+_{\bar q}, 2^-_q, 3^+_g, 4^+_g) \;=\;& 4\,\frac{(s_{13}+s_{23}+s_{34})}{\mathrm{tr}_-\{\not p_3 \not p_{12} \not p_4 \not p_{12}\}}\left[\frac{\langle 2\,4\rangle\ \langle 2\,3\rangle\ [4\,3]}{\langle 2\,1\rangle\ \langle 4\,3\rangle^2} + \frac{[1\,4]\ [1\,3]}{[2\,1]\ \langle 4\,3\rangle}\right] \\ -\;& 8\,\frac{\langle 2|(1+4)|3]^2\ \langle 1|(2+4)|3]}{\langle 2\,1\rangle\ \langle 1\,4\rangle\ \langle 4|(1+2)|3]\ (s_{13}+s_{23}+s_{34})\, s_{124}} \end{aligned} \tag{4.4.9}$$

4.4.2 Coefficients for $H_4^{34}(\bar q^+, q^-, g^-, g^+; h)$

The coefficients that must be computed for this amplitude are shown in Table 4.5.

4.4.2.1 Boxes

$\boldsymbol{d_{3\times 21\times 4}}$

$$\begin{aligned} d^{(0)}_{3\times 21\times 4}(1^+_{\bar q}, 2^-_q, 3^-_g, 4^+_g) = &\frac{1}{2}\ \langle 3|(1+2)|4]\left[\frac{[1\,4]^2}{[2\,1]\ [4\,3]} - \frac{\langle 2\,3\rangle^2}{\langle 1\,2\rangle\ \langle 3\,4\rangle}\right] \\ &+ 2\,m^2\,\frac{\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]}\left[\frac{\langle 2\,4\rangle\ \langle 2\,3\rangle}{\langle 1\,2\rangle\ \langle 3\,4\rangle} - \frac{[1\,3]\ [1\,4]}{[1\,2]\ [3\,4]}\right] \end{aligned} \tag{4.4.10}$$

$\boldsymbol{d_{4\times 3\times 21}}$

$$d^{(0)}_{4\times3\times21}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = \frac{1}{2}\,\frac{s_{123}}{\langle 4|(1+2)|3]}\left[\frac{\langle 2\,3\rangle^2\;[3\,4]}{\langle 1\,2\rangle} + \frac{[1\,4]^2\;\langle 3\,4\rangle}{[1\,2]} - 4\,\frac{[1\,3]\;\langle 4|(2+3)|1]\;s_{34}\,s_{123}}{[1\,2]\;\langle 4|(1+2)|3]^2}\right] - 2\,\frac{m^2}{\langle 4|(1+2)|3]}\left[\frac{3\;[1\,3]\;\langle 3|(1+2)|4]\;\langle 4|(2+3)|1]}{[1\,2]\;\langle 4|(1+2)|3]} + \frac{\langle 2\,3\rangle\;\langle 2|(1+3)|4]}{\langle 1\,2\rangle}\right] \tag{4.4.11}$$

4.4.2.2 Triangles

$\boldsymbol{c^{(0)}_{12\times 34}, c^{(2)}_{12\times 34}}$

$$\begin{aligned}
&c^{(0)}_{12\times34}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = 8\,(s_{124}-s_{123})\,(s_{12}+s_{34}+2\,s_{13}+2\,s_{23})\\
&\times\frac{\langle 2\,4\rangle\;[1\,3]\;\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]^2\,\Delta_3(1,2,3,4)}\\
&+\Big((9\,s_{13}-7\,s_{23}-s_{14}-s_{24}+4\,s_{34})\;\langle 2\,4\rangle\;[1\,4]\\
&-(9\,s_{14}-7\,s_{24}-s_{13}-s_{23}+4\,s_{34})\;\langle 2\,3\rangle\;[1\,3]\Big)\times\frac{1}{\langle 4|(1+2)|3]^2}\\
&+12\,\frac{s_{1234}\,((s_{13}+s_{23})^2-(s_{14}+s_{24})^2)\;\langle 2|(3+4)|1]\;\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)^2}\\
&+4\Big(\Big\{3\,(s_{12}+s_{34})+4\,(s_{13}+s_{23}+s_{14})\Big\}\;[1\,3]\;\langle 2\,3\rangle\\
&-\Big\{3\,(s_{12}+s_{34})+4\,(s_{13}+s_{24}+s_{14})\Big\}\;[1\,4]\;\langle 2\,4\rangle\Big)\times\frac{\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]\;\Delta_3(1,2,3,4)}\\
&-24\,\frac{[1\,3]\;\langle 2\,4\rangle\;\langle 3|(1+2)|4]^2}{\langle 4|(1+2)|3]\;\Delta_3(1,2,3,4)}-8\,\frac{[1\,4]\;\langle 2\,3\rangle\;\langle 3|(1+2)|4]}{\Delta_3(1,2,3,4)}+8\,\frac{[1\,4]\;\langle 2\,3\rangle}{\langle 4|(1+2)|3]}\\
&+\Bigg\{\frac{2\;\langle 2\,4\rangle^2\;[3\,4]\;(s_{14}+s_{24})^2}{\langle 1\,2\rangle\;\langle 4|(1+2)|3]^3}+\frac{[1\,3]\;\langle 2\,4\rangle\;(s_{14}+s_{24})\,(4\,s_{124}-2\,s_{34})}{\langle 4|(1+2)|3]^3}\\
&+\frac{(s_{13}+s_{23})\;\langle 2\,3\rangle\;\langle 2\,4\rangle\;(s_{14}+s_{24}-s_{13}-s_{23})}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\langle 4|(1+2)|3]^2}+\frac{2\;\langle 2\,3\rangle\;\langle 2\,4\rangle\;[3\,4]\;(s_{14}+s_{24})}{\langle 1\,2\rangle\;\langle 4|(1+2)|3]^2}\Bigg\}\\
&-\Bigg\{1\leftrightarrow 2, 3\leftrightarrow 4, \langle\,\rangle\leftrightarrow[\,]\Bigg\}
\end{aligned}\tag{4.4.12}$$

$$c^{(2)}_{12\times34}(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g) = \Bigg\{\frac{4\;\langle 2\,3\rangle^2}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\langle 4|(1+2)|3]} - \frac{4\;\langle 2\,4\rangle^2\;\langle 3|(1+2)|4]}{\langle 1\,2\rangle\;\langle 3\,4\rangle\;\langle 4|(1+2)|3]^2}$$

$$
\begin{aligned}
&- 8\,\frac{(s_{13}+s_{23}+s_{14}+s_{24})\,\langle 2\,3\rangle\,\langle 2\,4\rangle\,\langle 3|(1+2)|4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)}\\
&- 16\,\frac{[1\,3]\,\langle 2\,3\rangle\,\langle 3|(1+2)|4]}{\langle 4|(1+2)|3]\,\Delta_3(1,2,3,4)}\Bigg\}\\
&-\Big\{1\leftrightarrow 2, 3\leftrightarrow 4, \langle\,\rangle\leftrightarrow [\,]\Big\} \qquad (4.4.13)
\end{aligned}
$$

$\boldsymbol{c^{(0)}_{4\times 123}, c^{(2)}_{4\times 123}}$

$$
\begin{aligned}
c^{(0)}_{4\times123}(1^+_{\bar q},2^-_q,3^-_g,4^+_g) = &-\frac{2\,(s_{14}+s_{24}+s_{34})\,[1\,3]\,\langle 4|(2+3)|1]\,s_{123}}{[1\,2]\,\langle 4|(1+2)|3]^3}\\
&-\frac{\langle 2\,3\rangle^2\,(s_{14}+s_{24}+s_{34})}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4|(1+2)|3]}-\frac{[1\,4]^2\,(s_{14}+s_{24}+s_{34})}{[1\,2]\,[3\,4]\,\langle 4|(1+2)|3]}\\
&+\frac{2\,[1\,4]^2\,[2\,4]}{[1\,2]\,[2\,3]\,[3\,4]} \qquad (4.4.14)
\end{aligned}
$$

$$
\begin{aligned}
c^{(2)}_{4\times123}(1^+_{\bar q},2^-_q,3^-_g,4^+_g) \;=\;& \frac{4\,(s_{14}+s_{24}+s_{34})}{\langle 4|(1+2)|3]^2}\left[\frac{\langle 2\,3\rangle\,\langle 2\,4\rangle}{\langle 1\,2\rangle\,\langle 3\,4\rangle}+\frac{[1\,3]\,[1\,4]}{[1\,2]\,[3\,4]}\right]\\
&-\;8\,\frac{[1\,4]^2\,[2\,4]}{[1\,2]\,[2\,3]\,[3\,4]\,(s_{14}+s_{24}+s_{34})} \qquad (4.4.15)
\end{aligned}
$$

4.4.3 Coefficients for $H_4^{34}(\bar q^+, q^-, g^+, g^-; h)$

We need only calculate one additional coefficient for this amplitude. Most can be obtained from those for $H_4^{34}(\bar q^+, q^-, g^-, g^+)$ by performing the following operation:

$$
1\leftrightarrow 2\,, \quad \langle\,\rangle\leftrightarrow [\,]\,, \qquad (4.4.16)
$$

exactly as for the scalar theory. Once we take into account the coefficients that are identical to the scalar case there is only one triangle coefficient that remains.

4.4.3.1 Triangle

$\boldsymbol{c^{(0)}_{4\times 123}, c^{(2)}_{4\times 123}}$

$$
\begin{aligned}
c^{(0)}_{4\times123}(1^+_{\bar q},2^-_q,3^+_g,4^-_g) = &-2\,\frac{\langle 3\,4\rangle\,\langle 2|(1+3)|4]^2\,s_{123}}{\langle 1\,2\rangle\,\langle 3|(1+2)|4]^3}+2\,\frac{\langle 2\,4\rangle\,\langle 2|(1+3)|4]\,s_{123}}{\langle 1\,2\rangle\,\langle 3|(1+2)|4]^2}\\
&-\frac{\langle 2\,4\rangle^2\,\langle 2|(1+3)|4]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3|(1+2)|4]}+\frac{[1\,3]^2\,(s_{14}+s_{24}+s_{34})}{[1\,2]\,[3\,4]\,\langle 3|(1+2)|4]}
\end{aligned}
$$

$$- \frac{\langle 2\,4\rangle^3}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 3\,4\rangle} \tag{4.4.17}$$

$$\begin{aligned} c^{(2)}_{4\times 123}(1^+_{\bar{q}}, 2^-_q, 3^+_g, 4^-_g) =& -4\frac{(s_{14}+s_{24}+s_{34})}{\langle 3|(1+2)|4]^2}\left[\frac{\langle 2\,3\rangle\;\langle 2\,4\rangle}{\langle 1\,2\rangle\;\langle 3\,4\rangle} + \frac{[1\,3]\;[1\,4]}{[1\,2]\;[3\,4]}\right] \\ &+ 8\,\frac{\langle 2\,4\rangle^3}{\langle 1\,2\rangle\;\langle 2\,3\rangle\;\langle 3\,4\rangle\;(s_{14}+s_{24}+s_{34})} \end{aligned} \tag{4.4.18}$$

4.5 Amplitude for $0 \to \bar{q}q\bar{q}h$

The results given in ref. [94], which we modified in section 3.7 for a scalar mediator, are in this case exactly the amplitude we require.

Starting with the tensor current for $0 \to ggh$ with two off-shell gluons (with momenta k_1 and k_2),

$$\mathcal{T}^{\mu_1\mu_2}(k_1,k_2) = -i\delta^{c_1c_2}\,\frac{g_s^2}{8\pi^2}\left(\frac{m^2}{v}\right)\left[F_T(k_1,k_2)\,T_T^{\mu_1\mu_2} + F_L(k_1,k_2)\,T_L^{\mu_1\mu_2}\right]. \tag{4.5.1}$$

The two tensor structures have been given in eq. (3.7.3), while the form factors in this case are[1],

$$\begin{aligned} F_T(k_1,k_2) =& -\frac{1}{\Delta(k_1,k_2)}\Big\{k_{12}^2\,(B_0(k_1;m)+B_0(k_2;m)-2B_0(k_{12};m) \\ &- 2k_1\cdot k_2\,C_0(k_1,k_2;m)) \\ &+ (k_1^2-k_2^2)\,(B_0(k_1;m)-B_0(k_2;m))\Big\} -\; k_1\cdot k_2\,F_L(k_1,k_2) \end{aligned} \tag{4.5.2}$$

$$\begin{aligned} F_L(k_1,k_2) =& -\frac{1}{\Delta(k_1,k_2)}\Big\{\Big[2-\frac{3k_1^2\,k_2\cdot k_{12}}{\Delta(k_1,k_2)}\Big]\,(B_0(k_1;m)-B_0(k_{12};m)) \\ &+ \Big[2-\frac{3k_2^2\,k_1\cdot k_{12}}{\Delta(k_1,k_2)}\Big]\,(B_0(k_2;m)-B_0(k_{12};m)) \\ &- \Big[4m^2+k_1^2+k_2^2+k_{12}^2-3\frac{k_1^2\,k_2^2\,k_{12}^2}{\Delta(k_1,k_2)}\Big]\,C_0(k_1,k_2;m)-2\Big\} \end{aligned} \tag{4.5.3}$$

where $k_{12} = k_1 + k_2$ and $\Delta(k_1,k_2) = k_1^2\,k_2^2 - (k_1\cdot k_2)^2$. As expected, the bubble and rational coefficients are identical to the case with a scalar mediator, eq. (3.7.5). As before, we then contract eq. (4.5.1) with currents for the quark-antiquark lines to

[1] Note that to produce our standard overall normalisation for the helicity amplitude we have changed the normalisation of the form factors by a factor of 2 with respect to ref. [94].

arrive at,

$$
\begin{aligned}
&H_4^{4q}(1_{\bar{q}}^+, 2_q^-, 3_{\bar{q}'}^+, 4_{q'}^-; h) = \\
&\qquad \left[\frac{\langle 2|(3+4)|1]\ \langle 4|(1+2)|3] + \langle 2\,4\rangle\ [1\,3]\ (2\,p_{12}.p_{34})}{s_{12}\, s_{34}}\right] F_T(p_{12}, p_{34}) \\
&\qquad + 2\ \langle 2\,4\rangle\ [1\,3]\ F_L(p_{12}, p_{34})\,.
\end{aligned}
\tag{4.5.4}
$$

From which all helicity combinations can be calculated.

4.6 Large mass limit

Results for these processes in the effective field theory, eq. (2.3.40), are already known, and can be obtained from our results by taking the limit $m_t \to \infty$.

Due to the form of the scalar integrals in the large mass limit, only the triangle integrals will contribute [106]. The large mass expansion for the triangle integral is given in eq. 3.3.7.

For the $0 \to hgggg$ process [107–109],

$$
m^2 H_4^{1234}(1^+, 2^+, 3^+, 4^+; h) \to \frac{2}{3} \frac{s_{1234}^2}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3\,4\rangle\ \langle 4\,1\rangle} \tag{4.6.1}
$$

$$
\begin{aligned}
m^2 H_4^{1234}(1^+, 2^+, 3^+, 4^-; h) \to \frac{2}{3}\Big[&- \frac{s_{1234}^2\ [1\,3]^4}{s_{134}\ [4\,3]\ [4\,1]\ \langle 2|(1+4)|3]\ \langle 2|(3+4)|1]} \\
&+ \frac{\langle 4|(2+3)|1]^3}{s_{234}\ \langle 2|(3+4)|1]\ \langle 3\,4\rangle\ \langle 2\,3\rangle} \\
&+ \frac{\langle 4|(2+1)|3]^3}{s_{124}\ \langle 2|(1+4)|3]\ \langle 1\,4\rangle\ \langle 2\,1\rangle}\Big]
\end{aligned}
\tag{4.6.2}
$$

$$
m^2 H_4^{1234}(1^+, 2^+, 3^-, 4^-; h) \to \frac{2}{3}\Big[\frac{[1\,2]^4}{[1\,2]\ [2\,3]\ [3\,4]\ [4\,1]} + \frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3\,4\rangle\ \langle 4\,1\rangle}\Big]\,. \tag{4.6.3}
$$

The amplitude for $H_4^{1234}(1^+, 2^-, 3^+, 4^-; h)$ can be calculated through the dual Ward identity, or photon decoupling identity, which arises due to eq (4.1.1) being valid for the gauge group $U(N_C) = SU(N_c) \times U(1)$ in this limit [110],

$$
H_4^{1234}(1^+, 2^-, 3^+, 4^-; h) + H_4^{1234}(3^+, 1^+, 2^-, 4^-; h) + H_4^{1234}(1^+, 3^+, 2^-, 4^-; h) = 0\,. \tag{4.6.4}
$$

The dual Ward identity implies that the sub-leading colour term in eq. (4.1.2) is no longer present. Similarly for $0 \to hggq\bar{q}$ in the large mass limit [108],

$$m^2\, H^{34}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^+; h) \to \frac{2}{3}\left[\frac{\langle 2|(1+4)|3]^2\ [1\,4]}{s_{124}\ \langle 2\,4\rangle}\left[\frac{1}{s_{12}} + \frac{1}{s_{14}}\right] - \frac{\langle 2|(1+3)|4]^2\,[1\,3]}{s_{123}\, s_{12}\,\langle 2\,3\rangle} + \frac{\langle 2|(3+4)|1]^2}{[1\,2]\ \langle 2\,3\rangle\ \langle 2\,4\rangle\ \langle 3\,4\rangle}\right] \tag{4.6.5}$$

$$m^2\, H^{34}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-; h) \to \frac{2}{3}\left[\frac{\langle 2\,4\rangle^3}{\langle 1\,2\rangle\ \langle 2\,3\rangle\ \langle 3\,4\rangle} - \frac{[1\,3]^3}{[1\,2]\ [1\,4]\ [3\,4]}\right] \tag{4.6.6}$$

$$m^2\, H^{34}(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+; h) \to \frac{2}{3}\left[\frac{\langle 2\,3\rangle^2\ \langle 1\,3\rangle}{\langle 1\,2\rangle\ \langle 1\,4\rangle\ \langle 3\,4\rangle} - \frac{[1\,4]^2\ [2\,4]}{[1\,2]\ [2\,3]\ [3\,4]}\right] \tag{4.6.7}$$

and for $0 \to h\bar{q}q\bar{q}'q'$,

$$m^2\, H^{4q}(1_{\bar{q}}^+, 2_q^-, 3_{\bar{q}'}^+, 4_{q'}^-; h) \to -\frac{2}{3}\left[\frac{[1\,3]^2}{[1\,2]\ [3\,4]} + \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\ \langle 3\,4\rangle}\right]. \tag{4.6.8}$$

The scalar mediated amplitudes in the large mass limit can be obtained by multiplying the above equations by a factor of $-1/2$, c.f. eq. (3.3.10).

4.7 Conclusion

In this chapter we have presented compact analytic results for all helicity amplitudes in the Higgs plus four parton processes where the interaction is mediated by a massive fermion, retaining all dependence on the fermion mass. To calculate the amplitudes we have decomposed them to sums of scalar integrals and used unitarity techniques to obtain the integral coefficients. The correspondence between the results here and those with a massive scalar mediator have reduced the number of integral coefficients that must be calculated in the fermion theory. In addition we used momentum twistors and reconstruction from high-precision numerical evaluations to simplify our analytic results. These techniques enabled us to produce results for every coefficient that are compact enough to print. All results were verified against an in-house implementation of the D-dimensional unitarity method [111] as well as against a previous unitarity-based calculation [73]. Complete agreement was found at the amplitude level. In addition our results for the squared matrix elements are in full agreement with those produced by OpenLoops 2 [88]. The previous implementation of the Higgs plus 2-jet process in MCFM [30–32] calculated the full matrix element squared at a similar speed to OpenLoops 2. After implementing these new, compact results the calculation time improves by at least an order of magnitude. These results are a key component of the NLO calculation of the Higgs plus 1-jet process and the NNLO inclusive cross section in the full theory, as well as being useful in themselves for studying the structure of gauge-theory amplitudes.

Chapter 5

Searching for Stop Squarks

5.1 Introduction

Experiments searching for new particles can be divided into two types: direct and indirect. Direct searches look for explicit evidence of a particle by detecting either the particle itself or measuring its decay products. Direct detection experiments are therefore model dependent, since we need to know exactly what we are looking for. Many models of new physics, however, have many possible decay chains and branching ratios for each particle, making direct searches more complicated since the analysis must be performed over multiple different parameter spaces. Indirect searches, on the other hand, look for measurements that don't fit our predictions, for example where a cross section may be increased due to an unknown particle acting as a propagator. These experiments are therefore model independent: we do not need to worry about the exact properties of the particle, only which particles it couples to. In general, direct searches give us increased precision, but only when we know the exact model we wish to study.

Gluon fusion production of the Higgs boson at hadron colliders gives us the opportunity to do model independent, indirect searches for the particles running in the loop. The only requirements are that the particle must be charged under $SU(3)$ (to couple to the gluons) and massive (to couple to the Higgs).

The stop squark, appearing as the super partner of the top quark in the Minimal Supersymmetric Standard Model (MSSM), is one such particle that can be probed in this way. Current indirect limits on the top squark are around $m_{\tilde{t}} \gtrsim 300$ GeV, much weaker than direct limits from specific decay chains, which are currently around the 1 TeV scale [112,113]. Searches for the stop squark have been comprehensive, however there is one region that is particularly difficult to study. Known as the "splinter" region, it is where $m_{\tilde{t}_1} \approx m_t$; the presence of stop squarks pollutes measurements of the top quark mass [114], making any prediction that depends on this parameter much more uncertain. Existing analysis of the effect of these loops has studied the inclusive and 1-jet processes [115–117], including the improved discrimination between the Standard Model and the MSSM that can be obtained from the 1-jet process in certain regions of parameter space [117].

It is therefore interesting to explore the Higgs plus 2-jet processes, to investigate whether any benefits can be gained over the inclusive and 1-jet processes. This analysis was published in ref. [2], and the cross sections are calculated using the results from chapters 3 and 4, adapted through crossing symmetry to obtain results for two partons in the initial state and a final state of two partons plus the Higgs boson.

All results presented in this chapter use the MMHT NLO PDF set [118], use the on-shell top quark mass of $m_t = 173.3$ GeV and consider the LHC operating at $\sqrt{s} = 14$ TeV. The normalisation and factorisation scales are set as

$$\mu_f = \mu_r = \frac{H_T'}{2} = \frac{1}{2}\left(\sqrt{m_h^2 + p_{T,h}^2} + \sum_i |p_{T,i}|\right) , \tag{5.1.1}$$

where the sum is over the jets present in the final state.

To begin with we will recap the stop squark sector of the MSSM in section 5.2 before reproducing the results for the inclusive and 1-jet processes in section 5.3. In sections 5.4 and 5.5 we present our results for the Higgs plus 2-jet process and discuss the feasibility of searching for stop quarks in Higgs boson production. We conclude in section 5.6.

5.2 Stop Squarks in the MSSM

The MSSM contains two scalars in the top squark sector, $\tilde{t}_1$ and $\tilde{t}_2$. We will assume that the Higgs boson already observed is the lightest within the MSSM and that it couples to the top squarks through a Lagrangian of the form in eq. (3.2.1). It is useful to parameterise this sector, following ref. [117], by

$$(m_{\tilde{t}_1}\ , \Delta m\ , \theta), \quad \Delta m = \sqrt{m^2_{\tilde{t}_2} - m^2_{\tilde{t}_1}}\,, \tag{5.2.1}$$

where $m_{\tilde{t}_1}$ is the mass of the lightest stop quark and θ is the mixing angle between the two states, taking values between $[-\pi/2, \pi/2]$.

For simplicity we work in a model where the mass of the MSSM pseudoscalar (A) is much greater than the weak scale, allowing us to work in the decoupling limit. This leads to particularly simple form of the couplings between the (lightest) Higgs and squarks:

$$\begin{aligned}
\lambda_{h\,\tilde{t}_1\tilde{t}_1} &= \frac{m_t^2}{v}\left(\alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta + 2 - \frac{(\Delta m)^2}{2m_t^2}\sin^2 2\theta\right)\ , \\
\lambda_{h\,\tilde{t}_2\tilde{t}_2} &= \frac{m_t^2}{v}\left(\alpha_1 \sin^2\theta + \alpha_2 \cos^2\theta + 2 + \frac{(\Delta m)^2}{2m_t^2}\sin^2 2\theta\right)\ ,
\end{aligned} \tag{5.2.2}$$

where

$$\alpha_1 = \frac{m_Z^2}{m_t^2}\cos 2\beta\left(1 - \frac{4}{3}\sin^2\theta_W\right), \quad \alpha_2 = \frac{4}{3}\frac{m_Z^2}{m_t^2}\cos 2\beta\sin^2\theta_W\ , \tag{5.2.3}$$

where β is defined in eq. (1.3.1). As an interesting aside, if $\tan\beta$ were very large, the contribution from sbottom squarks would be significant. Here we will consider the effect of stop squarks only.

For the Higgs plus 1-jet case we define jets by a minimum p_T and the rapidity requirement $|y(\text{jet})| < 2.4$. When studying Higgs plus 2-jets we must implement a full jet clustering algorithm; which we choose to be the anti-k_T algorithm [119] with a jet resolution parameter of $R = 0.5$.

5.3 Inclusive and 1-jet results

To begin with we will reproduce, and analyse, the results present in the current literature [115–117].

5.3.1 Inclusive cross section

Before doing a numerical study, it is useful to analytically derive the features we hope to find. Using the notations introduced in eqs. (3.3.4, 3.3.5) to abbreviate Standard Model and supersymmetric amplitudes,

$$\begin{aligned}\mathcal{M}^{SM} = \mathcal{H}_2^{gg} &= C\,F_{1/2}(4m_t^2/m_h^2)\,, \\ \mathcal{M}^{SUSY} &= \mathcal{A}_2^{gg}(\tilde{t}_1) + \mathcal{A}_2^{gg}(\tilde{t}_2) \\ &= C\left[\left(\frac{v}{2m_{\tilde{t}_1}^2}\right)\lambda_{h\tilde{t}_1\tilde{t}_1}F_0(4m_{\tilde{t}_1}^2/m_h^2) + \left(\frac{v}{2m_{\tilde{t}_2}^2}\right)\lambda_{h\tilde{t}_2\tilde{t}_2}F_0(4m_{\tilde{t}_2}^2/m_h^2)\right]\,,\end{aligned} \tag{5.3.1}$$

where $\mathcal{M}^{\text{SUSY}}$ contains the stop squark contributions only. In this analysis the value of C is unimportant, but can easily be determined. We are most interested in the areas of parameter space where the SUSY contribution is very close to zero, as these are the hardest to probe. For simplicity in the following derivation we can work under the assumption that the heavy quark EFT in eq. (2.3.40) is always valid, $m_t, m_{\tilde{t}_1}, m_{\tilde{t}_2} \gg m_h$, so that F_0 and $F_{\frac{1}{2}}$ take their asymptotic values. This is broadly true for the regions we will study, though our numerical results are calculated in the full theory. We can therefore also drop the terms proportional to α_1 and α_2, as these are suppressed by m_Z^2/m_t^2. This brings us to,

$$\begin{aligned}\mathcal{M}^{SM} &= -\frac{4C}{3}\,, \\ \mathcal{M}^{SUSY} &= -\frac{C}{3}\left[\left(\frac{v}{2m_{\tilde{t}_1}^2}\right)\lambda_{h\tilde{t}_1\tilde{t}_1} + \left(\frac{v}{2m_{\tilde{t}_2}^2}\right)\lambda_{h\tilde{t}_2\tilde{t}_2}\right]\,,\end{aligned} \tag{5.3.2}$$

and substituting eqs. (5.2.2),

$$\mathcal{M}^{SUSY} = -\frac{C}{6}\left[\left(\frac{m_t^2}{m_{\tilde{t}_1}^2}\right)\left(2 - \frac{(\Delta m)^2}{2m_t^2}\sin^2 2\theta\right) + \left(\frac{m_t^2}{m_{\tilde{t}_2}^2}\right)\left(2 + \frac{(\Delta m)^2}{2m_t^2}\sin^2 2\theta\right)\right]$$

$$= \quad -\frac{C}{3}\left[\frac{m_t^2}{m_{\tilde{t}_1}^2}+\frac{m_t^2}{m_{\tilde{t}_2}^2}-\frac{1}{4}\sin^2 2\theta\frac{(\Delta m)^4}{m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2}\right], \tag{5.3.3}$$

after the simplifications discussed above have been applied (c.f. eq. (2.15) of ref. [117]). This can be rewritten as

$$\mathcal{M}^{SUSY} = \frac{C}{12m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2}\left[\sin^2 2\theta(\Delta m)^4 - 4m_t^2(\Delta m)^2 - 8m_t^2 m_{\tilde{t}_1}^2\right], \tag{5.3.4}$$

so that the numerator depends only on the three parameters in eq. (5.2.1). We are looking for values of Δm where $\mathcal{M}^{SUSY} = 0$,

$$\Delta m = m_t \times \sqrt{2} \times \sqrt{\frac{1+\sqrt{1+2\sin^2 2\theta\, m_{\tilde{t}_1}^2/m_t^2}}{\sin^2 2\theta}}, \tag{5.3.5}$$

for $\theta > 0$. This gives regions where the mixing between the two SUSY states effectively cancel and only the SM contribution to the amplitude remains. When $\theta = 0$ there is no mixing between the squark states.

Due to the nature of the cross section, containing the amplitude squared, there is a second area of parameter space where the SM and MSSM results are equivalent. That is when

$$\begin{aligned}
\left(\mathcal{M}^{SM}+\mathcal{M}^{SUSY}\right)^2 &= |\mathcal{M}^{SM}|^2, \\
\Rightarrow 2\mathcal{M}^{SM}+\mathcal{M}^{SUSY} &= -\frac{C}{3}\left[\frac{m_t^2}{m_{\tilde{t}_1}^2}+\frac{m_t^2}{m_{\tilde{t}_2}^2}-\frac{1}{4}\sin^2 2\theta\frac{(\Delta m)^4}{m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2}+8\right] \\
&= 0,
\end{aligned} \tag{5.3.6}$$

since in this case the interference between the SM and SUSY amplitudes cancels out the contribution from $|\mathcal{M}^{SUSY}|^2$. After manipulating eq. (5.3.6) as above to depend on the three parameters in eq. (5.2.1),

$$\sin^2 2\theta(\Delta m)^4 - 4(m_t^2+8m_{\tilde{t}_1}^2)(\Delta m)^2 - 8m_{\tilde{t}_1}^2(m_t^2+4m_{\tilde{t}_1}^2) = 0. \tag{5.3.7}$$

This equation has no solutions for small Δm, however for large Δm the solution is

$$\Delta m \approx 2\frac{\sqrt{m_t^2+8m_{\tilde{t}_1}^2}}{\sin 2\theta}, \tag{5.3.8}$$

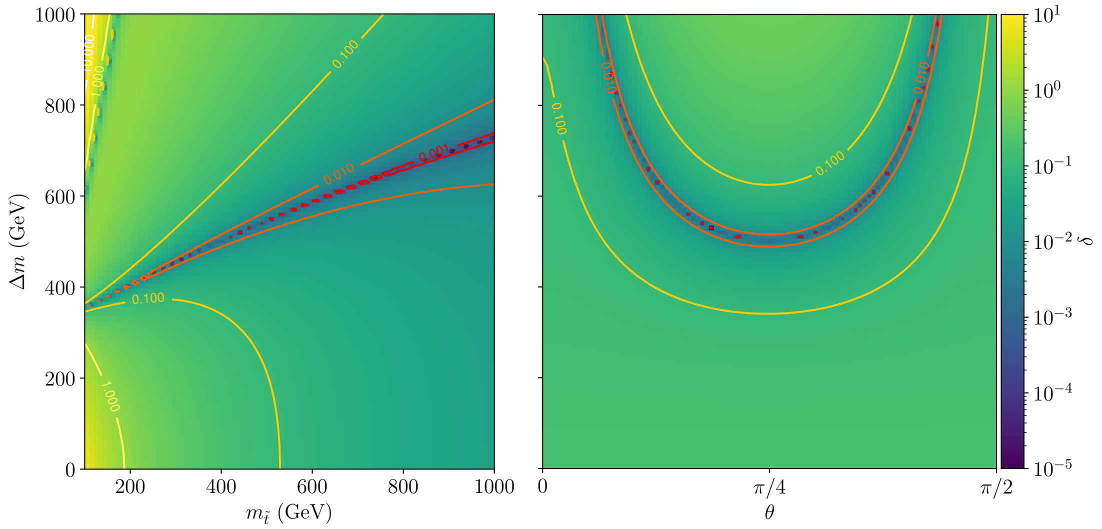


Figure 5.1: The deviation of the inclusive Higgs cross section from the SM case, measured by δ defined in eq. (5.3.9), in the full theory. The left panel shows δ as a function of $m_{\tilde{t}_1}$ and Δm with stop squarks mixing in a maximal fashion ($\theta = \frac{\pi}{4}$). In the right panel δ is shown as a function of θ and Δm with $m_{\tilde{t}_1} = 400$ GeV. In both panels $\tan\beta = 10$.

again for $\theta > 0$.

Following the analysis in ref. [117], we quantise the difference between the SM and MSSM results with,

$$\delta = \left| \frac{\sigma^{SM} - \sigma^{SUSY}}{\sigma^{SM}} \right| . \tag{5.3.9}$$

This allows us to search for sections of the parameter space where the SM and MSSM cross sections are close to identical.

We know, from eq. (5.3.3), that for $\theta = 0$ the stop squark contribution is non-zero and will increase the overall cross section, while $\theta = \pi/4$ gives maximal mixing. We will work in this case of maximal mixing so that we can focus on the ($m_{\tilde{t}_1}$,Δm) parameter space. However, as can be seen from fig. (5.1 (right)) there are regions of the (Δm,θ) parameter space away from $\theta = \pi/4$ where we still see δ vanishing. The analysis in this chapter therefore is not dependent on maximal mixing in the stop squark section.

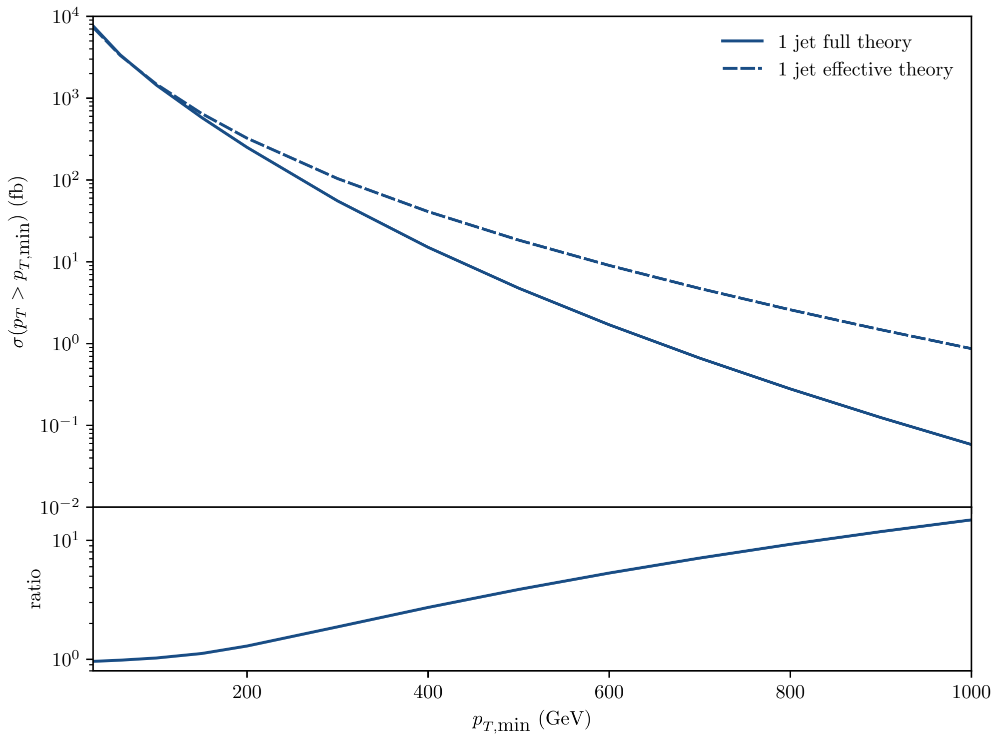


Figure 5.2: Lowest order predictions for Higgs plus 1-jet production in the SM, computed in the EFT (dashed) and in the full theory (solid). The lower panel shows the ratio $\sigma^{\mathrm{EFT}}/\sigma^{\mathrm{full}}$.

In fig. (5.1 (left)) we can see the expected vanishing of δ for

$$\Delta m \approx m_t \sqrt{2\left(1 + \sqrt{2}\, m_{\tilde{t}_1}/m_t\right)}\,, \tag{5.3.10}$$

as observed in ref. [117], across the centre of the plot. Additionally, the region expected from eq. (5.3.8) can be seen in the top left corner, corresponding to $\Delta m \approx 6m_{\tilde{t}_1}$, since we are focusing on the parameter space where $m_{\tilde{t}_1} \approx m_t$. These results are useful to confirm that, although our analytic derivation focused on asymptotic values, it was sufficient to highlight the main features of the parameter space.

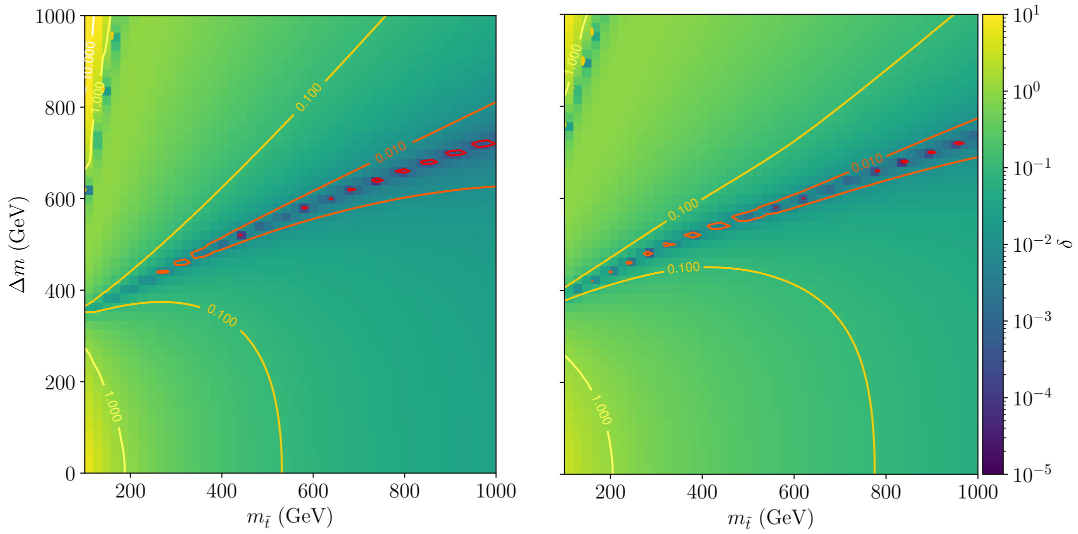


Figure 5.3: The deviation of the Higgs+1-jet cross section from the SM case, measured by δ defined in eq. (5.3.9), as a function of $m_{\tilde{t}_1}$ and Δm. These results are calculated in the full theory. Top squarks mix in a maximal fashion $(\theta = \frac{\pi}{4})$ and $\tan\beta = 10$. Results are shown for two choices of jet p_T: 30 GeV (left) and 600 GeV (right).

5.3.2 Higgs plus 1-jet

When the Higgs is produced in conjunction with a jet, the EFT only holds for low p_T of the final state particles. Once the p_T of the jet, or Higgs, becomes high enough, the EFT result begins to deviate from the full analytic cross section, shown for the SM in fig. (5.2). By comparing the ratio of the cross sections in the EFT and full SM, lower panel of fig. (5.2), we can see how quickly the two calculations diverge once $p_{T,\min} > m_t$. The same applies in the MSSM, though the relevant scales are given by $m_{\tilde{t}_1}$ and Δm. This means that, when the transverse momentum is less than the relevant scales, the variation between the SM and MSSM cases is very similar to the inclusive cross section result. We can see this by comparing fig. (5.3 (left)) for the Higgs plus 1-jet case with $p_T > 30$ GeV to fig. (5.1 (left)) for the inclusive case. This is due to the jet (or equivalently the Higgs boson) having too little energy to resolve the coloured particles in the loop; the same reason for the validity of the EFT.

We expect, therefore, any difference between the inclusive and 1-jet processes to become visible when the transverse momentum is of order $m_{\tilde{t}_1}$ or larger. A comparison of the two plots in fig. (5.3) shows the effect of a higher cut on the transverse momentum. With $p_T > 600$ GeV, the jet is able to resolve the top quark as well as the stop squark with masses of $m_{\tilde{t}}$ up to a similar scale. This region, where the EFT is no longer valid, breaks the equivalence with the inclusive cross section and leads to larger deviations from the SM prediction. This is demonstrated by the δ contours in fig. (5.3 (right)), which are more constraining than for $p_T > 30$ GeV (fig. (5.3 (left))). These results are discussed in detail in ref. [117].

This opens the possibility of studying the splinter region by comparison of the Higgs plus 1-jet results at a large minimum p_T cut with the inclusive results.

5.4 Higgs Boson plus 2-Jet Production

We have seen that the ability to distinguish the SM and MSSM cases is related to the breakdown of the EFT, and therefore the ability of the final state particles to resolve the loop propagators. When there are more than two particles in the final state, as for Higgs plus 2 jets, the breakdown of the EFT is related to the p_T of the hardest particle [87].

Applying a large minimum p_T cut to the jets causes the cross section to fall off quickly, since we are requiring two of the three final state particles to have high energy, as shown in fig. (5.4). This also leads to the difference between the full theory and EFT being less pronounced than for the 1-jet case. If however we require the jets to have $p_T > 30$ GeV and enforce a higher p_T cut on the Higgs boson, we are only requiring one hard final state particle. We therefore sacrifice less of the cross section and the deviation from the EFT looks similar to the 1-jet case.

Using this p_T cut on the Higgs boson we can study the effect of the scalar mass on the breakdown of the EFT. For this we will consider a single scalar particle, with no top

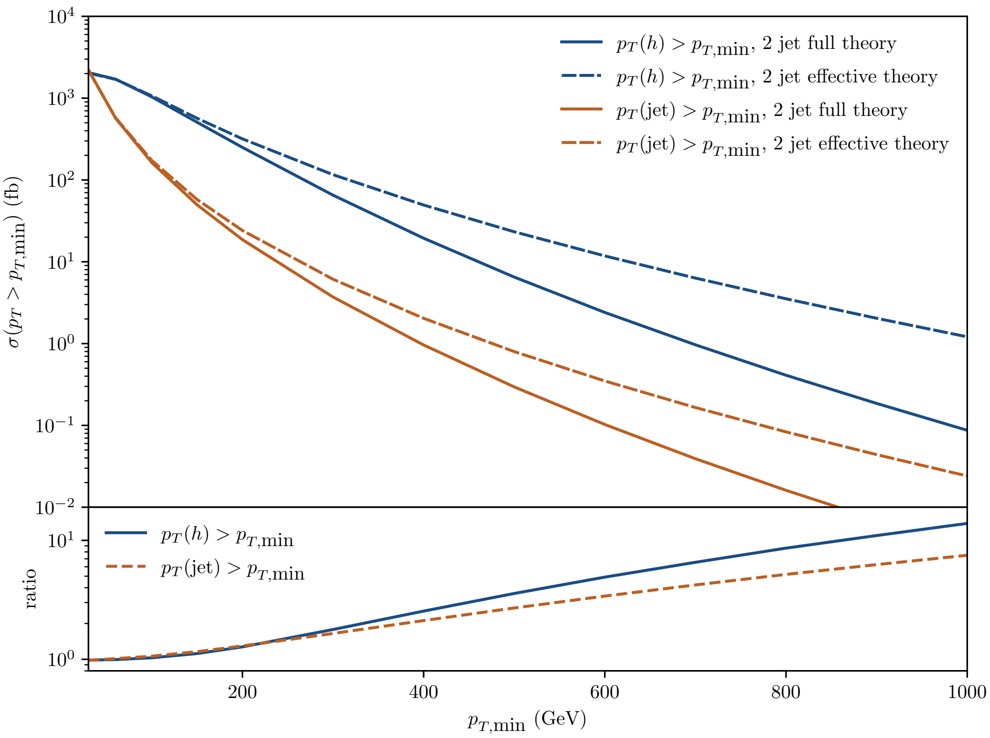


Figure 5.4: Rates for Higgs plus 2-jet production in the SM, as a function of a minimum p_T, computed in the full theory (solid) and EFT (dashed). Jets are either subject to this minimum p_T themselves (orange), or they are only required to satisfy a 30 GeV cut and the minimum p_T cut is applied to the Higgs boson p_T (blue). The lower panel shows the ratio $\sigma^{\text{EFT}}/\sigma^{\text{full}}$ for both cases.

quark contribution. In fig. (5.5) we plot the cross section with $m_{\text{scalar}} = m_t = 173.3$ GeV, for a direct comparison with the fermion theory, and $m_{\text{scalar}} = 600$ GeV. The breakdown of the EFT can be clearly seen for $p_{T,\text{min}} \gtrsim m_{\text{scalar}}$. The larger scalar particle mass significantly reduces the cross section for the process, thus making analysis of results much more difficult. This method of indirect searching is unable to place limits on the stop squark mass that are competitive with the results from direct searches.

An analysis of the effectiveness of the EFT in both the fermion and scalar mediated theories is shown in fig. (5.6). Both lines follow a very similar pattern, with the EFT showing to be marginally better in the scalar theory at high $p_T(\text{min})$. The reverse is

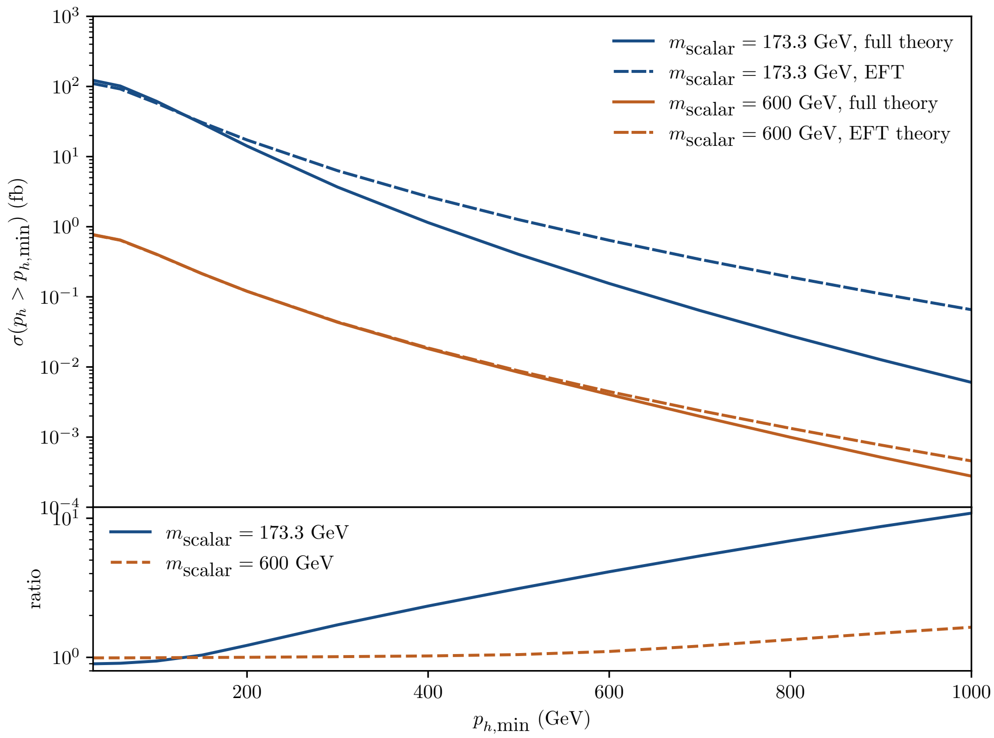


Figure 5.5: Rates for Higgs plus 2-jet production through a scalar loop, as a function of a minimum p_T applied to the Higgs boson, computed in the full theory (solid) and EFT (dashed). The scalar mediator mass is either 173.3 GeV (blue) or 600 GeV (orange). The lower panel shows the ratio $\sigma^{\text{EFT}}/\sigma^{\text{full}}$ for both examples of m_{scalar}.

true for low $p_T(\text{min})$, as the fermion EFT is closer to the full theory. The EFT for both quickly becomes less accurate once $p_T(\text{min}) \gtrsim m$. It is easy to calculate the expected values of each as $p_T(\text{min}) \to 0$ from eqs. (3.3.4, 3.3.5),

$$\begin{aligned}\sigma^{\text{full}}(gg \to h)/\sigma^{\text{EFT}}(gg \to h)|_{\text{fermion}} &= \left[F_{1/2}(4m_t^2/m_h^2)/(-4/3)\right]^2 = 1.065\,,\\ \sigma^{\text{full}}(gg \to h)/\sigma^{\text{EFT}}(gg \to h)|_{\text{scalar}} &= \left[F_0(4m_t^2/m_h^2)/(-1/3)\right]^2 = 1.157\,.\end{aligned} \tag{5.4.1}$$

These values are not equal to exactly 1 due to $F_{1/2}(\tau)$ and $F_0(\tau)$ never reaching their respective asymptotic limits. The two curves will never reach the values in eq. (5.4.1), though the Higgs plus 2-jet final state with two very soft gluons resembles the final state with a lone Higgs.

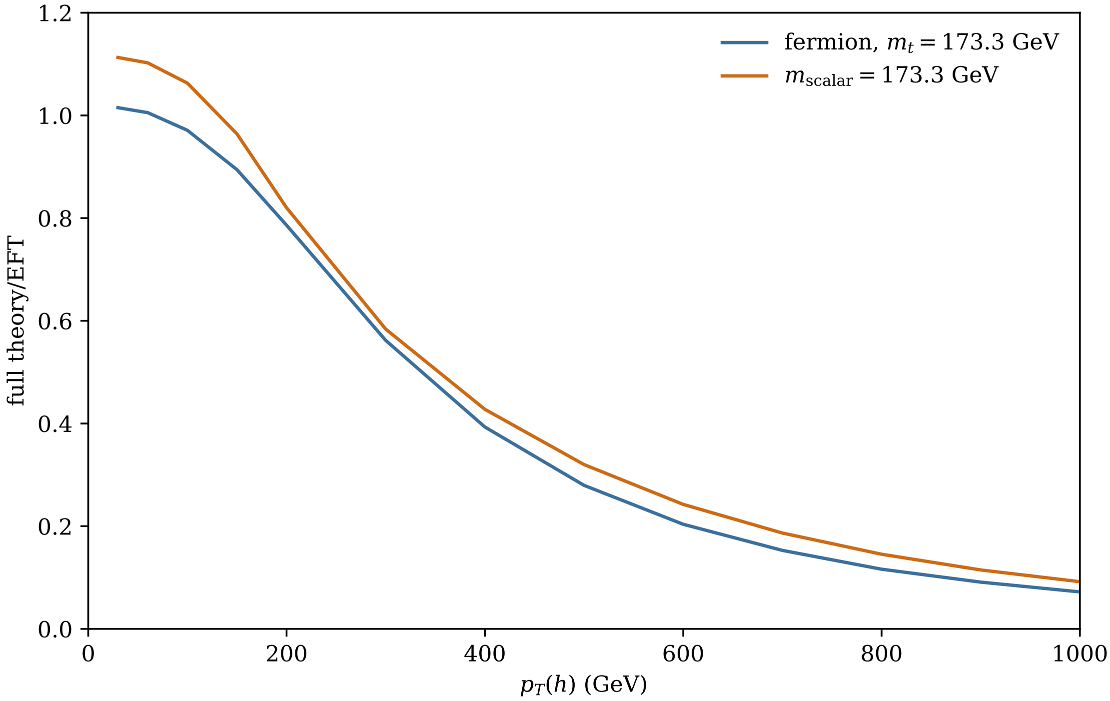


Figure 5.6: The ratio of the cross section in the full theory to that in the effective theory, for Higgs plus 2-jet production through a fermion (blue) and scalar (orange) loop, as a function of a minimum p_T applied to the Higgs boson. The mass of the mediator is set to 173.3 GeV in both cases.

The accuracy of the EFT for varying mediator mass, with a cut of $p_T > 300$ GeV applied to the Higgs, is shown in fig. (5.7). As expected, for very large mediator mass the EFT is a nearly perfect description.

Setting $m_{\tilde{t}} = 600$ GeV and keeping to the case of maximal stop squark mixing, fig. (5.8) shows the dependence of δ on the p_T cut applied to the Higgs boson for the 2-jet process. As a comparison the inclusive results are also shown. The deviations in the 2-jet case are larger than for the inclusive case, however when compared to the 1-jet case, c.f. fig. (2) of ref. [117], the results are very similar. This is further illustrated in fig. (5.9), which shows that the deviations from the SM are almost identical to the 1-jet results in fig. (5.3).

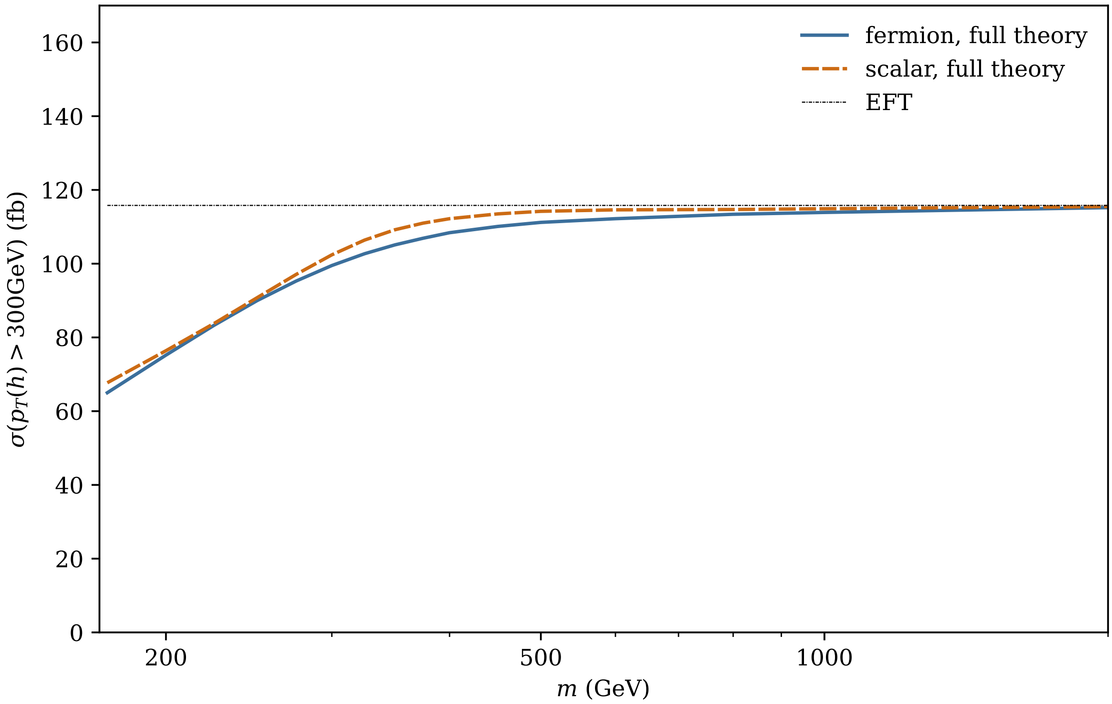


Figure 5.7: The cross section for Higgs plus 2-jet production through a fermion (solid) and scalar (dashed) loop, as a function of the mediator mass m, after application of a cut $p_T(h) > 300$ GeV. The EFT result is shown as a horizontal dotted line.

5.5 Discussion

Our ability to study these processes is dependent on getting enough experimental data. For $p_T(\text{jet}) > 30$ GeV, the leading order cross sections for the Higgs plus 0-, 1- and 2-jet processes are,

$$\begin{aligned}
\sigma(gg \to h) &= 16240 \text{ fb}\,, \\
\sigma(gg \to h + 1 \text{ jet}) &= 7640 \text{ fb}\,, \\
\sigma(gg \to h + 2 \text{ jets}) &= 2230 \text{ fb}\,, \\
\sigma(gg \to h + 1 \text{ jet}) &= 2 \text{ fb} \qquad (p_T(h) > 600 \text{ GeV})\,, \\
\sigma(gg \to h + 2 \text{ jets}) &= 2 \text{ fb} \qquad (p_T(h) > 600 \text{ GeV})\,.
\end{aligned} \tag{5.5.1}$$

Therefore, although the 1- and 2-jet cases have larger deviations from the Standard Model compared to the inclusive cross section, it is unlikely that the LHC will be

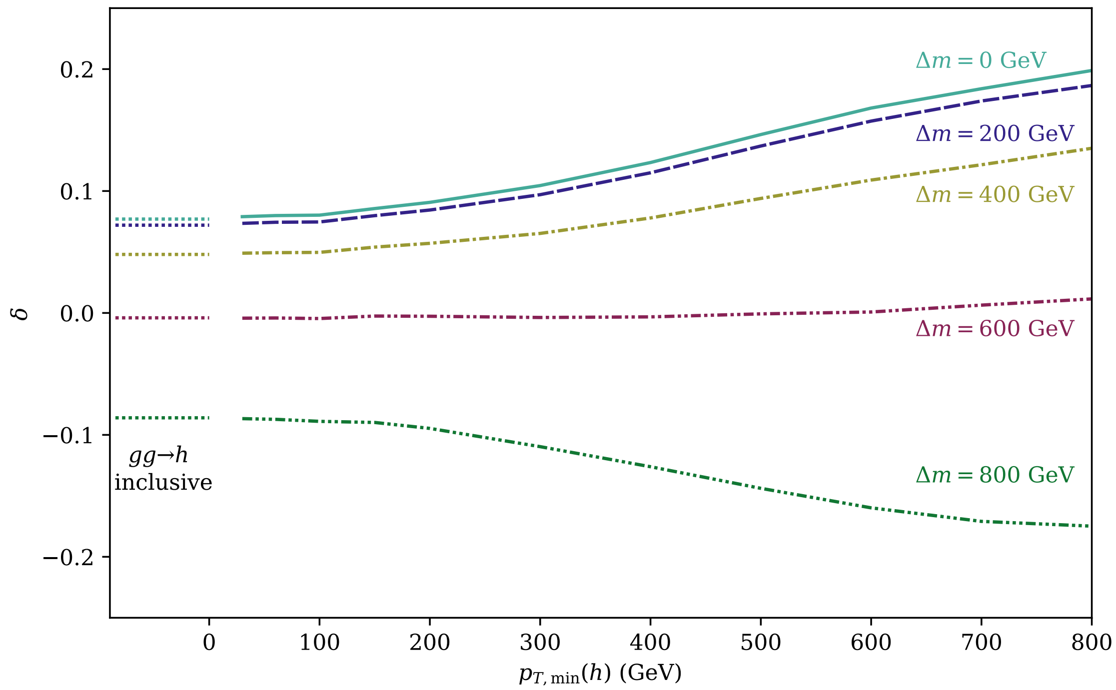


Figure 5.8: 2-jet calculation of δ as a function of the cut on the Higgs boson p_T, for the parameters $m_{\tilde{t}_1} = 600$ GeV, $\theta = \frac{\pi}{4}$ and $\tan\beta = 10$. The corresponding inclusive result is shown as a dashed line on the left of the plot.

able to discriminate between the two models. This applies even after considering that $h \to b\bar{b}$ decays may be identifiable for the highly-boosted Higgs; the number of events would still be orders of magnitude smaller than for non-boosted Higgs. The reduction in statistical power would still be too great to overcome.

Given the potential for higher order QCD corrections to be significant it is worth considering the impact they will have on these processes. We can rescale our results by corrections computed in the EFT,

$$\begin{aligned} K(gg \to h) &= 1.91\,, \\ K(gg \to h + 1 \text{ jet }) &= 1.78\,, \\ K(gg \to h + 2 \text{ jets }) &= 1.72\,, \\ K\left(gg \to h + 1 \text{ jet}, \, p_T(h) > 600\text{GeV}\right) &= 1.85\,, \\ K\left(gg \to h + 2 \text{ jets}, \, p_T(h) > 600\text{GeV}\right) &= 1.42\,, \end{aligned} \tag{5.5.2}$$

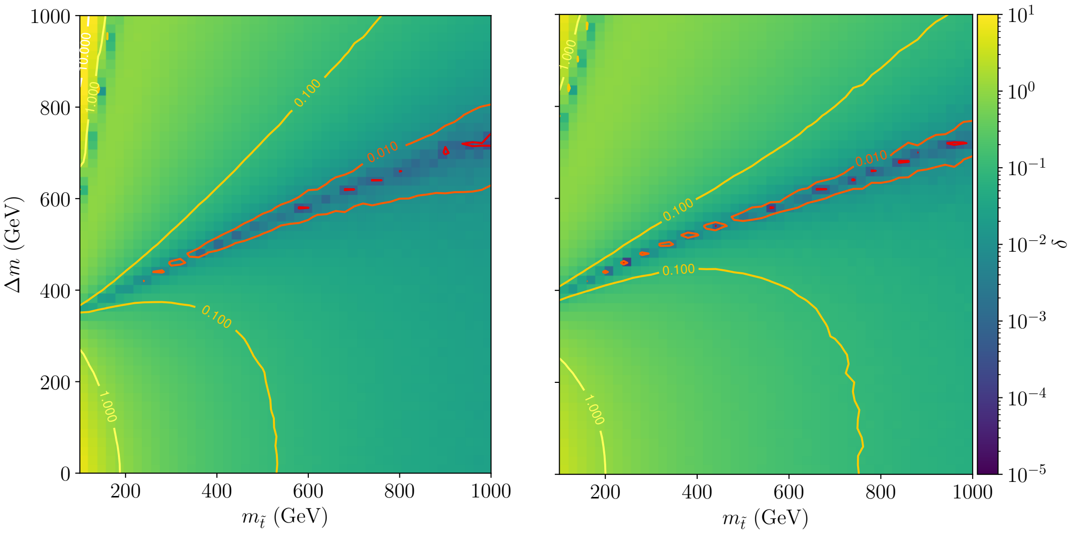


Figure 5.9: The deviation of the Higgs+2-jet cross section from the SM case, measured by δ defined in eq. (5.3.9), as a function of $m_{\tilde{t}_1}$ and Δm. These results are calculated in the full theory. Top squarks mix in a maximal fashion ($\theta = \frac{\pi}{4}$) and $\tan\beta = 10$. Results are shown in the cases of no additional cut (left) and $p_T(h) > 600$ GeV (right).

where $K = \sigma^{\text{NLO}}/\sigma^{\text{LO}}$. These values in fact suggest that higher order corrections will increase the gap in event numbers due to the K-factors for the 1- and 2-jet processes being smaller than for the inclusive cross section.

Alongside our consideration of statistical uncertainties we must also take systematic uncertainties into account. Events with a boosted Higgs, and therefore high energy decay products, have smaller systematic uncertainties. Theoretical predictions, however, have larger systematic uncertainties for boosted Higgs production. For inclusive production the uncertainties are at the percent level in the expansion up to N^{3}LO [59, 61], rising to 10% in the boosted case at NNLO [65, 66, 120–122]. It is therefore evident that a good understanding of both statistical and systematic errors is required to assess the value of Higgs plus multi-jet processes in the search for supersymmetry.

5.6 Conclusion

We have used the analytic expressions presented in chapters 3 and 4 to analyse the potential of the Higgs plus 2-jet process to discriminate between the SM and MSSM at the LHC. Analysis has previously been done for both the inclusive and 1-jet processes, demonstrating the improved sensitivity of the 1-jet case over the inclusive analysis.

We have found that the 2-jet analysis does not offer an improvement over the 1-jet case, giving very similar results. A high p_T cut on at least one of the final state particles was required in order to effectively probe the loop particles. This leads to a final state dominated by two hard particles and one soft, mimicking the 1-jet final state.

It is unlikely that the relatively small gain in the 1- and 2-jet analyses will be able to overcome the loss of statistical power at the high transverse momentum needed in both cases in order to probe the loop mediators. If, however, a deviation from the Standard Model prediction for Higgs boson production were to be observed, a thorough understanding of both SM and beyond the SM predictions would be required.

Appendix A

Numerical value of coefficients at a given phase-space point

The following tables contain numerical results for the integral coefficients for the $ggggh$ and $\bar{q}qggh$ amplitudes in both the fermion- and scalar-mediated theories at the phase-space point ($p = (E, p_x, p_y, p_z)$)

$$
\begin{aligned}
p_1 &= (-15\kappa, -10\kappa, +11\kappa, +2\kappa)\,, \\
p_2 &= (-9\kappa, +8\kappa, +1\kappa, -4\kappa)\,, \\
p_3 &= (-21\kappa, +4\kappa, -13\kappa, +16\kappa)\,, \\
p_4 &= (-7\kappa, +2\kappa, -6\kappa, +3\kappa)\,, \\
p_h &= (+52\kappa, -4\kappa, +7\kappa, -17\kappa)\,,
\end{aligned}
\tag{A.0.1}
$$

with $\kappa = 1/\sqrt{94}$ and $p_h = -p_1 - p_2 - p_3 - p_4$. This fixes $s_{1234} = 25$, $m_h = 5$ and we further choose $m = 1.5$.

Helicities	Coefficient	Real Part	Imaginary Part	Absolute Value
$++++$	$\tilde{d}_{1\times2\times34}$	-0.9840613828	-0.5144323508	1.1104131883
	$\tilde{d}_{1\times23\times4}$	-3.3548957407	-4.8432206981	5.8916985803
	$\tilde{d}_{1\times2\times3}$	-6.7445910748	-15.4663942318	16.8730216411
	$\tilde{c}_{1\times234}$	-10.6368762164	-31.6829840771	33.4208709592
	$\tilde{r}$	-6.4366316747	-19.1721417745	20.2237792595
$+++-$	$\tilde{d}_{1\times2\times34}$	23.4451295603	18.5996441921	29.9269254046
	$\tilde{d}_{1\times4\times32}$	20.5071688388	27.4451393815	34.2604677355
	$\tilde{d}_{2\times1\times43}$	-4.9009936782	42.1225176136	42.4066767047
	$\tilde{d}_{2\times34\times1}$	-44.3845463184	-38.3339964812	58.6471076705
	$\tilde{d}_{4\times3\times21}$	-7.1203811993	0.6886216537	7.1536024635
	$\tilde{d}_{1\times23\times4}$	-1.8005835535	1.5351129014	2.3661514646
	$\tilde{d}_{2\times3\times4}$	0.8206155641	1.4735210192	1.6866161680
	$\tilde{d}_{1\times2\times3}$	-19.2397847846	-1.4762925832	19.2963405429
	$\tilde{d}_{3\times4\times1}$	-0.3316788675	1.6114692592	1.6452489309
	$\tilde{c}_{3\times4}$	-0.0041638038	0.0115576710	0.0122848289
	$\tilde{c}_{2\times34}$	3.1035163815	-0.1080335333	3.1053961381
	$\tilde{c}_{1\times43}$	6.9656648763	-0.8139894264	7.0130639492
	$\tilde{c}_{4\times123}$	-11.0616538761	-1.7916339105	11.2058082504
	$\tilde{c}_{1\times234}$	18.9646702722	24.4510167733	30.9436736633
	$\tilde{c}_{2\times341}$	-8.9934514290	11.1934355822	14.3588010899
	$\tilde{c}_{12\times34}$	-3.7461389306	21.0493483972	21.3800988032
	$\tilde{b}_{34}$	-0.0409808246	0.1015477837	0.1095051613
	$\tilde{b}_{234}$	0.2936947594	-0.0490382211	0.2977605730
	$\tilde{b}_{1234}$	-0.9341272666	-0.1920882562	0.9536727156
	$\tilde{r}$	-3.8872487587	10.3025699409	11.0115235230

Table A.1: Numerical values of coefficients of the $ggggh$ process with gluon helicities of for $++++$ and $+++-$ mediated by a massive coloured scalar at the kinematic point defined in eq. (A.0.1).

A.1 Coefficient Values in the Scalar-Mediated Theory

The numerical values of the coefficients and the rational terms are given in Tables A.1, A.2 and A.3.

At this kinematic point, once the integrals have been evaluated, the values of the full colour-ordered sub-amplitudes are,

$$A^{1234}(1^+, 2^+, 3^+, 4^+; h) = -26.50523303 - 3.722078577\, i,$$

Helicities	Coefficient	Real Part	Imaginary Part	Absolute Value
$+-+-$	$\tilde{d}_{4\times3\times21}$	-6.9368235764	-13.4220769362	15.1086621053
	$\tilde{d}_{1\times23\times4}$	-5.4005161311	3.8281939917	6.6197162870
	$\tilde{d}_{1\times2\times3}$	-21.0997803781	-62.3608308275	65.8336840341
	$\tilde{c}_{3\times4}$	-0.0785670511	0.1424216217	0.1626551563
	$\tilde{c}_{2\times34}$	5.9965313107	-4.5593453199	7.5329952546
	$\tilde{c}_{12\times34}$	-39.7403340718	22.2104113517	45.5257786814
	$\tilde{c}_{1\times234}$	3.9682125956	13.4813791531	14.0532663489
	$\tilde{b}_{34}$	0.0682767006	0.0433975227	0.0809015007
	$\tilde{b}_{234}$	2.3825660060	-0.8884219110	2.5428162074
	$\tilde{b}_{1234}$	-4.2679248420	3.0120566624	5.2237599288
	$\tilde{r}$	2.3606586680	0.8025702116	2.4933568319
$++--$	$\tilde{d}_{1\times2\times34}$	-0.0267530609	-1.1100908623	1.1104131883
	$\tilde{d}_{1\times4\times32}$	22.6518970482	-458.1248398611	458.6845074097
	$\tilde{d}_{2\times34\times1}$	64.2316548189	-59.0233562841	87.2322306708
	$\tilde{d}_{1\times23\times4}$	-5.7450785528	3.2885735727	6.6197162870
	$\tilde{d}_{1\times2\times3}$	-10.8954346530	-12.8836471165	16.8730216411
	$\tilde{c}_{2\times3}$	-41.9249189131	-23.3819669075	48.0043248295
	$\tilde{c}_{1\times23}$	-1036.7850502032	-480.4884415677	1142.7127297817
	$\tilde{c}_{23\times41}$	1075.3186068541	747.6290891424	1309.6791061854
	$\tilde{c}_{1\times234}$	36.8856220760	-309.1172377677	311.3101601314
	$\tilde{b}_{23}$	-5.3092820284	-9.1916846550	10.6148736429
	$\tilde{b}_{234}$	-0.8234906782	5.4869406052	5.5483920285
	$\tilde{b}_{1234}$	26.7533037866	15.0643874405	30.7030134100
	$\tilde{r}$	1.3340510134	0.8053633104	1.5583010518

Table A.2: Numerical values of coefficients of the $ggggh$ process with gluon helicities of $+-+-$ and $++--$ mediated by a massive coloured scalar at the kinematic point defined in eq. (A.0.1).

Helicities	Coefficient	Real Part	Imaginary Part	Absolute Value
$+-++$	$\tilde{d}_{3\times21\times4}$	370.4335392027	1300.4704659852	1352.1998520434
	$\tilde{d}_{4\times3\times21}$	0.9220079194	-4.0077609078	4.1124501332
	$\tilde{c}_{3\times21}$	-73.7590711176	-242.0029027576	252.9936867102
	$\tilde{c}_{12\times34}$	-13.7672899406	-4.3871539915	14.4494080313
	$\tilde{c}_{4\times123}$	25.1014609317	92.8813584055	96.2134610133
	$\tilde{c}_{3\times412}$	67.1532044289	252.1462326711	260.9353857094
	$\tilde{b}_{12}$	1.4197098901	0.3520351648	1.4627046624
	$\tilde{b}_{124}$	-0.5233075583	-1.0058683561	1.1338527023
	$\tilde{b}_{123}$	-0.7954324907	-2.6428744805	2.7599815882
	$\tilde{b}_{1234}$	-0.1009698411	3.2967076718	3.2982535351
	$\tilde{r}$	-5.4119652752	0.7153882121	5.4590428130
$+--+$	$\tilde{d}_{3\times21\times4}$	20.8960073185	19.9656478672	28.9010417911
	$\tilde{d}_{4\times3\times21}$	-0.4267971904	-3.8149302668	3.8387301002
	$\tilde{c}_{3\times21}$	-0.8475265952	2.4513967233	2.5937708504
	$\tilde{c}_{3\times4}$	0.0094395944	0.0336984573	0.0349955993
	$\tilde{c}_{12\times34}$	-9.0987353955	-7.6314351405	11.8754279123
	$\tilde{c}_{4\times123}$	1.9450686855	1.9314994054	2.7411643775
	$\tilde{b}_{12}$	-0.1514791222	-0.2049070241	0.2548191770
	$\tilde{b}_{34}$	0.0053272676	0.1024178437	0.1025562991
	$\tilde{b}_{124}$	0.2906270969	-0.1175213822	0.3134890504
	$\tilde{b}_{123}$	0.6201039140	0.1805231188	0.6458463135
	$\tilde{b}_{1234}$	-0.7645791563	0.0394874437	0.7655981613
	$\tilde{r}$	-0.2305425036	1.4755649890	1.4934663983
$+-+-$	$\tilde{c}_{4\times123}$	5.2050785566	1.1864337667	5.3385829453
	$\tilde{b}_{123}$	-1.0956687877	0.8092787161	1.3621388082

Table A.3: Numerical values of coefficients of the $\bar{q}qggh$ process mediated by a massive coloured scalar at the kinematic point defined in eq. (A.0.1).

$$
\begin{aligned}
|A^{1234}(1^+,2^+,3^+,4^+;h)| &= 26.76529930\,,\\
A^{1234}(1^+,2^+,3^+,4^-;h) &= 10.00550042 + 10.39130252\,i,\\
|A^{1234}(1^+,2^+,3^+,4^-;h)| &= 14.42529746\,,\\
A^{1234}(1^+,2^-,3^+,4^-;h) &= 2.105330472 - 3.500785469\,i,\\
|A^{1234}(1^+,2^-,3^+,4^-;h)| &= 4.085084491\,,\\
A^{1234}(1^+,2^+,3^-,4^-;h) &= -0.788758613 + 0.151525137\,i,\\
|A^{1234}(1^+,2^+,3^-,4^-;h)| &= 0.803181185\,.
\end{aligned}
\tag{A.1.1}
$$

$$
\begin{aligned}
A^{34}(1^+,2^-,3^+,4^+;h) &= -3.151452974 + 5.766222683\,i,\\
|A^{34}(1^+,2^-,3^+,4^+;h)| &= 6.571223621\,,\\
A^{34}(1^+,2^-,3^-,4^+;h) &= 1.375544184 + 1.088612645\,i,\\
|A^{34}(1^+,2^-,3^-,4^+;h)| &= 1.754194771\,,\\
A^{34}(1^+,2^-,3^+,4^-;h) &= 3.032201250 - 1.275260855\,i,\\
|A^{34}(1^+,2^-,3^+,4^-;h)| &= 3.289458111\,.
\end{aligned}
\tag{A.1.2}
$$

$$
\begin{aligned}
A^{4q}(1^+,2^-,3^+,4^-;h) &= 1.583011630 - 1.072246795\,i,\\
|A^{4q}(1^+,2^-,3^+,4^-;h)| &= 1.911972544\,.
\end{aligned}
\tag{A.1.3}
$$

A.2 Coefficient Values in the Fermion-Mediated Theory

Due to the correspondence of results between the scalar and fermion loop cases, many of these values have already been listed in section A.1. Tables A.4 and A.5 therefore contain only the coefficients that differ from the scalar case.

At this kinematic point, once the integrals have been evaluated, the values of the full colour-ordered sub-amplitudes are,

$$
\begin{aligned}
H^{1234}(1^+,2^+,3^+,4^+;h &= +29.24088185 - 46.63892079\,i,\\
|H^{1234}(1^+,2^+,3^+,4^+;h)| &= 55.04741687\\
H^{1234}(1^+,2^+,3^+,4^-;h) &= -28.10008864 + 9.836858255\,i,\\
|H^{1234}(1^+,2^+,3^+,4^-;h)| &= 29.77211383\\
H^{1234}(1^+,2^-,3^+,4^-;h) &= +4.580787288 + 7.498254006\,i,\\
|H^{1234}(1^+,2^-,3^+,4^-;h)| &= 8.786775593\\
H^{1234}(1^+,2^+,3^-,4^-;h) &= +0.369177073 - 1.815728344\,i,\\
|H^{1234}(1^+,2^+,3^-,4^-;h)| &= 1.852879146
\end{aligned}
\tag{A.2.1}
$$

$$
\begin{aligned}
H^{34}(1^+,2^-,3^+,4^+;h) &= -8.998796972 - 13.02970981\,i,\\
|H^{34}(1^+,2^-,3^+,4^+;h)| &= 15.83514081\\
H^{34}(1^+,2^-,3^-,4^+;h) &= -3.850947633 + 1.791151530\,i,\\
|H^{34}(1^+,2^-,3^-,4^+;h)| &= 4.247119197\\
H^{34}(1^+,2^-,3^+,4^-;h) &= -0.412185752 + 7.682564596\,i,\\
|H^{34}(1^+,2^-,3^+,4^-;h)| &= 7.693613966
\end{aligned}
\tag{A.2.2}
$$

Helicities	Coefficient	Real Part	Imaginary Part	Absolute Value
$++++$	$d_{1\times 2\times 34}$	1.7494424584	0.9145464014	1.9740678903
	$d_{1\times 23\times 4}$	5.9642590946	8.6101701300	10.4741308095
	$d_{1\times 2\times 3}$	11.9903841330	27.4958119676	29.9964829174
	$c_{1\times 234}$	18.9100021625	56.3253050259	59.4148817052
$+++-$	$d_{1\times 2\times 34}$	-24.3908884307	-34.1026538098	41.9273948071
	$d_{1\times 4\times 32}$	-22.2037730441	-29.7427434881	37.1165505885
	$d_{2\times 1\times 43}$	16.2217246906	-62.9923572563	65.0475320412
	$d_{2\times 34\times 1}$	-66.4392574700	12.9335349956	67.6864185834
	$d_{4\times 3\times 21}$	8.2313626631	-0.7960661671	8.2697673869
	$d_{1\times 23\times 4}$	2.0815256682	-1.7746340633	2.7353382179
	$d_{2\times 3\times 4}$	-0.9920798783	-1.5084323993	1.8054336843
	$d_{1\times 2\times 3}$	22.2417370205	1.7066361068	22.3071170816
	$d_{3\times 4\times 1}$	0.8741489856	-5.3830902459	5.4536040418
	$c_{4\times 123}$	12.7875856866	2.0711796271	12.9542322327
	$c_{1\times 234}$	-41.8343835373	-39.3169799861	57.4102827129
	$c_{2\times 341}$	-0.0578594858	-18.9964204402	18.9965085545
	$c_{12\times 34}$	12.4596639704	-35.5399553316	37.6607441672
$+-+-$	$d_{4\times 3\times 21}$	-7.6953556408	-6.4085129013	10.0143664825
	$d_{1\times 23\times 4}$	-2.3752436126	1.8890031582	3.0348171528
	$d_{1\times 2\times 3}$	-14.9620839628	-39.7624750054	42.4843309359
	$c_{12\times 34}$	-22.6495761599	21.1031361652	30.9571584004
	$c_{1\times 234}$	-9.2908649214	-2.0570320613	9.5158579166
$++--$	$d_{1\times 2\times 34}$	-0.0125227093	-0.5196169994	0.5197678754
	$d_{1\times 4\times 32}$	8.4132295790	-459.7920528912	459.8690186715
	$d_{2\times 34\times 1}$	62.3890431832	-51.7566409441	81.0625844095
	$d_{1\times 23\times 4}$	-3.4240833144	4.6410747884	5.7674883386
	$d_{1\times 2\times 3}$	-5.4385640586	-6.5811803202	8.5375589853
	$c_{23\times 41}$	1080.7316959848	740.3414428401	1309.9948284984
	$c_{1\times 234}$	22.0281339875	-305.0638529285	305.8581256899

Table A.4: Numerical values of coefficients of the $ggggh$ process, that were not included in tables A.1 and A.2, mediated by a massive fermion at the kinematic point defined in eq. (A.0.1).

Helicities	Coefficient	Real Part	Imaginary Part	Absolute Value
$+-++$	$d_{4\times3\times21}$	4.0685161820	-4.0500901147	5.7407363517
	$d_{4\times21\times3}$	425.5033072909	1294.6650310348	1362.7951449502
	$c_{12\times34}$	14.7023801790	-6.9563781545	16.2650293561
	$c_{4\times123}$	32.6756691373	92.0151860555	97.6447326711
	$c_{3\times412}$	87.6696567417	249.0681532719	264.0471807982
$+--+$	$d_{4\times3\times21}$	1.0782715488	-4.7280903169	4.8494852900
	$d_{4\times21\times3}$	13.3402061977	-3.4340877490	13.7751246842
	$c_{12\times34}$	-3.7289304305	-10.8894201371	11.5101864919
	$c_{4\times123}$	1.7886984296	-1.6881229718	2.4595123988
	$c_{3\times412}$	1.8223566626	0.0772529014	1.8239933708
$+-+-$	$c_{4\times123}$	0.0795879764	-1.9432491013	1.9448782264

Table A.5: Numerical values of the coefficients for the $+-++$, $+--+$ and $+-+-$ helicities of the $\bar{q}qggh$ process, that were not included in table A.3, mediated by a massive fermion at the kinematic point defined in eq. (A.0.1).

$$
\begin{aligned}
H^{4q}(1^+,2^-,3^+,4^-;h) &= 0.620045806 + 4.7030845622\, i, \\
|H^{4q}(1^+,2^-,3^+,4^-;h)| &= 4.743781319
\end{aligned}
\tag{A.2.3}
$$

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