CONTROL AND INVERSE PROBLEMS FOR THE WAVE EQUATION ON METRIC GRAPHS

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Abstract

This thesis focuses on control and inverse problems for the wave equation on finite metric graphs. The first part deals with the control problem for the wave equation on tree graphs. We propose new constructive algorithms for solving initial boundary value problems on general graphs and boundary control problems on tree graphs. We demonstrate that the wave equation on a tree is exactly controllable if and only if controls are applied at all or all but one of the boundary vertices. We find the minimal controllability time and prove that our result is optimal in the general case. The second part deals with the inverse problem for the wave equation on tree graphs. We describe the dynamical Leaf Peeling (LP) method. The main step of the method is recalculating the response operator from the original tree to a peeled tree. The LP method allows us to recover the connectivity, potential function on a tree graph and the lengths of its edges from the response operator given on a finite time interval. In the third part we consider the control problem for the wave equation on graphs with cycles. Among all vertices and edges we choose certain active vertices and edges, and give a constructive proof that the wave equation on the graph is exactly controllable if Neumann controllers are placed at the active vertices and Dirichlet controllers are placed at the active edges. The control time for this construction is determined by the chosen orientation and path decomposition of the graph. We indicate the optimal time that guarantees the exact controllability for all systems of a described class on a given graph. While the choice of the active vertices and edges is not unique, we find the minimum number of controllers to guarantee the exact controllability as a graph invariant.

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Dedication

This thesis is dedicated to my mother Rongfang Ding and father Jianjiang Zhao. They instilled in me a love of life long learning.

Chapter 1: General Introduction

This thesis addresses some control and inverse problems for the wave equation on metric graphs. A graph is called a metric graph if every edge is identified with an interval with a positive length. A quantum graph is a metric graph equipped with differential operators and differential equations satisfying appropriate vertex conditions [*Berkolaiko and Kuchment*, 2013]. Many problems of applied science and engineering depend on detailed models of partial differential equations on network-like structures. For example, gas propagation in networks and traffic flow can be modelled by first order hyperbolic equation systems [*Steinbach*, 2007; *Herty et al.*, 2010; *Garavello and Piccoli*, 2006]; groundwater movement can be described by convection-diffusion models [*Oppenheimer*, 2000; *García et al.*, 2015]; The behaviour of carbon nano-structure can be described by diffusion models [*Kuchment and Post*, 2006]; Diffusion equations on graphs are also used to compute voltage phase angles and voltage frequencies in power networks [*Cheng and Scherpen*, 2018]. The propagation of electric signals in a cable transmission lines [*Alam et al.*, 2022] are computed through the telegrapher's equations; quantum mechanical behavior of materials at atomic scale can be modelled by the Schrödinger equation defined on a graph [*Duca*, 2021]. Wave equations are used to describe mechanical systems constituted by coupled flexible or elastic elements as strings, beams, and membranes [*Dáger and Zuazua*, 2006].

Control problems of differential equations on graphs tune input parameters or boundary conditions to obtain a desired behaviour of the system at specific times. Examples of such control problems on graphs include [*Gugat et al.*, 2011; *Zlotnik et al.*, 2015; *Alam et al.*, 2021]. Controllability properties of the wave equation is a fundamental topic in the control theory for partial differential equations. Various methods have being used by different researchers to investigate the controllability of the wave equation system on tree graphs: the energy estimates together with the Hilbert uniqueness method in [*Dáger and Zuazua*, 2006; *Lagnese et al.*, 1994]; the methods of moments and the theory of vector valued exponentials in [*Avdonin and Ivanov*, 1995], the propagation of singularities in the wave equation in [*Belishev and Vakulenko*, 2006]. In these works it was proved that the system is exactly controllable if the control functions act at all or at all but one of its boundary vertices. However, none of the known work in this direction contained algorithms solving control problems for differential equations on graphs.

Very little was known about controllability of the wave equation (or any other partial differential equation) on graphs with cycles. It was only proved (see, e.g. [*Avdonin and Ivanov*, 1995; *Avdonin et al.*, 2009, 2010a]) that the wave equation on graphs with cycles is never exactly controllable from the boundary. It may be spectrally controllable (a weaker formation than exact controllable) from the boundary, but this property is very unstable with respect to small perturbations of the system parameters.

The inverse problem on quantum graphs is to recover the unknown coefficients of differential equations on the edges from spectral or dynamical Dirichlet to Neumann (DtN) data at the boundary. There are deep connections between the controllability and identifiability of a dynamical system with distributed parameters. It was shown that inverse spectral and scattering problems for differential equations on graphs with cycles do not have in general a unique solution [*Gutkin and Smilansky*, 2001; *Kurasov and Nowaczyk*, 2005]. Current positive uniqueness results concerning boundary inverse problems are mainly on tree graphs [*Brown and Weikard*, 2005; *Yurko*, 2005; *Freiling and Yurko*, 2007; *Belishev*, 2004, 2006]. The Leaf Peeling (LP) method was proposed in [*Avdonin and Kurasov*, 2008] and developed in [*Avdonin et al.*, 2017a] to recover the graph topology, edge lengths, and potentials for tree graphs from their spectral or dynamical DtN data. Both versions have recursive procedures. In each step, one recalculates the DtN data on the smaller tree obtained by removing boundary edges (except for the root edge) from the original tree. The paper [*Avdonin et al.*, 2017a] however, did not describe and justify all steps of the dynamical version in all details.

In this thesis, the second chapter addresses controllability of the wave equation system on tree graphs. We give constructive algorithms to solve the exact shape, velocity and exact control problems on a tree graph. We show the time required to guarantee for an arbitrary tree to reach any arbitrary terminal shape or velocity is less than all the minimum control times described in the literature (see for example, [*Dáger and Zuazua*, 2006; *Lagnese et al.*, 2012; *Belishev and Vakulenko*, 2006]).

In the third chapter we give a complete and rigorous presentation of the dynamical LP method. To our best knowledge it was the first solution of the inverse problem for general tree graphs by pure dynamical methods. The method is based on the powerful boundary control (BC) method in inverse theory, which was proposed in [*Belishev*, 1987] using DtN data to recover distributed parameters in the wave equation in a given domain. Its main property is locality, which allows one to recover the topology and other parameters of a subset only from data related to that subset instead of the entire domain. In [*Avdonin and Zhao*, 2021a] algorithms solving forward and control problems for the wave equation on graphs were proposed, which significantly simplifies the presentation of the dynamical LP version and solution of the inverse problem. The dynamical version of the LP method is much more complicated for theoretical justification than the spectral one. However, it combines the strength from both the BC method and the simplicity of the algorithm in [*Avdonin and Zhao*, 2021a], therefore has great potentials to serve as a basis for developing effective numerical algorithms solving inverse problems on quantum graphs.

The fourth chapter addresses controllability of the wave equation system on a general graph with cycles. We generalize the idea of controllability on trees in [*Avdonin and Zhao*, 2021a]. While the choice of control functions is not unique, we find the minimum number of controllers to guarantee the exact controllability as a graph invariant.

The main results of the thesis were published at [*Avdonin and Zhao*, 2021a], [*Avdonin and Zhao*, 2021b], and [*Avdonin and Zhao*, 2022]. In the first paper, my independent results are the folding ruler formulae and the constructive proof of the shape controllability of the wave equation on tree graphs. In the second paper, I used the folding ruler formulae to present a rigorous proof of the dynamical version of the leaf peeling theorem. In the last paper, the main result of controllabiliy of the wave equation on graphs with cycles are joint results with Dr. Avdonin. Based on the examples of control problems in [*Avdonin*, 2019] and the shape controllability result in [*Avdonin and Zhao*, 2021a], I developed a general approach to construction shape control functions on a general undirected graph. I further characterised the minimum number of controllers to guarantee the exact controllability as a graph invariant.

Chapter 2: Control Problems for the Wave Equation on Metric Tree Graphs

2.1 Introduction

Control problems for the wave equation on trees have important applications in science and engineering and were studied in many papers (see the monographs [*Avdonin and Ivanov*, 1995]; [*Dáger and Zuazua*, 2006]; [*Lagnese et al.*, 1994]; the surveys [*Avdonin*, 2008]; [*Zuazua*, 2013]; and references therein). They also have deep connection with inverse problem on trees, see, e.g. [*Avdonin and Kurasov*, 2008], [*Belishev and Vakulenko*, 2006]. In this chapter we consider exact controllability for the wave equation

$$u_{tt} - u_{xx} + qu = 0$$

on trees with boundary controls. The so-called Kirchhoff-Neumann (KN) conditions are assumed at all internal vertices. This problem was studied, e.g. in [*Belishev and Vakulenko*, 2006], [*Dáger and Zuazua*, 2006], [*Lagnese et al.*, 1994], (in those papers the problem was stated in slightly different forms). It was proved that the system is exactly controllable if the control functions act at all or at all but one of the boundary vertices.

Unfortunately, none of the known work in this direction contained algorithms solving control problems for differential equations on graphs. Numerical solutions of control and even forward problems on graphs are very difficult (see, e.g. [*Arioli and Benzi*, 2017])

We give constructive algorithms to solve the exact shape, velocity and full control problems on a tree graph Ω. Let U be a union of disjoint paths (except for the end points) on Ω. Each path P starts from a controlled boundary vertex and ends anywhere on Ω, and $\cup_{P \in U} P = \Omega$. We construct shape/velocity control functions by controlling the terminal shape/velocity on one path at a time. The full control function is constructed by combining the shape and velocity control functions via the moment method. We show the minimal guaranteed time to reach any arbitrary terminal shape or velocity for an arbitrary tree is

$$T_s = \min_U \max_{P \in U} \operatorname{length} P.$$

The minimum time required for the full controllability is $2T_s$.

This chapter is organized as follows. Section 2.2 introduces the forward and control problems for the wave equation on quantum graphs. Section 2.3 solves the forward and control problems on a finite interval. For the solution of the forward problem, we extend the well known D'Alembert's formula to the case with variable potential coefficients. This representation is a useful tool for solving the forward, control, and inverse problems on graphs. Section 2.4 uses the KN condition and the result from Section 2.3 to solve the forward and control problems on a star-shaped neighborhood subgraph, which consists of an internal vertex and its incident edges and adjacent vertices. Section 2.5 proposes an algorithm to solve the forward problem on a general graph. Section 2.6 gives constructive proofs for the exact shape, velocity, and full controllability of the wave equation on finite metric tree graphs using boundary control functions.

2.2 Preliminaries

Let $\Omega = (V, E)$ be a metric graph with finitely many vertices and edges, where V and E are the sets of vertices and edges respectively. Every edge $e_j \in E$ is identified with an interval of finite length on the real line. We denote the set of boundary vertices (the vertices with degree one) by $\Gamma(\Omega)$ or just Γ; $V \setminus \Gamma$ is the set of internal vertices.

The graph Ω naturally determines the Hilbert space of square integrable functions

$$\mathcal{H} := L^2(\Omega) = \oplus_{e \in E} L^2(e),$$

in which $\|\phi\|^2_{\mathcal{H}} = \sum_{e \in E} \|\phi\|^2_{L^2(e)} < \infty$. The space $\mathcal{H}^1$ consists of continuous functions ϕ on Ω such that $\phi|_e$ belongs to the Sobolev space $H^1(e)$ for each edge e and $\phi|_\Gamma = 0$. The space

$\mathcal{H}^{-1}$ is defined as the dual space of $\mathcal{H}^1$.

Let Γ_1 and Γ_0 be two disjoint subsets of Γ such that $\Gamma_1 \cup \Gamma_0 = \Gamma$. We now associate the following initial boundary value problem (IBVP) for the wave equation to the graph Ω:

$$u_{tt} - u_{xx} + qu = 0 \quad \text{in } \{\Omega \setminus V\} \times (0, T), \tag{2.1}$$

$$\sum_{e_j \sim v} \partial_j u(v, t) = 0 \text{ at each vertex } v \in V \setminus \Gamma, t \in [0, T],$$

$$u(\cdot, t) \text{ is continuous at each vertex } v, \tag{2.2}$$

$$u = f \text{ on } \Gamma_1 \times [0, T], u = 0 \text{ on } \Gamma_0 \times [0, T] \tag{2.3}$$

$$u|_{t \leq 0} = 0 \text{ in } \Omega. \tag{2.4}$$

In (2.2) $\partial_j u(v, t)$ means the derivative of u taken at v along the edge e_j in the direction outward of v and the sum is taken over all edges incident to v. The notation $e_j \sim v_j$ means the edge e_j is incident to the vertex v. The vertex conditions in (2.2) are called the Kirchhoff-Neumann (KN) conditions. We agree to extend the functions u and f to negative time by 0, and it is convenient to present the initial conditions $u(x, 0) = u_t(x, 0) = 0$ as $u|_{t \leq 0} = 0$ in (2.4). This forward problem is well defined, as stated in Proposition 2.1 below. For different conditions on the potential q the proof of this proposition can be found in [*Ali Mehmeti*, 1994], [*Ali Mehmeti and Meister*, 1989], [*Avdonin and Nicaise*, 2015], [*Dáger and Zuazua*, 2006], and [*Lagnese et al.*, 1994].

Proposition 2.1. *Let* $m := |\Gamma_1|$, $f \in \mathcal{F}^T := L^2([0, T]; \mathbb{R}^m)$ *and* $q \in L^1(\Omega)$. *There exists a unique solution* $u = u^f(x, t)$ *to the IBVP* (2.1)–(2.4) *(understood in a weak sense) and*

$$u \in C([0, T]; \mathcal{H}) \cap C^1([0, T]; \mathcal{H}^{-1}), \tag{2.5}$$

that is, $u(\cdot,t) \in \mathcal{H}$, $u_t(\cdot,t) \in \mathcal{H}^{-1}$ *for all* $t \in [0,T]$, *and both* $u(\cdot,t)$ *and* $u_t(\cdot,t)$ *are continuous with respect to* t *in corresponding norms. If* $f \in C^2([0,T];\mathbb{R}^m)$, $f(0) = f'(0) = 0$ *and* $q(x) \in C(\overline{\Omega})$, *the IBVP* (2.1)–(2.4) *has a unique classical solution.*

In what follows we assume $f \in \mathcal{F}^T$. The solution for (2.1)-(2.4) on Ω is denoted as $u^{f,\Gamma_1}(\cdot,t)$. In the case when Γ_1 only contains one boundary vertex γ, we denote the solution of (2.1)-(2.4) as $u^{f,\gamma}(\cdot,t)$. When $f(t) = (f_1(t), f_2(t), \ldots, f_m(t))$, by the superposition principle

$$u^{f,\Gamma_1}(\cdot,t) = \sum_{i=1}^{m} u^{f_i,\gamma_i}(\cdot,t). \tag{2.6}$$

Let G be a subgraph of Ω containing all actively controlled boundary vertices of Ω. We denote the solution of (2.1)-(2.4) on graph G as $u_G^f(\cdot,t)$. Note that $u_G^f(\cdot,t)$ is not the same as $u^f(\cdot,t)|_G$, which is $u^f(\cdot,t)$ on Ω restricted to G. The reason for that is the boundary conditions are different for $u_G^f(\cdot,t)$ and $u^f(\cdot,t)|_G$. When we solve (2.1)-(2.4) on G, all boundary vertices in $\Gamma_0(G)$ are assumed to be fixed, however some of the boundary vertices in $\Gamma_0(G)$ are internal vertices on Ω. We denote $u|_{e_j}$, the solution of (2.1)-(2.4) on Ω restricted to an edge e_j as u_j.

The exact controllability for the system (2.1)–(2.4) in this chapter is defined as following:

Definition 2.2. *Let* $T > 0$. *We say that the system* (2.1)–(2.4) *is:*

1. *exactly shape controllable in time* T *if for any* $y \in \mathcal{H}$, *there exists* $f \in \mathcal{F}^T$ *such that* $u^{f,\Gamma_1}(\cdot,T) = y(\cdot)$;
2. *exactly velocity controllable in time* T *if for any* $z \in \mathcal{H}^{-1}$, *there exists* $f \in \mathcal{F}^T$ *such that* $u_t^{f,\Gamma_1}(\cdot,T) = z(\cdot)$;
3. *exactly controllable in time* T *if for any* $y \in \mathcal{H}$ *and* $z \in \mathcal{H}^{-1}$, *there exists* $f \in \mathcal{F}^T$ *such that* $u^{f,\Gamma_1}(\cdot,T) = y(\cdot)$ *and* $u_t^{f,\Gamma_1}(\cdot,T) = z(\cdot)$.

There are other types of controllability on graphs such as the spectral controllability and approximate controllability. We refer to [*Avdonin*, 2008] for their definitions.

2.3 The forward and control problems for the wave equation on a finite length interval

In this section we consider the IVBP (2.1)-(2.4) on a graph consisting of one edge (v_1, v_2), of length l. We identify (v_1, v_2) with the interval $[0, l]$, where v_1 is identified with $x = 0$, and v_2 is identified with $x = l$. We consider two cases where an active control function is placed at v_1 or v_2 respectively, and give algorithms to compute $u^{f,v_1}(x,t)$ or $u^{f,v_2}(x,t)$ respectively. In the first case, (2.1)-(2.4) becomes

$$\begin{cases} u_{tt} - u_{xx} + q(x)u = 0, \quad 0 < x < l, \ 0 < t < T \\ u|_{t \le 0} = 0, \quad u(0,t) = f(t), \quad u(l,t) = 0. \end{cases} \tag{2.7}$$

When $T \le l$ and $f \in L^2(0,T)$, it is well known that the system (2.7) has a unique generalized solution u^{f,v_1} presented in the form

$$u^{f,v_1}(x,t) = \begin{cases} 0, \quad 0 < t < x \\ f(t-x) + \int_x^t w(x,s) f(t-s)\, ds, \quad x \le t. \end{cases} \tag{2.8}$$

Here $w(x,t)$ is a solution to the Goursat problem

$$\begin{cases} w_{tt} - w_{xx} + q(x)w = 0, \quad 0 < x < t \\ w(0,t) = 0, w(x,x) = -\frac{1}{2}\int_0^x q(s)\, ds \end{cases} \tag{2.9}$$

that can be found by standard iteration method (see, e.g., [*Tikhonov and Samarskii*, 2013] for smooth q and [*Avdonin and Mikhaylov*, 2010] for $q \in L^1(0,l)$).

We now solve (2.7) for $T > l$. We extend the potential $q(x)$ to the semi-axis $x > 0$ by the rule $q(2nl \pm x) = q(x)$ for all $n \in \mathbb{N}$ (See Figure 2.1). Let w be a solution to the Goursat problem (2.9) with such an extended potential q. By direct substitution one can prove the following proposition.

Proposition 2.3. *For $q \in L^1(0,l)$ and $f \in L^2(0,T)$, the system (2.7) has a unique generalized solution $u^{f,v_1} \in C([0,T];L^2(0,l))$. When $0 \le t \le x$, $u^{f,v_1}(x,t) = 0$. When $t > x$,*

$$
\begin{aligned}
& u^{f,v_1}(x,t) \\
= \; & f(t-x) + \int_x^t w(x,s) f(t-s)\, ds \\
& - f(t-2l+x) - \int_{2l-x}^t w(2l-x,s) f(t-s)\, ds \\
& + f(t-2l-x) - \int_{2l+x}^t w(2l+x,s) f(t-s)\, ds \\
& - f(t-4l+x) - \int_{4l-x}^t w(4l-x,s) f(t-s)\, ds + \dots \\
= \; & \sum_{n=0}^{\lfloor \frac{t-x}{2l} \rfloor} \left(f(t-2nl-x) + \int_{2nl+x}^t w(2nl+x,s) f(t-s)\, ds \right) \\
& - \sum_{n=1}^{\lfloor \frac{t+x}{2l} \rfloor} \left(f(t-2nl+x) + \int_{2nl-x}^t w(2nl-x,s) f(t-s)\, ds \right)
\end{aligned}
\tag{2.10}
$$

where $\lfloor \cdot \rfloor$ is the floor function.

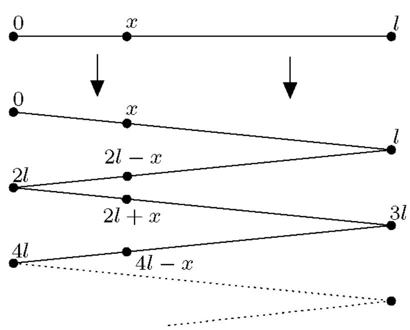

Figure 2.1: Extend $q(x)$ on a finite interval like a folding ruler. When the ruler is folded, the overlapping points have the same potential value. That is, $q(2nl \pm x) = q(x)$.

Next we consider the IVBP on $[0,l]$ with the control function $f(t)$ applied at the right endpoint:

$$\begin{cases} u_{tt} - u_{xx} + q(x)u = 0, \quad 0 < x < l, t > 0 \\ u|_{t\le 0} = 0, \quad u(0,t) = 0, \quad u(l,t) = f(t). \end{cases} \tag{2.11}$$

We construct $p(x) = q(l-x)$ and extend p by letting $p(2nl \pm x) = p(x)$. Let $k(x,t)$ be the solution to the Goursat problem associated with the extended $p(x)$:

$$\begin{cases} k_{tt} - k_{xx} + p(x)k = 0, \quad 0 < x < t \\ k(0,t) = 0, \quad k(x,x) = -\frac{1}{2}\int_0^x p(\eta)\,d\eta. \end{cases} \tag{2.12}$$

By changing the coordinates in (2.10) one can prove the following proposition.

Proposition 2.4. *For* $q \in L^1(0,l)$ *and* $f \in L^2(0,T)$*, the IBVP* (2.11) *has a unique generalized solution* $u^{f,v_2} \in C([0,T];L^2(0,l))$*. When* $0 \le t \le l-x$*,* $u^{f,v_2}(x,t) = 0$*. When* $t > l-x$ *the solution for* (2.11) *is*

$$\begin{aligned} & u^{f,v_2}(x,t) \\ = \ & f(t-l+x) + \int_{l-x}^{t} k(l-x,s)f(t-s)\,ds \\ & - f(t-l-x) - \int_{l+x}^{t} k(l+x,s)f(t-s)\,ds + \dots \\ = \ & \sum_{n=0}^{\lfloor \frac{t+x-l}{2l} \rfloor} f(t-2nl-l+x) + \sum_{n=0}^{\lfloor \frac{t+x-l}{2l} \rfloor} \int_{2nl+l-x}^{t} k(2nl+l-x,s)f(t-s)\,ds \\ & - \sum_{n=0}^{\lfloor \frac{t-x-l}{2l} \rfloor} f(t-2nl-l-x) - \sum_{n=0}^{\lfloor \frac{t-x-l}{2l} \rfloor} \int_{2nl+l+x}^{t} k(2nl+l+x,s)f(t-s)\,ds. \end{aligned} \tag{2.13}$$

We refer to (2.10) and (2.13) as *Folding Ruler Formulae.*

The classical results on exact controllability of the system (2.7) are summarized in the following proposition. They will be used in solving control problems in Section 2.6.

Proposition 2.5. *Let $q \in L^1(0,l)$ and $T \geq l$,*

1. *For any function $\phi \in \mathcal{H}$, there exists a control $f \in L^2(0,T)$ such that $u^{f,v_1}(x,T) = \phi(x)$. If $T = l$ then f is unique.*

2. *For any function $\psi \in \mathcal{H}^{-1}$, there exists a control $f \in L^2(0,T)$ such that $u_t^{f,v_1}(x,T) = \psi(x)$.*

3. *For any $\phi \in \mathcal{H}$ and any $\psi \in \mathcal{H}^{-1}$ there exists a control $f \in L^2(0,2T)$ such that $u^{f,v_1}(x,2T) = \phi(x)$, $u_t^{f,v_1}(x,2T) = \psi(x)$.*

Proof. This proposition can be proved using various methods. However since it is a part of a constructive proof of the controllability on tree graphs, we give the proof of Part 1 and 2 through a dynamical approach.

For Part 1, let $\tilde{f}(t)$, $0 \leq t \leq l$, be the unique solution to the Volterra integral equation of the second kind:

$$\phi(x) = \tilde{f}(l-x) + \int_x^l w(x,s)\tilde{f}(t-s)\,ds,$$

then

$$f(t) = \begin{cases} 0, & 0 \leq t \leq T-l \\ \tilde{f}(t+l-T), & T-l < t \leq T \end{cases}$$

solves the control problem

$$u^{f,v_1}(x,T) = \phi(x).$$

For Part 2, We first show the mapping

$$(f,t) \mapsto u_t^{f,v_1}(x,t)$$

is a continuous mapping from $L^2(0,T)\times(0,T]$ to $H^{-1}(0,l)$. First assume $f\in C^2$. Since

$$\begin{aligned}
u_t^{f,v_1}(x,t) = &\sum_{n=0}^{\lfloor\frac{t-x}{2l}\rfloor} \left(f'(t-2nl-x)+w(2nl+x,2nl+x)f(t-2nl-x)\right)\\
&+\sum_{n=0}^{\lfloor\frac{t-x}{2l}\rfloor}\int_{2nl+x}^{t} w_s(2nl+x,s)f(t-s)\,ds\\
&-\sum_{n=1}^{\lfloor\frac{t+x}{2l}\rfloor} \left(f'(t-2nl+x)+w(2nl-x,2nl-x)f(t-2nl+x)\right)\\
&-\sum_{n=1}^{\lfloor\frac{t+x}{2l}\rfloor}\int_{2nl-x}^{t} w_s(2nl-x,s)f(t-s)\,ds,
\end{aligned}$$

for any $g\in H_0^1(0,l)$,

$$\begin{aligned}
&\int_0^l u_t^{f,v_1}(x,t)g(x)\,dx = \int_0^l \left(\sum_{n=0}^{\lfloor\frac{t-x}{2l}\rfloor} f(t-2nl-x)+\sum_{n=1}^{\lfloor\frac{t+x}{2l}\rfloor} f(t-2nl+x)\right)g'(x)\,dx\\
&+\int_0^l\sum_{n=0}^{\lfloor\frac{t-x}{2l}\rfloor}\left(w(2nl+x,2nl+x)f(t-2nl-x)+\int_{2nl+x}^{t} w_s(2nl+x,s)f(t-s)\,ds\right)g(x)\,dx\\
&-\int_0^l\sum_{n=1}^{\lfloor\frac{t+x}{2l}\rfloor}\left(w(2nl-x,2nl-x)f(t-2nl+x)+\int_{2nl-x}^{t} w_s(2nl-x,s)f(t-s)\,ds\right)g(x)\,dx
\end{aligned} \tag{2.14}$$

The right side of (2.14) extends to a continuous mapping from $L^2(0,T)\times(0,T]$ to $H^{-1}(0,l)$, and serves as a definition of the action of $u_t(\cdot,t)$ on $H_0^1(0,l)$.

Similar to the proof of Part 1, to show velocity controllability at time $T\ge l$, we will find a control function $\tilde f(t)$, $0\le t\le l$, such that

$$u_t^{\tilde f,v_1}(x,l)=\psi(x).$$

Then

$$f(t)=\begin{cases}0, \quad 0\le t\le T-l\\ \tilde f(t+l-T), \quad T-l<t\le T\end{cases}$$

is a solution to the control problem

$$u_t^{f,v_1}(x,T)=\psi(x).$$

To find $\tilde f(t)$, by (2.14)

$$\begin{aligned}&\int_0^l u_t^{\tilde f,v_1}(x,l)g(x)\,dx\\ &=\int_0^l \tilde f(l-x)g'(x)\,dx+\int_0^l\left(w(x,x)\tilde f(l-x)+\int_x^l w_s(x,s)\tilde f(l-s)\,ds\right)g(x)\,dx\\ &=\int_0^l\left[\tilde f(l-x)+\int_x^l\left(w(\xi,\xi)\tilde f(l-\xi)+\int_\xi^l w_s(\xi,s)\tilde f(l-s)\,ds\right)\,d\xi\right]g'(x)\,dx\\ &=\int_0^l\left[\tilde f(l-x)+\int_x^l\left(w(s,s)+\int_x^s w_s(\xi,s)\,d\xi\right)\tilde f(l-s)\,ds\right]g'(x)\,dx\end{aligned} \tag{2.15}$$

Suppose $z\in H^{-1}(0,l)$. Then by Riesz Representation, there exists $Z\in H_0^1(0,l)$ such that

$$z(g)=<Z,g>=\int_0^l Z'(x)g'(x)dx,\ \forall g\in H_0^1(0,l).$$

Let $\tilde f(t)$ be the solution to the Volterra integral equation

$$\tilde f(l-x)+\int_x^l\left(w(s,s)+\int_x^s w_s(\xi,s)\,d\xi\right)\tilde f(l-s)\,ds=Z'(x),$$

then $u_t^{f,v_1}(\cdot,l)=z(\cdot)$.

Part 3 is a classical result in control theory. For example, in [*Avdonin and Edward*, 2018] and [*Avdonin and Edward*, 2019], the full controllability at $T\ge 2l$ is deduced from the shape and velocity controllability at $T \geq l$. We will use the same principle to prove the full controllability on tree graphs in Section 2.6.

□

2.4 The forward and control problems in a star-shaped neighborhood graph of an internal vertex

Let u be the solution for (2.1)-(2.4). Section 2.3 suggests, if the trace of u on vertices $u|_V$ is known, one can use (2.10) and (2.13) and the superposition principle to compute u. Since $u|_\Gamma$ is given in (2.3), it remains to find $u|_{V\setminus\Gamma}$. In this section we state and solve the forward problem and a control problem on the neighborhood graph of an internal vertex v (denoted as $\mathcal{N}_\Omega(v)$), which consist of v and all of its adjacent vertices and incident edges. Note that if Ω is a star graph and v is the internal vertex, we have $\mathcal{N}_\Omega(v) = \Omega$. The algorithm to find the trace of u on the entire $V \setminus \Gamma$ is presented in section 2.5.

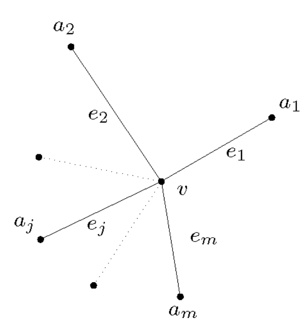

Figure 2.2: The neighborhood graph of v

Let v be an internal vertex of Ω. Label all vertices in $\mathcal{N}_\Omega(v) \setminus \{v\}$ by $a_1, \dots, a_m$, and each edge (v, a_j) by e_j[1], see Figure 2.2. Let l_j be the length of e_j. We identify each e_j with the interval $[0, l_j]$, where v is identified with 0 and a_j is identified with l_j. Let q_j be real valued $L^1(0, l_j)$ functions and $T > 0$. We put $u_j := u|_{e_j}$, $j = 1, 2, \dots, m$ and consider the following

[1] In a tree graph there is only one edge between v and a_j. The case where there are multiple edges between two vertices can be solved in the same way.

IBVP on $\mathcal{N}_v(\Omega)$:

$$(u_j)_{tt} - (u_j)_{xx} + q_j(x)u_j = 0, \quad x \in e_j,\ t \in (0,T), \tag{2.16}$$

$$\sum_{j=1}^{m} \partial u_j(0,t) = 0, \quad t \in (0,T), \tag{2.17}$$

$$u_i(0,t) = u_j(0,t), \quad i,j = 1,2,\ldots,m,\ t \in (0,T), \tag{2.18}$$

$$u_j(l_j,t) = f_j(t), \quad t \in (0,T), \tag{2.19}$$

$$u_j(\cdot,t)|_{t\le 0} = 0. \tag{2.20}$$

In the forward problem (2.16)-(2.20) we assume all $f_j(t)$ are known. We solve the forward problem by finding the common value $g(t) := u_j(0,t)$ first. The initial condition (2.20) indicates $g(0) = 0$. According to the superposition principle, for every $j = 1,2,\ldots,m$

$$u_j(x,t) = u_j^{f_j,v_j}(x,t) + u_j^{g,v}(x,t). \tag{2.21}$$

Using the Folding Ruler Formulae (2.10) and (2.13) for $j = 1,2,\ldots,m$, we get

$$\begin{aligned} &{u_j}^{g,v}(x,t) \\ &= \sum_{n=0}^{\lfloor \frac{t-x}{2l_j} \rfloor} g(t-2nl_j-x) + \sum_{n=0}^{\lfloor \frac{t-x}{2l_j} \rfloor} \int_{2nl_j+x}^{t} w_j(2nl_j+x,s)g(t-s)\,ds \\ &\quad - \sum_{n=1}^{\lfloor \frac{t+x}{2l_j} \rfloor} g(t-2nl_j+x) - \sum_{n=1}^{\lfloor \frac{t+x}{2l_j} \rfloor} \int_{2nl_j-x}^{t} w_j(2nl_j-x,s)g(t-s)\,ds, \end{aligned}$$

and

$$
\begin{aligned}
&{u_j}^{f_j,v_j}(x,t) \\
&= \sum_{n=0}^{\lfloor \frac{t+x-l_j}{2l_j} \rfloor} f_j(t-2nl_j-l_j+x) + \sum_{n=0}^{\lfloor \frac{t+x-l_j}{2l_j} \rfloor} \int_{2nl_j+l_j-x}^{t} k_j(2nl_j+l_j-x,s) f_j(t-s)\,ds \\
&\quad - \sum_{n=0}^{\lfloor \frac{t-x-l_j}{2l_j} \rfloor} f_j(t-2nl_j-l_j-x) - \sum_{n=0}^{\lfloor \frac{t-x-l_j}{2l_j} \rfloor} \int_{2nl_j+l_j+x}^{t} k_j(2nl_j+l_j+x,s) f_j(t-s)\,ds.
\end{aligned}
$$

Substituting (2.21) to the KN condition (2.17) we obtain

$$
mg'(t) - \int_0^t G(s) g(t-s)\,ds = F(t), \tag{2.22}
$$

where

$$
G(s) = \sum_{j=1}^{m} \partial_x w_j(0,s), \tag{2.23}
$$

and

$$
\begin{aligned}
&F(t) = \\
&2\sum_{j=1}^{m} \sum_{n=0}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} f_j'(t-(2n+1)l_j) + 2\sum_{j=1}^{m} \sum_{n=0}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} k_j((2n+1)l_j,(2n+1)l_j) f_j(t-(2n+1)l_j) \\
&- 2\sum_{j=1}^{m} \sum_{n=0}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} \int_{(2n+1)l_j}^{t} \partial_x k_j((2n+1)l_j,s) f_j(t-s)\,ds - 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} g'(t-2nl_j) \\
&- 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} w_j(2nl_j,2nl_j) g(t-2nl_j) + 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} \int_{2nl_j}^{t} \partial_x w_j(2nl_j,s) g(t-s)\,ds
\end{aligned} \tag{2.24}
$$

Equation (2.22) can be transformed into a delay Volterra integral equation with respect to $g'(t)$. We make the transformation

$$\int_0^t G(s)g(t-s)\,ds = \int_0^t G(s)\int_0^{t-s} g'(\xi)\,d\xi ds$$
$$= \int_0^t g'(\xi)\int_0^{t-\xi} G(s)\,dsd\xi = \int_0^t g'(\xi)H(\xi,t)\,d\xi, \tag{2.25}$$

where

$$H(\xi,t) = \int_0^{t-\xi} G(s)\,ds. \tag{2.26}$$

So (2.22) becomes

$$mg'(t) - \int_0^t H(s,t)g'(s)\,ds = F(t) \tag{2.27}$$

The right hand side of equation (2.27) contains terms of g and g' with delayed arguments. So (2.27) can be solved in steps. Let us consider a simple example of Ω consist of three edges. We assume $l_1 > l_2 > l_3 > \frac{l_2}{2}$, and $f_2(t) = f_3(t) = 0$. When $t < l_1$, $g(t)$=0. When $l_1 \le t \le l_1 + 2l_3$, (2.27) becomes

$$3g'(t) - \int_0^t H(s,t)g'(s)\,ds = 2f_1'(t-l_1) + 2k_1(l_1,l_1)f_1(t-l_1) - 2\int_{l_1}^t \partial_x k_1(l_j1,s)f_j(t-s)\,ds.$$

Solving the above equation gives $g'(t)$ for $t \le l_1 + 2l_3$ and since $g(0) = 0$, $g(t) = \int_0^t g'(\xi)\,d\xi$. On the next time interval $l_1 + 2l_3 < t \le l_2 + 2l_2$, (2.27) becomes

$$\begin{aligned} 3g'(t) - \int_0^t H(s,t)g'(s)\,ds = \\ 2f_1'(t-l_1) + 2k_1(l_1,l_1)f_1(t-l_1) - 2\int_{l_1}^t \partial_x k_1(l_j1,s)f_j(t-s)\,ds \\ - 2g'(t-2l_3) - 2w_3(2l_3,2l_3)g(t-2l_3) + 2\int_{2l_3}^t \partial_x w_3(2l_3,s)g(t-s)\,ds. \end{aligned}$$

The right hand side includes values of $g'(t)$ and $g(t)$ for $l_1 < t \le l_1 + 2l_2 - 2l_3$, which is known from previous calculations. Once we compute $g'(t)$ and $g(t)$ for this time interval, we can compute $g(t)$ on the next time interval.

The above steps to compute $g(t)$ are generalized in the following proposition.

Proposition 2.6. *Suppose that $f_j(t) \in L_{2,loc}(0,\infty)$ for $j = 1, 2, \ldots, m$ in the IBVP* (2.16)–(2.19)*, then $g(t)$ can be computed from* (2.27)*, and $g \in L_{2,loc}(0,\infty)$.*

Proof. Let $l_{min} = \min_{j=1,2,\ldots,m} l_j$. If $g(s)$ for $0 \le s \le t - 2l_{min}$ is known, $F(s)$ for $0 \le s \le t$ is known. Then $g'(s)$ for $s \in [t - 2l_{min}, t]$ can be calculated by solving equation (2.27) as an integral Volterra equation of the second kind (IVSK equation). Since the initial value of $g(t)$ known to be zero, $g(t)$ can be obtained by integrating $g'(t)$. We can therefore calculate $g(t)$ in steps. □

Next we consider a problem concerning the controllability of $g(t)$ from one adjacent vertex.

Proposition 2.7. *On $\mathcal{N}_\Omega(v)$, suppose $m-1$ of the m boundary condition (all f_j's for $j = 1, \ldots, i-1, i+1, \ldots, m$) are known and each $f_j \in L_{2,loc}(0,\infty)$. Then for any target $g \in L^2(l_i, T)$, there is a unique $f_i \in L^2(0, T - l_i)$ such that the KN conditions* (2.22) *at v is satisfied and $u(v,t) = g(t)$, $t \in [l_i, T]$. In other words, $g(t)$ can be controlled through $f_i(t)$.*

Proof. One can transform (2.22) into a delay Volterra Second Kind integral equation with respect to $f_i'(t)$:

$$
\begin{aligned}
2(f_i)'(t-l_i) + 2\int_0^{t-l_i} & K_i(\xi,t)(f_i)'(\xi)\,d\xi = \\
& -2\sum_{j\neq i}(f_j)'(t-l_j) - 2\sum_{j\neq i} k_j(l_j,l_j)f_j(t-l_i) \\
& + 2\sum_{j\neq i}\int_{l_j}^{t} \partial_x k_j(l_j,s) f_j(t-s)\,ds - 2\sum_{j=1}^{m}\sum_{n=1}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} (f_j)'(t-(2n+1)l_j) \\
& - 2\sum_{j=1}^{m}\sum_{n=1}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} k_j((2n+1)l_j,(2n+1)l_j) f_j(t-(2n+1)l_i)
\end{aligned}
$$

$$+ 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} \int_{(2n+1)l_j}^{t} \partial_x k_j((2n+1)l_j, s) f_j(t-s)\, ds$$

$$+ mg'(t) - \sum_{j=1}^{m} \int_0^t \partial_x w_j(0, s) g(t-s)\, ds$$

$$+ 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} g'(t-2nl_j) + 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} w_j(2nl_j, 2nl_j) g(t-2nl_j)$$

$$- 2\sum_{j=1}^{m} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} \int_{2nl_j}^{t} \partial_x w_j(2nl_j, s) g(t-s)\, ds \tag{2.28}$$

where

$$K_i(\xi, t) = k_i(l_i, l_i) - \int_0^{t-l_i-\xi} \partial_x k_i(l_i, s+l_i)\, ds$$

For any given time t_0, the right hand side of (2.28) depends on $f_i(t)$ for $0 \leq t \leq t_0 - 2l_i$. So if $f_i(t)$ for $0 \leq t \leq t_0 - 2l_i$ is already known, $f_i'(t)$ for $t \in [t_0 - 2l_i, t_0]$ can be calculated by solving equation (2.28) as a Volterra integral equation of the second kind. We can therefore calculate $f_i(t)$ chronologically in time step $\Delta t = 2l_i$. □

2.5 Solving the forward problem for wave equations on general graphs

Suppose a finite metric graph Ω contains M vertices and N edges. As mentioned in the beginning of Section 2.4, solving the forward problem (2.1)-(2.4) on Ω can be reduced to finding the trace $u|_V$. Let $g_i(t) = u(v_i, t)$ for each $v_i \in V$ and let $g(t) = [g_1(t), g_2(t), \ldots, g_M(t)]^T$. The values of $g(t)$ on Γ are given as boundary conditions in (2.3). What's left to compute is $g(t)$ on $V \setminus \Gamma$. In this section we use the KN condition to derive a system of $|V| - |\Gamma|$ delay Voterra integral equations with respect to $g'(t)$ on $V \setminus \Gamma$, which can be solved in steps. This algorithm is a generalization of the algorithm given in Proposition 2.6.

Let $e_k = (v_i, v_j)$ be an edge in Ω which is identified with the interval $[0, l]$. Suppose v_i is identified with 0 and v_j is identified with l (we say e_k goes from v_i to v_j). Define operators

$W_k^{\pm} : L^2(0,T) \mapsto C([0,T]; L^2(0,l))$:

$$(W_k^-(f))(x,t) = u_k^{f,v_i}(x,t), \quad \text{and} \quad (W_k^+(f))(x,t) = u_k^{f,v_j}(x,t).$$

where u_k^{f,v_i} and u_k^{f,v_j} are obtained using (2.10) and (2.13) respectively.

Operators ∂^- and ∂^+ on $C([0,T]; L^2([0,l])$ are defined for taking derivative of the wave function along an edge outward of its vertices: $\partial^-(u) = \partial_x u(0,\cdot)$ and $\partial^+(u) = -\partial_x u(l,\cdot)$. We have four combinations of $\partial^{\pm}$ and $W_k^{\pm}$ on e_k:

$$\begin{aligned}(\partial^- W_k^-)(f) = &-f'(t) + \int_0^t \partial_x w(0,s) f(t-s)\, ds \\ &- 2f'(t-2l) - 2w(2l,2l)f(t-2l) + 2\int_{2l}^t \partial_x w(2l,s) f(t-s)\, ds \\ &- 2f'(t-4l) - 2w(4l,4l)f(t-4l) + 2\int_{4l}^t \partial_x w(4l,s) f(t-s)\, ds - \dots\end{aligned} \tag{2.29}$$

$$\begin{aligned}(\partial^+ W_k^-)(f) = &\, 2f'(t-l) + 2w(l,l)f(t-l) - 2\int_l^t \partial_x w(l,s) f(t-s)\, ds \\ &+ 2f'(t-3l) + 2w(3l,3l)f(t-3l) - 2\int_{3l}^t \partial_x w(3l,s) f(t-s)\, ds + \dots\end{aligned} \tag{2.30}$$

$$(\partial^- W_k^+)(f) = 2f'(t-l) + 2k(l,l)f(t-l) - 2\int_l^t \partial_x k(l,s)f(t-s)\,ds \\ + 2f'(t-3l) + 2k(3l,3l)f(t-3l) - 2\int_{3l}^t \partial_x k(3l,s)f(t-s)\,ds + \dots \tag{2.31}$$

$$(\partial^+ W_k^+)(f) = -f'(t) + \int_0^t \partial_x k(0,s)f(t-s)\,ds \\ - 2f'(t-2l) - 2k(2l,2l)f(t-2l) + 2\int_{2l}^t \partial_x k(2l,s)f(t-s)\,ds \\ - 2f'(t-4l) - 2k(4l,4l)f(t-4l) + 2\int_{4l}^t \partial_x k(4l,s)f(t-s)\,ds - \dots \tag{2.32}$$

We next define an $N \times M$ matrix operator U, such that U has one column for each vertex and one row for each edge. The entries of U are defined in analogy to the entries in the incident matrix of Ω: if there is an edge e_k from v_i to v_j, then $U_{k,i} = W_k^-$ and $U_{k,j} = W_k^+$. All other entries in U are zeros. As one can see, Ug gives us a column vector, where the k^{th} entry equals to $u_k(x,t)$ on the edge e_k. If $g(t)$ is known then Ug represents the solution for (2.1)-(2.4).

Operator K is defined as an $M \times N$ matrix operator. Its entries are defined in analogy to the transpose of the incident matrix of Ω: if there is an edge e_k from vertex v_i to v_j, then $K_{ik} = \partial^-$ and $K_{jk} = \partial^+$. All other entries in K are zeros. Now KUg is a column vector of M entries. The i^{th} entry represents the derivative sum of u at a vertex v_i (over all incident edges, outwards of v_i).

Since the KN conditions (2.2) hold only at the internal vertices, we use an $M \times M$ diagonal matrix D to pick out the interior vertices. Let $D_{ij} = 1$ if $i = j$ and $v_i \in V \setminus \Gamma$,

$D_{ij} = 0$ otherwise. So the KN conditions on $V \setminus \Gamma$ can be represented by

$$DKUg = 0 \tag{2.33}$$

Equation (2.33) is a system of $|V| - |\Gamma|$ equations similar to Equation (2.22). With a transformation similar to (2.25), we have $|V| - |\Gamma|$ delay Volterra integral equations of the form

$$\deg(v_i)g_i'(t) - \int_0^t H_i(s,t)g_i'(s)\,ds = F_i(t). \tag{2.34}$$

where for all $i = 1, \dots, |V| - |\Gamma|$, $H_i(s,t)$ is known and $F_i(t)$ depends on the vector function g with arguments delayed by at least $\Delta := \min_{i=1,\dots,N}(l_i)$. So if g on $[0, t-\Delta]$ is known, we can calculate g' on $[t-\Delta, t]$ using (2.33). Since $g(0) = 0$, g on $V \setminus \Gamma$ can be calculated in steps.

2.6 Controllability on a tree graph

In this section let Ω be a finite metric tree. We prove exact shape, velocity, and full controllability for the system (2.1)-(2.4) on Ω by constructing the corresponding boundary control functions. Shape and velocity boundary control functions are constructed using pure dynamical method. The boundary control functions for full controllability are constructed using combined dynamical and moment method approach. First we state that for exact controllability the number of controlled boundary vertices cannot be less than $|\Gamma| - 1$.

Proposition 2.8. *If* $|\Gamma_1| < |\Gamma| - 1$, *the system* (2.1)-(2.4) *is not exactly shape, velocity controllable or exactly controllable in any time* T.

Proof. If the number of boundary control functions is less than $|\Gamma| - 1$, there is a path connecting two boundary vertices with zero Dirichlet boundary conditons. The proposition follows from the fact that the system of serially connected strings:

$$(u_j)_{tt} - (u_j)_{xx} + q_j(x)(u_j) = 0, \quad x \in (0, l_j), t > 0, j = 1, \dots N$$

$$u_j(l_j, t) = u_{j+1}(0, t) = f_j(t), \quad j = 1, \ldots N - 1$$

$$u_1(0, t) = u_N(l_n, t) = 0$$

$$u_j|_{t \leq 0} = 0, \quad j = 1, \ldots, N$$

is not exactly controllable in any time interval $[0, T]$. The proof of this fact for the full controllability is presented in [*Avdonin and Ivanov*, 1995] Section 7.2 and [*Dáger and Zuazua*, 2006] Section 6.3 for the case $q_j = 0$ for all j, but it can be easily extended to the case of nontrivial potentials and to the case of the shape or velocity controllability. □

For tree graphs with at least $|\Gamma| - 1$ actively controlled boundary vertices, we control the terminal shape/velocity one path at a time.

Definition 2.9. ***Path:*** *The definition of a path on a metric graph is similar to the definition of a path on a combinatorial graph, it is an alternating sequence of vertices and edges* $(v_n, e_n, v_{n-1}, e_{n-1}, \ldots, v_1, e_1, a)$ *with no repeated edges or vertices. The main difference is in our definition, the last point* a *is not necessarily a vertex, it can be any point in the graph (in which case* $e_1 = (v_1, a)$ *is a section of an edge). Since two points on a tree uniquely determine a path, we denote the path* $(v_n, e_n, v_{n-1}, e_{n-1}, \ldots, v_1, e_1, a)$ *as* $P(v_n, a)$.

We represent Ω with a union of disjoint paths (except for the end points). The starting point of each path must have an active control function. We first show such path union representations exist if $|\Gamma_1| \geq |\Gamma| - 1$.

Lemma 2.10. *A tree graph* Ω *has a union representation* U,

$$\Omega = \cup_{\gamma \in \Gamma_1, a \in \Omega} P(\gamma, a),$$

of disjoint (except the end points) paths (they are referred to as U*-paths) if and only if* $|\Gamma_1| = |\Gamma|$ *or* $|\Gamma_1| = |\Gamma| - 1$.

Proof. Whether $|\Gamma_1| = |\Gamma|$ or $|\Gamma_1| = |\Gamma| - 1$, the existence of a path union representation of Ω can be proved by induction on the number of edges. Suppose Ω contains one edge e and two vertices γ_1 and γ_2. If $\Gamma_1 = \{\gamma_1, \gamma_2\}$, one path union representation is $\Omega = P(\gamma_1, a) \cup P(\gamma_2, a)$, where a is a point in the interior of e; two other path union representations are $\Omega = P(\gamma_1, \gamma_2)$ and $\Omega = P(\gamma_2, \gamma_1)$. If Γ_1 contains only γ_1, then the only path union representation is $\Omega = P(\gamma_1, \gamma_2)$.

Assume path union representations exist for all trees of up to $N-1$ edges and $|\Gamma_1| \geq |\Gamma| - 1$ ($N \geq 2$). Suppose Ω contains N edges and $|\Gamma_1| \geq |\Gamma| - 1$, then Γ_1 contains at least one boundary vertex γ_1. Let v be the interior vertex adjacent to γ_1, then $\Omega = P(\gamma_1, v) \cup (\Omega \setminus P(\gamma_1, v))$, where $\Omega \setminus P(\gamma_1, v)$ is a tree graph with at most $N - 1$ edges, and has a path union representation by the induction hypothesis. Thus Ω has a path union representation. Note that we did not assume our tree graphs to be series-reduced (a series reduced tree is a tree graph that contains no vertices of degree two, see [*Bergeron et al.*, 1998]).

Now suppose there are at least two boundary vertices $\gamma_1, \gamma_2 \in \Gamma_0$ in Ω and a path union representation exist for Ω, then γ_1 and γ_2 are the end vertices of some U-paths $P(\gamma_3, \gamma_1)$ and $P(\gamma_4, \gamma_2)$. Since $P(\gamma_3, \gamma_1)$ and $P(\gamma_4, \gamma_2)$ are disjoint but Ω is connected, there is a path $P(v_1, v_2)$ from $P(\gamma_3, \gamma_1)$ to $P(\gamma_4, \gamma_2)$. Since v_1, v_2 are interior points, $P(v_1, v_2) \notin U$. The path $P(v_1, v_2)$ cannot be a segment of a U-path since it intersects both $P(\gamma_3, \gamma_1)$ and $P(\gamma_4, \gamma_2)$. Neither can $P(v_1, v_2)$ contain the end point of two U-paths, since if both U-paths start at some boundary vertices, and end on some point on $P(v_1, v_2)$, at least one path intersects $P(\gamma_2, \gamma_1)$ or $P(\gamma_4, \gamma_3)$ at a vertex that is not its end point. So such a U-path union representation of Ω does not exist.

□

Definition 2.11. ***Watershed and Tributary***: *Let U be a path union representation of Ω*

$$\Omega = \cup_{\gamma \in \Gamma_1} P(\gamma, a), \quad a \in \Omega.$$

If $P(\gamma_1, v) \cap P(\gamma_2, a) = \{v\}$ and $v \neq a$, then path $P(\gamma_1, v)$ is called a ***tributary*** *of path $P(\gamma_2, a)$. We call a path that is not a tributary to another path a* ***level-****1 path or mainstream. A tributary of level i paths is called a level $i+1$ path for all $i \geq 1$.*

We call the union of $P(\gamma_2, a)$, its tributaries, and tributaries of tributaries of all levels the ***watershed*** *of path $P(\gamma_2, a)$, or $W(P(\gamma_2, a))$.*

In the following lemma, we show that the terminal shape of one path can be controlled from its beginning boundary vertex. This gives the possibility of controlling the terminal shape of the whole graph one path at a time.

Lemma 2.12. *Let $P = (v_n, e_n, v_{n-1}, e_{n-1}, \ldots, v_1, e_1, a)$ be a path in Ω where $v_n \in \Gamma_1$ and $a \in \Omega$.*[2] *Let l_i be the length of e_i for each i and $l_P = \sum_{i=1}^{n} l_i$ (See Figure 2.3.) Suppose all vertices in $\Gamma \setminus \{v_n\}$ are fixed. That is, $f(t)|_\gamma = 0$ for $\gamma \neq v_n$. Then for any $\phi \in L^2(P)$, there exists a unique function $f \in L^2(0, l_P)$ such that $u^{f, v_n}(\cdot, l_P)|_P = \phi(\cdot)|_P$.*

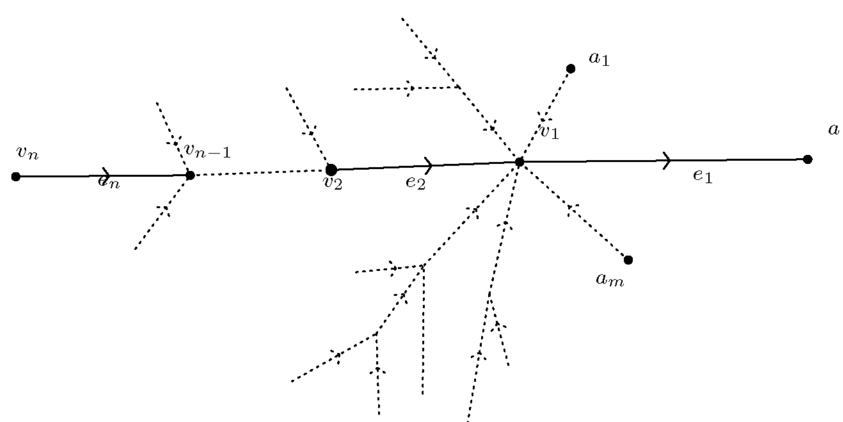

Figure 2.3: Control on path $P(v_n, a)$

[2]Although v_n is a boundary vertex, we label it v_n instead of γ_n, so we can refer to vertices on P as v_i's together.

Proof. Denote $u|_{e_i}(x,t) = u_i(x,t)$, $\phi|_{e_i}(x) = \phi_i(x)$, $u(a,t) = g_0(t)$, $u(v_i,t) = g_i(t)$ for $i = 1,\dots,n$, and let $f(t) = g_n(t)$. Note that only $f(t)$ is a boundary control and all $g_i(t)$ for $t = 0,\dots,n-1$ are viewed as virtual controls. The plan is to choose appropriate $f(t)$ so that through KN conditions at each v_i, $u^{f,v_n}(v_i,t) = g_i(t)$, and for each $i = 1,\dots,n$,

$$u_i^{g_i,v_i}(\cdot,l_P) + u_i^{g_{i-1},v_{i-1}}(\cdot,l_P) = \phi_i(\cdot).$$

Since the wave propagates on P at the unit speed, the minimal time it takes for the wave generated by f to reach a is l_P. So the shape controllability is not achievable in a time less than l_P. We will show f is uniquely determined if we require f to vanish outside $[0,l_P]$.

The construction of f is by constructing g_i's in the ascending order of i. Since we require $\text{supp}(f) = [0,l_P]$, and all g_i's are the results of wave propagation generated by f, $\text{supp}(g_i) = \left(\sum_{j=i+1}^n l_j, l_P\right]$ for $i = 0,\dots,n-1$. It follows immediately $g_0(t) = 0$. By Proposition 2.5 (1), there is a unique square integrable $g_1(t) : \text{supp}(g_1) = \left(\sum_{j=2}^n l_j, l_P\right]$ such that

$$u_1^{g_1,v_1}(\cdot,l_P) = \phi_1(\cdot).$$

Since we cannot place $g_1(t)$ at v_1 directly, we have to construct $g_2(t)$ at v_2 such that $g_2(t)$ controls $g_1(t)$ through Proposition 2.7. Another constraint for g_2 is the terminal shape on e_2. We label all vertices in $\mathcal{N}_\Omega(v_1) \setminus \{v_2, v_1, a\}$ from a_1 to a_m and set $h_j(t) = u(a_j,t)$. For $j = 1,\dots,m$, pick $\gamma_j \in \Gamma$ such that $a_j \in P(\gamma_j, v_1)$, and denote G_j the watershed of $P(\gamma_j, v_1)$. Note that on each watershed G_j, the vertex v_1 is the only boundary vertex with an active input function, and all other boundary vertices are fixed. So $h_j(t) = u_{G_j}^{g_1,v_1}(a_j,t)$, which can be computed using the method described in Section 2.5. On $\mathcal{N}_\Omega(v_1)$, now we have $g_0 = 0$, g_1 and $h_1, \dots h_m$ are all known, by Proposition 2.7, there is a unique square integrable function $g_2^1(t) : \text{supp}(g_2^1) = \left(\sum_{j=3}^n l_j, l_P - l_2\right]$ such that the KN condition at v_1 is satisfied and $u(v_1,t) = g_1(t)$.

To satisfy the second constraint, we construct another component $g_2^2(t)$ of g_2. Since $g_2^1(t)$

generates $g_1(t)$ and $g_1(t)$ controls the terminal shape of e_1, $g_2^2(t)$ should not have an impact to $g_1(t)$, therefore $g_2^2(t) = 0$ for $t < l_P - l_2$. By Proposition 2.5 (1), there is a unique square integrable $g_2^2(t) : \text{supp}(g_2^2) = (l_P - l_2, l_P]$ such that

$$u_2^{g_2^2, v_2}(\cdot, l_P) = \phi_2(\cdot) - u_2^{g_2^1, v_2}(\cdot, l_P) - u_2^{g_1, v_1}(\cdot, l_P).$$

Let $g_2(t) = g_2^1(t) + g_2^2(t)$, so $g_2(t) \in L^2\left(\sum_{j=3}^n l_j, l_P\right)$. Since the two components of $g_2(t)$ are supported on disjoint intervals, and each is uniquely determined on its support, $g_2(t)$ is the unique function satisfying both constraints imposed by $g_1(t)$ and $\phi_2(\cdot)$.

Similarly, we will find $g_3(t) = g_3^1(t) + g_3^2(t) \in L^2\left(\sum_{j=4}^n l_j, T\right)$, such that $\text{supp}(g_3^1) = \left(\sum_{j=4}^n l_j, l_P - l_3\right]$ and g_3^1 generates g_2; $\text{supp}(g_3^2) = (l_P - l_3, l_P]$ and $u_3^{g_3^2, v_3}(\cdot, l_P) = \phi_3(\cdot) - u_3^{g_3^1, v_3}(\cdot, l_P) - u_3^{g_2, v_2}(\cdot, l_P)$. Repeat this process until $f = g_n \in L^2(0, l_P)$ is found, such that $u^{f, v_n}(\cdot, T)|_P = \phi(\cdot)|_P$. From the construction process we conclude f is uniquely determined.

□

Corollary 2.13. *Let P be a path in Ω as defined in Lemma 2.12. For any $\psi \in \mathcal{H}^{-1}$, There exists a function $f \in L^2(0, l_P)$ such that $u_t^{f, v_n}(\cdot, l_P)|_P = \psi(\cdot)|_P$.*

Proof. Similar to Proposition 2.5 Part 2, the map $(f, t) \mapsto u_t^{f, v_n}(\cdot, t)$, where

$$< u_t^{f, v_n}(\cdot, T), g(\cdot) >= \sum_{j=1}^{|E|} \int_0^{l_j} \partial_t u_j(x, T) g(x)\, dx$$

extends from a continuous mapping from $C^2(0, T) \times (0, T]$ to $\mathcal{H}^{-1}$ to a continuous mapping from $L^2(0, T) \times (0, T]$ to $\mathcal{H}^{-1}$. Proof of this Corollary is then similar to the proof of Lemma 2.12. Instead of using Proposition 2.5 (1) and Proposition 2.7 to obtain the shape control, we use Proposition 2.5 (2) and Proposition 2.7 to obtain the velocity control. □

We now use Lemma 2.12 (respectively Corollary 2.13) to prove shape (respectively velocity) controllability on tree graphs, one path at a time.

Theorem 2.14. *Let Ω be a tree graph where $|\Gamma_1| \geq |\Gamma| - 1$. Let U be a path union representation of Ω*

$$\Omega = \cup_{\gamma \in \Gamma_1} P(\gamma, a), \quad a \in \Omega,$$

of disjoint (except the end points) paths $P(\gamma, a)$. Let $T = \max_{P \in U} \operatorname{length} P(\gamma, a)$. Then

1. *For any $\phi \in \mathcal{H}$, there exists a boundary control $f \in \mathcal{F}^T$ such that $u^{f,\Gamma_1(U)}(\cdot, T) = \phi(\cdot)$.*
2. *For any $\psi \in \mathcal{H}^{-1}$, there exists a boundary control $f \in \mathcal{F}^T$ such that $u_t^{f,\Gamma_1(U)}(\cdot, T) = \psi(\cdot)$.*
3. *Let the minimal time for both shape and velocity controllability be T_s, then*

$$T_s = \min_{U} \max_{P \in U} \operatorname{length} P(\gamma, a).$$

Proof. **Part 1**: We first control the shape on mainstreams, then on higher level paths successively. Let the set of mainstreams in the union U be $\{P(\gamma_i, a_i)\}$, where $i = 1, \ldots, m$. For each i, let $l_{P_i} = \operatorname{length}(P(\gamma_i, a_i))$, and denote the watersheds of $P(\gamma_i, a_i)$ as W_i. All W_i's are disjoint except for the endpoint of the mainstreams, and $\cup_{i_1}^{m} W_i = \Omega$. By Lemma 2.12, for each $i = 1, \ldots, m$, we can find a unique function $f_i \in L^2(T - l_{P_i}, T)$ such that

$$u^{f_i,\gamma_i}|_{P(\gamma_i,a_i)} = \phi(\cdot)|_{P(\gamma_i,a_i)}.$$

Since the time for the wave generated by f_i to propagate to a_i is l_{P_i}, $u(a_i, T) = 0$, and f_i has no impact on watersheds different from W_i. That is,

$$u^{f_i,\gamma_i}|_{\Omega \setminus W_i}(\cdot, T) = 0.$$

So the shape control on level-1 paths $P(\gamma_i, a_i)$ is obtained through f_i for $i = 1, \ldots, m$.

The same method is then used to obtain the shape control on level-2 paths. For instance, suppose $P(\gamma_j, a_j)$ is a tributary of a mainstream $P(\gamma_i, a_i)$. Then $W(P(\gamma_j, a_j)) \subset W(P(\gamma_i, a_i))$. By Lemma 2.12 we can find$f_j(t) \in L^2(T - T_j, T)$ such that

$$u^{f_j,\gamma_j}|_{P(\gamma_j,a_j)} = \phi(\cdot)|_{P(\gamma_j,a_j)} - u^{f_i,\gamma_i}|_{P(\gamma_i,a_i)}.$$

Again f_j has no impact on $\Omega \setminus W(P(\gamma_j, a_j))$.

Since there are a finite number of paths in the union U, repeat this process we have the final shape control on paths of all levels within time T.

Part 2: The proof for velocity controllability is similar to the proof of the shape controllability in Part 1. Using Corollary 2.13 we first obtain the velocity on mainstreams, then on higher level paths successively.

Part 3: Let U_{min} be the union representation such that

$$\max_{P \in U_{min}} \text{length}\, P(\gamma, a) = \min_{U} \max_{P \in U} \text{length}\, P(\gamma, a).$$

By Part 1 and Part 2 we can construct boundary control functions to have shape or velocity controllability at $T = \max_{P \in U_{min}} \text{length}\, P(\gamma, a)$. On the other hand, if $T < \max_{P \in U_{min}} \text{length}\, P(\gamma, a)$ one can easily construct examples of the graphs which are not shape or velocity controllable at that T. We can consider, for example, an interval with one or two boundary controls, a star graph with equal length edges and controls at all boundary vertices.

Therefore

$$T_s = \max_{P \in U_{min}} \text{length}\, P(\gamma, a)$$

gives the minimal shape/velocity controllability time guaranteed for all trees.

□

Remark 1. *The statement in Part 3 of Theorem 2.14 is a general statement for all tree graphs. In some particular cases, one can have shape controllability at a time less than* $\max_{P \in U_{min}} \operatorname{length} P(\gamma, a)$.

For example, consider a tree graph as shown in Figure 2.4, where the lengths of edges e_0, e_1, e_2 *are* $l_0 = 1$, $l_1 = 1$, *and* $l_2 = 4$ *(the common point* v *is identified as* 0*). Assume* γ_0 *is fixed and all three edges have zero potentials. Let the target functions be* $\phi_0(x), \phi_1(x)$, $\phi_2(x)$ *on* e_0, e_1, e_2 *respectively. Let*

$$f_1(t) = \begin{cases} \frac{3}{2}\left(\phi_2(t) - \phi_0(t)\right), & 0 \le t \le 1 \\ \frac{3}{2}\phi_2(2-t), & 1 < t \le 2 \\ \phi_1(t-2), & 2 < t \le 3. \end{cases}$$

be placed at γ_1,

$$f_2(t) = \begin{cases} \phi_2(t+1) - \phi_2(1-t) + \phi_0(1-t), & 0 \le t \le 1 \\ \phi_2(t+1), & 1 < t \le 3 \end{cases}$$

be placed at γ_2, *then at time* $T = 3$ *the tree graph has shape controllability on all three edges and*

$$T < \max_{P \in U_{min}} \operatorname{length} P(\gamma, a) = 4.$$

In Figure 2.4, the section of the graph filled by wave propagated from γ_1 *is dashed, the section filled by wave propagated from* γ_2 *is dotted, and the section filled by wave propagated from both* γ_1 *and* γ_2 *is solid.*

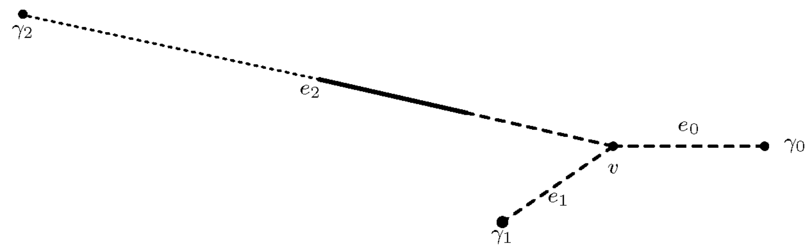

Figure 2.4: Shape controllability at $T = 3$

Remark 2. *Let T_f be the least time it takes for the propagation of waves generated at the controlled boundary vertices to fill Ω. From the geometry one can easily derive the inequalities:*

$$T_f \leq \max_{P \in U_{min}} \text{length}\, P(\gamma, a) \leq \text{diameter}(\Omega), \tag{2.35}$$

which are valid for any tree graph. The example from Figure 2.4 is also an example where the inequalities in (2.35) *are strict.*

We still assume the lengths of edges e_0, e_1, e_2 are $l_0 = 1$, $l_1 = 1$, and $l_2 = 4$, and γ_0 is fixed. In the control function $f = (f_1, f_2)$, $f_1(t)$ is placed at γ_1, f_2 is placed at γ_2. One can see that $T_f = 2.5$. Indeed, in Figure 2.5, we dash the section of the graph filled by wave propagated from γ_1, and dot the section filled by wave propagated from γ_2. Since the optimal path union is $\{\gamma_1, e_1, v, e_0, \gamma_0\} \cup \{\gamma_2, e_2, v\}$, $\max_{P \in U_{min}} \text{length}\, P(\gamma, a) = 4$. On the other hand $\text{diameter}(\Omega) = l_0 + l_2 = 5$. *So, we have the strict inequalities*

$$T_f < \max_{P \in U_{min}} \text{length}\, P(\gamma, a) < \text{diameter}(\Omega).$$

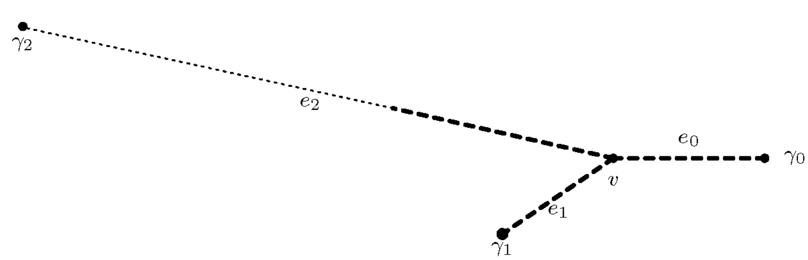

Figure 2.5: $T_f = 2.5$

Theorem 2.15. *Suppose the system* (2.1)-(2.4) *has both shape and velocity controllability within time* T*. Then for any* $(\phi, \psi) \in \mathcal{H} \times \mathcal{H}^{-1}$*, there exists a boundary control* $f \in \mathcal{F}^{2T}$ *such that* $u^{f,\Gamma_1}(\cdot, 2T) = \phi(\cdot)$ *and* $u_t^{f,\Gamma_1}(\cdot, 2T) = \psi(\cdot)$*.*

Proof. To prove the full controllability of the system (2.1)-(2.4) we combine the dynamical approach used for proving the shape and velocity controllability with the spectral methods. We consider the following eigenvalue problem on the graph Ω:

$$-\varphi''(x) + q(x)\varphi(x) = \omega^2\varphi(x),$$

$$\sum_{e_j \sim v} \partial_j \varphi(v) = 0 \quad \forall\, v \in V \setminus \Gamma;$$

φ is continuous on Ω.

$$\varphi|_\Gamma = 0.$$

It is known that the spectrum $\{\omega_n\}_{n\in\mathbb{N}}$ of this problem is purely discrete and the eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ form an orthonormal basis in $\mathcal{H}$. The solution $u(\cdot, t)$ of (2.1)-(2.4) can be represented in a form of a series with respect to $\{\phi_n\}$ (see, for example, [*Avdonin and Nicaise*, 2015], [*Avdonin and Ivanov*, 1995], [*Belishev and Vakulenko*, 2006]).

Control problems are reduced to moment problems using the Fourier method. The shape controllability is equivalent to the solvability of the moment problem (see [*Avdonin and Ivanov*, 1995], Chapter 3)

$$a_n = \langle f, \mathfrak{s}_n \rangle_{\mathcal{F}^T}, \quad n \in \mathbb{N}, \quad \mathfrak{s}_n(t) = \frac{\varphi_n'|_{\Gamma_1}}{\omega_n} \sin \omega_n t. \tag{2.36}$$

Solvability means that for any $\{a_n\} \in l^2$, there exists $f \in \mathcal{F}^T$ satisfying (2.36). For simplicity we assume here that $\omega_n \neq 0$. If $\omega_n = 0$ we use t to replace $\frac{\sin \omega_n t}{\omega_n}$ in the expression of (2.36).

The velocity controllability in time T is equivalent to the solvability of the moment problem

$$b_n = \langle g, \mathfrak{c}_n \rangle_{\mathcal{F}^T}, \quad n \in \mathbb{N}, \quad \mathfrak{c}_n(t) = \frac{\varphi_n'|_{\Gamma_1}}{\omega_n} \cos \omega_n t. \tag{2.37}$$

Denote by $f_-(t)$ the odd extension of $f(t)$ from $[0, T]$ to $[-T, T]$ and by $g_+(t)$ the even extension of $g(t)$. We observe that the function

$$h(t) = \frac{f_-(t) + g_+(t)}{2}$$

solves both moment problems

$$a_n = \langle h, \mathfrak{s}_n \rangle_{\tilde{\mathcal{F}}^T}, \quad b_n = \langle h, \mathfrak{c}_n \rangle_{\tilde{\mathcal{F}}^T}, \tag{2.38}$$

where $\tilde{\mathcal{F}}^T := L^2([-T, T]; \mathcal{R}^m)$. It means that the moment problem (2.38) is solvable for any sequences $\{a_n\}, \{b_n\} \in l^2$. Therefore, the system (2.1)-(2.4) is fully controllable in the time interval $[-T, T]$. Since the system is linear with time independent coefficients, it is also fully controllable in the time interval $[0, 2T]$.

□

Chapter 3: Inverse Problem for the Wave Equation on Graphs

3.1 Introduction

Quantum graphs or differential equation networks (DENs) are metric graphs with differential equations defined on the edges coupled by matching conditions at the vertices. DENs play a fundamental role in many problems of science and engineering and give rise to challenging problems in various areas of mathematics from combinatoric graph theory to PDE and spectral theory. Inverse theory of DENs constitutes an important part of the rapidly developing area of applied mathematics — analysis on graphs. Inverse spectral and scattering problems for quantum graphs (almost exclusively, for trees – graphs without cycles) were considered in many papers (see, e.g., [*Gerasimenko and Pavlov*, 1988; *Berkolaiko and Kuchment*, 2013; *Carlson*, 1999; *Pivovarchik*, 1999; *Belishev*, 2004; *Yurko*, 2005; *Brown and Weikard*, 2005] and references therein).

In [*Avdonin and Kurasov*, 2008] inverse dynamical, spectral and scattering problems for the Schrödinger equation on quantum trees were studied and the leaf peeling (LP) was proposed. This method was extended to boundary inverse problems for various types of PDEs on trees and various matching conditions in a series of subsequent papers ([*Avdonin et al.*, 2010b,c; *Avdonin and Bell*, 2015; *Avdonin et al.*, 2015, 2017b, 2019]). The LP method allows one to recover not only coefficients in the equations on the graphs edges but also the lengths of the edges and topology (connectivity) of the graph.

There are two versions of the method: spectral, where the Titchmarsh–Weyl (TW) matrix function serves as inverse data, and dynamical with the response operator as inverse data. Both identification procedures are recursive — they allow recalculating the TW-function (or response operator) from the original tree to the smaller trees, "pruning" leaves step by step up to the rooted edge. The dynamical version of the LP method is much more complicated for theoretical justification than the spectral one, but the main steps of the dynamical algorithm contain only Volterra type equations of the second kind and triangular matrix equations.

Therefore, this version may serve as a basis for developing effective numerical algorithms solving inverse problems on quantum graphs. The most comprehensive exposition of the LP method is presented in [*Avdonin et al.*, 2017a], but even there not all steps of the dynamical version are described and justified in all details. The first main goal of this chapter is to give a complete and rigorous presentation of this version.

Using the LP method we solve the dynamical boundary inverse problem for the wave equation on a tree graph, i.e. recover the topology of the graph, lengths of the edges and coefficients in the equations on the edges by the given boundary Dirichlet-to-Neumann map (response operator). We do not use any spectral methods at any step of our solution. To our best knowledge it is the first solution for the inverse problem for general tree graphs by pure dynamical methods. We present the algorithm solving the problem and indicate the time required for this procedure.

The LP method is based on the powerful boundary control method in inverse theory, which uses deep connections between controllability and identifiability of dynamical systems. In the current chapter we use our recent results presented in [*Avdonin and Zhao*, 2021a], where new algorithms solving forward and control problems for the wave equation on graphs were proposed. These results allow us to significantly simplify the presentation of the dynamical LP version and solution of the inverse problem.

This chapter is organized as follows. Section 3.2 introduces a finite compact metric tree graph Ω and the wave equation associated with Ω. Section 3.3 presents several useful formulae for solving the forward initial boundary value problem on the graph and Duhamel's principle. Section 3.4 introduces the response function and the dynamical inverse problem. In Section 3.5 we present our main result, the Leaf Peeling theorem, and complete the solution of the dynamical inverse problem.

3.2 Preliminaries

Let Ω be a finite compact rooted tree graph with edges $E = \{e_1, \ldots, e_N\}$ and vertices $V = \{v_1, \ldots, v_{N+1}\}$; every edge e_j is identified with a real interval $(0, l_j)$ where l_j is referred to as the length of e_j. On each e_j, we identify the further vertex from the root vertex as $x = 0$, and the closer vertex to the root vertex as $x = l_j$. The set of boundary (pendant) vertices $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is a subset of V consisting of vertices with degree 1. The root vertex is labelled as γ_m. The other pendant vertices $\{\gamma_1, \ldots, \gamma_{m-1}\}$ are called **leaf vertices**. We put $\Gamma_0 := \{\gamma_m\}$ and $\Gamma_1 := \{\gamma_1, \ldots, \gamma_{m-1}\}$. The edges incident to the leaf (respectively root) vertices are called the leaf (respectively root) edges. For $v_1, v_2 \in V$, we denote the path length between v_1 and v_2 as $\operatorname{dist}(v_1, v_2)$, which is well defined since there is a unique path between any two vertices on a tree graph.

The graph Ω naturally determines the Hilbert space of square integrable functions

$$\mathcal{H} := L^2(\Omega) = \oplus_{e \in E} L^2(e).$$

The space $\mathcal{H}^1$ consists of continuous functions ϕ on Ω such that $\phi|_e$ belongs to the Sobolev space $H^1(e)$ for each edge e and $\phi|_\Gamma = 0$. The space $\mathcal{H}^{-1}$ is defined as the dual space of $\mathcal{H}^1$.

For convenience we label $u|_{e_j}$ and $q|_{e_j}$ as u_j and q_j. Let $\partial u_j(v)$ be the derivative of u at the vertex v taken along the edge e_j in the direction outward of v. For $\gamma \in \Gamma$, since there is only one edge incident to γ, we denote the derivative of u outward of γ as $\partial u(\gamma)$. By $E(v)$ we denote the set of edges incident to v. We associate the following initial boundary value problem (IBVP) on the graph Ω with the potential function q such $q_j \in C[0, l_j]\ \forall j$:

$$u_{tt} - u_{xx} + qu = 0 \quad \text{in } \{\Omega \setminus V\} \times (0, T), \tag{3.1}$$

$$u|_{t \le 0} = 0 \text{ in } \Omega, \tag{3.2}$$

$$\begin{cases} \sum_{e_j \in E(v)} \partial u_j(v,t) = 0, \\ u_i(v,t) = u_j(v,t), \end{cases} \quad v \in V \setminus \Gamma,\ e_i, e_j \in E(v),\ t \in [0,T], \tag{3.3}$$

$$u = f \text{ on } \Gamma_1 \times [0,T], u = 0 \text{ on } \Gamma_0 \times [0,T] \tag{3.4}$$

Here $T > 0$ and $f \in \mathcal{F}^T := L^2([0,T);\mathbb{R}^{m-1})$. There is a unique generalized solution to the problem (3.1)–(3.4) (the forward problem). Under different conditions on q it was proved (see, e.g. [*Ali Mehmeti and Meister*, 1989; *Ali Mehmeti*, 1994; *Lagnese et al.*, 1994; *Dáger and Zuazua*, 2006; *Avdonin and Nicaise*, 2015]) that

$$u^f \in C([0,T];\mathcal{H}) \cap C^1([0,T];\mathcal{H}^{-1}).$$

3.3 The forward problem and the Duhamel's principle

In Propositions 3.16 and 3.17 below we cite our results from [*Avdonin and Zhao*, 2021a] that are used in the present chapter. Proposition 3.16 contains several useful formulae regarding solutions and end point responses of the forward problem for the equation (3.1) on the simplest graph – interval $[0,l]$.

Proposition 3.16. *Let $w(x,t)$ be the solution to the Goursat problem*

$$\begin{cases} w_{tt} - w_{xx} + q(x)w = 0, \quad 0 < x < t < \infty \\ w(0,t) = 0, w(x,x) = -\frac{1}{2}\int_0^x q(s)\,ds \end{cases} \tag{3.5}$$

in which the $q(x)$ is an extension of the potential function $q(x)$ in (3.1) *from $[0,l]$ to $[0,\infty)$, following the rule $q(2nl \pm x) = q(x)$. Let $k(x,t)$ be the solution to the Goursat problem*

$$\begin{cases} k_{tt} - k_{xx} + p(x)k = 0, \quad 0 < x < t < \infty \\ k(0,t) = 0, \quad k(x,x) = -\frac{1}{2}\int_0^x p(\eta)\,d\eta. \end{cases} \tag{3.6}$$

in which $p(x)$ is an extension of the potential function $q(x)$ in (3.1) from $[0,l]$ to $[0,\infty)$, following the rule $p(2nl \pm x) = q(l-x)$ for $0 \le x \le l$.

Let $u^{f,-}$ be the solution to the equation (3.1) on the interval $[0,l]$ with the initial condition (3.2), boundary conditions $u(0,t) = f(t)$ and $u(l,t) = 0$; let $u^{f,+}$ be the solution to the equation (3.1) on the interval $[0,l]$ with the initial condition (3.2), boundary conditions $u(0,t) = 0$ and $u(l,t) = f(t)$. Then: $u^{f,-}$ and $u^{f,+}$ can be expressed in terms of w and k :

$$\begin{aligned} u^{f,-}(x,t) = f(t-x) + \int_x^t w(x,s)f(t-s)\,ds \\ - f(t-2l+x) - \int_{2l-x}^t w(2l-x,s)f(t-s)\,ds \\ + f(t-2l-x) - \int_{2l+x}^t w(2l+x,s)f(t-s)\,ds \\ - f(t-4l+x) - \int_{4l-x}^t w(4l-x,s)f(t-s)\,ds + \dots \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} u^{f,+}(x,t) = f(t-l+x) + \int_{l-x}^t k(l-x,s)f(t-s)\,ds \\ - f(t-l-x) - \int_{l+x}^t k(l+x,s)f(t-s)\,ds + \dots r \\ + f(t-3l+x) + \int_{3l-x}^t k(3l-x,s)f(t-s)\,ds \\ - f(t-3l-x) - \int_{3l+x}^t k(3l+x,s)f(t-s)\,ds + \dots . \end{aligned} \tag{3.8}$$

We notice that if $q \in C[0,l]$, then the kernels w and k are continuously differentiable [*Avdonin and Mikhaylov*, 2010]. Furthermore, from (3.7) and (3.8) we obtain the following formulae for responses at end points of the interval (these formulae will be discussed in Sections 3.4 and 3.5):

$$
\begin{aligned}
\partial_x u^{f,-}(0,t) = -f'(t) + \int_0^t \partial_x w(0,s) f(t-s)\, ds \\
- 2f'(t-2l) - 2w(2l,2l) f(t-2l) + 2\int_{2l}^t \partial_x w(2l,s) f(t-s)\, ds \\
- 2f'(t-4l) - 2w(4l,4l) f(t-4l) + 2\int_{4l}^t \partial_x w(4l,s) f(t-s)\, ds - \ldots,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
-\partial_x u^{f,-}(l,t) = 2f'(t-l) + 2w(l,l) f(t-l) - 2\int_l^t \partial_x w(l,s) f(t-s)\, ds \\
+ 2f'(t-3l) + 2w(3l,3l) f(t-3l) - 2\int_{3l}^t \partial_x w(3l,s) f(t-s)\, ds + \ldots,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\partial_x u^{f,+}(0,t) = 2f'(t-l) + 2k(l,l) f(t-l) - 2\int_l^t \partial_x k(l,s) f(t-s)\, ds \\
+ 2f'(t-3l) + 2k(3l,3l) f(t-3l) - 2\int_{3l}^t \partial_x k(3l,s) f(t-s)\, ds + \ldots,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
-\partial_x u^{f,+}(l,t) = -f'(t) + \int_0^t \partial_x k(0,s) f(t-s)\, ds \\
- 2f'(t-2l) - 2k(2l,2l) f(t-2l) + 2\int_{2l}^t \partial_x k(2l,s) f(t-s)\, ds \\
- 2f'(t-4l) - 2k(4l,4l) f(t-4l) + 2\int_{4l}^t \partial_x k(4l,s) f(t-s)\, ds - \dots .
\end{aligned}
\tag{3.12}
$$

The next proposition uses formulas (3.9)–(3.12), and the Kirchhoff-Neumann (KN) condition (3.3) at an internal vertex of Ω. It was proved in [*Avdonin and Zhao*, 2021a].

Proposition 3.17. *Let $o^f(t) := \left(u^f|_{V\setminus\Gamma}\right)(t)$ be the vector function of the trace of u^f on the interior vertices of Ω. Let $o_i(t) = \left(u^f|_{v_i}\right)(t)$ for some $v_i \in V \setminus \Gamma$, and $h_j(t)$ for $j = 1, \dots, \deg(v_i)$ be the trace functions of u on the adjacent vertices of v_i (depending on the location of v_i, some of its adjacent vertices may be boundary vertices, some of them may be interior vertices). For each edge incident to v_i, we identify the edge with an interval of its length $(0, l_i)$, where v_i is identified with 0, and the other vertex is identified with l_i (see Figure 3.1).*

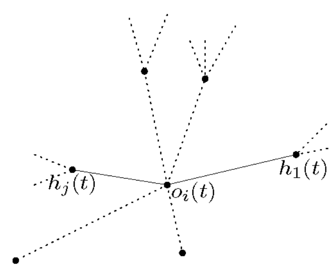

Figure 3.1: The neighborhood of v_i

Then we can express the relationship of the trace functions $o_i(t)$ and $h_j(t)$ ($j = 1, \dots, \deg(v_i)$) in the form of the integro-differential equation:

$$\deg(v_i)o_i'(t) - \int_0^t G_i(s)o_i(t-s)\,ds = F_i(t), \tag{3.13}$$

where

$$G_i(s) = \sum_{j=1}^{\deg(v_i)} \partial_x w_j(0,s) \tag{3.14}$$

and

$$\begin{aligned}
F_i(t) =& 2\sum_{j=1}^{\deg(v_i)} \sum_{n=0}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} h_j'(t-(2n+1)l_j) \\
&+ 2\sum_{j=1}^{\deg(v_i)} \sum_{n=0}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} k_j((2n+1)l_j,(2n+1)l_j)h_j(t-(2n+1)l_j) \\
&- 2\sum_{j=1}^{\deg(v_i)} \sum_{n=0}^{\lfloor \frac{t-l_j}{2l_j} \rfloor} \int_{(2n+1)l_j}^t \partial_x k_j((2n+1)l_j,s)h_j(t-s)\,ds \\
&- 2\sum_{j=1}^{\deg(v_i)} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} o_i'(t-2nl_j) \\
&- 2\sum_{j=1}^{\deg(v_i)} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} w_j(2nl_j,2nl_j)o_i(t-2nl_j) \\
&+ 2\sum_{j=1}^{\deg(v_i)} \sum_{n=1}^{\lfloor \frac{t}{2l_j} \rfloor} \int_{2nl_j}^t \partial_x w_j(2nl_j,s)o_i(t-s)\,ds.
\end{aligned} \tag{3.15}$$

Here w_j are k_j are solutions to (3.5) and (3.6) on each incident edge of v_i.

Remark 3. *Since (3.13) holds for every entry of o^f, we have a system of $|V \setminus \Gamma|$ equations for each component of o^f. In [Avdonin and Zhao, 2021a] we proved that the system can be solved in steps for any positive time T, and each component of o^f belongs to $L^2(0,T)$. Once we know o^f, together with the given boundary conditions, we can find u_j^f for $j = 1, \ldots, |E|$ with the help of formulas (3.7) and (3.8).*

The above remark gives an algorithm to solve the forward problem (3.1)–(3.4). By Duhamel's principle, the solution u^f admits the representation

$$u^f(x,t) = \sum_{i=1}^{m-1} f_i(t) * u^i(x,t),$$

where $*$ stands for the convolution with respect to t, and $u^i(x,t)$ is the solution to the problem (3.1)–(3.3) with boundary conditions

$$u(\gamma_i, t) = \delta(t), \quad u(\gamma_j, t) = 0 \text{ for } j \neq i, \tag{3.16}$$

where $\delta(t)$ is the Dirac delta function.

Denote $o^i(t) = \left(o^i_1, \ldots, o^i_{|V\setminus\Gamma|}\right) := u^i|_{V\setminus\Gamma}$. Since $o^i(t)$ is the result of propagation of the wave generated by $\delta(t)$ on Ω, we have the following statement regarding the type and location of discontinuities on each entry of $o^i(t)$.

Proposition 3.18. *For every fixed $i = 1, \ldots, |\Gamma|$ and $j = 1, \ldots, |V|$, let $\{\xi_{i,j,r}\}_{r=1}^{\infty}$ be the set of all distinct lengths of all walks on Ω from γ_i to v_j. Then all discontinuities of $o^i_j(t)$ are located on $\{\xi_{i,j,r}\}_{r=1}^{\infty}$. The discontinuities have the forms of either δ function (spikes) or jumps. That is, we can represent o^i_j in the following form:*

$$o^i_j(t) = \theta_{i,j}(t) + \sum_{r=1}^{\infty} a_{i,j,r}\delta(t - \xi_{i,j,r}). \tag{3.17}$$

where $\theta_{i,j}(t)$ is a piecewise continuous function with possible jump discontinuities at $\{\xi_{i,j,r}\}_{r=1}^{\infty}$, and $a_{i,j,r} \in \mathbb{R}$.

Proof. To obtain equation (3.17), we track the propagation of the waves generated by the $\delta(t)$ from γ_i to v_j at the unit speed. Due to the unit speed of wave propagation, the set $\{\xi_{i,j,r}\}_{r=1}^{\infty}$ can be viewed as the set of time points when a new wavefront arrives at v_j.

We will illustrate this fact by solving (3.13)–(3.15) on $V \setminus \Gamma$ for a few steps. We first arrange the set of time points when the wavefronts arrive at all internal vertices $\Xi_i :=$

$\cup_{v_j \in V\setminus\Gamma} \cup_r \xi_{i,j,r}$ in ascending order, then solve (3.13)–(3.15) on the intervals defined by the first terms of Ξ_i.

The explicit expressions of first terms in Ξ_i is determined by the connectivity and edge lengths of Ω, so we have to make some assumptions here, while noting that the steps solving (3.13)–(3.15) hold the same for other assumptions. Since the Dirac delta function is placed at γ_i, we denote the vertex adjacent to γ_i as v_1 and let $l_1 := \mathrm{dist}(\gamma_i, v_1)$. We next denote the closest interior vertex to v_1 as v_2, the second closest interior vertex to v_1 as v_3. Note that v_3 may be connected to either v_1 or v_2, as illustrated in Figure 3.2. Let $l_2 := \mathrm{dist}(v_1, v_2)$ and $l_3 := \mathrm{dist}(v_1, v_3)$. For simplicity we assume $l_3 > 2l_2$, thus the first few terms of Ξ_i are $l_1, l_1 + l_2, l_1 + 2l_2, l_1 + l_3, \ldots$. We will show below the steps of solving (3.1)–(3.4) on the time intervals $[0, l_1), [l_1, l_1 + l_2), [l_1 + l_2, l_1 + 2l_2)$, and $[l_1 + 2l_2, l_1 + l_3)$.

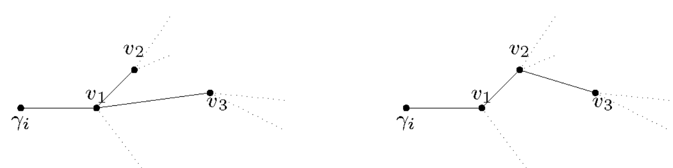

Figure 3.2: The propagation of $\delta(t)$ from γ_i. Vertex v_3 may be adjacent to either v_1 or v_2.

When $t \in [0, l_1])$, $o^i_j = 0$ for all $j = 1, \ldots, |V \setminus \Gamma|$. When $t \in [l_1, l_1 + l_2)$, we apply Proposition 3.17 and obtain the equation for $o^i_1(t)$ at v_1:

$$\begin{aligned}\deg(v_1)(o^i_1)'(t) - \int_0^t G_1(s) o^i_1(t-s)\, ds = \\ 2 \sum_{n=0}^{\lfloor \frac{t-l_1}{2l_1} \rfloor} \delta'(t - (2n+1)l_1) \\ + 2 \sum_{n=0}^{\lfloor \frac{t-l_1}{2l_1} \rfloor} k_1((2n+1)l_1, (2n+1)l_1)\delta(t - (2n+1)l_1) \\ - 2 \sum_{n=0}^{\lfloor \frac{t-l_1}{2l_1} \rfloor} \partial_x k_1((2n+1)l_1, t). \end{aligned} \tag{3.18}$$

We recall that for continuous q_j, the integral kernels $G_1, \partial_x k_1$ are continuous functions, see [*Avdonin and Mikhaylov*, 2010]. The right hand side contains three kinds of discontinuities: δ' (the dipoles), δ (the spikes) and jumps, all are located at $(2n+1)l_1$ for $n \geq 0$. Thus the left hand side must contain all discontinuities at the same locations. Moreover, the term $\deg(v_1)(o_1^i)'(t)$ must contain all the δ' dipoles and δ spikes. Thus $o_1^i(t)$ contains δ terms and jump discontinuities at $(2n+1)l_1$ for $n \geq 0$. The other entries $o_2^i, \ldots, o_{|V\setminus\Gamma|}^i$ are all zeros.

When $t \in [l_1 + l_2, l_1 + 2l_2)$, o_1 still satisfies (3.18) and o_2 satisfies

$$\begin{aligned}
&\deg(v_2)(o_2^i)'(t) - \int_0^t G_2(s) o_2^i(t-s)\, ds = \\
&2 \sum_{n=0}^{\lfloor \frac{t-l_2}{2l_2} \rfloor} (o_1^i)'(t-(2n+1)l_2) + 2 \sum_{n=0}^{\lfloor \frac{t-l_2}{2l_2} \rfloor} k_2((2n+1)l_2, (2n+1)l_2) o_1^i(t-(2n+1)l_2) \\
&- 2 \sum_{n=0}^{\lfloor \frac{t-l_2}{2l_2} \rfloor} \int_{(2n+1)l_2}^t \partial_x k_2((2n+1)l_2, s) o_1^i(t-s)\, ds. \qquad (3.19)
\end{aligned}$$

During this time $o_1^i(t)$ is piecewise continuous, with spikes and jumps at $\{(2n+1)l_1 | \, n \geq 0\}$; $o_2^i(t)$ is piecewise continuous, with spikes and jumps at $\{\{(2n+1)l_1 + (2k+1)l_2 | \, n, k \geq 0\}$. The other entries $o_3^i, \ldots, o_{|V\setminus\Gamma|}^i$ are all zeros.

When $t \in [l_1 + 2l_2, l_1 + l_3)$, o_1^i satisfies

$$
\begin{aligned}
\deg(v_1)(o_1^i)'(t) - \int_0^t G_1(s) o_1^i(t-s)\, ds = \\
& 2 \sum_{n=0}^{\lfloor \frac{t-l_1}{2l_1} \rfloor} \delta'(t - (2n+1)l_1) \\
& + 2 \sum_{n=0}^{\lfloor \frac{t-l_1}{2l_1} \rfloor} k_1((2n+1)l_1, (2n+1)l_1) \delta(t - (2n+1)l_1) \\
& - 2 \sum_{n=0}^{\lfloor \frac{t-l_1}{2l_1} \rfloor} \partial_x k_1((2n+1)l_1, t) - 2 \sum_{n=1}^{\lfloor \frac{t}{2l_2} \rfloor} (o_1^i)'(t - 2nl_2) \\
& - 2 \sum_{n=1}^{\lfloor \frac{t}{2l_2} \rfloor} w_2(2nl_2, 2nl_2) o_1^i(t - 2nl_2) \\
& + 2 \sum_{n=1}^{\lfloor \frac{t}{2l_2} \rfloor} \int_{2nl_2}^t \partial_x w_2(2nl_j 2, s) o_1^i(t-s)\, ds
\end{aligned} \tag{3.20}
$$

and $o_2^i(t)$ still satisfies (3.19). Therefore $o_1^i(t)$ is piecewise continuous, with spikes and jumps at $\{\{(2n+1)l_1 + 2kl_2 | \, n, k \geq 0\}$; $o_2^i(t)$ is piecewise continuous, with spikes and jumps at $\{\{(2n+1)l_1 + (2k+1)l_2 | \, n, k \geq 0\}$. The other entries $o_3^i, \ldots, o_{|V \setminus \Gamma|}^i$ are all zeros.

With this process, we solve each o_j^i in each time interval. As we add more locations of discontinuities for each o_j^i as time goes on, eventually every o_j^i are piecewise continuous, with spikes and jumps on $\{\xi_{i,j,r}\}_{r \geq 1}$. □

Remark 4. *In the expression of o_j^i in equation* (3.17)*, we separate out the δ spike discontinuity but not the jump discontinuity. This is driven by the need in Section 3.5 and will become clear in that section.*

3.4 The response function and the inverse problem

We now introduce the *dynamical response operator* for system (3.1)–(3.4) by the rule

$$(R^T\{f\})(t) = \partial u^f(\cdot, t)|_\Gamma, \quad t \in [0, T].$$

The response operator can be expressed through convolution

$$(R^T\{f\})(t) = (\bar{\mathbf{R}} * f)(t), \quad t \in [0, T]$$

where $\bar{\mathbf{R}}(t)$ is a $m \times m$ matrix valued response function with entries $R_{ij}(t) = \partial u^i(\gamma_j, t)$. Using Propositions 3.17, 3.18 and equations (3.9)–(3.12) we conclude that each R_{ij} has a representation

$$R_{ij} = \theta_{i,j}(t) + \sum_{r=1}^{\infty} a_{i,j,r}\delta(t - \xi_{i,j,r}) + \sum_{r=1}^{\infty} b_{i,j,r}\delta'(t - \xi_{i,j,r}), \tag{3.21}$$

where θ_{ij} is a piecewise continuous function, $a_{i,j,r}, b_{i,j,r} \in \mathbb{R}$, and $\{\xi_{i,j,r}\}_{r=1}^{\infty}$ is the set of distinct lengths of all walks from γ_i to γ_j. We call $\mathbf{R}(t) := \{R_{ij}\}_{i,j=1}^{m-1}$ the reduced response matrix function.

The dynamical inverse problem for the Schrödinger operator on Ω is to recover the connectivity, lengths of edges, and potential on Ω from $R(t)$. When Ω is an interval of length l, $\mathbf{R}(t)$ contains only one entry $R_{11}(t)$. It is proved in [*Avdonin et al.*, 1992] that if $R_{11}(t)$ for $0 \leq t \leq 2l$ is known, one can use the Boundary Control (BC) method to recover $q(x)$ for $0 \leq x \leq l$. For a tree graph, the following Proposition first appeared in [*Avdonin and Kurasov*, 2008]:

Proposition 3.19. *Let Ω be a metric tree graph and $\mathbf{R}(t)$ on Ω for $t \geq 2l_m$ is known (l_m is the maximum length of the leaf edges), then one can recover the lengths and potentials of all its leaf edges. One can separate the leaf edges into groups, where all leaf edges in one group are incident to the same internal vertex. Moreover, we can find the degrees for those* *internal vertices.*

Proof. Denote the length of the edge incident to γ_j as l_j. Since $\mathbf{R}(t)$ is given and its each entry has the expression of (3.21), for each j we have $l_i = \frac{\xi_{i,i,2}}{2}$. Once the length is known, one can use the BC method described in [*Avdonin et al.*, 1992] to recover the potential functions on all leaf edges.

The grouping of leaf edges based on their internal vertex can be done by the following method: if $\xi_{i,j,1} = \xi_{j,i,1} = 2(l_i + l_j)$ then e_i and e_j are incident to the same internal vertex.

From (3.13) we deduce that $b_{i,i,1}$ in (3.21) equals to $\frac{2}{m_i}$, where m_i is the degree of the internal vertex that the boundary vertex v_i is adjacent to. □

The lengths and potentials of the non-leaf edges and connectivity of the graph will be recovered through the LP method described in Section 3.5.

3.5 Leaf peeling method on a rooted tree

In this section we start with a rooted tree graph Ω and its reduced response matrix $\mathbf{R}$, remove (peel) all leaf edges from a sheaf on Ω (see definition below), and calculate the reduced response matrix for the peeled tree.

Definition 3.20. *We call a star shaped subgraph of a tree graph a* ***sheaf*** *if it contains all edges of the graph incident to some internal vertex and if all but one of its edges are leaf edges. The center vertex of the star is called the* ***abscission vertex*** *of the sheaf. The one edge that is not a leaf edge is called the* ***stem edge*** *of the sheaf. An example is shown in Figure 3.3. The statement below is a slightly modified version of a similar result from [Avdonin et al., 2015].*

Lemma 3.21. *A tree graph contains at least one sheaf.*

Proof. Let Ω be a tree graph. If Ω contains only one internal vertex, it is a star graph. All edges except for the root edge are leaf edges, therefore Ω is a sheaf. If Ω contains more than

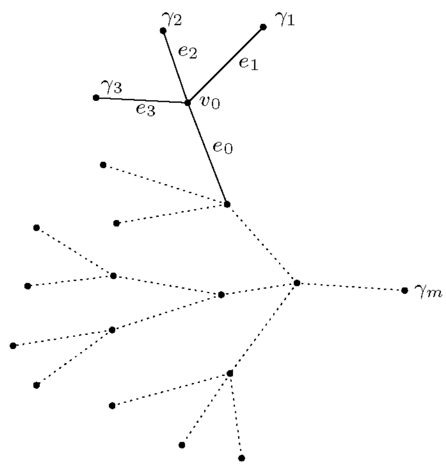

Figure 3.3: A sheaf on a tree graph rooted at γ_m (the sheaf is in solid lines), in which v_0 is the abscission vertex and e_0 is the stem edge.

one internal vertices, let v, v' be the two internal vertices such that the path length between v and v' is the maximum distance between any two internal vertices. Since Ω is a tree graph, there is a unique path P between v and v'. Neither v nor v' can be adjacent to another internal vertex that is not on P. Indeed, without loss of generality, say v' is adjacent to an internal vertex v'' and v'' is not on P, then the distance between v and v'' is greater than the distance between v and v', which contradicts the assumption. On the other hand, at least one of v, v' is not adjacent to the root vertex, say v. Then all but one of the incident edges of v are leaf edges. Thus v and its incident edges is a sheaf of Ω. □

Without loss of generality we assume that γ_1 belongs to a sheaf S of Ω, and e_1 is incident to γ_1. and let $\tilde{\Omega}$ be the new tree graph formed by removing the leaf edges of S from Ω, thus the stem edge of S (denoted by e_0) becomes a new leaf edge in $\tilde{\Omega}$. We label the abscission vertex of S (which becomes a new leaf vertex of $\tilde{\Omega}$) as γ_0 and the stem edge as e_0. Let $\tilde{\mathbf{R}}(t)$ be the reduced response function of $\tilde{\Omega}$. For consistency we keep the indices of the "unpeeled" leaf vertices from Ω the same in $\tilde{\Omega}$. Therefore $\Gamma(\tilde{\Omega}) = \{\gamma_0\} \cup (\{\gamma_1, \ldots, \gamma_m\} \setminus S)$, and the entries of $\tilde{R}$ are denoted as $\tilde{R}_{ij}$, where $i, j \in \tilde{I} := \{0, \ldots, m-1 |\, \gamma_i, \gamma_j \notin S\}$. (As a comparison, the entries of $\mathbf{R}$ are R_{ij}, where $i, j \in I := \{1, \ldots, m-1\}$.)

For each $i \in I$, denote $g^i(t) := o_0^i = u^i(\gamma_0, t)$ and $A^i(t) := \partial u_0^i(\gamma_0, t)$. We recall that u^i is defined in (3.16). For each $j \in \tilde{I}$, we set

$$\rho_{ij}(t) := \begin{cases} A^i(t) & j = 0 \\ R_{ij} & j \neq 0. \end{cases}$$

The matrix $(\rho_{ij})_{i\in I, j\in\tilde{I}}$ is an intermediate step between $\mathbf{R}$ and $\tilde{\mathbf{R}}$: it contains the responses at all leaf vertices of the peeled tree $\tilde{\Omega}$ to the boundary input (3.16) on Ω.

Lemma 3.22. *For each $i \in I$ and $j \in \tilde{I}$, one can deduce $g^i(t)$ and (ρ_{ij}) from the reduced response matrix $\mathbf{R}$ on Ω.*

Proof. Using R_{ii} , for each $i \in I$, we first recover l_i according to Proposition 3.19 and $q_i(x)$ using the BC method as described in [*Avdonin et al.*, 1992]. Denote the boundary condition $u(\gamma_1, t)$ as $h(t)$. If $i = 1$, $h(t) = \delta(t)$, otherwise $h(t) = 0$. By the superposition principle,

$$R_{i1}(t) = \partial_x u_1^{h,-}(0,t) + \partial_x u_1^{g^i,+}(0,t) \tag{3.22}$$

Using (3.9) and (3.11) in equation (3.22), we get a delay Volterra integral equation with respect to $\frac{d}{dt}g^i$, which can be solved in steps. Since the initial value of g^i is zero, we can compute g^i.

When g^i is known, $A^i(t)$ can be obtained by using (3.10) and (3.12) in (3.3):

$$A^i(t) = \partial_x u_1^{h,-}(l_1,t) + \sum_{e_j\in E(\gamma_0),\, j\neq 0} \partial_x u_j^{g^i,+}(l_j,t)$$

Since all other entries of (ρ_{ij}) are known, it follows we now have the complete matrix (ρ_{ij}). □

Since we will peel off the leaves of S and $\gamma_1 \in S$, without loss of generality, we use $g^1(t)$ and ρ_{1j} to compute $\tilde{R}_{0j}$ for $j \in \tilde{I}$, and then use g^i and ρ_{ij} to compute $\tilde{R}_{ij}$ for $i \in \tilde{I}\setminus\{0\}$ and $j \in \tilde{I}$ (thus we have all entries of $\tilde{\mathbf{R}}$). The following combinatorial/connectivity properties on Ω and $\tilde{\Omega}$ will be used in the peeling process.

Let $\{\mu_r\}_{r=1}^{\infty}$ be the set of distinct lengths of all walks on Ω from γ_1 to γ_0; For every $j \in \tilde{I}$, let $\{\nu_{j,r}\}_{r=1}^{\infty}$ be the set of distinct lengths of all walks on Ω from γ_1 to γ_j. let $\{\lambda_{j,r}\}_{r=1}^{\infty}$ be the set of distinct lengths of all walks on Ω from γ_0 to γ_j (when $j = 0$, such walks are closed). In contrast to $\{\lambda_{j,r}\}_{r=1}^{\infty}$, let $\{\tilde{\lambda}_{j,r}\}_{r=1}^{\infty}$ be the set of distinct lengths of all walks on $\tilde{\Omega}$ from γ_0 to γ_j. All sets $\{\mu_r\}_{r=1}^{\infty}$, $\{\nu_{j,r}\}_{r=1}^{\infty}$, $\{\lambda_{j,r}\}_{r=1}^{\infty}$, and $\{\tilde{\lambda}_{j,r}\}_{r=1}^{\infty}$ are sorted in ascending orders.

Lemma 3.23. *The sets $\{\mu_r\}_{r=1}^{\infty}$, $\{\nu_{j,r}\}_{r=1}^{\infty}$, $\{\lambda_{j,r}\}_{r=1}^{\infty}$, and $\{\tilde{\lambda}_{j,r}\}_{r=1}^{\infty}$ have the following relationships:*

$$\{\tilde{\lambda}_{j,r}\} \subsetneq \{\lambda_{j,r}\}; \tag{3.23}$$

$$\nu_{j,r} \leq \mu_r + \lambda_{j,s}, \quad \nu_{j,r} \leq \mu_s + \lambda_{j,r} \quad \text{for every } r, s \geq 1; \tag{3.24}$$

$$\nu_{j,r} = \mu_1 + \lambda_{j,r} \quad \text{for every } r \geq 1. \tag{3.25}$$

Proof. Equation (3.23) is obvious since every walk from γ_0 to γ_j on $\tilde{\Omega}$ is a walk from γ_0 to γ_j on Ω but the converse is not true.

Since γ_0 is on every walk from γ_1 to γ_j, a walk from γ_1 to γ_j can be partitioned into a walk from γ_1 to γ_0 and a walk from γ_0 to γ_j, so

$$\{\nu_{j,r}\}_{r\geq 1} = \{\mu_r + \lambda_{j,s}\}_{r,s\geq 1}. \tag{3.26}$$

Since all $\{\mu_r\}, \{\nu_{j,r}\}, \{\lambda_{j,r}\}$ are in ascending order, we have (3.24).

Furthermore, since γ_0 is the second vertex on every walk from γ_1 to γ_j, every walk from γ_1 to γ_j can be partitioned into the path (γ_1, γ_0) and a walk from γ_0 to γ_j, so

$$\{\nu_{j,r}\}_{r\geq 1} = \{\mu_1 + \lambda_{j,r}\}_{r\geq 1}.$$

Since both $\{\nu_{j,r}\}$ and $\{\lambda_{j,r}\}$ are in ascending order, we have (3.25).

$\square$

We now prove the Leaf Peeling theorem.

Theorem 3.24. ***(Leaf Peeling Theorem)*** *The reduced response matrix function $\tilde{\boldsymbol{R}}(t)$ on $\tilde{\Omega}$ can be computed from $\mathbf{R}(t)$ on Ω.*

Proof. With $\mathbf{R}(t)$ given, we will first compute $\tilde{R}_{0j}$ for all $j \in \tilde{I}$, then $\tilde{R}_{ij}$ for all $i \in \tilde{I} \setminus \{0\}$ and $j \in \tilde{I}$. Thus we will get all entries of $\tilde{\mathbf{R}}(t)$.

For $\tilde{R}_{0j}$ $(j \in \tilde{I})$, we first compute $g^1(t)$ and $\rho_{1j}(t)$ through Lemma 3.22. By (3.17),

$$g^1(t) = \psi(t) + \sum_{r=1}^{\infty} \alpha_r \delta(t - \mu_r), \quad \alpha_1 \neq 0, \tag{3.27}$$

and by (3.17), (3.9) and (3.11),

$$\rho_{1j}(t) = \phi_j(t) + \sum_{r=1}^{\infty} a_{j,r} \delta(t - \nu_{j,r}) + \sum_{i=1}^{\infty} b_{j,r} \delta'(t - \nu_{j,r}), \tag{3.28}$$

where $\alpha_r, a_{j,r}, b_{j,r} \in \mathbb{R}$, $\psi(t), \phi_j(t)$ are piecewise continuous.

Let $\tilde{u}^0$ be the solution of (3.1)–(3.3) on $\tilde{\Omega}$ with the boundary conditions $u(\gamma_0, t) = \delta(t)$ and $u(\gamma_i, t) = 0$ for $\gamma_i \notin S$. Then by (3.21) the unknown function $\tilde{R}_{0j}$ has the form

$$\tilde{R}_{0j}(t) = \partial \tilde{u}^0(\gamma_0, t) = \chi_j(t) + \sum_{r=1}^{\infty} \tilde{c}_{j,r} \delta(t - \tilde{\lambda}_{j,r}) + \sum_{r=1}^{\infty} \tilde{d}_{j,r} \delta'(t - \tilde{\lambda}_{j,r}) \tag{3.29}$$

for some $\tilde{c}_{j,r}, \tilde{d}_{j,r}, \in \mathbb{R}$ and piecewise continuous $\chi_j(t)$.

Due to (3.23), instead of using Equation (3.29), we can represent $\tilde{R}_{0j}(t)$ as

$$\tilde{R}_{0j}(t) = \chi_j(t) + \sum_{r=1}^{\infty} \tilde{c}_{j,r} \delta(t - \lambda_{j,r}) + \sum_{r=1}^{\infty} \tilde{d}_{j,r} \delta'(t - \lambda_{j,r}) \tag{3.30}$$

Equation (3.30) has all terms from (3.29) and some extra zero terms: if $\lambda_{j,r} \notin \{\tilde{\lambda}_{j,r}\}$, then $\tilde{c}_{j,r} = \tilde{d}_{j,r} = 0$.

By Duhamel's principle

$$\rho_{1j}(t) = g^1(t) * \tilde{R}_{0j} \tag{3.31}$$

We plug (3.27), (3.28) and (3.30) into (3.31):

$$\begin{aligned}(\psi * \chi_j)(t) + \sum_{r=1}^{\infty} \tilde{c}_{j,r}\psi(t-\lambda_{j,r}) + \sum_{r=1}^{\infty} \tilde{d}_{j,r}\psi'(t-\lambda_{j,r}) + \sum_{r=1}^{\infty} \alpha_r \chi_j(t-\mu_r) \\ + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \alpha_r \tilde{c}_{j,s}\delta(t-\mu_r-\lambda_{j,s}) + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \alpha_r \tilde{d}_{j,s}\delta'(t-\mu_r-\lambda_{j,s}) \\ = \phi_j(t) + \sum_{r=1}^{\infty} a_{j,r}\delta(t-\nu_{j,r}) + \sum_{r=1}^{\infty} b_{j,r}\delta'(t-\nu_{j,r})\end{aligned} \tag{3.32}$$

where $\{\tilde{c}_{j,r}\}_{r=1}^{\infty}$, $\{\tilde{d}_{j,r}\}_{r=1}^{\infty}$ and $\chi_j(t)$ are unknown. Since $\phi_j(t), \delta(t), \delta'(t)$ vanish for $t < 0$, at any finite t, all series in (3.32) contain finitely many nonzero terms.

We collect the terms containing δ' from (3.32) to obtain

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \alpha_r \tilde{d}_{j,s}\delta'(t-\mu_r-\lambda_{j,s}) = \sum_{r=1}^{\infty} b_{j,r}\delta'(t-\nu_{j,r}) \tag{3.33}$$

Let N be the largest integer such that $\nu_{j,N} \leq t$ and M be an integer greater than N. By (3.24), $t < \nu_{j,M} \leq \mu_M + \lambda_{j,r}$ and $t < \nu_{j,M} \leq \mu_r + \lambda_{j,M}$ for any $r \geq 1$. Thus

$$\delta'(t-\mu_M-\lambda_{j,r}) = \delta'(t-\mu_r-\lambda_{j,M}) = \delta'(t-\nu_{j,M}) = 0$$

for any $M > N$ and $r \geq 1$. Equation (3.33) then becomes

$$\sum_{r=1}^{N}\sum_{s=1}^{N} \alpha_r \tilde{d}_{j,s}\delta'(t-\mu_r-\lambda_{j,s}) = \sum_{r=1}^{N} b_{j,r}\delta'(t-\nu_{j,r})$$

which can be represented by a matrix equation

$$M\vec{d} = \vec{b} \tag{3.34}$$

where $\vec{d} = [\tilde{d}_{j,1}, \ldots, \tilde{d}_{j,N}]^T$; $\vec{b} = [b_{j,1}, \ldots, b_{j,N}]^T$; The entries of M are determined row by row: for every $r = 1, \ldots, N$, $M_{rs} = \alpha_m$ if $\mu_m + \lambda_{j,s} = \nu_{j,r}$ for some $m \geq 1$, otherwise $M_{rs} = 0$. By (3.25), all diagonal entries M_{rr} are equal to α_1. For all $s > r$, $M_{rs} = 0$. In fact, since $\lambda_{j,s} > \lambda_{j,r}$, if there is some μ_m such that

$$\mu_m + \lambda_{j,s} = \mu_1 + \lambda_{j,r} = \nu_{j,r},$$

μ_m must be less than μ_1, which is impossible. Therefore M is lower triangular with nonzero diagonal entries, thus invertible. We can find all $\tilde{d}_{j,s}$ for $s \leq N$ for any $N < \infty$.

The next step is to collect the terms in (3.32) containing δ. Since $\psi(t)$ is piecewise continuous, ψ' may contain terms of impulse functions, so

$$\sum_{r=1}^{\infty} \tilde{d}_{j,r} \psi'(t - \lambda_{j,r}) = \Psi(t) + \sum_{r=1}^{\infty} \beta_r \delta(t - \xi_r)$$

where $\Psi(t)$ is piecewise continuous and $\beta_r, \xi_r \in \mathbb{R}$. So from (3.32) we have

$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \alpha_r \tilde{c}_{j,s} \delta(t - \mu_r - \lambda_{j,s}) = \sum_{r=1}^{\infty} a_{j,r} \delta(t - \nu_{j,r}) - \sum_{r=1}^{\infty} \beta_r \delta(t - \xi_r), \tag{3.35}$$

By (3.26), $\{\mu_r + \lambda_{j,s}\}_{r,s \geq 1} = \{\nu_{j,r}\}_{r \geq 1}$, so $\{\xi_r\} \subset \{\nu_{j,r}\}$, therefore (3.35) can be written as

$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \alpha_r \tilde{c}_{j,s} \delta(t - \mu_r - \lambda_{j,s}) = \sum_{r=1}^{\infty} \tilde{a}_{j,r} \delta(t - \nu_{j,r}) \tag{3.36}$$

for some known $\tilde{a}_{j,r} \in \mathbb{R}$. We can convert (3.36) to a matrix equation similar to (3.34),

$$M\vec{c} = \vec{a},$$

and find $\tilde{c}_{j,r}$.

The piecewise continuous terms in (3.32) forms an integral equation with respect to χ_j, which can be solved in steps:

$$\sum_{r=1}^{\infty} \alpha_r \chi_j(t-\mu_r) + (\psi * \chi_j)(t) = \phi_j(t) - \sum_{r=1}^{\infty} \tilde{c}_{j,r} \psi(t-\lambda_{j,r}) - \Psi(t). \tag{3.37}$$

Thus we have found the entries $\tilde{R}_{0j}(t)$. What is left to compute are the entries $\tilde{R}_{ij}(t)$, where $i \in \tilde{I} \setminus \{0\}$ and $j \in \tilde{I}$. Again by Duhamel's principle,

$$\rho_{ij} = \tilde{R}_{ij} + g^i(t) * \tilde{R}_{0j}, \tag{3.38}$$

where ρ_{ij} and $g^i(t)$ are computed in Lemma 3.22, and $\tilde{R}_{0j}$ was computed in the previous step. Substituting ρ_{ij} and $g^i(t)$ into (3.38) we obtain $\tilde{R}_{ij}$. □

Corollary 3.25. *One can recover the edge lengths, connectivity, and potential of a finite metric tree graph* Ω *from* $\mathbf{R}(t)$ *for a finite time interval* $t \in (0,T)$ *with* $T > 2L$*, where* L *is the length of the longest path from a leaf vertex to the root vertex.*

Proof. For the new tree $\tilde{\Omega}$, the connectivity and geometry of the sheaves that does not include e_0 is already recovered from Ω using Proposition 3.19. To recover the connectivity and geometry of the sheaf on $\tilde{\Omega}$ including e_0, one needs to obtain $\tilde{\lambda}_{0,2}$ (note that $\tilde{\lambda}_{0,1} = 0$) and $\tilde{\lambda}_{j,1}$ for $j \in \tilde{I} \setminus \{0\}$.

Since M is a lower triangular matrix, equation (3.34) becomes:

$$\begin{bmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ * & \alpha_1 & 0 & \dots & 0 \\ * & * & \alpha_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ * & * & * & \dots & \alpha_1 \end{bmatrix} \cdot \begin{bmatrix} d_{j,1} \\ d_{j,2} \\ \dots \\ \dots \\ d_{j,N} \end{bmatrix} = \begin{bmatrix} b_{j,1} \\ b_{j,2} \\ \dots \\ \dots \\ b_{j,N} \end{bmatrix}, \tag{3.39}$$

in which $\alpha_1 = \frac{2}{m_0}$, where m_0 is the degree of v_0.

Suppose $\tilde{\lambda}_{0,2} = \lambda_{0,K}$ for some $K \geq 2$ ($K > 2$ indicates some of the leaf edges on S are shorter than e_0), one can obtain that $b_{0,k} = -\alpha_k$ for $1 \leq k < K$, and $b_{0,K} \neq -\alpha_K$. Thus the above triangular system indicates $d_{0,1} = -1$, $d_{0,k} = 0$ for $1 < k < K$, and $d_{0,K} \neq 0$. Thus $\tilde{\lambda}_{0,2}$ (which is the same as $\lambda_{0,K}$), is explicitly included in the expression of $\tilde{R}_{00}$.

Let the path between v_0 and γ_j contain K vertices and denote the degree of each vertex as m_k for $k = 1, \ldots, K$. Since Ω is a tree graph, it is easy to see $\tilde{\lambda}_{j,1} = \lambda_{j,1}$, and $b_{j,1} = 2 \prod_{k=1}^{K-1} \frac{2}{m_k}$. Therefore $d_{j,1} \neq 0$, and $\tilde{\lambda}_{j,1}$ is explicitly included in $\tilde{R}_{0j}$.

Using $\tilde{\lambda}_{0,0}$, $\tilde{\lambda}_{j,1}$, and Proposition 3.19, we can identify the geometry and connectivity of the sheaf e_0 is on on $\tilde{\Omega}$. Repeat this procedures and Theorem 3.24 one can solve the dynamical inverse problem for the Schrödinger operator on Ω. Since a time interval of length $2l_i$ is needed to recover the potential function on any edge e_i, overall we need a time interval of length T, $T > 2L$ to recover the potential, connectivity of the whole tree, and the lengths of the edges. □

Chapter 4: Control Problems for the Wave Equations on Graphs with Cycles

4.1 Introduction

Controllability properties of the wave equation is a fundamental topic in the control theory for partial differential equations. Many powerful methods were used to prove controllability of the wave equation in various spatial domains under the action of various types of controls (see, e.g. [*Lasiecka and Triggiani*, 2000; *Lions*, 1988; *Russell*, 1978; *Zuazua*, 2007] and references therein). Control problems for the wave equation on graphs have important applications in science and engineering and were studied in many papers (see the monographs [*Avdonin and Ivanov*, 1995; *Dáger and Zuazua*, 2006; *Lagnese et al.*, 2012]; the surveys [*Avdonin*, 2008; *Zuazua*, 2013]; and references therein). They also have deep connection with inverse problem on graphs, see, e.g. [*Avdonin and Kurasov*, 2008; *Belishev and Vakulenko*, 2006] for tree graph examples. In this chapter we consider the exact controllability problem for the equation

$$u_{tt} - u_{xx} + qu = 0$$

on metric graphs with the Kirchhoff-Neumann matching conditions. For graphs without cycles (trees) this problem was studied (in slightly different but essentially equivalent forms) in, e.g. [*Avdonin and Mikhaylov*, 2008; *Belishev and Vakulenko*, 2006; *Dáger and Zuazua*, 2006; *Lagnese et al.*, 1994, 2012]. It was proved that the system is exactly controllable if Dirichlet controllers act at all or at all but one of the boundary vertices. In [*Avdonin and Zhao*, 2021a] a constructive proof for controllability of the wave equation on a tree graph was proposed, in which the graph was represented as a union of disjoint paths with each path started from a controlled boundary vertex. The sharp controllability time was also indicated in that paper. Similar construction for control functions can be made in other hyperbolic systems, see [*Alam et al.*, 2021] for an example.

Very little was known about controllability of the wave equation (or any other partial differential equation) on graphs with cycles. It was only proved that the wave equation on graphs with cycles is never exactly boundary controllable [*Avdonin and Ivanov*, 1995]; it may be spectrally (boundary) controllable, but this property is very unstable with respect to small perturbations of the system parameters (see, e.g. [*Avdonin et al.*, 2009, 2010a]).

Historically, the first work on controllability of the wave equation on graphs was the paper by S. Rolewicz [*Rolevicz*, 1970]. He considered the case of Dirichlet conditions at all boundary and internal vertices, i.e. the system consisted of separate strings connected by common controls at the internal vertices. Mathematically it means that instead of the conditions (4.11), (4.12) (see Sec. 2 below) he considered the condition $u_j(v_i,t) = f_i(t),\ j \in J(v_i),\ i \in I$. Rolewicz proved the exact controllability of such a system in the case when a graph has no cycles and lack of the exact controllability in the case when a graph has more than one cycle. His statement about controllability when a graph has one cycle and the optical lengths of the edges are commensurable is incorrect: such a system is not exactly controllable for any lengths of the edges. Complete solution of the problem of exact and spectral controllability for the system of strings with Dirichlet controls at all vertices was obtained in [*Avdonin and Ivanov*, 1995].

In this chapter we give a constructive proof for the controllability of the wave equation on a general graph with the Kirchhoff-Neumann matching conditions. Our construction here generalizes the idea of controllability on trees in [*Avdonin and Zhao*, 2021a]. However, to prove controllability of systems on graphs with cycles we need to use not only boundary but also internal controls, as was proposed in [*Avdonin*, 2019]. Given an undirected graph, we first give it an acyclic orientation. We then construct an active set of vertices and edge directions based on the chosen orientation, such that the active vertices include all source vertices specific to the orientation; the active edge directions include all but one of the outgoing directions at every vertex. Then Neumann controllers are placed at the active vertices, and Dirichlet controllers are placed at the active edge directions. The shape and velocity controllability are proved using constructive dynamical method. The exact controllability is proved by combining the shape and velocity control functions via the method of moments.

This chapter is organized as follows. Section 4.2 introduces the quantum graph, the orientation of the graph, the observation and control problems for the wave equation on the quantum graph, the solutions to the forward problem for the wave equation with Dirichlet and Neumann conditions on an interval and on general graphs. Section 4.3 proposes an algorithm solving the control problem on a general graph, discusses the control time and the minimal number of controllers as an invariant independent of the graph orientation, and links this invariant to the maximal multiplicity of the spectrum of the Schrödinger operator on the graph obtained in [*Kac and Pivovarchik*, 2011].

4.2 Preliminaries

4.2.1 Metric graphs and Hilbert spaces on graphs

Let $\Omega(V, E)$ be a finite, undirected, connected graph, where V, E are the sets of vertices and edges of Ω respectively. Each edge in E is associated with two vertices in V called its endpoints. Here we require both $V = \{v_i : i \in I\}$ and $E = \{e_j : j \in J\}$ to be nonempty; and the two endpoints of an edge to be distinct. That is, our graphs contain no loops; we will use this assumption in Section 4.3 when introduce acyclic orientations on graphs. This assumption is not restrictive. Since we admit multiple edges between two vertices, a loop can be considered as two edges between two vertices. In this chapter it is convenient to consider I and J the disjoint subsets of $\mathbb{N}$.

An undirected edge e_j between v_i and v_k is denoted as $e_j(v_i v_k)$. The set of indices of the edges incident to v_i is denoted by $J(v_i)$. The set, $\Gamma = \{v_i \in V : |J(v_i)| = 1\}$, plays the role of the graph boundary. Here we use $|\cdot|$ to denote the cardinality of a set of features on a graph. Figure 4.1 shows a finite undirected graph Ω. In which we have $I = \{1, \ldots, 13\}$ and $J = \{14, \ldots, 34\}$.

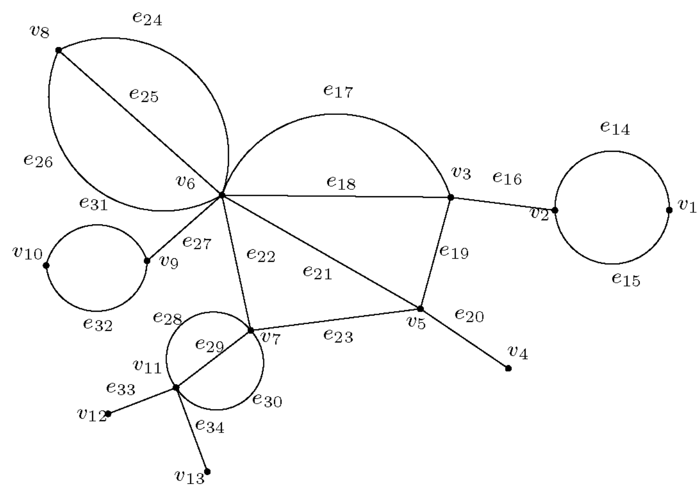

Figure 4.1: An example graph Ω.

We recall that a graph is called a **metric graph** if every edge $e_j \in E$ is identified with an interval $(0, l_j)$ of the real line with a positive length l_j. The graph Ω determines naturally the Hilbert space of square integrable functions $\mathcal{H} = L^2(\Omega)$. We define the space $\mathcal{H}^1$ of continuous functions y on Ω such that $y_j := y|_{e_j} \in H^1(e_j)$ for every $j \in J$. Let $\mathcal{H}^{-1}$ be the dual space of $\mathcal{H}^1$. We further introduce the space $\mathcal{H}^2$ of continuous functions y on Ω such that $y_j \in H^2(e_j)$ for every $j \in J$, and for every $i \in I$, the equality

$$\sum_{j \in J(v_i)} \partial y_j(v_i) = 0 \tag{4.1}$$

holds. Here (and everywhere later on) $\partial y_j(v_i)$ denotes the derivative of y at the vertex v_i taken along the edge e_j in the direction outwards the vertex. Vertex conditions (4.1) (together with continuity at v) are called the **standard** or **Kirchhoff-Neumann** (KN) conditions for the internal vertices. For the boundary vertices they are just Neumann conditions.

Let q be a real valued function (potential) such that $q|_{e_j} \in C[0, l_j]$. We define the Schrödinger operator on the graph Ω as the operator $L = -\frac{d^2}{dx^2} + q$ in $\mathcal{H}$ with the domain $\mathcal{H}^2$.

4.2.2 Observation and control problems of the wave equation

Consider the following initial boundary value problem (IBVP):

$$w_{tt} - w_{xx} + q(x)w = 0 \quad \text{in } \{\Omega \setminus V\} \times (0,T), \tag{4.2}$$

$$w_j(v_i,t) = w_k(v_i,t) \text{ for } j,k \in J(v_i), v_i \in V,\ t \in [0,T], \tag{4.3}$$

$$\sum_{j \in J(v_i)} \partial w_j(v_i,t) = 0 \text{ at each vertex } v_i \in V,\ t \in [0,T], \tag{4.4}$$

$$w|_{t=0} = w^0,\ w_t|_{t=0} = w^1 \quad \text{in } \Omega. \tag{4.5}$$

Here $T > 0,\ w^0 \in \mathcal{H},\ w^1 \in \mathcal{H}^{-1}$. Note that (4.4) includes the boundary condition, $\partial w|_\Gamma = 0$. Using the Fourier method, one can show, similarly to [*Avdonin and Nicaise*, 2015], that this IBVP has a unique generalized solution such that for any $i \in I, j \in J$,

$$w \in C([0,T];\mathcal{H}),\ w_t \in C([0,T];\mathcal{H}^{-1}),\ w(v_i,\cdot) \in L^2(0,T),\ \partial w_j(v_i,\cdot) \in H^{-1}(0,T) \tag{4.6}$$

To formulate the observation problem for the system (4.2)–(4.5) we define a set of active vertices $V^* = \{v_i : i \in I^*\}$ as a subset of V, where we put observers for the trace $w(v,\cdot)$. The set V^* may include some boundary vertices and we put $\Gamma^* = V^* \cap \Gamma$. For each vertex v_i we define a set of active edges, where we put observers for directional derivatives $\partial w_j(v_i,\cdot),\ j \in J^*(v_i)$. Here $J^*(v_i)$ is the index set of the active edges of v_i. Note that $J^*(v_i)$ may be empty for a vertex v_i. On the other hand, due to (4.4), we may assume the set $J^c(v_i) := J(v_i) \setminus J^*(v_i)$ contains at least one index for every $i \in I$. Therefore if $v_i \in \Gamma$, $J(v_i) = J^c(v_i)$ and $J^*(v_i) = \emptyset$. In this chapter, we assume that one edge has only one observer from one of its two ends. That is, for an edge $e_j(v_iv_k)$, either $j \in J^*(v_i)$ or $j \in J^*(v_k)$ but not both. This assumption is for notation convenience only, it is not restrictive. Since we do not exclude vertices of degree two, if there are two observers at the two ends of an edge, we just add a vertex at the middle of the edge. Let $J^* := \{j : j \in J^*(v_i), i \in I\}$. We call $\{I^*, J^*\}$ the **active set**. We say that the system (4.2)–(4.5) with the active set $\{I^*, J^*\}$ is **observable** in time T if there is a positive constant C, independent of w_0, w_1, such that

$$\sum_{i\in I^*} \|w(v_i,\cdot)\|^2_{L^2(0,T)} + \sum_{j\in J^*} \|\partial w_j(v_i,\cdot)\|^2_{H^{-1}(0,T)} \geq C\left\{\|w^0\|^2_{\mathcal{H}} + \|w^1\|^2_{\mathcal{H}^{-1}}\right\} \tag{4.7}$$

for every $w^0 \in \mathcal{H},\ w^1 \in \mathcal{H}^{-1}$. The observation problem is to find $\{I^*, J^*\}$ and T which make the system observable.

Now we derive the control problem as the dual to the observation problem. Let w^0 and w^1 be smooth functions on Ω, so w is a classical solution to the IBVP (4.2)–(4.5), and for every j, u_j be a smooth function in $[0, l_j] \times [0, T]$ such that

$$(u_j)_{tt} - (u_j)_{xx} + q_j(x)u_j = 0, \quad 0 < x < l_j,\ 0 < t < T, \quad u_j|_{t=0} = (u_t)_j|_{t=0} = 0.$$

Substituting $w_j(x, T-t)$ and $u_j(x,t)$ into the identity

$$0 = \sum_{j=1}^{|E|} \int_0^{l_j} \int_0^T [(w_j)_{tt}(x, T-t) - (w_j)_{xx}(x, T-t) + q_j(x)w_j(x, T-t)]u_j(x,t)\, dt\, dx$$

and integrating by parts, we get

$$
\begin{aligned}
0 &= \sum_{j=1}^{|E|} \int_0^{l_j} [-(w_j)_t(x,0)u_j(x,T) - w_j(x,0)(u_j)_t(x,T)]\, dx \\
&\quad + \sum_{j=1}^{|E|} \int_0^T [-(w_j)_x(l_j,T-t)u_j(l_j,t) + (w_j)_x(0,T-t)u_j(0,t) \\
&\quad + w_j(l_j,T-t)(u_j)_x(l_j,t) - w_j(0,T-t)(u_j)_x(0,t)]\, dt \\
&= -\sum_{j=1}^{|E|} \int_0^{l_j} [w_j^1(x)u_j(x,T) + w_j^0(x)(u_j)_t(x,T)]\, dx \\
&\quad + \int_0^T \left[\sum_{i\in I, j\in J(v_i)} \partial w_j(v_i,T-t)\, u_j(v_i,t) - \sum_{i\in I, j\in J(v_i)} w_j(v_i,T-t)\, \partial u_j(v_i,t) \right] dt, \qquad (4.8)
\end{aligned}
$$

Taking into account the KN conditions (4.3), (4.4), this equality can be written in the form

$$
\int_\Omega [w^1(x)u(x,T) + w^0(x)u_t(x,T)]\, dx
$$

$$
= \int_0^T \left[\sum_{i\in I, j\in J(v_i)} \partial w_j(v_i,T-t)\, [u_j(v_i,t) - u_k(v_i,t)] - \sum_{i\in I} w(v_i,T-t) \left(\sum_{j\in J(v_i)} \partial u_j(v_i,t) \right) \right], \qquad (4.9)
$$

where $w(v_i,t)$ is the common value of $w_j(v_i,t)$, $j \in J(v_i)$, k is an arbitrary index from $J(v_i)$ and u is a function on $\Omega \times [0,T]$ such that $u_j = u|_{e_j}$.

The control system dual to (4.2)–(4.5) with the active set $\{I^*, J^*\}$ can be written as follows:

$$
u_{tt} - u_{xx} + q(x)u = 0 \quad \text{in } \{\Omega \setminus V\} \times (0,T), \qquad (4.10)
$$

$$
\sum_{j\in J(v_i)} \partial u_j(v_i,t) = \begin{cases} 0, & i \in I \setminus I^*, \\ f_i'(t), & i \in I^*, \end{cases} \qquad (4.11)
$$

$$\begin{cases} u_j(v_i,t) - u_r(v_i,t) = f_j(t), & i \in I,\ j \in J^*(v_i),\ r \in J^c(v_i), \\ u_j(v_i,t) - u_r(v_i,t) = 0, & i \in I,\ j,r \in J^c(v_i), \end{cases} \tag{4.12}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in} \ \ \Omega. \tag{4.13}$$

In (4.11) the notation $f_i'(t)$ is used for convenience, it means the derivative of $f_i(t)$. In this chapter we assume that $f_i \in H^1(0,T),\ f_i(0) = 0$ for all $i \in I$. The same assumption is extended also to functions $f_j, j \in J$, from the conditions (4.12).

Integral identity (4.9) can now be written as

$$\int_\Omega [w^1(x)u(x,T) + w^0(x)u_t(x,T)]\, dx$$

$$= \int_0^T \left[\sum_{j\in J^*} \partial w_j(v, T-t)\, f_j(t) - \sum_{i \in I^*} w(v_i, T-t)\, f_i'(t) \right] dt. \tag{4.14}$$

If we assume that in (4.11) and (4.12) $f_j \in H_0^1(0,T)$ and $f_i' \in L^2(0,T)$ $\forall i,j$, then using the inclusions (4.6) this identity can be extended to the broader classes of functions:

$$\langle w^1, u(\cdot,T)\rangle_{\mathcal{H}^1} + \langle w^0, u_t(\cdot,T)\rangle_{\mathcal{H}}$$

$$= \sum_{j\in J^*} \langle \partial w_j(v, T-\cdot), f_j\rangle_{H^1} - \sum_{i\in I^*} \langle w(v_i, T-\cdot), f_i'\rangle_{L^2}. \tag{4.15}$$

Here $\langle\cdot,\cdot\rangle_{\mathcal{H}^1}$ means the pairing $\mathcal{H}^{-1} - \mathcal{H}^1$, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ – scalar product in $\mathcal{H}$, $\langle\cdot,\cdot\rangle_{H^1}$ – the pairing $H^{-1}(0,T) - H_0^1(0,T)$, and $\langle\cdot,\cdot\rangle_{L^2}$ – scalar product in $L^2(0,T)$.

Initial boundary value problems like (4.10)–(4.13) were studied by many authors (see, e.g. [*Ali Mehmeti*, 1994; *Ali Mehmeti and Meister*, 1989; *Avdonin and Nicaise*, 2015; *Avdonin and Zhao*, 2021a; *Dáger and Zuazua*, 2006; *Lagnese et al.*, 1994]) mostly with L^2-controls acting only on the boundary. It was proved that such a problem is well defined and the solution has the following regularity: for the Dirichlet L^2-controls, $u \in C([0,T];\mathcal{H}) \cap C^1([0,T];\mathcal{H}^{-1})$ and for the Neumann L^2-controls, $u \in C([0,T];\mathcal{H}^1) \cap C([0,T];\mathcal{H})$. (For the Dirichlet problem the definition of $\mathcal{H}^1$ includes the condition $u|_\Gamma = 0$.) The case of the Dirichlet L^2 internal controls was considered in [*Avdonin and Nicaise*, 2015] using the Fourier method.

In this chapter we consider H^1 Dirichlet internal controls and L^2 Neumann boundary and internal controls. In Section 4.2.5 we demonstrate that

$$u \in C([0,T];\mathcal{H}^1_c) \cap C([0,T];\mathcal{H}).$$

Here $\mathcal{H}^1_c$ is the space of of functions y on Ω such that $y_j \in H^1(e_j)$ for every $j \in J$, and for every $i \in I$, $y_j(v_i) = y_k(v_i)$ if $j,k \in J^c(v_i)$.

It is convenient to extend $f_j(t)$ and $f_i'(t)$ by 0 to $t \le 0$ and define the following spaces:

$$H^1_*(0,T) := \{y \in H^1(-\infty,T), y(t) = 0, \text{ for } t \le 0\},$$

$$H^1_0(0,T) := \{y \in H^1_*, y(T) = 0\},$$

$$L^2_*(0,T) := \{y \in L^2(-\infty,T), y(t) = 0, \text{ for } t \le 0\},$$

$$\mathcal{F}^T := \prod_{j \in J^*} \{f_j \in H^1_*(0,T)\} \times \prod_{i \in I^*} \{f_i' \in L^2_*(0,T)\},$$

$$\mathcal{F}^T_0 := \prod_{j \in J^*} \{f_j \in H^1_0(0,T)\} \times \prod_{i \in I^*} \{f_i' \in L^2_*(0,T)\}.$$

Let $\mathbf{f}$ be a vector function with all controls in IBVP (4.10)–(4.13) as the entries. We assume $\mathbf{f} \in \mathcal{F}^T$ and refer to the corresponding solution as $u^{\mathbf{f}}$. The main results of this chapter concern controllability of the system (4.10)–(4.13). In Section 4.3 we obtain the conditions for the shape, velocity and exact controllability of the system.

Definition 4.26. *Let $T > 0$. We say that the system* (4.10)–(4.13) *is:*

1. *Shape controllable in time T if for any $y \in \mathcal{H}^1$, there exists $\mathbf{f} \in \mathcal{F}_0^T$ such that $u^{\mathbf{f}}(\cdot, T) = y$.*

2. *Velocity controllable in time T if for any $z \in \mathcal{H}$, there exists $\mathbf{f} \in \mathcal{F}^T$ such that $u_t^{\mathbf{f}}(\cdot, T) = z$.*

3. *Exactly controllable in time T if for any $y \in \mathcal{H}^1$ and $z \in \mathcal{H}$, there exists $\mathbf{f} \in \mathcal{F}_0^T$ such that $u^{\mathbf{f}}(\cdot, T) = y$ and $u_t^{\mathbf{f}}(\cdot, T) = z$.*

Observability of the system (4.2)–(4.5) defined in (4.7) follows from the exact controllability of the system (4.10)–(4.13) and the duality between two systems expressed by the identity (4.15).

4.2.3 Directed acyclic graphs and linear ordering of vertices

Let $\vec{\Omega} = (V, \vec{E})$ be a directed graph obtained by orienting edges in Ω. We denote a directed edge e_j from v_i to v_k as $e_j(v_i, v_k)$. When $e_j(v_i, v_k)$ is identified with the interval $(0, l_j)$, v_i is identified with $x = 0$ and v_k is identified with $x = l_j$. We denote the sets of indices for all outgoing edges of v_i by $J^+(v_i)$ and for all incoming edges by $J^-(v_i)$. Since orienting Ω does not add or delete vertices or edges to/from the graph, we refer to the vertices and edges in $\vec{\Omega}$ by the same names as in Ω.

Lemma 4.27. *There is a directed acyclic graph (DAG), $\vec{\Omega} = (V, \vec{E})$ based on Ω.*

Proof. Assign a distinct positive integer to each vertex in V, then direct each edge from the vertex with larger assigned number to the vertex with smaller assigned number. Such an orientation clearly contains no cycles. □

For the rest of the text we assume all directed graphs are acyclic. See Figure 4.2 for a DAG graph $\vec{\Omega}$ based on Ω in Figure 4.1.

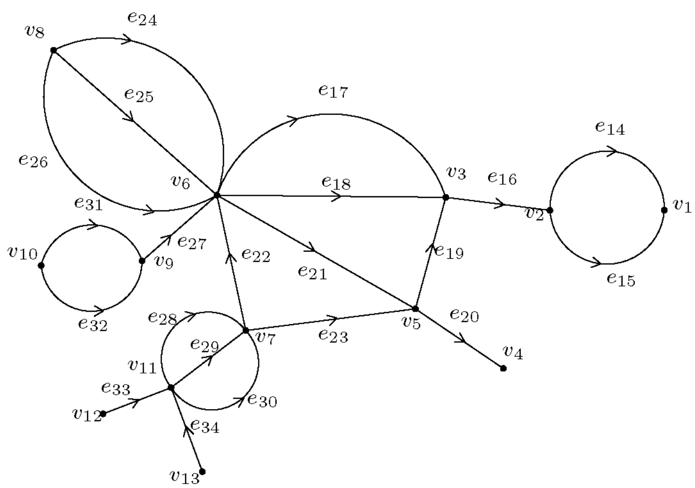

Figure 4.2: A DAG $\vec{\Omega}$ based on Ω.

In a directed graph $\vec{\Omega}$, a **source** is a vertex without incoming edges; a **sink** is a vertex without outgoing edges. We denote the sets of sources and sinks by $\vec{\Omega}^+$ and $\vec{\Omega}^-$ respectively. The following lemma ensures a linear ordering of vertices that is consistent with edge directions exists. Such a linear ordering is critical in solving the control problem.

Lemma 4.28. *The DAG $\vec{\Omega}$ has at least one linear ordering $W(V, <)$ of the set V such that for every directed path from v_i to v_k, $v_k < v_i$ in the ordering.*

Proof. Since $\vec{\Omega}$ is finite and acyclic, it has at least one sink, v_1. Let v_1 be the first element in W. Let $\vec{\Omega} \setminus \{v_1\}$ be the graph formed by deleting v_1 and its incident edges from $\vec{\Omega}$. Clearly $\vec{\Omega} \setminus \{v_1\}$ is acyclic and has at least one sink, say v_2. Let v_2 be the second element in W and repeat this process. We will have a linear ordering W such that for every directed path from v_i to v_k, $v_k < v_i$ in the ordering. $\square$

For less complicated graphs, one can identify a linear ordering without explicit algorithms. For example, in Figure 4.2, a natural linear ordering of all vertices is $W = (V, <)$, where $v_1 < \cdots < v_{13}$, such that the edges are directed from vertices with higher orders to vertices with lower orders. For more complex graphs, readers are referred to [*Knuth*, 1997] Section 2.2.3 for algorithms to find such a linear orderings.

4.2.4 The forward problem on an interval

In this section we discuss several IBVP problems of the wave equation on an interval $[0, l]$ with Dirichlet or Neumann boundary conditions. The solutions will be used in solving forward and control problems on general graphs later. We first consider the following problem:

$$\begin{cases} (u)_{tt} - (u)_{xx} + q(x)u = 0, \quad 0 < x < l, \ 0 < t < T \\ u|_{t \le 0} = 0, \quad u(0,t) = f(t), \quad u(l,t) = 0. \end{cases} \tag{4.16}$$

We will refer to f as the Dirichlet control function or simply, Dirichlet control. For $f \in H_*^1(0,T)$, the system (4.16) has a unique solution $u^{f,DD} \in C([0,T]; H^1(0,l))$. This solution can be presented by the so-called folding ruler formula [*Avdonin and Zhao*, 2021a]:

$$\begin{aligned}
& u^{f,DD}(x,t) \\
= \;& f(t-x) + \int_x^t \omega(x,s) f(t-s)\, ds \\
& - f(t-2l+x) - \int_{2l-x}^t \omega(2l-x,s) f(t-s)\, ds \\
& + f(t-2l-x) + \int_{2l+x}^t \omega(2l+x,s) f(t-s)\, ds \\
& - f(t-4l+x) - \int_{4l-x}^t \omega(4l-x,s) f(t-s)\, ds + \dots \\
= \;& \sum_{n=0}^{\lfloor \frac{t-x}{2l} \rfloor} \left(f(t-2nl-x) + \int_{2nl+x}^t \omega(2nl+x,s) f(t-s)\, ds \right) \\
& - \sum_{n=1}^{\lfloor \frac{t+x}{2l} \rfloor} \left(f(t-2nl+x) + \int_{2nl-x}^t \omega(2nl-x,s) f(t-s)\, ds \right)
\end{aligned} \tag{4.17}$$

where $\lfloor \cdot \rfloor$ is the floor function; $\omega(x,t)$ is a solution to the Goursat problem

$$\begin{cases} (\omega)_{tt} - (\omega)_{xx} + q(x)\omega = 0, \quad 0 < x < \infty, \\ \omega(0,t) = 0, \omega(x,x) = -\frac{1}{2}\int_0^x q(s)\,ds, \end{cases} \tag{4.18}$$

in which the potential $q(x)$ is extended to the semi-axis $x > 0$ by the rule $q(2nl \pm x) = q(x)$ for all $n \in \mathbb{N}$.

The folding ruler formula gives convenient presentations of the solution with various boundary conditions and will be used in many constructions of this chapter. More details about this formula can be found in [*Avdonin and Zhao*, 2021a]. When the Dirichlet control function $f \in H^1_*(0,T)$ is applied at $x = l$, the IBVP

$$\begin{cases} (u)_{tt} - (u)_{xx} + q(x)u = 0, \quad 0 < x < l, t > 0 \\ u|_{t \le 0} = 0, \quad u(0,t) = 0, \quad u(l,t) = g(t). \end{cases} \tag{4.19}$$

can be solved by changing of variables in (4.16). We put $p(x) = q(l-x)$ and extend p by letting $p(2nl \pm x) = p(x)$. Let $\bar{\omega}(x,t)$ be the solution to the Goursat problem (4.18) where $q(x)$ is replaced with $p(x)$, then

$$\begin{aligned} u^{DD,f}(x,t) \quad &= \quad f(t-l+x) + \int_{l-x}^{t} \bar{\omega}(l-x,s) f(t-s)\,ds \\ &\quad - f(t-l-x) - \int_{l+x}^{t} \bar{\omega}(l+x,s) f(t-s)\,ds + \dots \end{aligned} \tag{4.20}$$

When Dirichlet control functions $f(t)$ and $g(t)$ from $H^1_*(0,T)$ are applied at $x = 0$ and $x = l$ respectively, the solution to for the wave equation (4.10) on e with zero initial condition is denoted as $u^{f,DD,g}$. By the superposition principle, $u^{f,DD,g}(t) = u^{f,DD}(t) + u^{DD,g}(t)$.

Next we present the folding ruler formula for the following IBVP:

$$\begin{cases} (u)_{tt} - (u)_{xx} + q(x)u = 0, \quad 0 < x < l, \ 0 < t < T \\ u|_{t \le 0} = 0, \quad \partial u(0,t) = f'(t), \quad \partial u(l,t) = 0. \end{cases} \tag{4.21}$$

For $f'(t) \in L^2_*(0,T)$, the system (4.16) has a unique solution $u^{f',NN} \in C([0,T]; H^1(0,l))$. Let $f(t) := \int_0^t f'(s)\, ds$, then

$$\begin{aligned} & u^{f',NN}(x,t) \\ = \quad & -f(t-x) - \int_x^t \mu(x,s) f(t-s)\, ds \\ & -f(t-2l+x) - \int_{2l-x}^t \mu(2l-x,s) f(t-s)\, ds \\ & -f(t-2l-x) - \int_{2l+x}^t \mu(2l+x,s) f(t-s)\, ds \\ & -f(t-4l+x) - \int_{4l-x}^t \mu(4l-x,s) f(t-s)\, ds + \dots \end{aligned} \tag{4.22}$$

where $\mu(x,t)$ is a solution to the Goursat problem

$$\begin{cases} (\mu)_{tt} - (\mu)_{xx} + q(x)\mu = 0, \quad 0 < x < t \\ \partial \mu(0,t) = 0, \mu(x,x) = -\frac{1}{2}\int_0^x q(s)\, ds \end{cases} \tag{4.23}$$

Here $q(x)$ is extended to the semi-axis $x > 0$ by the same rule as in (4.18).

For $u^{f',ND}$, $u^{f,DN}$, we omit expressing their corresponding IBVP problems and just give the solutions. By direct substitution one can verify that

$$\begin{aligned} & u^{f',ND}(x,t) \\ = \quad & -f(t-x) - \int_x^t \mu(x,s)f(t-s)\,ds \\ & + f(t-2l+x) + \int_{2l-x}^t \mu(2l-x,s)f(t-s)\,ds \\ & + f(t-2l-x) + \int_{2l+x}^t \mu(2l+x,s)f(t-s)\,ds \\ & - f(t-4l+x) - \int_{4l-x}^t \mu(4l-x,s)f(t-s)\,ds + \dots \end{aligned} \tag{4.24}$$

$$\begin{aligned} & u^{f,DN}(x,t) \\ = \quad & f(t-x) + \int_x^t \omega(x,s)f(t-s)\,ds \\ & + f(t-2l+x) + \int_{2l-x}^t \omega(2l-x,s)f(t-s)\,ds \\ & - f(t-2l-x) - \int_{2l+x}^t \omega(2l+x,s)f(t-s)\,ds \\ & - f(t-4l+x) - \int_{4l-x}^t \omega(4l-x,s)f(t-s)\,ds + \dots \end{aligned} \tag{4.25}$$

The solutions $u^{NN,f'}$, $u^{ND,f}$, and $u^{DN,f'}$ can be computed by changing variables. The solutions $u^{f',NN,g'}$, $u^{f',ND,g}$, $u^{f,DN,g'}$ can be found using the superposition principle. In all these problems we assume that the Dirichlet control functions belong to $H^1_*(0,T)$, and all Neumann control functions belong to $L^2_*(0,T)$. The solutions are in the space $C([0,T];H^1(0,l))$.

4.2.5 Solution to the forward problem on a general graph

In this section, we discuss the solution of the problem (4.10)–(4.13) under the action of control $\mathbf{f}$ placed on the active set $\{I^*, J^*\}$. Here we consider a general case of arbitrary active set. In Section 4.3, we discuss a more specific case of the active set $\{I^*, J^*\}$. The scheme of our solution is similar to the presented in [*Avdonin and Zhao*, 2021a], but there we considered only Dirichlet controls that were placed only on the boundary edges.

Theorem 4.29. *Let $\vec{\Omega}$ be a connected DAG with more than one edge. There is a unique solution $u^{\mathbf{f}}$ to the IBVP* (4.10)–(4.13) *and*

$$u^{\mathbf{f}} \in C([0,T];\mathcal{H}_c^1) \cap C^1([0,T];\mathcal{H}). \tag{4.26}$$

Proof. Let $\mathbf{g}$ be a vector function, such that for any $v_i \in V \setminus \Gamma$, $g_i(t) = u_j(v_i,t)$, $j \in J^c(v_i)$. We find $\mathbf{g}$ by using the KN condition (4.11) on $V \setminus \Gamma$ to derive and solve a system of $|V| - |\Gamma|$ Volterra integral equations with respect to each entry of $\mathbf{g}'(t)$. The details are explained below.

Extend $\mathbf{f}$ to all nonactive boundary vertices vertices by 0, i.e., if $v_i \in \Gamma \setminus V^*$, let $f_i'(t) = 0$. Let $\overline{\mathbf{g}}$ be a vector function of $|V|$ entries, where $\bar{g}_i'(t) = f_i'(t)$ for $v_i \in \Gamma$, and $\bar{g}_i(t) = g_i(t)$ for $v_i \notin \Gamma$. For $v_i \notin \Gamma$, let

$$h_{ji}(t) = \begin{cases} \bar{g}_i(t) + f_j(t), & j \in J^*(v_i), \\ \bar{g}_i(t), & j \in J^c(v_i). \end{cases} \tag{4.27}$$

On each edge $e_j(v_i,v_k)$, define operators $W_j^{\pm}$ from $H_*^1(0,T)$ to $C([0,T];H^1(0,l_j))$, where

$$W_j^{+}\bar{g}_i = \begin{cases} u_j^{\bar{g}_i',ND}, & v_i \in \Gamma, v_k \notin \Gamma, \\ u_j^{h_{ji},DD}, & v_i \notin \Gamma, v_k \notin \Gamma, \\ u_j^{h_{ji},DN}, & v_i \notin \Gamma, v_k \in \Gamma, \end{cases} \tag{4.28}$$

and

$$W_j^{-}\bar{g}_k = \begin{cases} u_j^{ND,h_{jk}}, & v_i \in \Gamma, v_k \notin \Gamma, \\ u_j^{DD,h_{jk}}, & v_i \notin \Gamma, v_k \notin \Gamma, \\ u_j^{DN,\bar{g}_k'}, & v_i \notin \Gamma, v_k \in \Gamma. \end{cases} \tag{4.29}$$

We next define the operator $\partial^{\pm}$: $C([0,T];H^1([0,l]) \to L^2(0,T)$ to take the derivative along an edge outward of its starting/finishing vertex. Depending on the control function and where it is located, we can express all four operators $\partial^{\pm}W_j^{\pm}$ explicitly. For example, if $W_j^+\bar{g}_i = u_j^{h_{ji},DD}$, then

$$\begin{aligned}\partial^+W_j^+\,\bar{g}_i &= -h'_{ji}(t) + \int_0^t \partial_x\omega_j(0,s)h_{ji}(t-s)\,ds \\ &\quad - 2h'_{ji}(t-2l_j) - 2\omega_j(2l_j,2l_j)h_{ji}(t-2l_j) + 2\int_{2l_j}^t \partial_x\omega_j(2l_j,s)h_{ji}(t-s)\,ds \\ &\quad - 2h'_{ji}(t-4l_j) - 2\omega_j(4l_j,4l_j)h_{ji}(t-4l_j) + 2\int_{4l_j}^t \partial_x\omega_j(4l_j,s)h_{ji}(t-s)\,ds - \dots\end{aligned}$$

We next define an $|E|\times|V|$ matrix operator U, such that U has one column for each vertex and one row for each edge. The entries of U are defined in analogue to the entries in the incident matrix of Ω: if there is an edge e_j from v_i to v_k, then $U_{j,i} = W_j^+$ and $U_{j,k} = W_j^-$. All other entries in U are zeros. As one can see, $U\overline{\mathbf{g}}$ gives us a column vector of $|E|$ entries, where each entry is a function in $C([0,T];H^1([0,l_j])$.

Operator K is defined as an $|V|\times|E|$ matrix operator. Its entries are defined in analogue to the transpose of the incident matrix of Ω: if there is an edge e_j from vertex v_i to v_k, then $K_{ij} = \partial^+$ and $K_{kj} = \partial^-$. All other entries in K are zeros. Now $KU\overline{\mathbf{g}}$ is a column vector of $|V|$ entries. The i^{th} entry represents the sum of derivatives of u in the directions outward of the vertex v_i.

We use an $|V|\times|V|$ diagonal matrix D to pick out the interior vertices. Let $D_{ij} = 1$ if $i=j$ and $v_i \in V\setminus\Gamma$, $D_{ij} = 0$ otherwise. So the KN conditions on $V\setminus\Gamma$ can be represented by

$$DKU\overline{\mathbf{g}} = z(t) \tag{4.30}$$

where $z(t)$ is a column vector with the components defined in the RHS of (4.11). Expression

(4.30) is a column of $|V| - |\Gamma|$ equations. Each equation can be written in the form

$$|J(v_i)|g_i'(t) - \int_0^t G_i(s) g_i(t-s)\, ds = F_i(t), \tag{4.31}$$

where $v_i \in V \setminus \Gamma$ and the form of $G_i(s)$ depends on the incident edges of v_i. For example, if none of the incident edges of v_i is a boundary edge,

$$G_i(s) = \sum_{j\in J^+(v_i)} \partial_x \omega_j(v_i, s) + \sum_{j\in J^-(v_i)} \partial_x \bar{\omega}_j(v_i, s). \tag{4.32}$$

Equations (4.31) can be transformed into

$$|J(v_i)|g_i'(t) - \int_0^t H_i(s,t) g_i'(s)\, ds = F_i(t), \quad v_i \in V \setminus \Gamma, \tag{4.33}$$

with the transformation

$$\begin{aligned} &\int_0^t G_i(s) g_i(t-s)\, ds = \int_0^t G_i(s) \int_0^{t-s} g_i'(\xi)\, d\xi ds \\ &= \int_0^t g_i'(\xi) \int_0^{t-\xi} G_i(s)\, ds d\xi = \int_0^t g_i'(\xi) H_i(\xi, t)\, d\xi, \end{aligned} \tag{4.34}$$

where

$$H_i(\xi, t) = \int_0^{t-\xi} G_i(s)\, ds. \tag{4.35}$$

Readers are referred to the Section 4 of [*Avdonin and Zhao*, 2021a] for the example of deriving equations (4.33) on a star graph. Although in [*Avdonin and Zhao*, 2021a] we considered the system with Dirichlet control functions, the process of deriving equations (4.33) is the same in both [*Avdonin and Zhao*, 2021a] and here.

The expression of $F_i(t)$ depends on the incoming edges of v_i. For example, assume $v_i \notin V^*$, $J^*(v_i)$ is empty, and none of the incident edges of v_i is a boundary edge. In the following expression, for convenience, $y_j(t)$ denotes h_{jk} where $e_j(v_i, v_k)$ or $e_j(v_k, v_i)$ is an incident edge of v_i :

$$
\begin{aligned}
F_i(t) =& 2\sum_{j\in J^+(v_i)}\sum_{n=0}^{\lfloor\frac{t-l_j}{2l_j}\rfloor} y_j'(t-(2n+1)l_j) + 2\sum_{j\in J^+(v_i)}\sum_{n=0}^{\lfloor\frac{t-l_j}{2l_j}\rfloor} \bar{\omega}_j((2n+1)l_j,(2n+1)l_j)y_j(t-(2n+1)l_j) \\
&- 2\sum_{j\in J^+(v_i)}\sum_{n=0}^{\lfloor\frac{t-l_j}{2l_j}\rfloor} \int_{(2n+1)l_j}^{t} \partial_x\bar{\omega}_j((2n+1)l_j,s)y_j(t-s)\,ds \\
&+ 2\sum_{j\in J^-(v_i)}\sum_{n=0}^{\lfloor\frac{t-l_j}{2l_j}\rfloor} y_j'(t-(2n+1)l_j) + 2\sum_{j\in J^-(v_i)}\sum_{n=0}^{\lfloor\frac{t-l_j}{2l_j}\rfloor} \omega_j((2n+1)l_j,(2n+1)l_j)y_j(t-(2n+1)l_j) \\
&- 2\sum_{j\in J^-(v_i)}\sum_{n=0}^{\lfloor\frac{t-l_j}{2l_j}\rfloor} \int_{(2n+1)l_j}^{t} \partial_x\omega_j((2n+1)l_j,s)y_j(t-s)\,ds \\
&- 2\sum_{j\in J^+(v_i)}\sum_{n=1}^{\lfloor\frac{t}{2l_j}\rfloor} g_i'(t-2nl_j) - 2\sum_{j\in J^+(v_i)}\sum_{n=1}^{\lfloor\frac{t}{2l_j}\rfloor} w_j(2nl_j,2nl_j)g_i(t-2nl_j) \\
&+ 2\sum_{j\in J^+(v_i)}\sum_{n=1}^{\lfloor\frac{t}{2l_j}\rfloor} \int_{2nl_j}^{t} \partial_x\omega_j(2nl_j,s)g_i(t-s)\,ds \\
&- 2\sum_{j\in J^-(v_i)}\sum_{n=1}^{\lfloor\frac{t}{2l_j}\rfloor} g_i'(t-2nl_j) - 2\sum_{j\in J^-(v_i)}\sum_{n=1}^{\lfloor\frac{t}{2l_j}\rfloor} \bar{\omega}_j(2nl_j,2nl_j)g_i(t-2nl_j) \\
&+ 2\sum_{j\in J^-(v_i)}\sum_{n=1}^{\lfloor\frac{t}{2l_j}\rfloor} \int_{2nl_j}^{t} \partial_x\bar{\omega}_j(2nl_j,s)g_i(t-s)\,ds\,. \qquad (4.36)
\end{aligned}
$$

In other cases, where $J^*(v_i)$ is not empty, and/or some boundary vertices are adjacent to v_i, the process of deriving the expressions for $F_i(t)$ is similar. Equation (4.36) shows that $F_i(t)$ depends on the vector function $\mathbf{g}$ with arguments delayed by at least $\Delta := \min_{i=1,\ldots,N}(l_i)$. So if $\mathbf{g}$ on $[0, t-\Delta]$ is known, we can calculate $\mathbf{g}'(t)$ on $[t-\Delta, t]$ using equations (4.31) or (4.33). First, since $\mathbf{g}(t) = 0$ for $t \le 0$, we can calculate $\mathbf{g}'(t)$ on $[0, \Delta]$. At this step, $F_i \in L^2_*(0,T)$ since $f_i'(t), h_{j,i}'(t) \in L^2_*(0,T)$ for all $i \in I, j \in J$, and all other terms in $F_i(t)$ are more regular. Thus, for $t \in [0,\Delta]$, $g_i' \in L^2_*(0,\Delta)$ for all $i \in I$. Then we increase t and calculate all entries of $g'(t)$ for the next time interval. At each step, $F_i \in L^2_*(0,T)$, each computed portion of $g_i'(t)$ belongs to the space $L^2_*(0, t - \Delta)$ and all other terms in $F_i(t)$ are more regular. Thus $g_i' \in L^2_*(0, T)$ for all $i \in I$. The vector function $\mathbf{g}$ can be found integrating $\mathbf{g}'$ and, for all $i \in I$, $g_i \in H^1_*(0, T)$.

We now define a function u on $\vec{\Omega}$ such that its restriction on each edge $e_j(v_i, v_k)$ is computed by

$$u_j(x,t) = \begin{cases} u_j^{f_i', ND, h_{jk}}(x,t), & v_i \in \Gamma, v_k \notin \Gamma, \\ u_j^{h_{ji}, DD, h_{jk}}(x,t), & v_i \notin \Gamma, v_k \notin \Gamma, \\ u_j^{h_{ji}, DN, f_k'}(x,t), & v_i \notin \Gamma, v_k \in \Gamma, \end{cases} \tag{4.37}$$

By construction, u satisfies all equations (4.10)–(4.13).

□

A more detailed discussion of this construction for a specific active set $\{I^*, J^*\}$ is presented in Section 4.36.

4.3 The forward and control problems on a DAG with controllers placed on a single-track active set

4.3.1 The tangle-free path union and single-track active set of a DAG

In [*Avdonin and Zhao*, 2021a] shape/velocity controllability on a tree graph with Dirichlet boundary control was proved by representing a tree graph with a union of paths, and controlling the final shape/velocity on one path at a time. Controllability problem on a general graph can also be approached by representing the graph as a union of paths.

Definition 4.30. *Let $\vec{\Omega}$ be a DAG of Ω. Let U be a union of directed paths. We say U is a* ***tangle-free (TF) path union*** *of Ω if U satisfies the conditions:*

1. *The direction of all edges are the same as the direction of all paths they are on.*

2. *All paths are disjoint except for the starting and finishing vertices.*

3. *If a finishing vertex v of a path is the starting vertex of another path, there must be an incoming edge of v that is not a finishing edge, and an outgoing edge of v that is not a starting edge.*

4. $\vec{\Omega} = \cup_{P \in U} P$.

Definition 4.31. *We say is I^* is the **single-track (ST) active set of vertex indices** of $\vec{\Omega}$ if*

$$I^* = \{i : v_i \in \Omega^+\}, \tag{4.38}$$

*J^* is a **ST active set of edge indices** if*

$$J^*(v_i) = \text{all but one elements of } J^+(v_i), i \in I. \tag{4.39}$$

*The set $\{I^*, J^*\}$ is then called a **ST active set.***

For a DAG, its ST active set of vertices is uniquely determined by its set of sources. However, there is more than one way to determine its ST active set of edge indices. Indeed, suppose a vertex v has n outgoing edges, then there are n ways to determine which $n-1$ of the outgoing edges of v are in J^*.

Suppose $\{I^*, J^*\}$ is a ST active set, we define the following **closure** *of J^*,*

$$\bar{J}^* := (\cup_{i \in I^*} J(v_i)) \cup J^*, \tag{4.40}$$

The ST active sets of edge indices are considered to be in the same equivalence class if they have the same closure.

Lemma 4.32. *Let $\vec{\Omega}$ be a DAG and V^* be its ST active set of vertices, then each TF path union U corresponds to a ST active set of edge indices J^* (up to the equivalence class $\bar{J}^*$). That is, for every path in U, the index of its starting edge is in $\bar{J}^*$. Conversely, every ST active set of edge indices J^* corresponds to a TF path union U, such that for every $j \in \bar{J}^*$, e_j is the starting edge of a path in U.*

Proof. Let U be a TF path union of $\vec{\Omega}$, and S be the set of all starting edges of paths in U. By Definition 4.30 conditions (1) and (4), S includes all incident (outgoing) edges of source vertices; by Definition 4.30 conditions (2) and (3), S includes all but one of the outgoing edges of non-source vertices. Let J^* include the indices of all but one of the outgoing edges of source vertices that are in S, and all outgoing edges of non-source vertices that are in S, then J^* contains the indices of all but one outgoing edge at every vertex in V. Equation(4.39) is satisfied. The closure of J^*, defined by (4.40), is precisely S.

Conversely, let J^* be a ST active set of edge indices, and $\bar{J}^*$ be its closure defined by (4.40). Denote $(\bar{J}^*)^c$ as the complement of $\bar{J}^*$ in J on $\vec{\Omega}$. We construct a union of paths U according to the rules in (1), where $\bar{J}^*$ are indices of the starting edges and $(\bar{J}^*)^c$ are indices of the non-starting edges. Starting from $n = 1$, suppose we already constructed $n - 1$ paths. The next step is to construct P_n. The edges in the constructed $P_1, \ldots, P_{n-1}$ are referred to as "used". We pick an unused edge $e_1(v_1, v_2)$ whose index is in $\bar{J}^*$ as the starting edge of P_n (If at this point all $\bar{J}^*$ indices are used, our construction is finished and our union has $n - 1$ paths). If v_2 has no unused outgoing edge indexed in $(\bar{J}^*)^c$, v_2 is the last vertex of P_n. If v_2 has an unused outgoing edge $e_3(v_2, v_3)$ indexed in $(\bar{J}^*)^c$, we add e_2 to P_n. We continue adding edges indexed in $(\bar{J}^*)^c$ to P_n until there are no unused edge indexed in $(\bar{J}^*)^c$ outgoing from the last vertex. Now we have P_n. Once we complete P_n we move on to construct P_{n+1}. We keep constructing paths until all indices in $\bar{J}^*$ are used. We will show at this point all $(\bar{J}^*)^c$ indices are used. Suppose there are several $(\bar{J}^*)^c$ indices that are not used. Due to the finiteness of the graph we can find an edge $e(v_1, v_2)$ indexed in $(\bar{J}^*)^c$ such that all incoming edges of v_1 are used in different paths (by (4.40) v_1 is not a source, so it has incoming edges). By (4.39), e is the only outgoing edge of v_1 that is not a starting edge. It should be added to one of the paths the incoming edges of v_1 are on. This contradicts the assumption that e is not used. Therefore the paths we constructed cover the entire graph. That is, condition (4) in Definition 4.30 is satisfied. Condition (2) and (3) are satisfied since each vertex has at most one outgoing edge indexed in $(\bar{J}^*)^c$, the edge can be added as a subsequent edge of one path. All other incoming edges are finishing edges and all other outgoing edges are indexed in $\bar{J}^*$ and can only be used as starting edges of paths. So U is a TF path union constructed from $\{I^*, J^*\}$.

□

Corollary 4.33. *Let $\vec{\Omega}$ be a DAG, $\{I^*, J^*\}$ be a ST active set, U be a TF path union determined from $\{I^*, J^*\}$, then the number of paths in U is $|V^*| + |J^*|$.*

Proof. By Lemma 4.32, the number of paths equals to $|\bar{J}^*|$. Since for each $i \in I^*$, there is only one element in $J(v_i)$ that is not in J^*, $|\bar{J}^*| = |I^*| + |J^*|$. □

Example 4.34. *For the DAG $\vec{\Omega}$ in Figure 4.2, a ST active set is $V^* = \{v_8, v_{10}, v_{12}, v_{13}\}$, $J^*(v_1) = \emptyset$, $J^*(v_2) = \{15\}$, $J^*(v_3) = \emptyset$, $J^*(v_4) = \emptyset$, $J^*(v_5) = \{19\}$, $J^*(v_6) = \{18, 21\}$, $J^*(v_7) = \{23\}$, $J^*(v_8) = \{25, 26\}$, $J^*(v_9) = \emptyset$, $J^*(v_{10}) = \{32\}$, $J^*(v_{11}) = \{29, 30\}$, $J^*(v_{12}) = \emptyset$, $J^*(v_{13}) = \emptyset$. See Figure 4.3.*

One ST path union associated with $\{I^*, J^*\}$ is then $P_1 = (v_8, e_{24}, v_6, e_{17}, v_3, e_{16}, v_2, e_{14}, v_1)$, $P_2 = (v_8, e_{26}, v_6)$, $P_3 = (v_8, e_{25}, v_6)$, $P_4 = (v_6, e_{18}, v_3)$, $P_5 = (v_5, e_{19}, v_3)$, $P_6 = (v_2, e_{15}, v_1)$, $P_7 = (v_{10}, e_{31}, v_9, e_{27}, v_6)$, $P_8 = (v_{10}, e_{32}, v_9)$, $P_9 = (v_{12}, e_{33}, v_{11}, e_{28}, v_7, e_{22}, v_6)$, $P_{10} = (v_{13}, e_{34}, v_{11})$, $P_{11} = (v_{11}, e_{29}, v_7)$, $P_{12} = (v_{11}, e_{30}, v_7)$, $P_{13} = (v_6, e_{21}, v_5, e_{20}, v_4)$, $P_{14} = (v_7, e_{23}, v_5)$.

Note that $|U| = |V^*| + |J^*| = 14$.

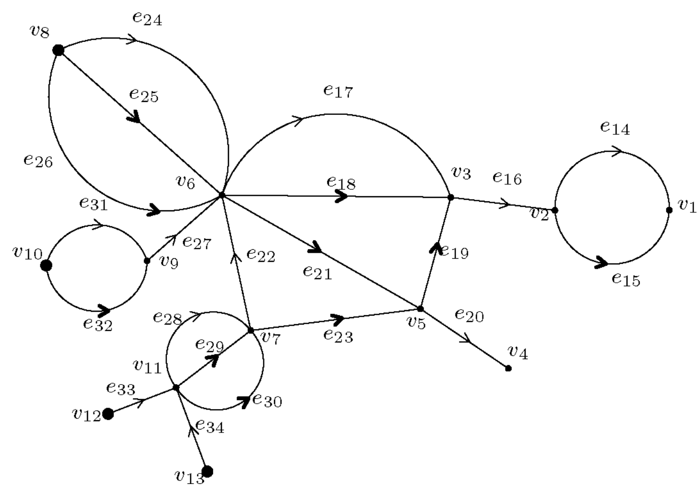

Figure 4.3: $\vec{\Omega}$ with ST active vertices and edges emphasized. The set $\bar{J}^*$ includes all edge indices in J^* and $10, 18, 20, 21$.

Definition 4.35. *Let U be a TF path union of a DAG $\vec{\Omega}$. We define **depth** of an edge $e_j(v_i, v_k)$ in $\vec{\Omega}, U$, recursively as*

$$\text{depth}(e_j(v_i, v_k)) = \begin{cases} l_j, & v_k \text{ is a finishing edge in } U, \\ l_j + \max_{r \in J^+(v_k)} \text{depth}(e_r), & \text{otherwise.} \end{cases} \tag{4.41}$$

Depth of an edge depends on the choice of orientation and TF path union. When multiple orientations and TF path unions are mentioned we write $\text{depth}(e_j)$ as $\text{depth}(e_j, \vec{\Omega}, U)$ to emphasis this dependency.

Intuitively, the depths of an edge can be viewed as the maximum length of the direct paths starting from the edge in the forest graph obtained from $\vec{\Omega}$, by disconnecting every path in U at its finishing vertex. See Figure 4.4 for example, where the depth of e_{17} is $\max(l_{17} + l_{16} + l_{15}, l_{17} + l_{16} + l_{14})$. The depth of e_{24} is $\max(l_{24} + l_{18}, l_{24} + l_{17} + l_{16} + l_{15}, l_{24} + l_{17} + l_{16} + l_{14}, l_{24} + l_{21} + l_{19}, l_{24} + l_{21} + l_{20})$.

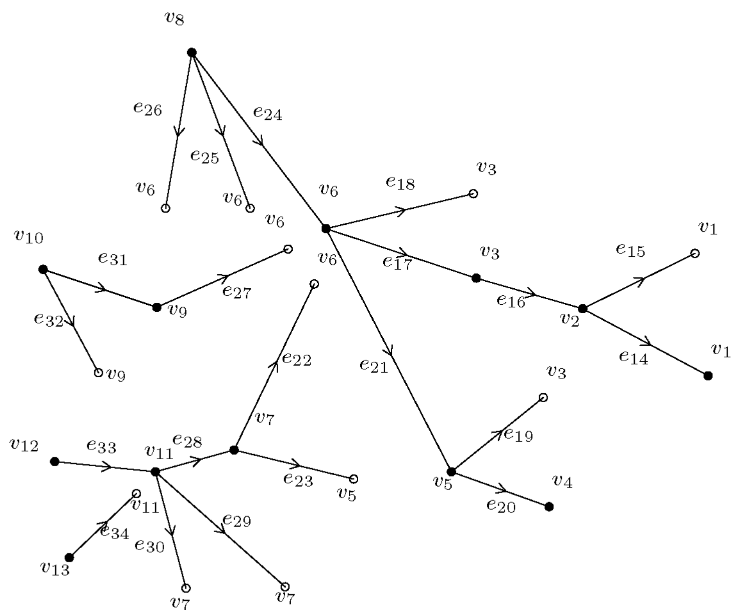

Figure 4.4: The forest graph obtained from $\vec{\Omega}$ by disconnecting every path in U at its finishing vertex.

Everywhere in Sections 4.36–4.43, $\vec{\Omega}$ is assumed to be a DAG, $\{I^*, J^*\}$ is assumed to be a ST active set, and U is assumed to be the TF-path union U corresponding to $\{I^*, J^*\}$.

4.3.2 The forward problem when $\{I^*, J^*\}$ is a ST active set

When $\{I^*, J^*\}$ is a ST active set, Theorem 4.29 can restated as:

Theorem 4.36. *Let $\vec{\Omega}$ be a DAG, and $\{I^*, J^*\}$ be an ST active set associated with $\vec{\Omega}$. Let $T > 0$, then for any $\mathbf{f} \in \mathcal{F}^T$ the system* (4.10)–(4.13) *has a unique solution $u^{\mathbf{f}}(\cdot, t) \in C([0,T]; \mathcal{H}_c^1) \cap C^1([0,T]; \mathcal{H})$.*

We give below more explicit expressions of equations (4.33) when $\{I^*, J^*\}$ is a ST active set, and when all potential functions $q_j = 0$. The expressions for nonzero potentials are are presented in the Section 4.4 Appendix. Those explicit equations will be used in the next section for solving the control problem.

Consider an edge $e_j(v_i, v_k)$ on Ω. Depending on whether v_i or v_k is a boundary vertex, there are four combinations. Case 1: $v_i \in \Gamma^*, v_k \in \Gamma$. In this case the graph contains only one edge, therefore $u_j(x,t)$ can be computed from

$$u_j(x,t) = u^{f'_i,NN}(x,t). \tag{4.42}$$

Case 2: $v_i \notin \Gamma, v_k \in \Gamma$, then

$$u_j(x,t) = u^{h_j,DN}(x,t), \tag{4.43}$$

where

$$u_j(v_i,t) = h_j(t) := \begin{cases} g_i(t), & j \notin J^*, \\ f_j(t) + g_i(t), & j \in J^*. \end{cases} \tag{4.44}$$

Case 3, $v_i, v_k \notin \Gamma$,

$$u_j(x,t) = u^{h_j,DD,g_k}(x,t), \tag{4.45}$$

where $h_j(t)$ is defined in (4.44) and

$$u_j(v_k,t) = g_k(t). \tag{4.46}$$

Case 4, $v_i \in \Gamma^*, v_k \notin \Gamma$,

$$u_j(x,t) = u^{f'_i,ND,g_k}(x,t). \tag{4.47}$$

.

Once we have the presentation of $u_j(x,t)$ for every $j \in J$, we can write explicit expressions for equations (4.31) for all $v_i \in V \setminus \Gamma$. In what follows we denote by $J^+(\Gamma^*)$ and $J^-(\Gamma)$ the set of indices of the outgoing and incoming edges of the boundary vertices.

Every $v_i \in V^* \setminus \Gamma$ has only outgoing edges. For all $j \in J^+(v_i) \cap J^-(\Gamma)$, $u_j(x,t) = u_j^{h_j,DN}$. For all $j \in J^+(v_i) \setminus J^-(\Gamma)$, $u_j(x,t) = u_j^{h_j,DD,g_{k(j)}}$, where $k(j)$ is the finishing vertex of e_j. Therefore

$$
\begin{aligned}
f_i'(t) = \sum_{j \in J^(v_i)} \partial u_j(v_i, t) \\
&= \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \left[-h_j'(t) - 2h_j'(t - 2l_j) - 2h_j'(t - 4l_j) + \dots \right] \\
&\quad + \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \left[2g_{k(j)}'(t - l_j) + 2g_{k(j)}'(t - 3l_j) + \dots \right] \\
&+ \sum_{j \in J^+(v_i) \cap J^-(\Gamma)} \left[-h_j'(t) + 2h_j'(t - 2l_j) - 2h_j'(t - 4l_j) + \dots \right]
\end{aligned}
\tag{4.48}
$$

Every $v_i \in (V \setminus V^*) \setminus \Gamma$ has incoming edges and possible outgoing edges. For all $j \in J^-(v_i) \cap J^+(\Gamma^*)$, $u_j(x,t) = u_j^{f_k', ND, g_i}$, where v_k is the starting vertex of e_j. For all $j \in J^-(v_i) \setminus J^+(\Gamma^*)$, $u_j(x,t) = u_j^{h_j, DD, g_i}$. If v_i has outgoing edges, for all $j \in J^+(v_i) \cap J^-(\Gamma)$, $u_j(x,t) = u_j^{h_j, DN}$. For all $j \in J^+(v_i) \setminus J^-(\Gamma)$, $u_j(x,t) = u_j^{h_j, DD, g_{k(j)}}$. Therefore

$$
\begin{aligned}
0 = \sum_{j\in J^-(v_i)\cap J^+(\Gamma^*)} & \left[-2f'_{k(j)}(t-l_j) + 2f'_{k(j)}(t-3l_j) - \dots\right] \\
& + \sum_{j\in J^-(v_i)\cap J^+(\Gamma^*)} \left[-g'_i(t) + 2g'_i(t-2l_j) - 2g'_i(t-4l_j) + \dots\right] \\
& + \sum_{j\in J^-(v_i)\setminus J^+(\Gamma^*)} \left[2h'_j(t-l_j) + 2h'_j(t-3l_j) + \dots\right] \\
& + \sum_{j\in J^-(v_i)\setminus J^+(\Gamma^*)} \left[-g'_i(t) - 2g'_i(t-2l_j) - 2g'_i(t-4l_j) + \dots\right] \\
& + \sum_{j\in J^+(v_i)\setminus J^-(\Gamma)} \left[-h'_j(t) - 2h'_j(t-2l_j) - 2h'_j(t-4l_j) + \dots\right] \\
& + \sum_{j\in J^+(v_i)\cap J^-(\Gamma)} \left[-h'_j(t) + 2h'_j(t-2l_j) - 2h'_j(t-4l_j) + \dots\right] \\
& + \sum_{j\in J^+(v_i)\setminus J^-(\Gamma)} \left[2g'_{k(j)}(t-l_j) + 2g'_{k(j)}(t-3l_j) + \dots\right]
\end{aligned}
\tag{4.49}
$$

For given control function $\mathbf{f}(t)$, each of $|V| - |\Gamma|$ equations (4.31) is in the form of either (4.48) or (4.49). This system of equations can be solved in steps to obtain $\mathbf{g}(t)$. Substituting $\mathbf{g}(t)$ into (4.42)–(4.47) we obtain the solution to the forward problem on all edges.

4.3.3 Shape and velocity controllability on an interval

In this section we discuss and compare the shape and velocity control problems for the IBVP (4.16) and the IBVP (4.21). Let $T_* = l$.

Proposition 4.37. *1. For any $T \geq T_*$ and any $\phi \in H^1(0,l)$ such that $\phi(l) = 0$, there exists a control function $f \in H^1_*(0,T)$ such that the solution to (4.16) satisfies the relationship $u^{f,DD}(x,T) = \phi(x)$.*

2. For any $T \geq T_*$ and $\psi \in L^2(0,l)$, there exists a control function $g \in H^1_*(0,T)$ such that $u_t^{g,DD}(x,T) = \psi(x)$.

Proof. It is sufficient to prove the case where $T = T_*$ for both parts. For Part 1, one can solve the following Volterra integral equation of the second kind to find $f(t)$.

$$\phi(x) = f(T - x) + \int_x^T w(x,s) f(T - s)\, ds.$$

For Part 2, one can solve the following Volterra integral equation of the second kind to find $g'(t)$, and obtain $g(t)$ through integration, since $g(0) = 0$.

$$\psi(x) = g'(T - x) + \int_x^T w(x,s) g'(T - s)\, ds.$$

□

Now we consider the control problem for (4.21). The target shape function ϕ is in $H^1(0,l)$. If $\phi(l) \neq 0$, then ϕ is not achievable in time $T = T_*$, since $u^{f',NN}(T,T) = 0$ for all $T \leq T_*$ and all $f' \in L^2(0,T)$.

Proposition 4.38. *1. For any $T > T_*$ and any function $\phi \in H^1(0,l)$, there exists a control function $f' \in L^2(0,T)$ such that the solution to (4.21) satisfies the relationship $u^{f',NN}(x,T) = \phi(x)$.*

2. For any $T \geq T_*$ and any function $\psi \in L^2(0,l)$, there exists a control function $g' \in L^2(0,T)$ such that $u_t^{g',NN}(x,T) = \psi(x)$.

Proof. For Part 1, it is sufficient to prove the case where $T_* < T < T_* + l$. Denote $\epsilon := T - T_*$. We first construct a control function $r(t)$ such that $u^{r',NN}(l,T) = \phi(l)$. Let $d(t)$ be a continuous function defined on $[0,T]$ such that it support is on $(0,2\epsilon)$, and $d(\epsilon) = 1$. Since the wave propagation speed is 1, $u^{d,NN}(l,T) \neq 0$. Let $c = \phi(l)/u^{d,NN}(l,T)$, and $r(t) = c \cdot d(t)$, then $u^{r,NN}(l,T) = \phi(l)$.

Let $b(t) = 0$ for $t \leq \epsilon$. Let the portion of $b(t)$ on the time interval $\epsilon \leq t \leq T$ be the unique solution to the Volterra integral equation of the second kind:

$$\phi(x) - u^{r',NN}(x,T) = -b(T-x) - \int_x^T \mu(x,s)b(T-s)\,ds,$$

then $f'(t) = r'(t) + b'(t)$ solves the control problem

$$u^{f',NN}(x,T) = \phi(x).$$

The support of $f'(t)$ is on $(0,2\epsilon) \cup (\epsilon,T) = (0,T)$. Since $r(t)$ is continuous, and $b'(t)$ is square integrable, $f'(t) \in L^2(0,T)$.

For Part 2, it is sufficient to prove the case where $T = T_*$. One can solve the following Volterra integral equation of the second kind to find $g'(t)$.

$$\psi(x) = -g'(T-x) - \int_x^T w(x,s)g'(T-s)\,ds.$$

□

A similar result in a slightly different problem was proved in [*Avdonin et al.*, 2022].

4.3.4 Shape and velocity controllability on graphs

In this section we prove the shape and velocity controllability on general graphs. For readability, the proof and examples are given for the case of zero potentials. The corresponding equations for nonzero potentials are presented in the Section 4.4 Appendix. We show that it is possible to find a control function $\mathbf{f}(t)$ such that the equations (4.48) or (4.49) are satisfied at every $v_i \in V \setminus \Gamma$, and the final shape or velocity of the solution computed by the folding ruler formulae equals to the target functions.

In Proposition 4.38, the wave equation system cannot attain the shape controllability at the minimum necessary time for the velocity controllability. This is true for many graphs. Although for many other graphs, shape controllability and velocity controllability take the same time.

Theorem 4.39. *Let $\vec{\Omega}$ be a DAG of Ω, $\{I^*, J^*\}$ be a ST active set, and U be the associated path union. Let*

$$T_* := \max_{j \in J} \text{depth}(e_j). \tag{4.50}$$

1. *For any $\phi \in \mathcal{H}^1$ and $T > T_*$, there exists $\boldsymbol{f} \in \mathcal{F}_0^T$ on $\{I^*, J^*\}$, such that $u_j(x, T) = \phi_j(x)$, where $u_j := u^{\boldsymbol{f}}|_{e_j}$ and $\phi_j(x) := \phi|_{e_j}(x)$ for all $j \in J$ and $x \in [0, l_j]$.*

2. *For any $\psi \in \mathcal{H}$ and $T \geq T_*$, there exists $\boldsymbol{f} \in \mathcal{F}^T$ such that $(u_j)_t(x, T) = \psi_j(x)$, where $u_j := u^{\boldsymbol{f}}|_{e_j}$ and $\psi_j(x) := \psi|_{e_j}(x)$ for all $j \in J$ and $x \in [0, l_j]$.*

3. *Let the greatest lower bound for both shape and velocity controllabilty on Ω be T_{inf} , then*

$$T_{inf} = \min_{\vec{\Omega}, U} \max_{j \in J} \text{depth}(e_j, \vec{\Omega}, U).$$

Proof. **Part 1**.

Let $\mathbf{g}$ and h_j be computed from $\mathbf{f}$ as in Theorem 4.36. We show $\mathbf{f}(t) \in \mathcal{F}_0^T$ can be constructed in such a way that if the four rules listed below are followed, $\mathbf{f}(t)$ is a solution to the shape control problem (4.10)–(4.13).

For every $v_i \in \Omega^-$, we pick one incoming edge $e_{m(i)}$ such that $l_{m(i)} = \min_{j \in J^-(v_i)} l_j$, and refer to $e_{m(i)}$ as the controlling edge of v_i. Note that for $v_i \in \Gamma \cap \Omega^-$, its controlling edge is its the incoming edge of this vertex; for $v_i \in \Omega^- \setminus \Gamma$, the pick of its controlling edge is not unique. Let

$$0 < \epsilon < \min_{j \in J} \{l_j, T - T_*\}. \tag{4.51}$$

The four rules for constructing shape control functions are:

1. If v_i is a sink with degree 1, let $e_j(v_k, v_i)$ be its only incoming edge, then $h_j(t) = 0$ when t is outside of $[T - l_j - \epsilon, T]$.

2. If v_i is a sink with degree more than 1, then $g_i(t)$ is a continuous function whose support is on $(T - \epsilon, T]$, and $g_i(T) = \phi(v_i)$. Let $e_{m(i)}$ be its controlling edge, then $h_{(m(i)}(t) = 0$

when t is outside of $[T - l_{m(i)} - \epsilon, T]$. For all other incoming edges e_j, $h_j(t) = 0$ when t is outside of $[T - l_j, T]$.

3. If v_i is not a sink, there is only one incoming edge (also denoted as $e_{m(i)}$) that is not a finishing edge of a path in U. For all incoming edges $e_j(v_k, v_i)$ that are not finishing edges of paths in U, $h_j(t) = 0$ when t is outside of $[T - l_j, T]$. Thus from all incoming edges of v_i, only $h_{m(i)}(t)$ has impact on $g_i(t)$.

4. On every $e_j(v_k, v_i)$ that $v_k \in \Gamma^*, v_i \in \Gamma$ (in this case a graph has only one edge)

$$u_j(x,T) = u_j^{h_j,NN}(x,T) = \phi_j(x), \quad x \in [0, l_j]. \tag{4.52}$$

On every $e_j(v_k, v_i)$ that $v_k \notin \Gamma^*, v_i \in \Gamma$,

$$u_j(x,T) = u_j^{h_j,DN}(x,T) = \phi_j(x), \quad x \in [0, l_j]. \tag{4.53}$$

On every $e_j(v_k, v_i)$ that $v_k \notin \Gamma^*$, $v_i \notin \Gamma$,

$$u_j(x,T) = u_j^{h_j,DD,g_i}(x,T) = \phi_j(x), \quad x \in [0, l_j]. \tag{4.54}$$

On every $e_j(v_k, v_i)$ that $v_k \in \Gamma^*, v_i \notin \Gamma$

$$u_j(x,T) = u_j^{f_k',ND,g_i}(x,T) = \phi_j(x), \quad x \in [0, l_j]. \tag{4.55}$$

Suppose the graph contains more than one edge. We now construct $h_j(t)$ for all $j \in J$, and derive entries of $\mathbf{f}(t)$ and $\mathbf{g}(t)$ from $h_j(t)$, following the linear ordering of V. Let $v_1, \ldots, v_n$ be a linear ordering of V by Lemma 4.28. At every v_i, we calculate $f_j(t)$ for $j \in J^*(v_i)$ and $f_i'(t)$ if $i \in I^*$ based on the known $h_j(t)$ for $j \in J^-(v_l)$ for all $v_l < v_i$. Then we construct $h_j(t)$ for $j \in J^-(v_i)$. The detailed steps are described below.

The first vertex, v_1, is a sink. If $v_1 \in \Gamma$, suppose $J^-(v_i) = \{j\}$. Similar to the process in Proposition 4.38 Part 1, we construct a function $h_j(t)$ supported on $(T - l_j - \epsilon, T]$ such that $u_j^{h_j,DN}(x, T) = \phi_j(x)$.

Now suppose v_1 is not a boundary vertex. Since it is a sink, we set the target $g_1(T) = \phi(v_1)$. This target is controlled from its controlling edge $e_{m(1)}$. Since $g_1(t)$ is supported on $[T - \epsilon, T]$, (4.49) becomes

$$2h'_{m(i)}(t - l_{m(i)}) = -\sum_{j \in J^-(v_1)} g'_1(t) \tag{4.56}$$

Solve (4.56) to get $h'_{m(i)}(t)$ for $t \in (T - l_{m(i)} - \epsilon, T - l_{m(i)})$. Since the initial value of $h'_{m(i)}(t)$ is 0, one can integrate to find $h_{m(i)}(t)$ on $[T - l_{m(i)} - \epsilon, T - l_{m(i)}]$. The portion of $h_{m(i)}(t)$ on $(T - l_{m(i)}, T)$ is computed through

$$h_{m(i)}(T - x) - h_{m(i)}(T - 2m(i) + x) + g_1(T - l_{m(i)} + x) = \phi_{m(i)}(x). \tag{4.57}$$

For every other incoming edge e_j, $j \neq m(i)$, $h_j(t)$ is supported on $[T - l_j, T]$. It can be computed through

$$h_j(T - x) + g_1(T - l_j + x) = \phi_j(x). \tag{4.58}$$

Extend all $h_j(t)$ to the negative axis by 0. Now we make the induction assumption that up to some $i \in \{2, \ldots, n\}$, for every $v_l < v_i$ and all its incoming edge $e_j(v_k, v_l)$, we have constructed $h_j(t)$, such that

$$h_j(t) \in H^1_*[T - \text{depth}^*(e_j), T], \tag{4.59}$$

where

$$\text{depth}^*(e_j(v_i, v_k)) := \begin{cases} l_j + \epsilon, & v_k \in \Omega^-, e_j \text{ is the controlling edge of } v_i, \\ l_j, & v_k \in \Omega^-, e_j \text{ is not the controlling edge of } v_i, \\ l_j, & v_k \notin \Omega^-, e_j \text{ is a finishing edge of a path in } U, \\ l_j + \max_{r \in J^+(v_k)} \text{depth}^*(e_r), & \text{otherwise.} \end{cases} \tag{4.60}$$

One can prove from (4.41) and (4.60) and by induction for any $j \in J$,

$$0 \leq \text{depth}^*(e_j) - \text{depth}(e_j) \leq \epsilon. \tag{4.61}$$

Thus by (4.50), (4.51), and (4.61),

$$T - \text{depth}^*(e_j) > 0 \text{ for any } j \in J. \tag{4.62}$$

Moreover, as a part of the induction assumption, we assume for all $v_l < v_i$ (when $i = 2$, the case is trivial), we have computed $f_j(t)$ for all $j \in J^*(v_l)$ and $f_l'(t)$ if $v_l \in V^*$, and

$$f_j(t) \in H_0^1(T - D^*(v_l), T), \quad f_l'(t) \in L_*^2(T - D^*(v_l), T),$$

where

$$D^*(v_l) := \max_{s \in J^+(v_l)} \text{depth}^*(e_s). \tag{4.63}$$

By (4.61) and (4.63),

$$T - D^*(v_i) > 0 \text{ for any } v_i \in V. \tag{4.64}$$

There are three cases at v_i:

Case 1, v_i is a sink. We compute $h_j(t)$ for all $j \in J^-(v_i)$ in the same way we compute $h_j(t)$ for all $j \in J^-(v_1)$.

Case 2, v_i is neither a sink nor a source. We first compute $f_j(t)$ for $j \in J^*(v_i)$. At this point we have computed $h_j(t)$ for all $j \in J^+(v_i)$, since each $j \in J^+(v_i)$ is in some $J^-(v_l)$ for some $v_l < v_i$. Since there is only one path in U through v_i, let the incoming edge of v_i on U be $e_{m(i)}$, and the outgoing edge of v_i in U be $e_{r(i)}$. By (4.12),

$$g_i(t) = h_{r(i)}(t).$$

For all $j \in J^*(v_i)$,

$$f_j(t) = h_j(t) - g_i(t).$$

Since each $h_s(t) = 0$ for $t \leq T - \text{depth}^*(e_s)$ for all $s \in J^+(v_i)$, $f_j(t) = 0$ for $t \leq T - D^*(v_i)$. Moreover, since $\phi(\cdot)$ is continuous at v_i, $h_j(T) = g_i(T) = \phi(v_i)$. Thus $f_j(T) = 0$. So

$$f_j(t) \in H_0^1(T - D^*(v_i), T). \tag{4.65}$$

Now we calculate the boundary conditions at the starting vertices of the incoming edges of v_i. We first discuss $e_{m(i)}$. Let $v_{k(m(i))}$ be its starting vertex of $e_{m(i)}$. For $0 \leq t \leq T$, if $v_{k(m(i))} \in V^*$, (4.49) becomes

$$\begin{aligned} 0 = &\left[-2f'_{k(m(i))}(t - l_{m(i)}) + 2f'_{k(m(i))}(t - 3l_{m(i)}) - \dots\right] \\ &+ \left[-g'_i(t) + 2g'_i(t - 2l_{m(i)}) - 2g'_i(t - 4l_{m(i)}) + \dots\right] \\ &+ \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \left[-h'_j(t) - 2h'_j(t - 2l_j) - 2h'_j(t - 4l_j) + \dots\right] \\ &+ \sum_{j \in J^+(v_i) \cap J^-(\Gamma)} \left[-h'_j(t) + 2h'_j(t - 2l_j) - 2h'_j(t - 4l_j) + \dots\right] \\ &+ \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \left[2g'_{k(j)}(t - l_j) + 2g'_{k(j)}(t - 3l_j) + \dots\right] \end{aligned} \tag{4.66}$$

in which $v_{k(j)}$ is the vertex of e_j other than v_i. If $v_k \notin V^*$, (4.49) becomes

$$
\begin{aligned}
0 = \big[2h'_{m(i)}(t - l_{m(i)}) + 2h'_{m(i)}(t - 3l_{m(i)}) + \dots\big] \\
+ \big[-g'_i(t) - 2g'_i(t - 2l_{m(i)}) - 2g'_i(t - 4l_{m(i)}) + \dots\big] \\
+ \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \big[-h'_j(t) - 2h'_j(t - 2l_j) - 2h'_j(t - 4l_j) + \dots\big] \\
+ \sum_{j \in J^+(v_i) \cap J^-(\Gamma)} \big[-h'_j(t) + 2h'_j(t - 2l_j) - 2h'_j(t - 4l_j) + \dots\big] \\
+ \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \big[2g'_{k(j)}(t - l_j) + 2g'_{k(j)}(t - 3l_j) + \dots\big]
\end{aligned}
\tag{4.67}
$$

In (4.66), the only unknown function is $f'_{(k(m(i))}(t)$. By the induction assumption, each $h_j(t) \in H^1_*[T - \text{depth}^*(e_j), T]$, thus the third and fourth lines of (4.66) together is in the space $L^2_*[T - \max_{j \in J^+(v_i)} (\text{depth}^*(e_j)), T]$.

For the fifth line of (4.66), since $g'_{k(j)}(t) \in L^2_*[T - \text{depth}^*(e_l), T]$ for some $l \in J^+(v_k(j))$, Thus the entire fifth line is in the space $L^2_*[T - \max_{j \in J^+(v_i)} (\text{depth}^*(e_j)), T]$. Therefore the $f'_{k(m(i))}(t)$ obtained from (4.66) is in the following space:

$$
L^2_*[T - \max_{j \in J^+(v_i)} \text{depth}^*(e_j) - l_{m(i)}, T - l_{m(i)}] = L^2_*[T - \text{depth}^*(e_{m(i)}), T - l_{m(i)}].
$$

By similar argument, if $v_{k(m(i))} \notin V^*$, we use (4.49) to obtain $h'_{m(i)}(t)$ in the space $L^2_*[T - \text{depth}^*(e_{m(i)}), T - l_{m(i)}]$. We integrate it to get $h_{m(i)}(t)$ in the space $H^1_*[T - \text{depth}^*(e_{m(i)}), T - l_{m(i)}]$.

The portion of $f'_{k(m(i))}(t)$, or $h_{m(i)}(t)$ on the interval $[T - l_{m(i)}, T)$ controls the final shape of $e_{m(i)}$ similar to (4.57). Therefore, to solve the control problem, if $v_{k(m(i))} \in V^*$, we solve

$$
u_{m(i)}^{f'_{k(m(i))}, ND, g_i}(x, T) = \phi_{m(i)}(x).
$$

If $v_{k(m(i))} \notin V^*$, we solve

$$u_{m(i)}^{h_{m(i)},DD,g_i}(x,T) = \phi_{m(i)}(x).$$

Now we have

$$f'_{k(m(i))}(t) \in L^2_*(T - \text{depth}^*(e_{m(i)}), T),$$

or

$$h_{m(i)}(t) \in H^1_*(T - \text{depth}^*(e_{m(i)}), T).$$

For all other $j \in J^-(v_i)$, where $j \neq m(i)$, let $v_{k(j)}$ be the starting vertex of e_j. If $j \in J^*$, we find $f'_{k(j)}(t)$ by solving

$$u_j^{f'_{k(j)},ND,g_i}(x,T) = \phi_j(x).$$

If $j \notin J^*$, we find $h_j(t)$ by solving

$$u_j^{h_j,DD,g_i}(x,T) = \phi_j(x).$$

Since by Principle 3, $f'_{k(j)}(t)$ or $h_j(t)$ are supported on $[T - l_j, T]$, the solution to either $f'_{k(j)}(t)$ or $h_j(t)$ on the interval $[T - l_j, T]$ is unique.

Case 3, v_i is a source, i.e., $i \in I^*$. Since all $h_j(t)$ for $j \in J^+(v_i)$ are known, we compute $f_j(t)$ for $j \in J^*(v_i)$ similar to Case 2, and $f_j(t) \in H^1_0(T - D^*(v_i), T)$. We compute $f'_i(t) \in L^2(T - D^*(v_i), T)$ from (4.48). We move on to the next vertex in order.

We continue this process until we are done with v_n. Then we obtain all entries of $\mathbf{f}(t)$, where $f'_i(t) \in L^2_*(T - D^*(v_i), T)$ and $f_j(t) \in H^1_0(T - D^*(v_i), T)$, in which v_i is the starting vertex of e_j. By (4.62) and (4.64), $\mathbf{f} \in \mathcal{F}_0^T$.

Part 2. The proof for the velocity controllability is parallel to the proof for the shape controllability, and is left to the reader. Since the target function is in $\mathcal{H}$ instead of $\mathcal{H}^1$, there is no need to control the final velocity at the sinks. Therefore the control time is T_* instead of T^*, as in Proposition 4.38.

Part 3. Let $\vec{\Omega}_{min}, U_{min}$ be the orientation and path union on the orientation such that

$$\max_{j\in J} \text{depth}(e_j, \vec{\Omega}_{min}, U_{min}) = T_{inf}.$$

By Part 1 and Part 2 we can construct control functions to have shape or velocity controllability at any $T > T_{inf}$. On the other hand, if $T < T_{inf}$, one can easily construct examples of the graphs which are not shape or velocity controllable at that T. We can consider, for example, an interval with one Neumann control, as described in Proposition 4.38, or the star graph examples below, or the laso graph example in [*Avdonin et al.*, 2022].

Therefore

$$T_{inf} = \min_{\vec{\Omega}, U} \max_{j\in J} \text{depth}(e_j, \vec{\Omega}, U).$$

□

Although not always possible, in many cases we can achieve shape controllability at $T = T_*$. See examples 4.40 and 4.41. Similar comparisons on a lasso graph can be found in [*Avdonin et al.*, 2022].

Example 4.40. *In the star graph in Figure 4.5, let all edges be directed away from v_1. Let $I^* = \{1\}$, and $J^* = \{6\}$. The TF path union associated with $\{I^*, J^*\}$ is then $U = \{(v_1, e_5, v_4, e_7, v_3), (v_4, e_6, v_2)\}$. So $\mathbf{f} = \{f_1', f_6\}$, and $T_* = l_5 + \max\{l_6, l_7\}$.*

If $l_6 = l_7$, $T_* = l_5 + l_6$. Let $\mathbf{f} \in \mathcal{F}^{T_*}$, $\mathbf{f}_1 = (f_1', 0)$ and $\mathbf{f}_6 = (0, f_6)$. Simple calculations show $u^{\mathbf{f}_1}(v_i, T_*) = 0$ for $i = 2, 3$, and $u^{\mathbf{f}_6}(v_2, T_*) = -2u^{\mathbf{f}_6}(v_3, T_*)$. By the superposition principle, $u^{\mathbf{f}}(v_2, T_*) = -2u^{\mathbf{f}}(v_3, T_*)$, hence the system is not shape controllable at time T_*.

If $l_6 \neq l_7$, the system is shape controllable at T_*. Let $0 < \epsilon < \min\left\{l_5, \frac{1}{2}|l_6 - l_7|\right\}$. We first construct a vector function $\tilde{\mathbf{f}} = \{\tilde{f}_1', \tilde{f}_6\}$, such that $\tilde{f}_1' = 0$, $\tilde{f}_6$ is a smooth function supported on $(T_* - l_6 - \epsilon, T_* - l_6 + \epsilon) \cup (T_* - l_7 - \epsilon, T_* - l_7 + \epsilon)$, and $u^{\tilde{\mathbf{f}}}(v_i, T_*) = \phi(v_i)$ for $i = 2, 3$. Let $\tilde{\phi}(\cdot) := \phi(\cdot) - u^{\tilde{\mathbf{f}}}(v_i, T_*)$.

We then construct the second vector function $\hat{\mathbf{f}} = \{\hat{f}_1', \hat{f}_6\}$, such that $u^{\hat{\mathbf{f}}}(\cdot, T_*) = \tilde{\phi}(\cdot)$. Using the process described in Part 1 of Theorem 4.39, we first construct $g \in H_*(T_* -$

$l_7, T_*)$, $\hat{f}_6 \in H^0_*(T_* - l_6, T_*)$, such that

$$g(T_* - x) = \tilde{\phi}_7(x), \quad 0 \le x \le l_7, \quad g(T_* - x) + \hat{f}_6(T_* - x) = \tilde{\phi}_6(x), \quad 0 \le x \le l_6.$$

We next find the portion of $\hat{f}_1'$ such that

$$\partial u_5^{\hat{f}_1', ND, g(t)}(v_4, t) - 2g'(t) - \hat{f}_6'(t) = 0$$

for $0 \le t \le T_*$. Here the expansion of the term $\partial u_5^{\hat{f}_1', ND, g(t)}(v_4, t)$ depends on l_5, l_6, and l_7, which are not specified. Suppose, say, $l_5 > \max\{l_6, l_7\}$, then $\partial u_5^{\hat{f}_1', ND, g(t)}(v_4, t) = -2f_1'(t - l) - g'(t)$. No matter how many reflections there are, this portion of $\hat{f}_1'$ is in $L^2_*(T_* - l_5, T_*)$. We then solve the equation

$$u_5^{\hat{f}_1', ND, g(t)}(x, T_*) = \tilde{\phi}_5(x).$$

Since g and the portion of $\hat{f}_1'$ on $(T_* - l_5, T_*)$ are known, we can uniquely obtain $\hat{f}_1'$ on the time interval $(0, l_5)$. Let $\mathbf{f} = \tilde{\mathbf{f}} + \hat{\mathbf{f}}$, then $u^{\mathbf{f}}(\cdot, T_*) = \phi(\cdot)$.

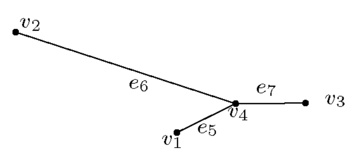

Figure 4.5: A star graph

Example 4.41. *We direct all edges in Figure 4.5 towards v_3. Let $I^* = \{1, 2\}$, and $J^* = \emptyset$, then $\mathbf{f} = \{f_1', f_2'\}$. Similar to Example 4.40, one can show the system is shape controllable at T_* if $l_6 < l_5$ or $l_6 > l_5 + l_7$, but not shape controllable at T_* if $l_5 \le l_6 \le l_5 + l_7$.*

4.3.5 Exact controllability on graphs

,

The proof of the full controllability of our system uses the spectral representation of the solution $u^{\mathbf{f}}$. Let $\{(\omega_n^2, \varphi_n) : n \geq 1\}$ be the eigenvalues and normalized eigenfunctions of the operator L, i.e. φ_n solves the following eigenvalue problem:

$$-\varphi_n'' + q(x)\varphi_n = \omega_n^2 \varphi_n \quad \text{on} \;\; \{\Omega \setminus \mathrm{V}\}, \tag{4.68}$$

$$\sum_{j \in J(v_i)}^{(p)} (\varphi_n)_j(v_i) = 0, \quad i \in I, \tag{4.69}$$

$$(\varphi_n)_j(v_i) = (\varphi_n)_k(v_i), \quad j, k \in J(v_i), i \in I. \tag{4.70}$$

where $(\varphi_n)_j$ is the restriction of φ_n to edge e_j.

The following properties of this spectral problem are well known, see, e.g. [*Berkolaiko and Kuchment*, 2013], [*Avdonin and Nicaise*, 2015].

Lemma 4.42. *The eigenvalues $\{\omega_n^2\}_1^\infty$ associated to* (4.68)-(4.70), *satisfy the relations*

$$cn \leq |\omega_n| + 1 \leq Cn,$$

with some positive constants c and C independent of n. The associated unit norm eigenfunctions satisfy the relations

$$|\varphi_n(v_i)| \leq c_1, \; |(\varphi_n)_j'(v_i)| \leq c_2 n, \quad i \in I, j \in J$$

with some positive constants c_1, c_2 independent of n, j, v.

Theorem 4.43. *Let $\vec{\Omega}$ be a DAG of Ω, and $\{I^*, J^*\}$ be a ST active set. For any $\phi \in \mathcal{H}^1$, $\psi \in \mathcal{H}$, and $T > T_*$, there exists $f \in \mathcal{F}_0^{2T}$ such that $u_j(x, 2T) = \phi_j(x)$, $(u_j)_t(x, 2T) = \psi_j(x)$ where $u_j := u^f|_{e_j}$, $\phi_j(x) := \phi|_{e_j}(x)$, and $\psi_j(x) := \psi|_{e_j}(x)$ for all $j \in J$ and $x \in [0, l_j]$.*

Proof. The solution of the IBVP (4.10)–(4.13) can be presented in a form of series

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x). \tag{4.71}$$

To find the coefficients a_n, let $w(x,t) = \mu(t)\varphi_n(x)$ in the integral identity (4.14), where μ is an arbitrary test function in $C_0^2[0,T]$. We obtain the initial value problem

$$a_n''(t) + \omega_n^2 a_n(t) = \sum_{j\in J^*} f_j(t)(\varphi_n)_j'(v_k(j)) - \sum_{i\in I^*} f_i'(t)\phi_n(v_i), \quad a_n(0) = a_n'(0) = 0.$$

where $v_{k(j)}$ is the starting vertex of e_j. By variation of parameters, we get

$$a_n(T) = \int_0^T \left[\sum_{j\in J^*} f_j(t)(\varphi_n)_j'(v_k(j)) - \sum_{i\in I^*} f_i'(t)\phi_n(v_i)\right] \frac{\sin\omega_n(T-t)}{\omega_n}\,dt. \tag{4.72}$$

We assume here that $\omega_n > 0$. If $\omega_n = 0$, instead of $\frac{\sin\omega_n(T-t)}{\omega_n}$ we put $T-t$, and if $\omega_n < 0$, we put $\frac{\sinh\omega_n(T-t)}{\omega_n}$.

Integrating by parts the term with f_j and using that $f_j \in H_0^1(0,T)$, we rewrite (4.72) in the form

$$a_n(T)\omega_n = \int_0^T \left\{-\left[\sum_{j\in J^*} f_j'(t)(\varphi_n)_j'(v_k(j))/\omega_n\right]\cos\omega_n(T-t) - \left[\sum_{i\in I^*} f_i'(t)\phi_n(v_i)\right]\sin\omega_n(T-t)\right\}dt. \tag{4.73}$$

Next, we differentiate (4.72) with respect to T, and then integrate by parts the term with $f_j(t)$ using that $f_j \in H_*^1(0,T)$. We obtain

$$\dot a_n(T) = \int_0^T \left\{\left[\sum_{j\in J^*} f_j'(t)(\varphi_n)_j'(v_k(j))/\omega_n\right]\sin\omega_n(T-t) - \left[\sum_{i\in I^*} f_i'(t)\phi_n(v_i)\right]\cos\omega_n(T-t)\right\}dt. \tag{4.74}$$

By the classical relations between the Fourier method, the moment problem, and control theory (see, e.g. [*Avdonin and Ivanov*, 1995]), Part 1 of Theorem 4.39 is equivalent to:

- For any sequence $\{a_n\omega_n\} \in \ell^2$, there exist $\mathbf{f}^0 \in \mathcal{F}_0^T$ such that the following equation holds.

$$a_n\omega_n = \int_0^T \left\{ -\left[\sum_{j\in J^*} (f^0)'_j(t)(\varphi_n)'_j(v_k(j))/\omega_n \right] \cos\omega_n(T-t) - \left[\sum_{i\in I^*} (f^0)'_i(t)\phi_n(v_i) \right] \sin\omega_n(T-t) \right\} dt \tag{4.75}$$

Part 2 of Theorem 4.39 is equivalent to:

- For any sequence $\{b_n\} \in \ell^2$, there exist $\mathbf{f}^1 \in \mathcal{F}^T$ such that the following equation holds

$$b_n = \int_0^T \left\{ \left[\sum_{j\in J^*} (f^1)'_j(t)(\varphi_n)'_j(v_k(j))/\omega_n \right] \sin\omega_n(T-t) - \left[\sum_{i\in I^*} (f^1)'_i(t)\phi_n(v_i) \right] \cos\omega_n(T-t) \right\} dt. \tag{4.76}$$

We extend the functions $\mathbf{f}_0, \mathbf{f}_1$ from the interval $[0,T]$ to $[0,2T]$ such that for $t \in [0,T]$ we have for all $i \in I^*$ and for $j \in J^*$,

$$(f^0)'_i(t) = -(f^0)_i(2T-t),\ (f^1)'_j(t) = -(f^1)'_j(2T-t),\ (f^1)'_i(t) = (f^1)'_i(2T-t),\ (f^0)'_j(t) = (f^0)'_j(2T-t)$$

and put

$$f'_i(t) = \frac{(f^0)'_i(t) + (f^1)'_i(t)}{2}, \quad f'_j(t) = \frac{(f^0)'_j(t) + (f^1)'_j(t)}{2}.$$

One can check that

$$a_n\omega_n = \int_0^{2T} \left\{ -\left[\sum_{j\in J^*} f'_j(t)(\varphi_n)'_j(v_k(j))/\omega_n \right] \cos\omega_n(T-t) - \left[\sum_{i\in I^*} f'_i(t)\phi_n(v_i) \right] \sin\omega_n(T-t) \right\} dt. \tag{4.77}$$

and

$$b_n = \int_0^{2T} \left\{ \left[\sum_{j \in J^*} f_j'(t) (\varphi_n)_j'(v_k(j))/\omega_n \right] \sin \omega_n (T-t) - \left[\sum_{i \in I^*} f_i'(t) \phi_n(v_i) \right] \cos \omega_n (T-t) \right\} dt. \tag{4.78}$$

Solvability of the moment problem (4.77), (4.78) is equivalent to solvability of another moment problem:

$$c_n = b_n + i a_n \omega_n = \int_0^{2T} \left\{ - \left[\sum_{i \in I^*} f_i'(t) \phi_n(v_i) \right] - i \left[\sum_{j \in J^*} f_j'(t) \phi_n'(v_{k(j)}/\omega_n \right] \right\} e^{i\omega_n(T-t)}\, dt. \tag{4.79}$$

$$d_n = b_n - i a_n \omega_n = \int_0^{2T} \left\{ - \left[\sum_{i \in I^*} f_i'(t) \phi_n(v_i) \right] + i \left[\sum_{j \in J^*} f_j'(t) \phi_n'(v_{k(j)}/\omega_n \right] \right\} e^{-i\omega_n(T-t)}\, dt. \tag{4.80}$$

for any sequences $\{c_n\}$, $\{d_n\} \in \ell^2$ and the same spaces of functions $f_i'(t), f_j'(t) \in L^2(0,2T)$, for all $i \in I^*, j \in J^*$. Clearly, in these equalities one can change T to $2T$ in the arguments of the exponentials. Finally, from $\exp\{\pm i\omega_n(2T-t)\}$ we can switch back to $\sin \omega_n(2T-t)$, $\cos \omega_n(2T-t)$. This allows us to claim solvability of the moment problem

$$a_n \omega_n = \int_0^{2T} \left\{ - \left[\sum_{j \in J^*} f_j'(t) (\varphi_n)_j'(v_k(j))/\omega_n \right] \cos \omega_n (2T-t) - \left[\sum_{i \in I^*} f_i'(t) \phi_n(v_i) \right] \sin \omega_n (2T-t) \right\} dt. \tag{4.81}$$

and

$$b_n = \int_0^{2T} \left\{ \left[\sum_{j \in J^*} f_j'(t) (\varphi_n)_j'(v_k(j))/\omega_n \right] \sin \omega_n (2T-t) - \left[\sum_{i \in I^*} f_i'(t) \phi_n(v_i) \right] \cos \omega_n (2T-t) \right\} dt. \tag{4.82}$$

For every $j \in J^*$, we put $f_j(t) = \int_0^t f_j'(s)\,ds$ and check that

$$2f_j(2T) = \int_0^{2T} (f^0)_j'(s)\,ds + \int_0^{2T} (f^1)_j'(s)\,ds = 2\int_0^{T} (f^0)_j'(s)\,ds = 0,$$

hence $f_j \in H_0^1(0, 2T)$. Also, evidently, $f_i'(t) \in L^2(0, 2T)$ for every $i \in I^*$. Now integrating by parts, we can rewrite (4.81), (4.82) in the form

$$a_n\omega_n = \int_0^{2T} \left\{ \left[\sum_{j\in J^*} f_j(t)(\varphi_n)_j'(v_k(j)) \right] - \left[\sum_{i\in I^*} f_i'(t)\phi_n(v_i) \right] \right\} \sin\omega_n(2T-t)\,dt. \qquad (4.83)$$

and

$$b_n = \int_0^{2T} \left\{ \left[\sum_{j\in J^*} f_j(t)(\varphi_n)_j'(v_k(j)) \right] - \left[\sum_{i\in I^*} f_i'(t)\phi_n(v_i) \right] \right\} \cos\omega_n(2T-t)\,dt. \qquad (4.84)$$

Thus for arbitrary sequences $\{a_n\omega_n\}$, $\{b_n\} \in \ell^2$, there exist $f_i' \in L^2(0, 2T)$, $f_j \in H_0^1(0, 2T)$ for all $i \in I^*$ and $j \in J^*$ satisfying equalities (4.83), (4.84). This is equivalent to the exact controllability of the IBVP (4.10)–(4.13). □

We remark in passing that Lemma 4.42 guarantees the appropriate regularity for the solution u defined by (4.71), (4.72).

The method proving Theorem 4.43 was used in [*Avdonin and Zhao*, 2021a] for tree graphs with boundary Dirichlet control, in [*Avdonin et al.*, 2022] for a lasso graph with mixed boundary and internal controls, and in [*Avdonin and Edward*, 2018] for a string with attached point masses.

4.3.6 Connectivity of the graph

Graph connectivity will be used to discuss the minimal number of controllers. We use the definitions concerning connectivity from [*West*, 2001] Section 4: a **cut-vertex** is a vertex whose deletion increases the number of connected components of a graph. The block-cut graph of a graph Ω is a graph whose vertices can be divided into two disjoint sets, one set consists of all cut-vertices of Ω, and the other set has one vertex for each block of Ω. Furthermore, every edge connects a vertex in the first set to a vertex in the second set. When Ω is connected, its block-cut graph is a tree graph, (see [*Harary*, 1969] Theorem 4.4 or [*West*, 2001] Exercise 4.1.34). It is therefore referred to as the **block-cut tree (BC tree)** of Ω. Figure 4.6 shows all blocks and cut-vertexs of Ω and Figure 4.7 shows the BC tree for Ω in Figure 4.1.

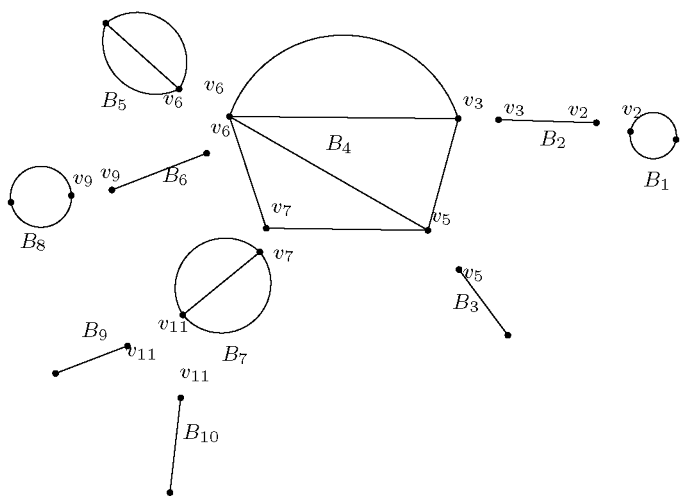

Figure 4.6: Cutpoints separate Ω into blocks.

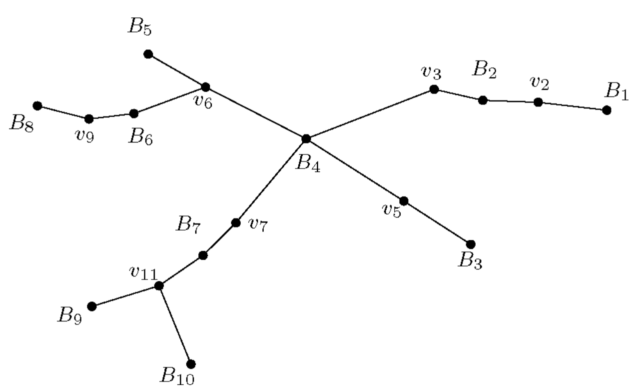

Figure 4.7: The BC-tree of Ω.

We denote the blocks sharing only one cut-vertex with its complement as **pendant blocks**. For examples, Ω in Figure 4.1 has six pendant blocks. They are B_1, B_3, B_5, B_8, B_9, and B_{10}, as marked in Figure 4.6. Each pendant block corresponds to a boundary vertex in the BC tree. The set of pendant blocks is denoted as B_Γ.

4.3.7 The number of controllers

In this section we discuss the smallest number of controllers needed in solving the control problem of the wave equation system (4.10)–(4.13) as described in Section 4.3.4, and compare this number with the maximum multiplicity of the spectrum of the operator L obtained in [*Kac and Pivovarchik*, 2011].

Theorem 4.44. *Let Ω be an undirected graph. Let $\kappa(\Omega)$ be the smallest number of Dirichlet and Neumann controllers used in the method described in Theorem 4.39, of all possible orientations of Ω and ST active set $\{I^*, J^*\}$ of that orientation. Then*

$$\kappa(\Omega) = \begin{cases} |E| - |V| + 2, & \Omega \text{ has one block;} \\ |E| - |V| + |B_\Gamma|, & \Omega \text{ has two or more blocks.} \end{cases} \tag{4.85}$$

Before proving the above Theorem, we introduce the following lemma [*Szwarcfiter et al.*, 1985].

Lemma 4.45. *Let G be a block of Ω and v, v' be any two vertices on G, then there is a bipolar orientation of G with respect to v, v', that is, an acyclic orientation on G such that v is the only source and v' is the only sink of G.*

Indeed, if G is a single edge, it certainly has a bipolar orientation with respect to its two end vertices. If G is not a single edge, [*Szwarcfiter et al.*, 1985] contains an algorithm to find a bipolar orientation with respect to any two arbitrary vertices.

Proof of Theorem 4.44:

To solve the control problem using the method described in Section 4.3.4, one needs $|I^*|$ Neumann controllers and $|J^*|$ Dirichlet controllers. Let $\vec{\Omega}$ be a DAG and $\{I^*, J^*\}$ be a ST active set based on $\vec{\Omega}$. By (4.38) and (4.39), the total number of controllers is:

$$|I^*| + |J^*| = |\vec{\Omega}^+| + \sum_{v \in V \setminus \vec{\Omega}^-} [|J^+(v)| - 1] = |E| - |V| + |\vec{\Omega}^+| + |\vec{\Omega}^-| \tag{4.86}$$

Here $|E|$ and $|V|$ are independent of the orientation of Ω, and $|\vec{\Omega}^+| + |\vec{\Omega}^-|$ depends on the orientation. We next show

$$\min_{\text{acyclic } \vec{\Omega} \text{ of } \Omega} (|\vec{\Omega}^+| + |\vec{\Omega}^-|) = \begin{cases} 2, & \text{if } \Omega \text{ is a single block} \\ |B_\Gamma|, & \text{otherwise.} \end{cases} \tag{4.87}$$

Suppose Ω contains only one block. Any acyclic orientation of Ω contains at least one source and one sink (distinct from the source). By Lemma 4.45 such a bipolar orientation is possible. So the minimal total number of sources and sinks is 2.

Now suppose Ω contains two or more blocks. In any orientation of Ω, for each block, if its source/sink is not its cut-vertex, the source/sink of the block is then the source/sink of Ω. Thus, to minimize the number of sources and sinks on Ω, one should make the sources/sinks of each block its cut-vertex whenever possible and make the cut-vertices not sources/sinks of the entire graph. On the other hand, there is at least one source or one sink on each pendant block, since it has only one cut-vertex.

We show below that it is possible to have an orientation of Ω such that all source and sinks are located on pendant blocks, and each pendant block has precisely one source or one sink (not both).

Let S be the BC tree of Ω. We denote the vertex in S that corresponds to a block G as v_G, denote the vertex in S that corresponds to a cut-vertex c as v_c. There is a one to one correspondence between the boundary vertices of S and pendant blocks of Ω. Let $\vec{S}$ be an orientation of S such that all sources and sinks are at its boundary vertices (such an orientation is possible since we can direct all edges towards one boundary vertex to form a rooted tree). Then each vertex corresponding to a cut-vertex in Ω has both incoming and outgoing edges.

Suppose G is a pendant block and c is the only cut-vertex on G. Lemma 4.45 guarantees a bipolar orientation exists for any pair of vertices of G. We orient G such that c is the source of G if v_G is a sink in $\vec{S}$, and c is the sink of G if v_G is a source in $\vec{S}$. We pick another point s on G as the sink/source of G if c is the source/sink. Since s is not a cut-vertex of G, it is a sink/source of Ω.

Now suppose G is a block with two or more cut-vertices. Then v_G has both incoming edges and outgoing edges on $\vec{S}$. We pick two cut-vertices c, c' on G such that there is an edge from v_c to v_G and an edge from v_G to $v_{c'}$ in $\vec{S}$. We then construct a bipolar orientation of G such that c is its source and c' is its sink.

We orient all blocks on Ω as described above, and thus obtain an orientation $\vec{\Omega}$. $\vec{\Omega}$ is acyclic since the orientations on each block are acyclic and all blocks are connected through cut-vertices. No cut-vertex is a sink/source of $\vec{\Omega}$. Indeed, if c is a cut-vertex of Ω, v_c on $\vec{S}$ has at least one incoming edge $e(v_G, v_c)$ and one outgoing edge $e(v_c, v_{G'})$. By our orientation method, c is possibly the sink of G and/or the source of G'. If c is both the sink of G and the source of G', it has incoming edges from G and outgoing edges from G'. If c is not the sink of G, it has both incoming and outgoing edges in G. Similarly, if c is not the source of G', it has both incoming and outgoing edges in G'. In all cases, c is not a source or sink of Ω. Therefore in $\vec{\Omega}$, all sources and sinks are located on pendant blocks, and each pendant block contains precisely one source or one sink. The smallest number of sources and sinks possible for an orientation of Ω is then $|B_\Gamma|$. □

Example 4.46. ***Orienting a graph to obtain the minimal number of sources and sinks.*** *Let Ω be the undirected graph in Figure 4.1 and S be its BC tree as shown in Figure 4.7. Let $\vec{S}$ be an orientation of S such that all its sinks and sources are at its boundary vertices (see Figure 4.8). Then $\vec{\Omega}$ in Figure 4.2 is orientated consistently with $\vec{S}$, where each block has a bipolar orientation that can be shown on $\vec{S}$.*

Indeed, block B_1 has v_2 as the source and v_1 as the sink; B_2 has v_3 as the source and v_2

as the sink; B_3 *has* v_5 *as the source and* v_4 *as the sink;* B_4 *has* v_7 *as the source and* v_3 *as the sink;* B_5 *has* v_8 *as the source and* v_6 *as the sink;* B_6 *has* v_9 *as the source and* v_6 *as the sink;* B_7 *has* v_{11} *as the source and* v_7 *as the sink;* B_8 *has* v_{10} *as the source and* v_9 *as the sink;* B_9 *has* v_{12} *as the source and* v_{11} *as the sink;* B_{10} *has* v_{13} *as the source and* v_{11} *as the sink.*

None of the cut-vertices are source and sinks of $\vec{\Omega}$. *Therefore* $\vec{\Omega}$ *has as many sources and sinks as the number of boundary vertices of* $\vec{S}$, *which is* $|B_\Gamma|$=*6.*

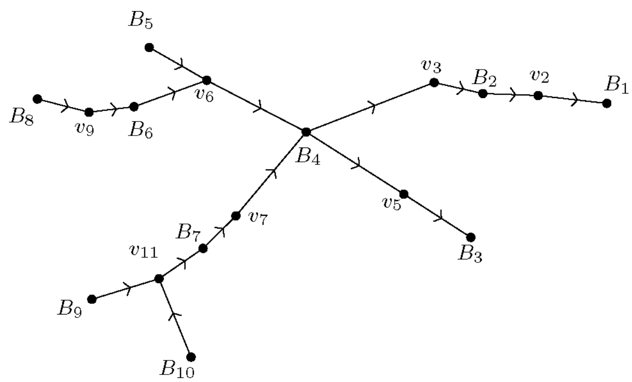

Figure 4.8: Orientation $\vec{S}$.

A **cyclically connected** graph, as defined [*Kac and Pivovarchik*, 2011], means for each pair of vertices $v_i, v_k \in \Omega$, a finite set of cycles $C_1, C_2, \ldots, C_n$ exist such that $v_i \in C_1, v_k \in C_n$ and each neighboring pair of cycles possesses at least one common vertex. For example, Ω in Figure 4.1 has three maximal cyclically connected subgraphs: the first one is B_1 in Figure 4.6. The second one is the union of B_4, B_5, and B_7. The third one is B_8.

Let $\mu = |E| - |V| + 1$ be the cyclomatic number of an undirected graph. Let p_Γ^T be the numbers of boundary vertices for the tree obtained by contracting all the cycles of the graph into vertices. This tree graph is denoted as S^T of the original graph. The S^T of Ω in Figure 4.1 is shown in Figure 4.9.

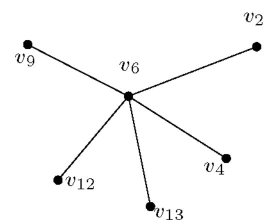

Figure 4.9: The S^T of Ω. A vertex from each contracted component is shown.

Denote by $\sigma(\Omega, q)$ the maximal multiplicity of an eigenvalue of the operator L on a quantum graph $\{\Omega, q\}$. Changing q we change the operator L and, therefore, its spectrum. *The maximal possible multiplicity of an eigenvalue of L is denoted by $\sigma(\Omega)$.* In [*Kac and Pivovarchik*, 2011] it is shown that

$$\sigma(\Omega) = \begin{cases} \mu + 1, & \Omega \text{ is cyclically connected,} \\ \mu + p_\Gamma^T - 1, & \Omega \text{ is not cyclically connected.} \end{cases} \tag{4.88}$$

Theorem 4.47. *For a general graph Ω, $\sigma(\Omega) \leq \kappa(\Omega)$. The equality takes place if Ω is cyclically connected and $|B_\Gamma| \leq 2$ or if Ω is not cyclically connected and $|B_\Gamma| = p_\Gamma^T$.*

Proof. Formula (4.85) can be rewritten in the form

$$\kappa(\Omega) = \begin{cases} \mu + 1, & \Omega \text{ has one block;} \\ \mu + |B_\Gamma| - 1, & \Omega \text{ has two or more blocks.} \end{cases} \tag{4.89}$$

We notice that each boundary vertex in S^T either corresponds to a boundary vertex or a cyclically connected component in Ω. If every such cyclically connected component contains precisely one pendant block, and all pendant blocks in Ω contract to boundary vertices in S^T, $|B_\Gamma| = p_\Gamma^T$. Otherwise $|B_\Gamma| > p_\Gamma^T$. The theorem follows immediately from this notice and formulas (4.88), (4.89). $\square$

Theorem 4.47 shows, whether $\sigma(\Omega) = \kappa(\Omega)$ or $\sigma(\Omega) < \kappa(\Omega)$ depends on the locations of the cyclically connected pendant blocks. See Example 4.48 below.

Example 4.48. *In Figure 4.10, the graphs A, B, C, and D are not cyclically connected. The S^T of A, B, C, D are shown in Figure 4.11.*

- *For graph A, $|B_\Gamma| = 2 = p_\Gamma^T$. Thus $\sigma(\Omega) = \kappa(\Omega) = 5$.*
- *For graph B, $|B_\Gamma| = 3$, but $p_\Gamma^T = 2$. There is one pendant block (the middle loop) contracts into an interior vertex in S^T. Thus $2 = \sigma(\Omega) < \kappa(\Omega) = 3$.*
- *For graph C, $|B_\Gamma| = 4$, but $p_\Gamma^T = 2$. There are two pendant blocks contracts into an interior vertex in S^T. Thus $4 = \sigma(\Omega) < \kappa(\Omega) = 6$.*
- *For graph D, $|B_\Gamma| = 3$, but $p_\Gamma^T = 2$. There are two pendant blocks contracts into the same boundary vertex in S^T. Thus $4 = \sigma(\Omega) < \kappa(\Omega) = 5$.*

The graphs E,F,G, and H are cyclically connected.

- *Graph E has 2 pendant blocks, and $\sigma(\Omega) = \kappa(\Omega) = 3$.*
- *Graph F has 2 pendant blocks, and $\sigma(\Omega) = \kappa(\Omega) = 4$.*
- *Graph G has 3 pendant blocks, and $4 = \sigma(\Omega) < \kappa(\Omega) = 5$.*
- *Graph H has 4 pendant blocks, and $6 = \sigma(\Omega) < \kappa(\Omega) = 8$.*

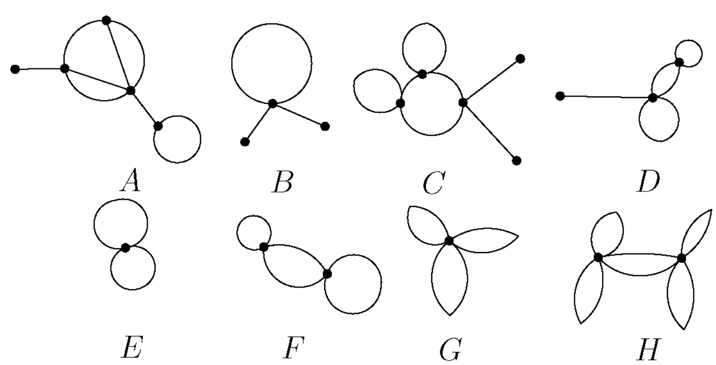

Figure 4.10: Several graph examples

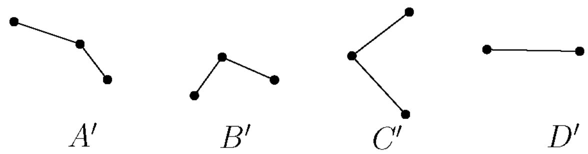

Figure 4.11: S^T of A, B, C, D.

We have proved that if the total number of Neumann and Dirichlet controllers, $N := |I^*| + |J^*|$ in the system (4.10)–(4.13) is greater than or equal to $\kappa(\Omega)$, then we can place the controllers in such a way that the system is exactly controllable in time greater than T_* independently of the potential q and lengths of the edges. On the other hand, $\sigma(\Omega)$ can serve as a lower bound for the necessary number of controllers. More exactly, the following statement is valid.

Proposition 4.49. *If the total number, N, of Neumann and Dirichlet controllers in the system* (4.10)–(4.13) *is less than $\sigma(\Omega)$, there exists a potential q such that the corresponding system is not approximately controllable for any finite $T > 0$.*

Proof. In Theorem 4.43 we reduced the controllability problem for the system (4.10)–(4.13) to the moment problem with respect to the family of vector functions

$$col\,\{(\varphi_n)'_j(v_k(j))/\omega_n,\ \phi_n(v_i)\}_{j\in J^*, i\in I^*}\ \sin\omega_n(2T-t) \tag{4.90}$$

and

$$col\,\{(\varphi_n)'_j(v_k(j))/\omega_n,\ \phi_n(v_i)\}_{j\in J^*, i\in I^*}\ \cos\omega_n(2T-t), \quad n\in\mathbb{N}, \tag{4.91}$$

in the space $L^2(0, 2T, \mathbb{R}^N)$ (see (4.81) and (4.82)). Let q be a potential such that the multiplicity of the corresponding operator L is $\sigma(\Omega)$. If $N < \sigma(\Omega)$, the families (4.90) and (4.91) are clearly linear dependent $L^2(0, 2T, \mathbb{R}^N)$. Therefore, the system (4.10)–(4.13) is not approximately controllable for any finite $T > 0$ (see [*Avdonin and Ivanov*, 1995] Theorem III.3.10).

□

Remark 5. *If $N < \sigma(\Omega)$, there are some cases in which the system may be spectrally controllable and even exactly controllable. Let us consider the wave equation with $q = 0$ on a 3-star graph with equal edge lengths $l_j = l$ and one boundary (Dirichlet or Neumann) control. For this graph, $\sigma(\Omega) = \kappa(\Omega) = 2$. We assume zero Dirichlet condition at one uncontrolled boundary vertex and zero Neumann condition at another. It is easy to prove that this system is shape and velocity controllable for $T \geq 3l$ and therefore, it is exactly controllable for $T \geq 6l$. However, controllability (exact and spectral) breaks down if we change a little the length of one of the uncontrolled edges.*

We conjecture that if $N < \sigma(\Omega)$, there is no system that is spectrally controllable and its spectral controllability is stable with respect to small pertubation of the system parameters (lengths of edges and potential). Similarly, if $\sigma(\Omega) \leq N < \kappa(\Omega)$, we conjecture there is no system that is exactly controllable and its exact controllability is stable with respect to small pertubation of the system parameters. These conjectures are confirmed by several examples from [Dáger and Zuazua, 2006] and [Avdonin et al., 2022]. The authors know only one general result of this type: the wave equation on a tree with all Dirichlet (or all Neumann) boundary conditions and boundary controls is not exactly controllable if the number of controllers is less than $|\Gamma| - 1$ [Avdonin and Zhao, 2021a]. We recall that for trees, $\sigma(\Omega) = \kappa(\Omega) = |\Gamma| - 1$.

4.4 Appendix

We give equations in Sections 4.3.4 and 4.36 for nonzero potentials. The solution to the forward problem for nonzero potentials are in parallel to the zero potentials case in Section 4.36, with equation (4.48) replaced by

$$f_i'(t) = \sum_{j \in J^+(v_i) \setminus J^-(\Gamma)} \left\{ -h_j'(t) + \int_0^t (\omega_j)_x(0, s) h_j(t-s)\, ds \right.$$

$$- 2h_j'(t-2l_j) - 2\omega_j(2l_j,2l_j)h_j(t-2l_j) + 2\int_{2l_j}^t (\omega_j)_x(2l_j,s)h_j(t-s)\,ds$$

$$-2h_j'(t-4l_j) - 2\omega_j(4l_j,4l_j)h_j(t-4l_j) + 2\int_{4l_j}^t (\omega_j)_x(4l_j,s)h_j(t-s)\,ds - \dots \Bigg\}$$

$$+ \sum_{j\in J^+(v_i)\setminus J^-(\Gamma)} \Bigg\{ 2g_{k(j)}'(t-l_j) + 2\bar{\omega}_j(l_j,l_j)g_{k(j)}(t-l_j) - 2\int_{l_j}^t (\bar{\omega}_j)_x(l_j,s)g_{k(j)}(t-s)\,ds$$

$$+2g_{k(j)}'(t-3_j) + 2\bar{\omega}_j(3_j,3_j)g_{k(j)}(t-3l_j) - 2\int_{3l_j}^t (\bar{\omega}_j)_x(3l_j,s)g_{k(j)}(t-s)\,ds + \dots \Bigg\}$$

$$+ \sum_{j\in J^+(v_i)\cap J^-(\Gamma)} \Bigg\{ -h_j'(t) + \int_0^t (\omega_j)_x(0,s)h_j(t-s)\,ds$$

$$+ 2h_j'(t-2l_j) + 2\omega_j(2l_j,2l_j)h_j(t-2l_j) - 2\int_{2l_j}^t (\omega_j)_x(2l_j,s)h_j(t-s)\,ds$$

$$-2h_j'(t-4l_j) - 2\omega_j(4l_j,4l_j)h_j(t-4l_j) + 2\int_{4l_j}^t (\omega_j)_x(4l_j,s)h_j(t-s)\,ds + \dots \Bigg\},$$

(4.49) replaced by

$$0 = \sum_{j\in J^-(v_i)\cap J^+(\Gamma^*)} \Bigg\{ -2f_{k(j)}'(t-l_j) - 2\mu_j(l_j,l_j)f_{k(j)}(t-l_j) + 2\int_{l_j}^t (\mu_j)_x(l_j,t)f_{k(j)}(t-s)\,ds$$

$$+2f_{k(j)}'(t-3l_j) + 2\mu_j(3l_j,3l_j)f_{k(j)}(t-3l_j) - 2\int_{3l_j}^t (\mu_j)_x(3l_j,t)f_{k(j)}(t-s)\,ds - \dots \Bigg\}$$

$$+ \sum_{j\in J^-(v_i)\cap J^+(\Gamma^*)} \Bigg\{ -g_i'(t) + \int_0^t (\bar{\omega}_j)_x(0,s)g_i(t-s)\,ds$$

$$+ 2g_i'(t-2l_j) + 2\bar{\omega}_j(2l_j,2l_j)g_i(t-2l_j) - 2\int_{2l_j}^t (\bar{\omega}_j)_x(2l_j,s)g_i(t-s)\,ds$$

$$-2g_i'(t-4l_j) - 2\bar{\omega}_j(4l_j,4l_j)g_i(t-4l_j) + 2\int_{4l_j}^t (\bar{\omega}_j)_x(4l_j,s)g_i(t-s)\,ds - \dots \Bigg\}$$

$$+ \sum_{j\in J^-(v_i)\setminus J^+(\Gamma^*)} \Bigg\{ 2h_j'(t-l_j) + 2\omega_j(l_j,l_j)h_j(t-l_j) - 2\int_{l_j}^t (\omega_j)_x(l_j,s)h_j(t-s)\,ds$$

$$+2h_j'(t-3l_j) + 2\omega_j(3l_j,3l_j)h_j(t-3l_j) - 2\int_{3l_j}^t (\omega_j)_x(3l_j,s)h_j(t-s)\,ds + \dots \Bigg\}$$

$$
\begin{aligned}
&+ \sum_{j\in J^-(v_i)\backslash J^+(\Gamma^*)} \Bigg\{ -g_i'(t) + \int_0^t (\bar{\omega}_j)_x(0,s) g_i(t-s)\, ds \\
&\qquad - 2g_i'(t-2l_j) - 2\bar{\omega}_j(2l_j,2l_j) g_i(t-2l_j) + 2\int_{2l_j}^t (\bar{\omega}_j)_x(2l_j,s) g_i(t-s)\, ds \\
&\qquad -2g_i'(t-4l_j) - 2\bar{\omega}_j(4l_j,4l_j) g_i(t-4l_j) + 2\int_{4l_j}^t (\bar{\omega}_j)_x(4l_j,s) g_i(t-s)\, ds - \dots \Bigg\} \\
&+ \sum_{j\in J^+(v_i)\backslash J^-(\Gamma)} \Bigg\{ -h_j'(t) + \int_0^t (\omega_j)_x(0,s) h_j(t-s)\, ds \\
&\qquad - 2h_j'(t-2l_j) - 2\omega_j(2l_j,2l_j) h_j(t-2l_j) + 2\int_{2l_j}^t (\omega_j)_x(2l_j,s) h_j(t-s)\, ds \\
&\qquad -2h_j'(t-4l_j) - 2\omega_j(4l_j,4l_j) h_j(t-4l_j) + 2\int_{4l_j}^t (\omega_j)_x(4l_j,s) h_j(t-s)\, ds - \dots \Bigg\} \\
&+ \sum_{j\in J^+(v_i)\cap J^-(\Gamma)} \Bigg\{ -h_j'(t) + \int_0^t (\omega_j)_x(0,s) h_j(t-s)\, ds \\
&\qquad + 2h_j'(t-2l_j) + 2\omega_j(2l_j,2l_j) h_j(t-2l_j) - 2\int_{2l_j}^t (\omega_j)_x(2l_j,s) h_j(t-s)\, ds \\
&\qquad -2h_j'(t-4l_j) - 2\omega_j(4l_j,4l_j) h_j(t-4l_j) + 2\int_{4l_j}^t (\omega_j)_x(4l_j,s) h_j(t-s)\, ds + \dots \Bigg\} \\
&+ \sum_{j\in J^+(v_i)\backslash J^-(\Gamma)} \Bigg\{ 2g_{k(j)}'(t-l_j) + 2\bar{\omega}_j(l_j,l_j) g_{k(j)}(t-l_j) - 2\int_{l_j}^t (\bar{\omega}_j)_x(l_j,s) g_{k(j)}(t-s)\, ds \\
&\qquad +2g_{k(j)}'(t-3_j) + 2\bar{\omega}_j(3_j,3_j) g_{k(j)}(t-3l_j) - 2\int_{3l_j}^t (\bar{\omega}_j)_x(3l_j,s) g_{k(j)}(t-s)\, ds + \dots \Bigg\}.
\end{aligned}
$$

The solution to the shape/velocity control problems for nonzero potentials are in parallel to the zero potentials case in Section 4.3.4, with equation (4.56) replaced by

$$
\begin{aligned}
&2h_{m(i)}'(t-l_{m(i)}) + 2\omega_{m(i)}(l_{m(i)},l_{m(i)}) h_{m(i)}(t-l_{m(i)}) - 2\int_{l_{m(i)}}^t (\omega_{m(i)})_x(l_{m(i)},s) h_{m(i)}(t-s)\, ds \\
&\qquad\qquad = \sum_{j\in J^-(v_1)} \left\{ -g_1'(t) + \int_0^t (\bar{\omega}_j)_(0,s) g_1(t-s)\, ds \right\},
\end{aligned}
$$

(4.57) replaced by

$$
\begin{aligned}
h_{m(i)}(T-x) &+ \int_x^T \omega_{m(i)}(x,s)h_{m(i)}(T-s)\,ds \\
&- h_{m(i)}(T-2m(i)+x) - \int_{2l_{m(i)}-x}^T \omega_{m(i)}(2l_{m(i)}-x,s)h_{m(i)}(T-s)\,ds \\
&\quad + g_1(T-l_{m(i)}+x) + \int_{l_{m(i)}-x}^T \bar{\omega}_{m(i)}(l_{m(i)}-x,s)g_1(T-s)\,ds = \phi_{m(i)}(x)\,,
\end{aligned}
$$

(4.58) replaced by

$$
h_j(T-x)+\int_x^T \omega_j(x,s)h_j(T-s)\,ds+g_1(T-l_j+x)+\int_{l_j-x}^T \bar{\omega}_j(l_j-x,s)g_1(T-s)\,ds = \phi_j(x)\,,
$$

(4.66) replaced by

$$
\begin{aligned}
0 \quad = \quad & \big\{-2f'_{k(m(i))}(t-l_{m(i)}) - 2\mu_{m(i)}(l_{m(i)},l_{m(i)})f_{k(m(i))}(t-l_{m(i)}) \\
& \qquad + 2\int_{l_{m(i)}}^t (\mu_{m(i)})_x(l_{m(i)},t)f_{k(m(i))}(t-s)\,ds \\
& + 2f'_{k(m(i))}(t-3l_{m(i)}) + 2\mu_{m(i)}(3l_{m(i)},3l_{m(i)})f_{k(m(i))}(t-3l_{m(i)}) \\
& \qquad -2\int_{3l_{m(i)}}^t (\mu_{m(i)})_x(3l_{m(i)},t)f_{k(m(i))}(t-s)\,ds - \dots \Big\} \\
& + \Big\{-g'_i(t) + \int_0^t (\bar{\omega}_{m(i)})_x(0,s)g_i(t-s)\,ds \\
& + 2g'_i(t-2l_{m(i)}) + 2\bar{\omega}_{m(i)}(2l_{m(i)},2l_{m(i)})g_i(t-2l_{m(i)}) \\
& \qquad - 2\int_{2l_{m(i)}}^t (\bar{\omega}_{m(i)})_x(2l_{m(i)},s)g_i(t-s)\,ds \\
& - 2g'_i(t-4l_{m(i)}) - 2\bar{\omega}_{m(i)}(4l_{m(i)},4l_{m(i)})g_i(t-4l_{m(i)}) \\
& \qquad +2\int_{4l_{m(i)}}^t (\bar{\omega}_{m(i)})_x(4l_{m(i)},s)g_i(t-s)\,ds - \dots \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j\in J^+(v_i)\setminus J^-(\Gamma)} \Bigg\{ -h_j'(t) + \int_0^t (\omega_j)_x(0,s) h_j(t-s)\,ds \\
&\qquad - 2h_j'(t-2l_j) - 2\omega_j(2l_j,2l_j)h_j(t-2l_j) + 2\int_{2l_j}^t (\omega_j)_x(2l_j,s)h_j(t-s)\,ds \\
&\qquad -2h_j'(t-4l_j) - 2\omega_j(4l_j,4l_j)h_j(t-4l_j) + 2\int_{4l_j}^t (\omega_j)_x(4l_j,s)h_j(t-s)\,ds - \dots \Bigg\} \\
&+ \sum_{j\in J^+(v_i)\cap J^-(\Gamma)} \Bigg\{ -h_j'(t) + \int_0^t (\omega_j)_x(0,s) h_j(t-s)\,ds \\
&\qquad + 2h_j'(t-2l_j) + 2\omega_j(2l_j,2l_j)h_j(t-2l_j) - 2\int_{2l_j}^t (\omega_j)_x(2l_j,s)h_j(t-s)\,ds \\
&\qquad -2h_j'(t-4l_j) - 2\omega_j(4l_j,4l_j)h_j(t-4l_j) + 2\int_{4l_j}^t (\omega_j)_x(4l_j,s)h_j(t-s)\,ds + \dots \Bigg\} \\
&+ \sum_{j\in J^+(v_i)\setminus J^-(\Gamma)} \Bigg\{ 2g_{k(j)}'(t-l_j) + 2\bar{\omega}_j(l_j,l_j)g_{k(j)}(t-l_j) - 2\int_{l_j}^t (\bar{\omega}_j)_x(l_j,s)g_{k(j)}(t-s)\,ds \\
&\qquad +2g_{k(j)}'(t-3_j) + 2\bar{\omega}_j(3_j,3_j)g_{k(j)}(t-3l_j) - 2\int_{3l_j}^t (\bar{\omega}_j)_x(3l_j,s)g_{k(j)}(t-s)\,ds + \dots \Bigg\},
\end{aligned}
$$

(4.67) replaced by

$$
\begin{aligned}
0 \quad = \quad & \big\{ 2h_{m(i)}'(t-l_{m(i)}) + 2\omega_{m(i)}(l_{m(i)},l_{m(i)})h_{m(i)}(t-l_{m(i)}) \\
&\qquad\qquad - 2\int_{l_{m(i)}}^t (\omega_{m(i)})_x(l_{m(i)},s)h_{m(i)}(t-s)\,ds \\
& + 2h_{m(i)}'(t-3l_{m(i)}) + 2\omega_{m(i)}(3l_{m(i)},3l_{m(i)})h_{m(i)}(t-3l_{m(i)}) \\
&\qquad\qquad -2\int_{3l_{m(i)}}^t (\omega_{m(i)})_x(3l_{m(i)},s)h_{m(i)}(t-s)\,ds + \dots \Bigg\} \\
& + \Bigg\{ -g_i'(t) + \int_0^t (\bar{\omega}_{m(i)})_x(0,s)g_i(t-s)\,ds \\
& - 2g_i'(t-2l_{m(i)}) - 2\bar{\omega}_{m(i)}(2l_{m(i)},2l_{m(i)})g_i(t-2l_{m(i)}) \\
&\qquad\qquad + 2\int_{2l_{m(i)}}^t (\bar{\omega}_{m(i)})_x(2l_{m(i)},s)g_i(t-s)\,ds \\
& - 2g_i'(t-4l_{m(i)}) - 2\bar{\omega}_{m(i)}(4l_{m(i)},4l_{m(i)})g_i(t-4l_{m(i)})
\end{aligned}
$$

$$+2\int_{4l_{m(i)}}^{t}(\bar{\omega}_{m(i)})_x(4l_{m(i)},s)g_i(t-s)\,ds-\dots\Bigg\}$$

$$+\sum_{j\in J^+(v_i)\setminus J^-(\Gamma)}\Bigg\{-h_j'(t)+\int_0^t(\omega_j)_x(0,s)h_j(t-s)\,ds$$

$$-2h_j'(t-2l_j)-2\omega_j(2l_j,2l_j)h_j(t-2l_j)+2\int_{2l_j}^{t}(\omega_j)_x(2l_j,s)h_j(t-s)\,ds$$

$$-2h_j'(t-4l_j)-2\omega_j(4l_j,4l_j)h_j(t-4l_j)+2\int_{4l_j}^{t}(\omega_j)_x(4l_j,s)h_j(t-s)\,ds-\dots\Bigg\}$$

$$+\sum_{j\in J^+(v_i)\cap J^-(\Gamma)}\Bigg\{-h_j'(t)+\int_0^t(\omega_j)_x(0,s)h_j(t-s)\,ds$$

$$+2h_j'(t-2l_j)+2\omega_j(2l_j,2l_j)h_j(t-2l_j)-2\int_{2l_j}^{t}(\omega_j)_x(2l_j,s)h_j(t-s)\,ds$$

$$-2h_j'(t-4l_j)-2\omega_j(4l_j,4l_j)h_j(t-4l_j)+2\int_{4l_j}^{t}(\omega_j)_x(4l_j,s)h_j(t-s)\,ds+\dots\Bigg\}$$

$$+\sum_{j\in J^+(v_i)\setminus J^-(\Gamma)}\Bigg\{2g_{k(j)}'(t-l_j)+2\bar{\omega}_j(l_j,l_j)g_{k(j)}(t-l_j)-2\int_{l_j}^{t}(\bar{\omega}_j)_x(l_j,s)g_{k(j)}(t-s)\,ds$$

$$+2g_{k(j)}'(t-3_j)+2\bar{\omega}_j(3_j,3_j)g_{k(j)}(t-3l_j)-2\int_{3l_j}^{t}(\bar{\omega}_j)_x(3l_j,s)g_{k(j)}(t-s)\,ds+\dots\Bigg\}\,.$$

Chapter 5: Conclusions

In this study we investigated control and inverse problem for the wave equation on metric graphs. In Chapter 2, we developed a novel approach to solve the wave equation with distributed parameters on graphs, and based on this approach, we give a constructive proof for the shape or velocity controllability of the wave equation on tree graphs. The full controllability is proved using shape and velocity controllablity via the moment method.

In Chapter 3, we utilized the algorithm presented in the first two chapters for the forward problem of the wave equation on graphs, gave the dynamical LP method to solve the inverse problem of the wave equation on tree graphs. Compared to the spectral version, the dynamical LP method is more practical, since it can recover the portion of the topology and distributed parameters of a subset of the system from only the data related to that subset, and doesn't need the whole set of data.

In Chapter 4, we extended the controllability results on tree graphs to a general graph with cycles. On graphs with cycles we need to use not only boundary but also internal controls.

Since our proofs of both controllability and identifiability are constructive, they provide good foundations for developing numerical algorithms to solve the control, inverse problems (and the forward problem) for the wave equation and other equations on graphs. Research on the numerical methods for solving the forward and control problems of the wave equation and Telegrapher's equation on graphs based on our theoretical foundations include [*Alam*, 2022].

Due to the connection between controllability and identifiability, our controllability result of the wave equation system on graphs with cycles can be extended to inverse problems on graphs with cycles. The wave equation system on a graph with cycles is never exactly controllable from a set of vertices. Correspondingly, the identification problem from the the traditional Dirichlet to Neumann map generally does not have a unique solution. The controllability from mixed use of Dirichlet and Neumann controls on a selected active set of vertices and edges suggests a mixed use of Dirichlet to Neumann and Neumann to Dirichlet map at selected edges and vertices should be used to uniquely solve the inverse problem. A natural next step is to solve the spectral and dynamical inverse problem of the Schrödinger operator on general graphs with cycles and prove necessary conditions for unique solution.

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