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Global stabilization for triangular formations under mixed distance and bearing constraints

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Abstract— **This paper addresses the triangular formation control problem for a system of three agents under mixed distance and bearing constraints. The main challenge is to find a fully distributed control law for each agent to guarantee the global convergence towards a desired triangular formation. To solve this problem, we invoke the property that a triangle can be uniquely determined by the lengths of its two sides together with the magnitude of the corresponding included angle. Based on this feature, we design a class of control strategies, under which each agent is only responsible for a single control variable, i.e., a distance or an angle, such that the control laws can be implemented in local coordinate frames. The global convergence is shown by analyzing the dynamics of the closed-loop system in its cascade form. Then we discuss some extensions on more general formation shapes and give the quadrilateral formation as an example. Simulation results are provided to validate the effectiveness of the proposed control strategies.**

I. INTRODUCTION

Distributed formation control is a fundamental problem in multi-agent systems, which has wide applications, such as search and rescue in hazardous environments, ocean data retrieval and sampling, satellite formation flying, etc [1]–[4]. The objective of formation control is to drive multiple agents to satisfy prescribed constraints on their states. In recent years, as distinguished by sensed and controlled variables, distance-based and bearing-based formation control for groups of mobile agents have gained a significant amount of attention [5]. Formation control laws with distance constriants have been proposed and the local and global stable systems are investigated in the literature [6]–[11]. The distance-based formation control means that the formation is specified by distance constraints. The most challenging problem is to drive the formation from undesired non-trivial equilibria to the desired one. It has been shown that there might be no global solution for a system with four or more agents [12]. Bearings have also been employed in formation control algorithms due to its easy access especially in vision-based control frameworks. There are several methods for bearing-only based fomation control in terms of different definitions of bearing. One denotes the bearing as the angle between the sensor link and the x-axis of the global coordinate frame [13]. Then [14] and [15] relax the definition of bearing to the angle between the sensor link and the x-axis of its local coordinate frame attached to the body. In [16], bearing is defined as the unit vector whose direction is along the sensing link of the pair agents in a global coordinate frame. However, these classes of bearing-only based formation control strategies result in an intrinsic problem: the scale of the formation is uncontrollable. To overcome this problem, some extra distance constraints can be added with the purpose of removing undesirable motion freedoms, and then one arrives at hybrid formation control with both distance and bearing constraints.

Formation control with a mixture of distance and bearing constraints is addressed in [17]–[19]. In [17], a class of control laws for achieving triangular formations with two angle and a single distance constraints is proposed and the local asymptotic stability of the closed-loop system has been proved. Later, different formation shapes with n agents are taken into consideration in an ambient two-dimensional space [18]. Although [19] achieves the global asymptotic stability for more than 3 agents in a two-dimensional space, the sensing graph of the system is limited to a 2-simple graph (the defination of 2-simple graph is given in [20]). When it comes to a three-agent systems, it requires the knowledge of all three distance constraints and one angle. This paper focuses on the problem of triangular formation control with a mix of distance and bearing constraints. A new class of control laws is proposed to ensure global convergence towards a desired triangular formation in a two-dimensional space. In the proposed control law, only two distance and one bearing constraints are used.

The main contribution of this work is threefold. On one hand, in comparison with the local stability achieved in [17] [18], one can obtain more applaudable global results by employing our proposed control laws. On the other hand, to relax the strong sensing requirements that agents need to measure both distances and angles, each agent is only required to measure one of these variables in our control scheme [19]. Our proposed method can reduce the overall hardware cost which is more preferable in practice. Furthermore, our control approach is free of the global coordinate frame due to the fact that the angle we used is the included angle with respect to its neighbors.

This paper is organized as follows. Section II gives some notations and formulates the problem. In Section III, the fully distributed control law is presented. Then the global stability

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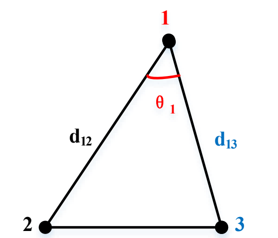

Fig. 1. A triangular formation.

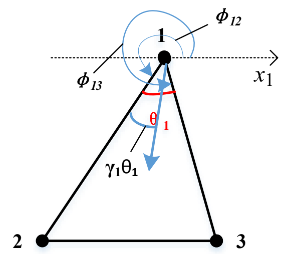

Fig. 2. The motion direction and some notions of agent 1.

analysis is provided. We then discuss some extensions to more general formation shapes and give the global quadrilateral formation with 4 agents as an example in Section IV. Section V presents simulation results to validate our theoretical results. Section VI concludes the paper.

II. NOTATION AND PROBLEM FORMULATION

A. Notation

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists a vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. A configuration in $\mathbb{R}^2$ is denoted by $p = [p_1^{\mathrm{T}}, \cdots, p_n^{\mathrm{T}}]^{\mathrm{T}}$ where $p_i \in \mathbb{R}^2 (i = 1, 2, \cdots, n)$. The neighbor set $\mathcal{N}_i \in \mathcal{V}$ represents the set of agents connected to agent i directly by an undirected edge. Agent j and agent k are two neighbor of agent i, namely $j, k \in \mathcal{N}_i$ and $j \neq k$.

The formation shape as well as its scale is characterized by the interior angle of agent i, θ_i and the distances between i and j, $d_{ij} = ||p_i - p_j|| (\forall j \in \mathcal{N}_i)$. Agent i can measure either the distance d_{ij} or the bearing $\phi_{ij} \in [0, -2\pi), \forall j \in \mathcal{N}_i$, defined as positive (counter-clockwise) or negative (clockwise) pointing from its local x-axis to agent j.

By introducing an auxiliary angle

$$\sigma_i = |\phi_{ij} - \phi_{ik}| \in [0, 2\pi), \tag{1}$$

the interior angle θ_i is given by

$$\theta_i = \begin{cases} \sigma_i & \text{if} \quad \sigma_i \leqslant \pi \\ 2\pi - \sigma_i & \text{otherwise} \end{cases}$$

with $\theta_i \in [0, \pi]$. When $\theta_i = 0$ or $\theta_i = \pi$, it means the three agents are collinear.

Define the desired distance between i and j by d_{ij}^* with $0 < d_{ij}^* < \infty$. The desired interior/included angle associated with agent i, denoted by θ_i^*, obeys the geometric relationship that the interior angles of a triangle add up to exactly π, namely $\theta_1^* + \theta_2^* + \theta_3^* = \pi$. Let the set of agents with distance constraints be $\mathcal{V}_D$ and the set of agents with bearing constraints $\mathcal{V}_B$. Note that $\mathcal{V} = \mathcal{V}_D \cup \mathcal{V}_B$ and we assume $\mathcal{V}_D \cap \mathcal{V}_B = \emptyset$ in this paper.

B. Problem Formulation

We consider a formation consisting of three agents ($n = 3$) whose positions are denoted by $p_1, p_2, p_3 \in \mathbb{R}^2$. A configuration is denoted as $p = [p_1^{\mathrm{T}}, p_2^{\mathrm{T}}, p_3^{\mathrm{T}}]^{\mathrm{T}}$. Assume that each agent can measure only one quantity (distance or bearing). Without loss of generality, in this paper, the agent, labelled as 1, is assumed to measure the bearing constraint θ_1 governed by the dynamics

$$\dot{p}_1 = u_1 \begin{bmatrix} \cos\beta_1 \\ \sin\beta_1 \end{bmatrix} \tag{2}$$

where u_1 and β_1 are both the control inputs for agent 1 to be determined. The two agents labelled as $i \in \{2, 3\}$, measuring the distances d_{1i}, are governed by the single-integrator dynamics

$$\dot{p}_i = u_i, \ \ i \in \{2, 3\}, \tag{3}$$

where $u_i \in \mathbb{R}^2$ is the control input for agent i to be designed.

The problem of triangular formation control with combined distance and bearing constraints is that for a framework $(\mathcal{G}, p)$, design a distributed control law (2) and (3) for each agent to reach the desired triangular formation globally. The desired formation are determined by the distances d_{12}^*, d_{12}^* and the angle θ_1^*.

We make the following assumptions.

Assumption 2.1: No two agents are initially coincident and the equation $\sum \theta_i^* = \pi (i = 1, 2, 3)$ holds. The case where $\theta_i^* = 0, \theta_j^* \neq 0$ and $\theta_k^* = \pi - \theta_j^*$ ($j, k \in \mathcal{N}_i$ and $j \neq k$) is excluded.

Assumption 2.2: The set of desired distances and angles associated with the desired formation are realizable.

The solution to this problem will be especially meaningful when one of the agents only has the capability of measuring the distance (angle) while the others can measure the angles (distances) in some practical applications. This control scheme enjoys the advantage that the desired formation can be achieved as long as each agent realizes its own control task. In addition, agents do not have to share a common coordinate frame and instead the measurements can be obtained in a fully local manner.

III. TRIANGULAR FORMATION CONTROL

In this section, we consider the triangular formation shown in Fig.1 as a representative. In this setup agent 1 measures the angle θ_1, agent 2 and 3 monitor the distance information d_{12} and d_{13} respectively.

A. Control Law Design

Agent 1 is governed by the dynamics (2), where β_1 is measured counter-clockwise from agent 1's local x-direction, defined by

$$\beta_1 = \begin{cases} \theta_1\gamma_1 + \frac{\pi}{2}, & \text{if} \quad \sigma_1 = 0 \\ \theta_1\gamma_1 + \min(\phi_{12}, \phi_{13}), & \text{if} \quad 0 < \sigma_1 \leqslant \pi \\ \theta_1\gamma_1 + \max(\phi_{12}, \phi_{13}), & \text{if} \quad \pi < \sigma_1 < 2\pi \end{cases} \tag{4}$$

where σ_1 is defined in (1) and $0 < \gamma_1 < 1$. Here $\sigma_1\gamma_1$ represents the motion direction of agent 1. γ_1 is allowed to be a function of time and it can be selected at run-time by agent 1. One illustrative example is given in Fig. 2.

We define the control input, u_1 as follows

$$u_1 = -k_1(\theta_1 - \theta_1^*) \tag{5}$$

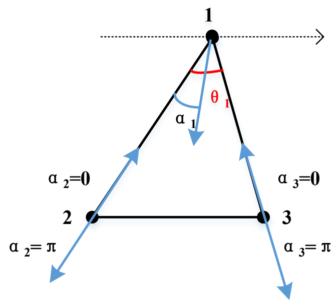

Fig. 3. The motion directions of agent i ($i \in \{1, 2, 3\}$).

where $k_1 > 0$ is a constant scalar. Denote by the angle stabilization error $e_1 = \theta_1 - \theta_1^*$. Agents 2 and 3 are governed by (3) and the control input u_i is given by

$$u_i = -k_i(p_i - p_1)\frac{||(p_i - p_1)||^2 - d_{1i}^{*2}}{||(p_i - p_1)||^2}. \tag{6}$$

When all agents keep moving, the dynamics of θ_1 can be given according to [14]

$$\dot{\theta}_1 = g_1 \cdot (\theta_1^* - \theta_1) - f_{12} \cdot (\theta_2^* - \theta_2) - f_{13} \cdot (\theta_3^* - \theta_3). \tag{7}$$

Let us denote $\dot{\hat{\theta}}_i$ ($i \in 1, 2, 3$) as the dynamics of the interior angle associated with agent i when the other two agents of the triangular are static. In (7),

$$\dot{\hat{\theta}}_1 = g_1 \cdot (\theta_1^* - \theta_1),$$

Since $\sum_{i=1}^{3} \dot{\hat{\theta}}_i = 0$ and $\dot{\hat{\theta}}_2$ and $\dot{\hat{\theta}}_3$ are given as follows by using the formula for the angular velocity in terms of the cross radial component of the velocity of agent i,

$$\dot{\hat{\theta}}_2 = -\frac{k_1}{d_{12}}\text{sin}(\alpha_1)(\theta_1^* - \theta_1),$$

$$\dot{\hat{\theta}}_3 = -\frac{k_1}{d_{13}}\text{sin}(\theta_1 - \alpha_1)(\theta_1^* - \theta_1),$$

where $\alpha_1 = \theta_1\gamma_1$, one can get the derivative of $\hat{\theta}_1$

$$\begin{aligned}\dot{\hat{\theta}}_1 &= \frac{k_1}{d_{12}}\text{sin}(\alpha_1)(\theta_1^* - \theta_1) + \frac{k_1}{d_{13}}\text{sin}(\theta_1 - \alpha_1)(\theta_1^* - \theta_1) \\ &= \frac{k_1(d_{12}\text{sin}(\theta_1 - \alpha_1) + d_{13}\text{sin}(\alpha_1))}{d_{12}d_{13}}(\theta_1^* - \theta_1).\end{aligned} \tag{8}$$

Then g_1 in (7) is given by

$$g_1 = k_1\frac{d_{12}\text{sin}(\theta_1 - \alpha_1) + d_{13}\text{sin}(\alpha_1)}{d_{12}d_{13}}. \tag{9}$$

It is known that $g_1 \geqslant 0$ as all of parameters in (9) are positive. To better illustrate the angle evolution, we present an example shown in Fig. 3. In (7), $-f_{1i} \cdot (\theta_i^* - \theta_i)$ with $i \in \{2, 3\}$ means the derivative of $\hat{\theta}_i$ when agent 1 and agent $i+1$ or $i-1$ are static, namely its angular velocity is zero. So f_{12} and f_{13} are defined as follows

$$f_{12} = \frac{1}{d_{12}}\text{sin}(\alpha_2), \tag{10}$$

$$f_{13} = \frac{1}{d_{13}}\text{sin}(\alpha_3), \tag{11}$$

where $\alpha_i (i \in \{2, 3\})$ denotes the angle i subtended from the movement direction of agent i to the edge $(1, i)$.

When agent 1 and agent 3 are static, the motion of agent 2 is along the direction of ϕ_{12} according to (6), the control law of agent 2. It is easy to see that $\alpha_2 = 0$ or $\alpha_2 = \pi$. Then $f_{12} = 0$. Similarly, we have $f_{13} = 0$. So (7) finally reduces to

$$\dot{\theta}_1 = -g_1 \cdot (\theta_1 - \theta_1^*). \tag{12}$$

B. Stability Analysis

In this section, we focus on the stability analysis of the overall system defined by (2) and (3).

The dynamics of the overall system is given by

$$\begin{cases} \dot{p}_1 = -k_1(\theta_1 - \theta_1^*)\begin{bmatrix}\cos\beta_1 \\ \sin\beta_1\end{bmatrix}, \\ \dot{p}_2 = -k_2(p_2 - p_1)\dfrac{||(p_2 - p_1)||^2 - d_{12}^{*2}}{||(p_2 - p_1)||^2}, \\ \dot{p}_3 = -k_3(p_3 - p_1)\dfrac{||(p_3 - p_1)||^2 - d_{13}^{*2}}{||(p_3 - p_1)||^2}. \end{cases} \tag{13}$$

where β_1 is given in (4).

In order to facilitate the stability analysis, we introduce the error $e \in \mathbb{R}^3$ as follows

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \theta_1 - \theta_1^* \\ ||p_2 - p_1||^2 - d_{12}^{*2} \\ ||p_3 - p_1||^2 - d_{13}^{*2} \end{bmatrix}. \tag{14}$$

Each $e_i \to 0$ ($i \in \{1, 2, 3\}$) implies that the distances and angle converge to their corresponding desired values. Note that the two distances of two sides, d_{12} and d_{13}, and the magnitude of their included angle θ_1 can uniquely determine a triangle. So the system (13) converges to a desired triangular formation asymptotically when each $e_i \to 0$ ($i \in \{1, 2, 3\}$).

The dynamics of e_1 is given by

$$\dot{e}_1 = \dot{\theta}_1 = -g_1 \cdot (\theta_1 - \theta_1^*) = -g_1 e_1. \tag{15}$$

The dynamics of e_i ($i \in \{2, 3\}$) are

$$\begin{aligned}\dot{e}_i =& 2(p_i - p_1)^{\mathrm{T}}(\dot{p}_i - \dot{p}_1) \\ =& -2k_i e_i + 2k_1(p_i - p_1)^{\mathrm{T}}\begin{bmatrix}\cos\beta_1 \\ \sin\beta_1\end{bmatrix}e_1.\end{aligned} \tag{16}$$

Define an auxiliary variable

$$h_{1i} = 2k_1(p_i - p_1)^{\mathrm{T}}\begin{bmatrix}\cos\beta_1 \\ \sin\beta_1\end{bmatrix}.$$

In view of (15) and (16), the dynamics of e can be written as

$$\dot{e} = \begin{bmatrix} -g_1 & 0 & 0 \\ h_{12} & -2k_2 & 0 \\ h_{13} & 0 & -2k_3 \end{bmatrix} e. \tag{17}$$

Now we analyze the equilibrium points of the overall system. From Assumptions 2.1 and 2.2, it is obvious that the equilibrium of the overall system (13) corresponds to those values of the p_i ($i \in \{1, 2, 3\}$) when $e_i = 0$ ($i \in \{1, 2, 3\}$

). Namely the equilibrium set of (13) is

$$\zeta = \{p : \theta_1 = \theta_1^*, d_{12} = d_{12}^*, d_{13} = d_{13}^*\}. \qquad (18)$$

Before presenting the main results, we introduce the following lemma.

Lemma 3.1: For system (13) with three agents, the distances between agent i ($i = \{2, 3\}$) and agent 1, $d_{1i} = ||p_i - p_1||$ are bounded.

Proof: We consider the following candidate Lyapunov function V_1 for agent 1:

$$V_1 = \frac{1}{2}k_1^2(\theta_1 - \theta_1^*)^2. \qquad (19)$$

The derivative of V_1 is

$$\dot{V}_1 = -k_1^2(\theta_1^* - \theta_1)\dot{\theta}_1 = -k_1^2 g_1(\theta_1^* - \theta_1)^2 \leqslant 0.$$

The inequality is obtained based on the fact that $g_1 \geqslant 0$. Since $V_1 \geqslant 0$ and $\dot{V}_1 \leqslant 0$, we have the conclusion that V_1 is bounded, namely there is a positive constant c so that $V_1 \leqslant c$ holds.

For agents 2 and 3, the candidate Lyapunov function V_i with $i \in \{2,3\}$ is given by

$$V_i = \frac{1}{2}k_i||p_i - p_1||^2. \qquad (20)$$

From (20), it can be seen that $V_i \geqslant 0$ and the derivative of V_i is

$$\begin{aligned}\dot{V}_i =& k_i(p_i - p_1)^{\mathrm{T}}(\dot{p}_i - \dot{p}_1)\\ =& k_i(p_i - p_1)^{\mathrm{T}}(-k_i(p_i - p_1)\frac{(||p_i - p_1||^2 - d_{1i}^{*2})}{||p_i - p_1||^2} +\\ & k_1(\theta_1 - \theta_1^*)\begin{bmatrix}\cos\beta_1\\ \sin\beta_1\end{bmatrix})\\ =& -k_i^2||p_i - p_1||^2 + k_i^2 d_{1i}^{*2} + k_1 k_i(p_i - p_1)^{\mathrm{T}}(\theta_1 - \theta_1^*)\cdot\\ & \begin{bmatrix}\cos\beta_1\\ \sin\beta_1\end{bmatrix}.\end{aligned} \qquad (21)$$

Due to the inequality condition $XY \leqslant \frac{1}{2\epsilon}X^2 + \frac{\epsilon}{2}Y^2$, $\dot{V}_i$ satisfies

$$\begin{aligned}\dot{V}_i \leqslant& -k_i^2||p_i - p_1||^2 + \frac{1}{2\epsilon_i}k_i^2||p_i - p_1||^2 + \frac{\epsilon_i}{2}k_1^2(\theta_1 - \theta_1^*)^2\\ & + k_i^2 d_{1i}^{*2}\\ =& -k_i^2(1 - \frac{1}{2\epsilon_i})||p_i - p_1||^2 + \frac{\epsilon_i}{2}k_1^2(\theta_1 - \theta_1^*)^2 + k_i^2 d_{1i}^{*2}\\ =& -2k_i(1 - \frac{1}{2\epsilon_i})V_i + \epsilon_i V_1 + k_i^2 d_{1i}^{*2}\\ \leqslant& -2k_i(1 - \frac{1}{2\epsilon_i})V_i + \epsilon_i c + k_i^2 d_{1i}^{*2}.\end{aligned} \qquad (22)$$

Let $\delta_i = 2k_i(1 - \frac{1}{2\epsilon_i})$ and $\rho_i = \epsilon_i c + k_i^2 d_{1i}^{*2}$. It follows that

$$\dot{V}_i \leqslant -\delta_i V_i + \rho_i. \qquad (23)$$

By solving the differential inequality (23), we get

$$V_i \leqslant V_i(0)e^{-\delta_i t} + \frac{\rho_i}{\delta_i}(1 - e^{-\delta_i t}) \leqslant \frac{\rho_i}{\delta_i}. \qquad (24)$$

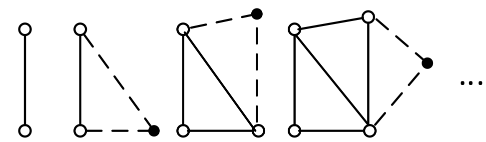

Fig. 4. Constructing a polygon based on vertex 2-addition.

It is known that V_i is bounded if $\delta_i > 0$ and $\rho_i > 0$. One can choose $\epsilon_i > \frac{1}{2}$ to ensure $\delta_i > 0$. So $d_{1i}^2 = ||p_i - p_1||^2$ is bounded because of the boundedness of V_i. Then the distance d_{1i} will be bounded. ■

Theorem 3.1: Suppose Assumptions 2.1 and 2.2 hold. Under the control law (13), the three-agent system converges to the desired triangular formation globally.

Proof: It follows from (15) that the angle error e_1 converges to 0 exponentially by taking into consideration the fact that g_1 is always positive. Based on Lemma 3.1, the distances d_{12} and d_{13} are bounded.Consider the following equation that

$$\begin{aligned}d_{12}^2 =& ||p_2 - p_1||^2\\ =& ((p_2 - p_1)^{\mathrm{T}}\begin{bmatrix}\cos\beta_1\\ \sin\beta_1\end{bmatrix})^2\\ =& h_{12}^2/k_1^2.\end{aligned}$$

Then we know h_{12} is also bounded. Similarly, we know that h_{13} is bounded as well. In light of (16), one can conclude from the input-to-state stable property $h_{12}e_1 \to 0$ as $t \to \infty$ by combining the boundedness of h_{12}. So $e_2 \to 0$ as $t \to \infty$. Similarly, we have the conclusion that $e_3 \to 0$ as $t \to \infty$. Then all e_i ($i \in \{1,2,3\}$) converge to 0, namely the states of the system converge to the equilibrium set (18) and the system achieves the desired triangular formation globally. ■

IV. DISCUSSION ON MORE GENERAL FORMATION SHAPES

In this section, we consider a general system with n agents aiming to give some indication on how to extend our proposed control methods for triangular formations to more general polygons in the plane.

Suppose we are given a desired planar formation shape, which can be globally determined by a set of angle and distance constraints. To properly construct the growing steps from a triangle to an arbitrary polygon. First, we present the definition of one type of Henneberg constructions [20].

Definition 4.1: Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a *vertex 2-addition of* k is the addition of one new vertex, k, and two new edges, (k, i) and (k, j) with $\{i, j\} \in \mathcal{E}$, creating the new graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$.

Given an arbitrary set of nodes, one can always construct a new graph based on vertex 2-addition connecting all the nodes, shown in Fig. 4. When a new vertex is added, one distance and one angle constraints need to be introduced so that the new triangle can be fixed based on Theorem 3.1. Starting from a stabilized triangle, the new triangles can be stabilized in sequence.

We give an example of a quadrangle presented in Fig. 5 obtained by adding a new vertex 4 on the basis of the triangle

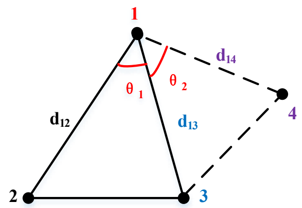

Fig. 5. An example of a quadrilateral formation.

in Fig. 1. The system is given by

$$\begin{cases} \dot{p}_1 = -(k_{11}(\theta_1 - \theta_1^*) + k_{12}(\theta_2 - \theta_2^*)) \begin{bmatrix} \cos\beta_1 \\ \sin\beta_1 \end{bmatrix}, \\ \dot{p}_i = -k_i(p_i - p_1)\dfrac{\|(p_i - p_1)\|^2 - d_{1i}^{*2}}{\|(p_i - p_1)\|^2} \quad i = \{2,3,4\}, \end{cases} \tag{25}$$

where β_1 is given by

$$\beta_1 = \begin{cases} (\theta_1 + \theta_2)\gamma_1 + \frac{\pi}{2} & \text{if } \sigma_1 + \sigma_2 = 0 \\ (\theta_1 + \theta_2)\gamma_1 + \min(\phi_{12},\phi_{14}) & \text{if } 0 < \sigma_1 + \sigma_2 \leqslant 2\pi \\ (\theta_1 + \theta_2)\gamma_1 + \max(\phi_{12},\phi_{14}) & \text{if } 2\pi < \sigma_1 + \sigma_2 < 4\pi, \end{cases} \tag{26}$$

and we choose $\gamma_1 = \frac{1}{2}$ for simplicity (γ_1 can be any scalar between 0 and 1).

Define the error $e \in \mathbb{R}^5$ as follows

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \begin{bmatrix} \theta_1 - \theta_1^* \\ \theta_2 - \theta_2^* \\ \|p_2 - p_1\|^2 - d_{12}^{*2} \\ \|p_3 - p_1\|^2 - d_{13}^{*2} \\ \|p_4 - p_1\|^2 - d_{14}^{*2} \end{bmatrix}. \tag{27}$$

Each $e_i \to 0$ ($i \in \{1, \cdots, 5\}$) implies that the measurable distances and angles go to their corresponding desired values. Based on Assumptions 2.1 and 2.2, the equilibrium set of (25) is

$$\zeta = \{p : \theta_1 = \theta_1^*, \theta_2 = \theta_2^*, d_{12} = d_{12}^*, d_{13} = d_{13}^*, d_{14} = d_{14}^*\}. \tag{28}$$

The dynamics of e can be written as

$$\dot{e} = \begin{bmatrix} -g_1 & 0 & 0 & 0 & 0 \\ 0 & -g_2 & 0 & 0 & 0 \\ h_{21} & h_{22} & -2k_2 & 0 & 0 \\ h_{31} & h_{32} & 0 & -2k_3 & 0 \\ h_{41} & h_{42} & 0 & 0 & -2k_4 \end{bmatrix} e, \tag{29}$$

where g_1 and g_2 are given respectively by

$$g_1 = k_{11}\frac{d_{12}\sin(\frac{\theta_1+\theta_2}{2}) + d_{13}\sin(|\frac{\theta_1-\theta_2}{2}|)}{d_{12}d_{13}},$$

$$g_2 = k_{12}\frac{d_{13}\sin(|\frac{\theta_1-\theta_2}{2}|) + d_{14}\sin(\frac{\theta_1+\theta_2}{2})}{d_{13}d_{14}},$$

and

$$h_{ij} = 2k_{1j}(p_i - p_1)^{\mathrm{T}} \begin{bmatrix} \cos\beta_1 \\ \sin\beta_1 \end{bmatrix},$$

for $i \in \{2,3,4\}$ and $j \in \{1,2\}$. Since $\theta_1, \theta_2 \in [0,\pi]$ and both k_{11} and k_{12} are positive, we have $g_1 \geqslant 0$ and $g_2 \geqslant 0$.

It is easy to know that $e_1 \to 0$ and $e_2 \to 0$. And according to Lemma 3.1, we can get the boundiness of h_{ij}. So we can conclude that $e_i \to 0$ for $i \in \{3,4,5\}$ as $t \to \infty$. As we know, the two distances of two sides, d_{12} and d_{13}, and the angle θ_1 between the two sides can form a unique triangle. A quadrangle can thus be uniquely determined when we add two extra constraints, the distance d_{14} and the angle θ_2. So the system (25) converges to a desired quadrilateral formation globally when each $e_i \to 0$ ($i \in \{1, \cdots, 5\}$).

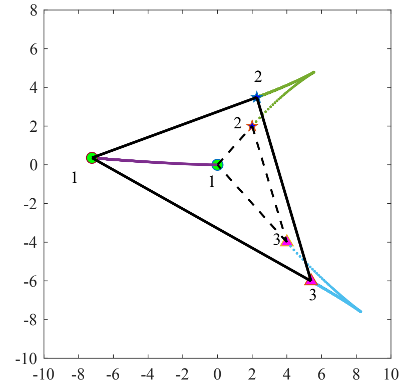

Fig. 6. The motion of the triangular formation with three non-collinear initial positions.

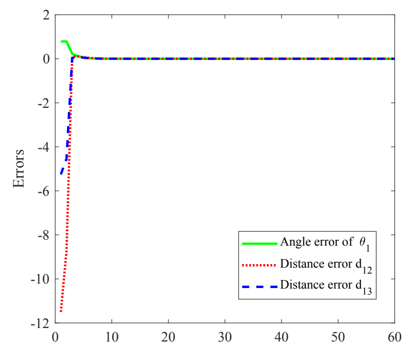

Fig. 7. The errors with desired constraints.

V. SIMULATION RESULTS

In this section, we present some simulations to validate our theoretical algorithms for distributed triangular formation control with a mixture of distance and bearing constraints.

Case 1: Three agents are not initially collinear.

The first example illustrates how the formation converges to a desired triangular formation given some random initial positions of agents 1, 2 and 3. The desired distances are set as $d_{12}^* = 10$ and $d_{13}^* = 10\sqrt{2}$ and the desired bearing is $\theta_1^* = \frac{\pi}{4}$. Fig. 6 shows the evolution of the triangular formation including the initial triangle and the geometric shape. In Fig. 7, the errors of each agent go to zero, which means that the desired formation is achieved. This simulation verifies that the proposed control laws can stabilize any initial non-degenerated triangles to desired formation shapes.

Case 2: Three agents are collinear.

Now we consider the case that three agents are initially collinear. Let the initial positions of agent 1, 2 and 3 be $p_{10} = [2, 2]^T$, $p_{20} = [1, 1]^T$ and $p_{30} = [3, 3]^T$ respectively.

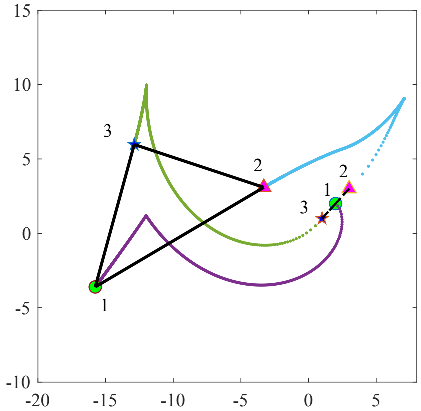

Fig. 8. The motion of the triangular formation with three collinear initial positions.

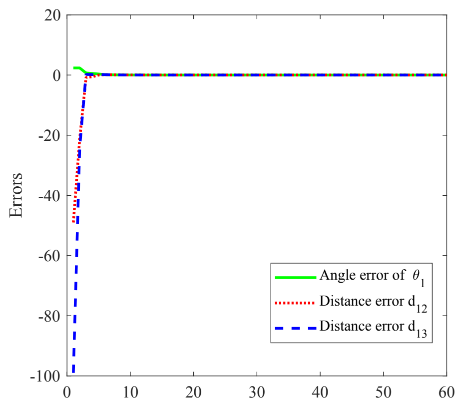

Fig. 9. The errors with desired constraints.

The desired distances are $d_{12}^* = 10$ and $d_{13}^* = 10\sqrt{2}$ and the desired bearing $\theta_1^* = \frac{\pi}{4}$. Fig. 8 shows the trajectory of the triangular formation from the colinear initial position to the stabilized position. The convergence of the errors is illustrated in Fig. 9. It can be seen from the simulation results that the desired formation can be achieved using our proposed control laws even if the agents are initially collinear.

VI. CONCLUSIONS

In this paper, we have investigated the problem of global stabilization for a triangular formation with a mixture of distance and bearing constraints. The key contribution in this paper is that we have realized the desired triangular formation in the sense of global stability using our proposed fully distributed control algorithms, in which each agent only needs to sense and control one variable. Compared with [19], the proposed control strategy enables the agents to measure less variables, which broadens its feasibility in practical applications. This is further enhanced by the property that in our proposed control strategy, agents are not required to share the global coordinate frame. One future work is to accomplish the triangular formation control with the alternative constraints, i.e., one distance and two angle constraints such that a more general global convergence result for triangular formation control can be established. We are also interested in extending our results to large-size systems with more than three agents.

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