A Measure Preserving Mapping for Structured Grassmannian Constellations in SIMO Channels

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***Abstract*—In this paper, we propose a new structured Grassmannian constellation for noncoherent communications over single-input multiple-output (SIMO) Rayleigh block-fading channels. The constellation, which we call Grass-Lattice, is based on a measure preserving mapping from the unit hypercube to the Grassmannian of lines. The constellation structure allows for on-the-fly symbol generation, low-complexity decoding, and simple bit-to-symbol Gray coding. Simulation results show that Grass-Lattice has symbol error rate performance close to that of a numerically optimized unstructured constellation, and is more power efficient than other structured constellations proposed in the literature.**

***Index Terms*—Noncoherent communications, Grassmannian constellations, SIMO channels, measure-preserving mapping.**

I. Introduction

In communications over fading channels, it is usually assumed that the channel state information (CSI) is typically estimated at the receiver side by sending a few known pilots and then used for decoding at the receiver and/or for precoding at the transmitter. These are known as coherent schemes. However, in fast fading scenarios or massive MIMO systems for ultra-reliable low-latency communications (URLLC), to obtain an accurate channel estimate would require pilots to occupy a disproportionate fraction of communication resources. These new scenarios that have emerged with 5G and B5G systems motivate the use of noncoherent communications schemes in which neither the transmitter nor the receiver have any knowledge about the instantaneous CSI.

Despite the receiver not having CSI, a large fraction of the coherent capacity can be achieved in noncoherent communication systems when the signal-to-noise ratio (SNR) is high, as shown in [1]–[4]. For the case of single-input multiple-output (SIMO) channels, which is the one we focus on in this paper, these works proved that at high SNR under additive Gaussian noise, assuming a Rayleigh block-fading SIMO channel with coherence time $T \geq 2$ symbol periods, the optimal strategy achieving the capacity is to transmit isotropically distributed unitary vectors belonging to the Grassmannian of lines.

An extensive research has been conducted on the design of noncoherent constellations as optimal packings on the Grassmann manifold [5]–[18]. Some experimental evaluation of Grassmannian constellations in noncoherent communications using over-the-air transmission has been reported in [19]. Existing constellation designs can be generically categorized into two groups: structured or unstructured. Among the unstructured designs we can mention the alternating projection method [6], the numerical methods in [7]–[10], which optimize certain distance measures on the Grassmannian (e.g., chordal or spectral), and the methods proposed in [11] and [12], which maximize the so-called diversity product [20].

On the other side, structured designs impose some kind of structure on the constellation points, facilitating low complexity constellation mapping and demapping. This is achieved through algebraic constructions such as the Fourier-based constellation in [13] or the analog subspace codes recently proposed in [14], designs based on group representations [15], [16], parameterized mappings of unitary matrices such as the Exp-Map design in [17] or structured partitions of the Grassmannian like the recently proposed Cube-Split constellation [18]. The Cube-Split constellation is of particular interest for this work as it is the design most related to our proposal. Cube-Split is based on a mapping from the unit hypercube to the Grassmann manifold such that the constellation points are distributed approximately uniformly on the Grassmannian. However, the Cube-Split mapping only achieves uniformly distributed points for $T = 2$. When $T > 2$, Cube-Split ignores the statistical dependencies between the components of the codewords and applies the same mapping derived for $T = 2$. These limitations are overcome with our proposed mapping, named Grass-Lattice, which is a *measure preserving mapping* between the unit hypercube and the Grassmannian for any value of $T \geq 2$. The fact that the Grass-Lattice mapping is measure preserving guarantees that any set of points uniformly distributed in the input space (the hypercube), is mapped onto another set of points or codewords uniformly distributed in

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the output space (the Grassmann manifold). The constellation structure allows for on-the-fly symbol generation, low-complexity decoding, and simple bit-to-symbol Gray coding.

Notation: Matrices are denoted by bold-faced upper case letters, column vectors are denoted by bold-faced lower case letters, and scalars are denoted by light-faced lower case letters. The Euclidean norm is denoted by $\|v\|$ and j denotes the imaginary unit. The superscripts $(\cdot)^{\mathrm{T}}$ and $(\cdot)^{\mathrm{H}}$ denote transpose and Hermitian conjugate, respectively. We denote by $\mathbf{I}_n$ the identity matrix of size n. $\mathcal{CN}(0,1)$ denotes a complex proper Gaussian distribution with zero mean and unit variance, $\mathbf{x} \sim \mathcal{CN}_n(\mathbf{0},\mathbf{R})$ denotes a complex Gaussian vector in $\mathbb{C}^n$ with zero mean and covariance matrix $\mathbf{R}$. For real variables we use $\mathbf{x} \sim \mathcal{N}_n(\mathbf{0},\mathbf{R})$. The Grassmannian of lines $\mathbb{G}\left(1,\mathbb{C}^T\right)$, also called the complex projective space, is the space of one-dimensional subspaces in $\mathbb{C}^T$. Points in $\mathbb{G}\left(1,\mathbb{C}^T\right)$ are denoted as $[\mathbf{x}]$.

II. System Model

We consider a noncoherent SIMO communication system where a single-antenna transmitter sends information to a receiver equipped with N antennas over a frequency-flat block-fading channel with coherence time T symbol periods. It is assumed that $T \geq 2$. Hence, the channel vector $\mathbf{h} \in \mathbb{C}^N$ stays constant during each coherence block of T symbols, and changes in the next block to an independent realization. The SIMO channel is assumed to be Rayleigh with no correlation at the receiver, i.e., $\mathbf{h} \sim \mathcal{CN}\left(0,\mathbf{I}_N\right)$, and unknown to both the transmitter and the receiver.

Within a coherence block the transmitter sends a signal $\mathbf{x} \in \mathbb{C}^T$, normalized as $\mathbf{x}^H\mathbf{x} = 1$, that is a unitary basis for the one-dimensional subspace $[\mathbf{x}]$ in $\mathbb{G}\left(1,\mathbb{C}^T\right)$. The signal at the receiver $\mathbf{Y} \in \mathbb{C}^{T\times N}$ is

$$\mathbf{Y} = \mathbf{x}\mathbf{h}^{\mathrm{T}} + \sqrt{\frac{1}{T\rho}}\mathbf{W}, \qquad (1)$$

where $\mathbf{W} \in \mathbb{C}^{T\times N}$ represents the additive Gaussian noise, with entries modeled as $w_{ij} \sim \mathcal{CN}\left(0,1\right)$, and ρ represents the signal-to-noise-ratio (SNR).

In a noiseless situation, Grassmannian signaling guarantees error-free detection without CSI because $\mathbf{x}$ and the noise-free vector on a receive antenna $\mathbf{y} = \mathbf{x}h$ represent the same point in $\mathbb{G}(1,\mathbb{C}^T)$.

For unstructured Grasmmannian codebooks, the optimal Maximum Likelihood (ML) detector (assuming equiprobable codewords) is given by

$$\tilde{\mathbf{x}} = \arg\max_{\mathbf{x}\in\mathcal{C}} \|\mathbf{Y}^{\mathrm{H}}\mathbf{x}\|^2, \qquad (2)$$

where $\mathcal{C}$ represents the codebook of K codewords. Each codeword carries $\log_2(K)$ bits of information.

The computational complexity of the ML detector increases with the number of codewords, K, since it is necessary to project the observation matrix onto each and every codeword. This is one of the main drawbacks of unstructured Grassmannian constellations especially when K is high. Another drawback of unstructured codes is how to solve the bit labeling problem, for which there are generally only suboptimal or computationally intensive solutions. In the following section we present a structured Grassmannian constellation, called *Grass-Lattice*, which solves the two problems of unstructured constellations: it can be decoded efficiently with a computational cost that does not grow with K, and it allows for a Gray-like bit-to-symbol mapping function.

III. Grass-Lattice Constellation

A. Overview

The Grass-Lattice constellation for SIMO channels is based on a measure preserving mapping from the unit hypercube (product of the interval $(0,1)$ with itself $2(T-1)$ times) to the Grassmann manifold $\mathbb{G}\left(1,\mathbb{C}^T\right)$

$$\vartheta : \mathcal{I} = \underbrace{(0,1)\times\cdots\times(0,1)}_{2(T-1)\text{ times}} \rightarrow \mathbb{G}\left(1,\mathbb{C}^T\right),$$

where recall that $T-1$ is the complex dimension of $\mathbb{G}\left(1,\mathbb{C}^T\right)$. Elements in $\mathcal{I}$ are denoted by

$$(\mathbf{a},\mathbf{b}) = (a_1,\ldots,a_{T-1},b_1,\ldots,b_{T-1}), \quad a_k,b_k \in (0,1).$$

Given the mapping ϑ, if we choose a set of input points uniformly distributed in the unit hypercube, the outputs points will be uniformly distributed in $\mathbb{G}\left(1,\mathbb{C}^T\right)$. The goal is to design structured codebooks that can be efficiently encoded (no need to store the constellation) and decoded (the real and imaginary parts a_j, b_j can be decoded independently). To this end, we quantize the (0,1) interval with 2^B equispaced points, where $B \geq 1$ is the number of bits per real component, and generate a Grass-Lattice constellation with $|\mathcal{C}| = 2^{2(T-1)B}$ codewords. The rate of the code is $R = 2^{2(T-1)B}/\,T$ b/s/Hz.

The Grass-Lattice mapping is composed of three consecutive mappings $\vartheta = \vartheta_3 \circ \vartheta_2 \circ \vartheta_1$, which are described in the following subsections.

B. Mapping ϑ_1

Mapping ϑ_1 maps points uniformly distributed in the unit hypercube $\mathcal{I}$ to points normally distributed in $\mathbb{C}^{T-1}$. The idea is to apply component-wise the inverse transform sampling method, which takes uniform samples on $[0,1]$ and returns the inverse of the cumulative distribution function with the desired distribution. More formally, we have the following classic result that is presented without proof.

Lemma 1 *Let a_k, b_k be independent random variables uniformly distributed in $[0,1]$: $a_k \sim \mathcal{U}[0,1]$ and $b_k \sim \mathcal{U}[0,1]$, and let $z_k = F^{-1}(a_k) + jF^{-1}(b_k)$ where*

$$F(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-s^2}\,ds. \qquad (3)$$

Then, both $\Re(z_k) = F^{-1}(a_k)$ and $\Im(z_k) = F^{-1}(b_k)$ are independent Gaussian random variables that follow a $\mathcal{N}(0,1/2)$ distribution, and hence $z_k \sim \mathcal{CN}(0,1)$.

C. Mapping ϑ_2

In Lemma 2 we describe the mapping ϑ_2, which maps normally distributed points in $\mathbb{C}^{T-1}$ to points uniformly distributed in the unit ball

$$\mathbb{B}_{\mathbb{C}^{T-1}}(0,1) = \{\mathbf{w} \in \mathbb{C}^{T-1}, \|\mathbf{w}\| < 1\}.$$

Lemma 2 *Let $\mathbf{z} = (z_1, \ldots, z_{T-1})^{\mathrm{T}}$ be a $(T-1)$-dimensional Gaussian vector with i.i.d. components $z_k \sim \mathcal{CN}(0,1)$. Moreover, let*

$$\begin{aligned} f_{T-1}(t) &= \frac{1}{t}\left(\frac{2(T-1)}{\Gamma(T)}\int_0^t s^{2(T-1)-1}e^{-s^2}\,ds\right)^{1/(2(T-1))} \\ &= \frac{1}{t}\left(1 - e^{-t^2}\sum_{k=0}^{T-2}\frac{t^{2k}}{k!}\right)^{1/(2(T-1))} \end{aligned} \tag{4}$$

Then, the random vector $\mathbf{w} = \vartheta_2(\mathbf{z}) = \mathbf{z} f_{T-1}(\|\mathbf{z}\|)$ is uniformly distributed in the unit ball $\mathbb{B}_{\mathbb{C}^{T-1}}(0,1)$.

PROOF. The proof is given in Appendix A. □

D. Mapping ϑ_3

In Lemma 3 we present the mapping ϑ_3, which maps uniformly distributed points in the unit ball $\mathbb{B}_{\mathbb{C}^{T-1}}(0,1)$ to points uniformly distributed in $\mathbb{G}\left(1, \mathbb{C}^T\right)$.

Lemma 3 *The mapping*

$$\begin{aligned} \vartheta_3 : \quad \mathbf{w} \in \mathbb{B}_{\mathbb{C}^{T-1}}(0,1) \quad &\to \quad \mathbb{G}\left(1, \mathbb{C}^T\right) \\ \mathbf{w} \quad &\mapsto \quad \begin{bmatrix} \sqrt{1-\|\mathbf{w}\|^2} \\ \mathbf{w} \end{bmatrix} \end{aligned}$$

is measure preserving. So in order to generate a uniform random element $[\mathbf{x}]$ in $\mathbb{G}\left(1, \mathbb{C}^T\right)$, one may generate a random uniform element $\mathbf{w}$ in $\mathbb{B}_{\mathbb{C}^{T-1}}(0,1)$ and output $[\sqrt{1-\|\mathbf{w}\|^2}, \mathbf{w}^{\mathrm{T}}]^{\mathrm{T}}$.

PROOF. The proof is given in Appendix B. □

E. Main result

The following theorem summarizes the measure preserving Grass-Lattice mapping for SIMO channels.

Theorem 1 *Let us consider a noncoherent SIMO communication system with coherence time $T \geq 2$ and let $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_{T-1}, b_1, \ldots, b_{T-1})$ be any point in the unit hypercube $\mathcal{I}$. Let the mapping $\vartheta : \mathcal{I} \to \mathbb{G}\left(1, \mathbb{C}^T\right)$ be given by:*

$$\vartheta(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} \sqrt{1-\|\mathbf{w}\|^2} \\ \mathbf{w} \end{bmatrix}$$

where:

- *$\mathbf{w} = \mathbf{z} f_{T-1}(\|\mathbf{z}\|)$, where f_{T-1} is defined in (4).*
- *$\mathbf{z} = (z_1, \ldots, z_{T-1})^{\mathrm{T}}$ with $z_k = F^{-1}(a_k) + jF^{-1}(b_k)$, where $F(x)$ is given in (3).*

Then, ϑ has a constant Jacobian and thus it is measure preserving.

PROOF. The proof is given in Appendix C. □

IV. Encoding and Decoding

As the measure preserving map is defined on an open interval $(0,1)^{2(T-1)}$, for a given number B of bits per real component, we consider 2^B equispaced points on the interval $[\alpha, 1-\alpha]$:

$$\hat{x}_p = \alpha + p\,\frac{1-2\alpha}{2^B-1}, \quad 0 \leq p \leq 2^B - 1, \tag{5}$$

where α is a parameter that can be optimized for performance (see Sec. V). The discretization of the real and imaginary (I/Q) components as in (5) allows us to use a simple bit-to-symbol Gray mapper. Therefore, the uniformly distributed points on the unit cube $a_1, b_1, \ldots, a_{T-1}, b_{T-1}$ are chosen randomly from the regular lattice defined by (5). The procedure for computing the codeword to be transmitted $\mathbf{x}$ for an input $a_1, b_1, \ldots, a_{T-1}, b_{T-1}$ is then:

1) Compute $z_k = F^{-1}(a_k) + jF^{-1}(b_k)$, $k = 1, \ldots, T-1$, where $F(x)$ is the cdf of a $\mathcal{N}(0, 1/2)$. The point $\mathbf{z}$ is isotropically distributed as $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{T-1})$.
2) Compute $\mathbf{w} = \mathbf{z} f_{T-1}(\|\mathbf{z}\|)$, where $f_{T-1}(\cdot)$ is given in (4). The point $\mathbf{w}$ is uniformly distributed in $\mathbb{B}_{\mathbb{C}^{T-1}}(0,1)$.
3) Output $\mathbf{x} = [\sqrt{1-\|\mathbf{w}\|^2}, \mathbf{w}^{\mathrm{T}}]^{\mathrm{T}}$. The point $[\mathbf{x}]$ with representative $\mathbf{x}$ is uniformly distributed in $\mathbb{G}\left(1, \mathbb{C}^T\right)$.

The cardinality of the structured Grassmannian constellation is $|\mathcal{C}| = 2^{2B(T-1)}$, and the spectral efficiency or rate is $R = \frac{2B(T-1)}{T} = 2B\left(1 - \frac{1}{T}\right)$ b/s/Hz.

For the Grass-Lattice decoding, let us first consider the case where the number of receive antennas is $N = 1$, so the received $T \times 1$ signal is $\mathbf{y} = \mathbf{x}h + \mathbf{n}$. Let $\mathbf{y} = (v_0, \mathbf{v})$, then the decoder performs the following sequence of steps:

1) Compute $\mathbf{w} = \mathbf{v}|v_0|/(v_0\|\mathbf{y}\|)$ (the chordal distance from $[\sqrt{1-\|\mathbf{w}\|^2}, \mathbf{w}^{\mathrm{T}}]^{\mathrm{T}}$ to $\mathbf{y}$ in $\mathbb{G}(1, \mathbb{C}^T)$ is minimal for this choice of $\mathbf{w}$).
2) Solve the equation $s f_{T-1}(s) = \|\mathbf{w}\|$, for instance by bisection, and let $\mathbf{z} = s\mathbf{w}/\|\mathbf{w}\|$. Denote by $z_1, \ldots, z_{T-1}$ its complex components.
3) Compute $\hat{a}_k = F(\Re(z_k))$, $\hat{b}_k = F(\Im(z_k))$, where $F(x)$ is the cdf of a $\mathcal{N}(0, 1/2)$.
4) Finally, $a_k = \lfloor \hat{a}_k \rceil$ and $b_k = \lfloor \hat{b}_k \rceil$ where $\lfloor x \rceil$ denotes the nearest point to x in the lattice (5).

For $N > 1$, we just perform a denoising step at the decoder before doing steps 1-4 above. To do so, we use the fact that the signal of interest $\mathbf{x}\mathbf{h}^{\mathrm{T}}$ in (1) is a rank-1 component of $\mathbf{Y}$. From the Eckart-Young theorem, the best rank-1 approximation in the Frobenius norm of $\mathbf{Y}$ is given by $\lambda_1 \mathbf{r}\mathbf{g}^{\mathrm{H}}$, where λ_1 is the largest singular value of $\mathbf{Y}$, and $\mathbf{r}$ and $\mathbf{g}$ are the corresponding left and right singular vectors. We then take $\mathbf{r} = (v_0, \mathbf{v})$ as a denoised $T \times 1$ vector of observations and compute the sequence of steps 1-4 above. Interestingly, $\mathbf{r}$ is the solution of

$$\arg \max_{\mathbf{r} \in \mathbb{C}^T:\ \|r\|^2 = 1} \|\mathbf{Y}^{\mathrm{H}}\mathbf{r}\|^2,$$

so it can be viewed as a relaxed version of the ML decoder presented in (2) where the discrete nature of the constellation has been relaxed. Therefore, $\mathbf{r}$ is a rough estimate of the transmitted symbol $\mathbf{x}$ on the unit sphere.

The encoding and decoding for the Grass-Lattice constellation can be performed on the fly, without the need to store the entire constellation. At the decoder, after performing steps 1-4 above, the complexity is that of a symbol-by-symbol detector per real component, similar to the decoding of a QAM constellation.

V. Performance Evaluation

In this section, we assess the performance of the proposed Grass-Lattice constellation, and compare it to other Grassmannian constellations. Since we compare constellations with different spectral efficiencies, we will show figures of SER or BER versus E_b/N_0 (SNR normalized by the spectral efficiency). Let us first evaluate the influence of α, which determines the length of the lattice used for each real component in (5), on the SER (same influence as on the BER). Fig. 1 shows the SER variation for a fixed SNR = 20 dB, $T \in \{4, 6\}$, $N = 2$ and $B \in \{1, 2\}$. As we can see, α may have a significant impact on the SER performance. Further, the SER varies significantly with the number of bits, B, used to encode each real component. For the rest of experiments in this section, we will choose the value of α that provides the lowest SER at SNR = 20 dB (the optimal value of α does not differ significantly for other SNRs). This value is precomputed offline and then used throughout the entire simulation.

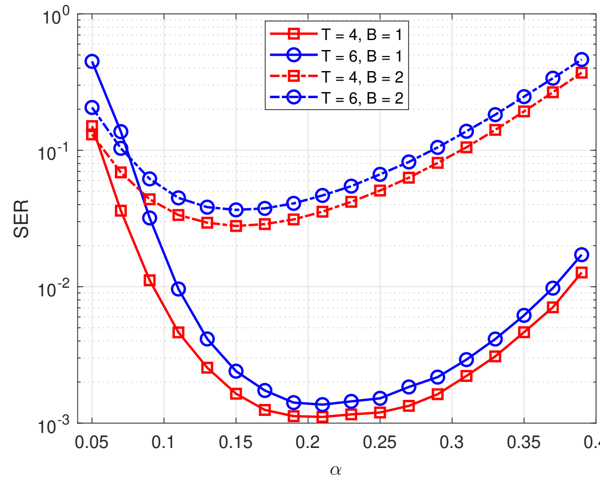

Fig. 1. SER as a function of α of the Grass-Lattice constellation for $T \in \{4, 6\}$, $N = 2$, $B \in \{1, 2\}$ and SNR = 20 dB.

Fig. 2 shows the SER as a function of E_b/N_0 for the proposed Grass-Lattice codebook for $T = 2$ symbol periods and $N = 1$ antenna. For comparison we include in the plot the structured Cube-Split [18] and Exp-Map [17] constellations, as well as the unstructured Grassmannian constellations proposed in [12] that minimize the asymptotic PEP union bound. For Grass-Lattice and Cube-Split we use $B \in \{2, 3\}$ bits per real component, while for UB-Opt and Exp-Map we choose constellations with the same spectral efficiency as the ones provided by Grass-Lattice. In Fig. 2 we can observe that Grass-Lattice outperforms the other structured constellations and it performs slightly worse than the unstructured UB-Opt constellation in terms of SER. Notice that UB-Opt uses the optimal ML detector in (2), whereas Grass-Lattice uses a suboptimal detector with much lower complexity.

Figs. 3 and 4 show the BER versus E_b/N_0 performance of Grass-Lattice constellations compared to Cube-Split and Exp-Map for $T = 2$, $N = 1$ and $B \in \{2, 3\}$ (Fig. 3) and $T = 4$, $N = 2$ and $B \in \{1, 2, 3\}$ (Fig. 4). For Grass-Lattice, we use a Gray encoding scheme that maps groups of B bits to I/Q symbols defined in (5). A Gray-like encoder is also used for Cube-Split and Exp-Map. Due to the lack of an optimal bit-to-symbol mapping for unstructured constellations, we omit UB-Opt in these figures. As we can see, Grass-Lattice constellations offer a superior performance in terms of BER than the other structured designs, which becomes more evident when the coherence time T is smaller.

Finally, Fig. 5 shows the spectral efficiency or rate in b/s/Hz against E_b/N_0 at BER=10^{-4} for different values of T and $N = 2$ for the Grass-Lattice and Cube-Split constellations. For given values of T and B, the spectral efficiency of the Grass-Lattice code is $\eta = (2B(T-1))\,/\,T$ and the spectral efficiency of Cube-split is given by $\eta = (\log_2 T + 2B(T-1))\,/\,T$. We notice from these two expressions that Cube-Split does not allow for a bit-to-symbol mapping when T is not a power of 2, so Grass-Lattice achieves a wider range of spectral efficiencies. For example, we can see in this figure that Grass-Lattice allows you to design constellations for $T \in \{3, 6, 14\}$. For values of $T \in \{2, 4, 8\}$, for which Grass-Lattice and Cube-Split constellations can be both designed, we see that Grass-Lattice is more power efficient than Cube-Split when T or B grows. This could be at least partially explained by the fact that Cube-Split ignores the statistical dependencies between the different components of the codeword $\mathbf{x}$ for $T > 2$.

VI. Conclusions

We have proposed a new Grassmannian constellation for noncoherent communications in SIMO channels based on a measure preserving mapping from the unit hypercube to the Grassmannian of lines. Thanks to its structure, the encoding and decoding steps can be performed on the fly with no need to store the whole constellation. Further, it allows for low-complexity and efficient decoding as well as for a simple Gray-like bit labeling. Simulation results show that this constellation outperforms other structured constellations in the literature in terms of SER and BER under Rayleigh block fading channels, in addition to being more power efficient. Further research will be done to study the extension of Grass-Lattice to the MIMO case.

Appendix

A. *Proof of Lemma 2*

Let us define $d = T - 1$. The function f_d is the unique solution of

$$f(t)^{2d-1}(f(t) + tf'(t)) = \frac{e^{-t^2}}{\Gamma(d+1)}, \quad \lim_{t\to\infty} tf(t) = 1,$$

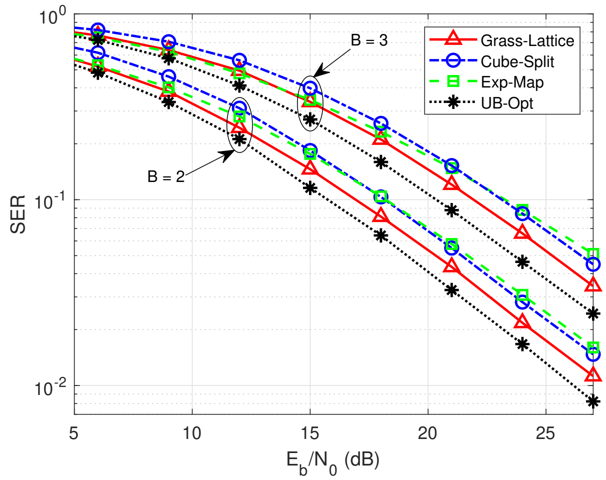

Fig. 2. Grass-Lattice SER curves in comparison with UB-Opt, Cube-Split and Exp-Map constellations for $T = 2$, $N = 1$ and $B \in \{2, 3\}$.

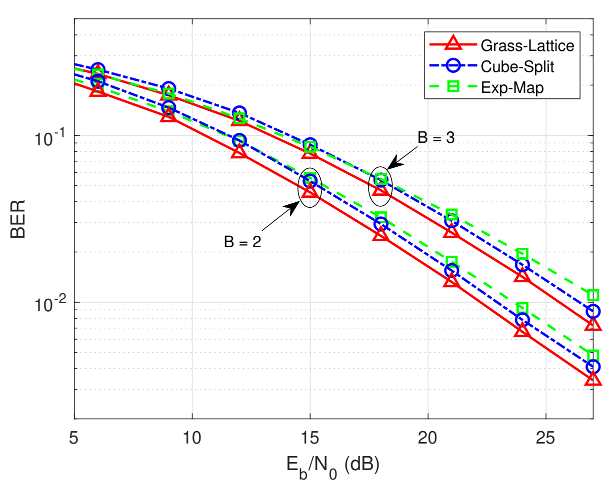

Fig. 3. Grass-Lattice BER curves in comparison with Cube-Split and Exp-Map constellations for $T = 2$, $N = 1$ and $B \in \{2, 3\}$.

which satisfies $tf(t) \in [0, 1)$ and can be written in terms of an incomplete Gamma function. It is easy to see that $\vartheta_2 : \mathbb{C}^d \to \mathbb{B}_{\mathbb{C}^d}(0, 1)$ is a diffeomorphism. Let us compute the Jacobian of ϑ_2: if $\dot{\mathbf{z}}$ is (real) orthogonal to $\mathbf{z}$ then

$$D\vartheta_2(\mathbf{z})\dot{\mathbf{z}} = \dot{\mathbf{z}} f_d(\|\mathbf{z}\|),$$

while for $\dot{\mathbf{z}} = \mathbf{z}/\|\mathbf{z}\|$ we have

$$\begin{aligned} D\vartheta_2(\mathbf{z})\frac{\mathbf{z}}{\|\mathbf{z}\|} &= \frac{\mathbf{z}}{\|\mathbf{z}\|} f_d(\|\mathbf{z}\|) + \mathbf{z} f'_d(\|\mathbf{z}\|) \\ &= \mathbf{z}\left(\frac{f_d(\|\mathbf{z}\|)}{\|\mathbf{z}\|} + f'_d(\|\mathbf{z}\|)\right). \end{aligned}$$

For any orthonormal basis of $\mathbb{C}^d \equiv \mathbb{R}^{2d}$ whose last vector is $\mathbf{z}/\|\mathbf{z}\|$ we have that the image by $D\vartheta_2$ is an orthogonal basis.

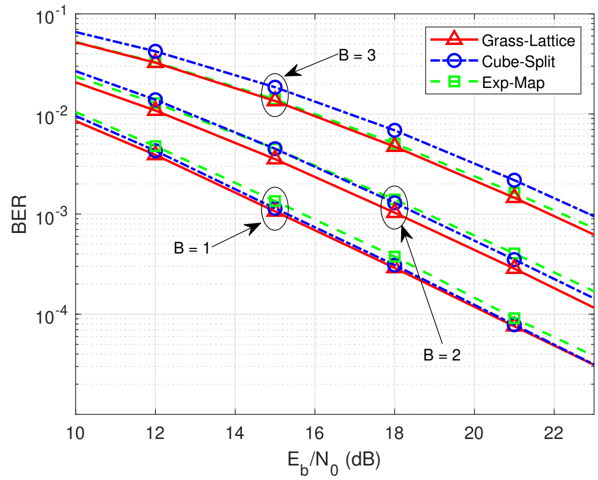

Fig. 4. Grass-Lattice BER curves in comparison with Cube-Split and Exp-Map constellations for $T = 4$, $N = 2$ and $B \in \{1, 2, 3\}$.

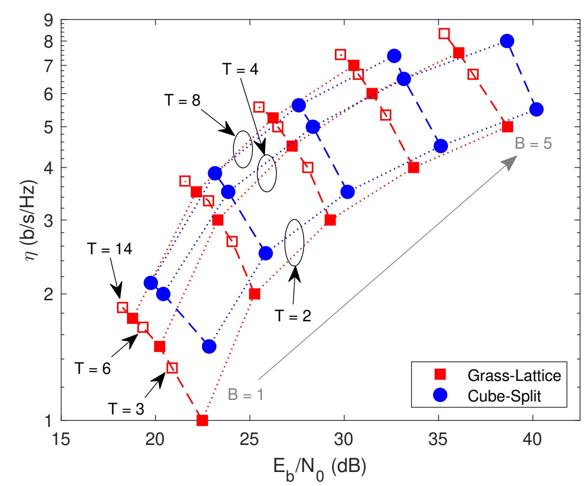

Fig. 5. Spectral efficiency as a function of E_b/N_0 at 10^{-4} BER for $T \in \{2, 3, 4, 6, 8, 14\}$ and $N = 2$.

The Jacobian of ϑ_2 at $\mathbf{z}$ is then just the product of the lengths of the resulting vectors:

$$Jac\,\vartheta_2(\mathbf{z}) = f_d(\|\mathbf{z}\|)^{2d-1}(f(\|\mathbf{z}\|)+\|\mathbf{z}\|f'(\|\mathbf{z}\|)) = \frac{e^{-\|\mathbf{z}\|^2}}{\Gamma(d+1)}.$$

Given any integrable mapping $g : \mathbb{B}_{\mathbb{C}^d}(0, 1) \to \mathbb{R}$, the expected value of $g(\mathbf{w})$ when $\mathbf{w}$ follows the distribution of the lemma is:

$$\begin{aligned} I &= \frac{1}{\pi^d}\int_{\mathbf{z}\in\mathbb{C}^d} g(\vartheta_2(\mathbf{z}))e^{-\|\mathbf{z}\|^2}\,d\mathbf{z} \\ &= \frac{\Gamma(d+1)}{\pi^d}\int_{\mathbf{z}\in\mathbb{C}^d} g(\vartheta_2(\mathbf{z}))Jac\,\vartheta_2(\mathbf{z})\,d\mathbf{z}, \end{aligned}$$

which by the Change of Variables Theorem equals

$$\frac{\Gamma(d+1)}{\pi^d}\int_{\mathbf{w}\in\mathbb{B}_{\mathbb{C}^d}(0,1)} g(\mathbf{w})\,d\mathbf{w}.$$

This is the expected value of g in $\mathbb{B}_{\mathbb{C}^d}(0,1)$, since the volume of $\mathbb{B}_{\mathbb{C}^d}(0,1)$ is precisely $\pi^d/\Gamma(d+1)$.

B. Proof of Lemma 3

Due to space limitations, we present here a sketch of the proof, relegating the full proof to a forthcoming journal paper. We will see that the Jacobian of ϑ_3 is constant and equal to 1, which proves the lemma from the Change of Variables Theorem. In order to compute the Jacobian, we denote the directional derivative of ϑ_3 along the direction $\dot{\mathbf{w}}_i$ by $D\vartheta_3(\mathbf{w})(\dot{\mathbf{w}}_i)$. Since the function ϑ_3 is not complex analytic, we need to consider its domain as a real space of dimension $2(T-1)$. Therefore, we choose orthonormal vectors $\dot{\mathbf{w}}_1,\ldots,\dot{\mathbf{w}}_{2T-4}$, which are complex orthogonal to $\mathbf{w}$, and complete the basis of $\mathbb{C}^{T-1}\equiv\mathbb{R}^{2(T-1)}$, with the two vectors $\dot{\mathbf{w}}_{2T-3}=j\mathbf{w}/\|\mathbf{w}\|$ and $\dot{\mathbf{w}}_{2T-2}=\mathbf{w}/\|\mathbf{w}\|$. The Jacobian of ϑ_3 is the volume of the parallelepiped spanned by $\dot{\mathbf{u}}_1,\ldots,\dot{\mathbf{u}}_{2T-2}$, where $\dot{\mathbf{u}}_i$ is the projection of $D\vartheta_3(\mathbf{w})(\dot{\mathbf{w}}_i)$ onto the orthogonal complement of $[\sqrt{1-\|\mathbf{w}\|^2},\mathbf{w}^{\mathrm{T}}]^{\mathrm{T}}$. A straightforward computation yields

$$\dot{\mathbf{u}}_i=\begin{bmatrix}0\\ \dot{\mathbf{w}}_i\end{bmatrix},\quad 1\le i\le 2T-4,$$
$$\dot{\mathbf{u}}_{2T-3}=j\begin{bmatrix}-\|\mathbf{w}\|\sqrt{1-\|\mathbf{w}\|^2}\\ \mathbf{w}(1-\|\mathbf{w}\|^2)/\|\mathbf{w}\|\end{bmatrix},$$
$$\dot{\mathbf{u}}_{2T-2}=\begin{bmatrix}-\|\mathbf{w}\|/\sqrt{1-\|\mathbf{w}\|^2}\\ \mathbf{w}/\|\mathbf{w}\|\end{bmatrix}.$$

These are all mutually orthogonal vectors and hence the parallelepiped they span has volume equal to the product of their norms, which is equal to 1.

C. Proof of Theorem 1

Let $G:\mathbb{G}\left(1,\mathbb{C}^T\right)\to\mathbb{C}$ be integrable. From Lemma 3,

$$\frac{1}{Vol(\mathbb{G}\left(1,\mathbb{C}^T\right))}\int_{[\mathbf{x}]\in\mathbb{G}(1,\mathbb{C}^T)}G([\mathbf{x}])\,d[\mathbf{x}]=$$
$$=\frac{1}{Vol(\mathbb{B}_{\mathbb{C}^{T-1}}(0,1))}\int_{\substack{\mathbf{w}\in\mathbb{C}^{T-1}\\ \|\mathbf{w}\|<1}}G\left(\begin{bmatrix}\sqrt{1-\|\mathbf{w}\|^2}\\ \mathbf{w}\end{bmatrix}\right)d\mathbf{w},$$

where $Vol(\mathbb{S})$ denotes the volume of the set $\mathbb{S}$. From Lemma 2, this equals

$$\frac{1}{\pi^{T-1}}\int_{\mathbf{z}\in\mathbb{C}^{T-1}}G\left(\begin{bmatrix}\sqrt{1-\|\mathbf{z}f_{T-1}(\|\mathbf{z}\|)\|^2}\\ \mathbf{z}f_{T-1}(\|\mathbf{z}\|)\end{bmatrix}\right)e^{-\|\mathbf{z}\|^2}\,d\mathbf{z},$$

which in turn from Lemma 1 equals

$$\int_{(\mathbf{a},\mathbf{b})\in\mathcal{I}}G\left(\begin{bmatrix}\sqrt{1-\|\mathbf{z}f_{T-1}(\|\mathbf{z}\|)\|^2}\\ \mathbf{z}f_{T-1}(\|\mathbf{z}\|)\end{bmatrix}\right)d(\mathbf{a},\mathbf{b}),$$

where

$$\mathbf{z}=(z_1,\ldots,z_{T-1})^{\mathrm{T}},\quad z_k=F^{-1}(a_k)+jF^{-1}(b_k).$$

All in one, we have proved that the point

$$\begin{bmatrix}\sqrt{1-\|\mathbf{w}\|^2}\\ \mathbf{w}\end{bmatrix},$$

with $\mathbf{w}=\mathbf{z}f_{T-1}(\|\mathbf{z}\|)$, is uniformly distributed in $\mathbb{G}\left(1,\mathbb{C}^T\right)$.

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