Journal of Humanistic Mathematics

Volume 13 | Issue 1 January 2023

The Mathematics of the Harp: Modeling the Classical Instrument and Designing Futuristic Ones

Cristina Carr
Montclair State University, Montclair, New Jersey

Daniel Chioffi
Northeastern University, Boston, Massachusetts

Maya Glenn
Columbia High School, Maplewood, New Jersey

Stefan O. Nita
Kinnelon High School, Kinnelon New Jersey

Vlad N. Nita
Montclair State University, Montclair, New Jersey

See next page for additional authors

Recommended Citation

Cristina Carr, Daniel Chioffi, Maya Glenn, Stefan O. Nita, Vlad N. Nita & Bogdan G. Nita, "The Mathematics of the Harp: Modeling the Classical Instrument and Designing Futuristic Ones," *Journal of Humanistic Mathematics*, Volume 13 Issue 1 (January 2023), pages 189-205. . Available at: https://scholarship.claremont.edu/jhm/vol13/iss1/15

The Mathematics of the Harp: Modeling the Classical Instrument and Designing Futuristic Ones

Cover Page Footnote

Cristina Carr gratefully acknowledges receiving a research stipend from the National Science Foundation through the Garden State LSAMP (NSF Award 1909824).

Authors

Cristina Carr, Daniel Chioffi, Maya Glenn, Stefan O. Nita, Vlad N. Nita, and Bogdan G. Nita

The Mathematics of the Harp: Modeling the Classical Instrument and Designing Futuristic Ones

Cristina Carr

Montclair State University, Montclair, New Jersey, USA
cvillatora@gmail.com

Daniel Chioffi

Northeastern University, Boston, Massachusetts, USA
danchioffi@gmail.com

Maya Glenn

Columbia High School, Maplewood, New Jersey, USA
mayagglenn@gmail.com

Stefan O. Nita

Kinnelon High School, Kinnelon, New Jersey, USA
nitas@students.kinnelon.org

Vlad N. Nita

Montclair State University, Montclair, New Jersey, USA
nitav1@montclair.edu

Bogdan G. Nita

Department of Mathematics, Montclair State University, New Jersey, USA
nitab@montclair.edu

Synopsis

We analyze and model the neck of the classical harp based on the length of the strings, their tension and density. We then use the results to design new and innovative harp shapes by adjusting the parameters of the model.

1. Introduction

Dating back to at least 3500 B.C.E. [12], the musical instrument we know today as the harp has a rich and captivating history. Throughout the centuries, it has evolved to reach an amazing beauty and complexity, from lap harps to concert pedal harps and even electric harps. In order to fully understand the evolution of this magnificent instrument, however, it is important to study the science, and in particular the mathematics, within its structure and function.

The shape of the harp's neck is highly mathematical and can be modeled by a harmonic curve. The equation of this curve is far from trivial and is dependent on several parameters including the length of the strings, the pitch at which they are tuned, the tension in each string and the string material (especially its density). Due to the ample flexibility in choosing these parameters, each harp model is unique and so is its shape.

A practical review of the literature on the mechanical and acoustical properties of the harp has been given by Waltham in [19]. He discusses the major acoustical components of the harp including the strings, the soundboard and the soundbox, their contributions to the sound of the harp and their historical evolution. There is a rich literature analyzing the mechanical properties of different strings (nylon, gut, etc) and how they influence the sound quality of a stringed instrument [10], [11], [16], [17], [22], [23]. The vibration of a harp's soundbox and its acoustical radiation was studied in [2], [3], [13]. Waltham and Kotlicki [20] have argued that a key factor for the musical quality of a harp is the soundboard and that the shape of the soundbord can be modified to optimize the balance between the strings. The resonance and higher air modes of the harp soundbox cavity have also been explored by Bell in [4].

In this paper we study the mathematical connection between the shape of the harp and the length of its strings, the pitch at which they are tuned, the tension in each string and the string material (especially its density). Our goal is to discover a formula that models the harmonic curve traced by the neck of the harp.[1] We base our model on a specific instrument, a 36-string harp tuned in the equal temperament scale, with regular diatonic tuning, in the key of F major and capable of producing notes in five octaves,

[1] For a similar exploration involving a cello instead, see [14].

from a low B♭1 to a high B♭6. Our particular goal is to find a formula for the length of the strings L as a function of x, where x is the distance in semitones from the note $A4$, i.e. $x = 0$ describes the note A4, $x = 1$ describes B♭4, $x = 3$ describes C5 and so on (see Figure 1). A complete table with all of the notes played by such a harp is shown in Table 1.

The shape of a harp's neck is a result of the variation in the length of each string. Accordingly, having a formula for $L(x)$, the length of the strings as a function of their relative distance in semitones from the reference note A4, is an effective way to model this curve.

The paper is structured as follows. In Section 2 we discuss the wave equation modeling a vibrating string and its solution, which leads to a relationship between the length of the strings, the fundamental frequency of vibration, the tension and the density of the string material. In Sections 3, 4 and 5 we discover formulas for the frequency, length and density as functions of x and use these parameters to develop the formula for the length $L(x)$ in Section 6. Lastly, we use the model of the classical harp to uncover new designs in Section 7 and draw conclusions in Section 8.

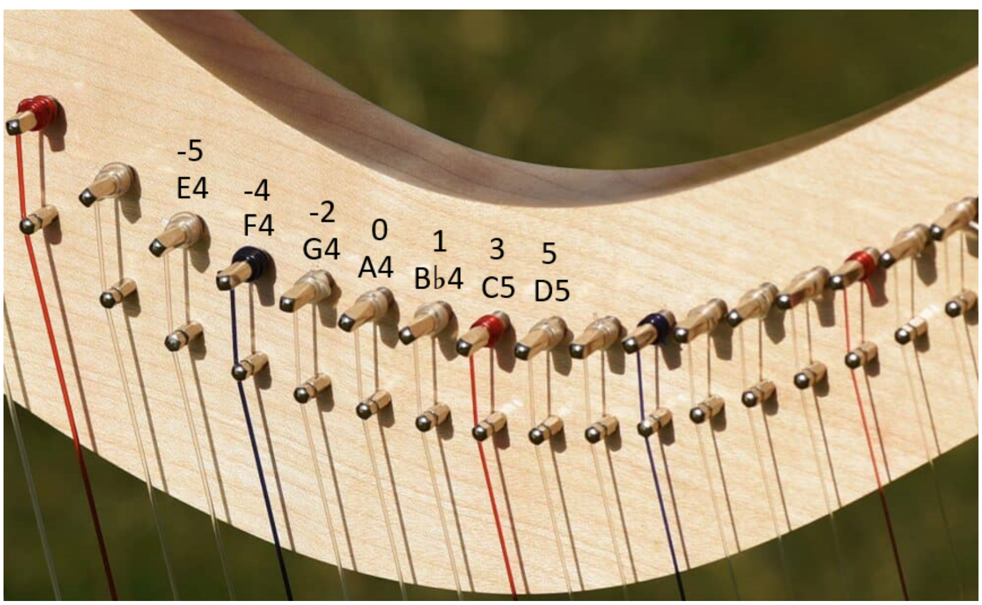


Figure 1: Section of the neck of the harp with the notes played by various strings and the respective value of the variable x. A complete list is given in Table 1.

x	Note	$\nu(x)$		x	Note	$\nu(x)$
25	B♭6	1864.66		-5	E4	329.63
24	A6	1760		-7	D4	293.66
22	G6	1567.98		-9	C4	261.63
20	F6	1396.91		-11	B♭3	233.08
19	E6	1318.51		-12	A3	220
17	D6	1174.66		-14	G3	196.00
15	C6	1046.50		-16	F3	174.61
13	B♭5	932.33		-17	E3	164.81
12	A5	880		-19	D3	146.83
10	G5	783.99		-21	C3	130.81
8	F5	698.46		-23	B♭2	116.54
7	E5	659.25		-24	A2	110
5	D5	587.33		-26	G2	98.00
3	C5	523.25		-28	F2	87.31
1	B♭4	466.16		-29	E2	82.41
0	A4	440		-31	D2	73.42
-2	G4	392.00		-33	C2	65.41
-4	F4	349.23		-35	B♭1	58.27

Table 1: The notes played by the 36-string harp modeled in this paper and the respective values of the variable x.

2. Length of a string in relation to frequency

The small transverse displacement of a vibrating string $u(y,t)$ is modeled by the one-dimensional wave equation (see for example [9])

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial y^2}$$

where y is the position variable along the string, t is time, and $c = \frac{T}{\rho}$ where T is the tension in the string and ρ is the linear density of the string. For a harp, a particular string of length L is fixed at both ends and therefore the displacement at $y = 0$ and $y = L$ is zero at all times

$$u(0,t) = 0, \quad u(L,t) = 0.$$

Using Fourier methods, we can solve this boundary value problem and write the solution as an infinite series of harmonics of the form [7]

$$u_n(y,t) = A_n \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi ct}{L} + \phi_n\right)$$

where A_n and ϕ_n are constants and $n \geq 1$ is an integer that indicates the respective term in the Fourier series. The fundamental frequency of vibration is obtained for $n = 1$ and is therefore

$$\nu = \frac{1}{2L}\sqrt{\frac{T}{\rho}},$$

and solving for L will produce the formula for the length of a string

$$L = \frac{1}{2\nu}\sqrt{\frac{T}{\rho}}. \tag{2.1}$$

Equation (2.1) describes the length of a string of density ρ tuned at a tension T that produces a note of frequency ν. This applies to all the strings of the harp along with those from other stringed instruments (guitar, piano, etc). In the next three sections, we will find expressions for ν, T and ρ as functions of x.

3. Equal Temperament Scale

In this section, we find a formula for $\nu(x)$, the fundamental frequency of vibration for a string that produces a musical note corresponding to the variable x described above. For this we examine a few facts related to music theory pertaining to scales.

One of the first scales discovered was the Pythagorean scale which is based on the most consonant musical intervals (the octave and the perfect fifth) [1, 6, 18]. These two intervals are represented using frequency ratios of the smallest integers: the octave by frequency ration of 2:1, and the fifth by a frequency ratio of 3:2. The scale can be constructed from a reference note, C4 in this example, by adding intervals equal to a perfect fifth to discover the next note in the scale (in this case G4). The frequency ratio between G4 and C4 is 3:2 (see Table 2 below).

Note	C4	D4	E4	F4	G4	A4	B4	C5
Ratio	1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Table 2: One octave of the Pythagorean scale of C major and the corresponding frequency ratios.

Adding another perfect fifth interval to G4, we obtain a frequency ratio of 9:4 (with respect to C4) which is greater than 2:1 and therefore outside one octave. Reducing this interval by an octave (by dividing this frequency ratio by 2), we obtain a ratio of 9:8 which is the third note in the scale (D4). Continuing this process, we can discover all the notes in the C major scale (see Table 2).

A common numerical representation for a scale is the system of cents [5]. We can convert frequency ratios to cents via the logarithmic function $12\log_2(r)$ which transforms any frequency ratio r into a real number description of that particular musical interval. For example, a perfect fifth interval can be described as having a length of $12\log_2\frac{3}{2} \approx 7.02$ cents, an octave, as having a length of $\log_2\frac{2}{1} = 12$ cents and so on (see Table 3).

Note	C4	D4	E4	F4	G4	A4	B4	C5
Real	0	2.04	4.08	4.98	7.02	9.06	11.10	12

Table 3: One octave of the Pythagorean scale of C major and the corresponding real number representation for each note.

In the cents representation, it is clear that the location of the main notes in the Pythagorean scale is relatively irregular, one of the consequences being that one full tone does not equal to two semitones [8]. This issue and others are addressed by the equal temperament scale which changes the location of the notes for the purpose of dividing the interval [0,12], one octave, into 12 equal intervals (semitones). The resulting partition defines the notes in the scale (see Table 4 below).

Note	C4	D4	E4	F4	G4	A4	B4	C5
Real	0	2	4	5	7	9	11	12

Table 4: One octave of the 12-note equal temperament scale of C major and their corresponding real number representations.

The equal temperament scale makes all the tones and semitones equal to each other. It is designed to correct the issues associated with the Pythagorean scale by distributing the dissonance occurring in different intervals throughout the entire scale [8]. It is obvious that the original intervals, like the perfect fifth, no longer have frequency ratios represented by small integers and therefore they are a little less consonant. However, the advantage is that these intervals sound the same (and still relatively consonant) in any key to the human ear. Because of its homogeneity and universality, most modern western instruments of today are tuned in this scale and typical western, modern music is based on musical intervals using the equal temperament scale.

Notice that the real representation for each note in Table 4 is a shifted version of the variable x that we had previously defined. Since we chose the reference note for our model to be A4, we calculate all frequency ratios with respect to A4 (440 Hz). Therefore, the relationship between the frequency $\nu(x)$ of a note played by a string in our harp and the value x is

$$x = 12 \log_2 \frac{\nu(x)}{440}.$$

Solving for the frequency we obtain

$$\nu(x) = 440 \cdot 2^{\frac{x}{12}}. \tag{3.1}$$

Equation (3.1) describes the fundamental frequency of vibration ν of the harp strings as a function of x, the distance in semitones from the reference note A4 (440 Hz).

4. Tension and harp strings

In this section, we look to establish a formula for the tension $T(x)$ applied to the harp strings as a function of the variable x described above.

String tension is a measure of the "looseness" of the strings: if the vibration of a string produces small transverse displacements then the tension is high, and if it produces large displacements then the tension is low. Tension affects the build, the volume, the resonance and the playability of a harp. This is a feature that varies among harp makers and even among harps from the same maker. Because of this, tension is very hard to quantify mathematically.

In this paper, we choose to use data measurements from a harp with similar characteristics to the harp used for our model described in Section 1, a 36-string harp tuned in the equal temperament scale, in the key of F major (see [21]). The data is listed in Table 5 for each string of the harp.

x	Note	$\nu(x)$	$T(x)$		x	Note	$\nu(x)$	$T(x)$
25	B♭6	1864.66	29.63		-5	E4	329.63	59.41
24	A6	1760	41.24		-7	D4	293.66	66.88
22	G6	1567.98	49.53		-9	C4	261.63	102.56
20	F6	1396.91	54.63		-11	B♭3	233.08	92.96
19	E6	1318.51	63.02		-12	A3	220	95.42
17	D6	1174.66	56.27		-14	G3	196.00	87.28
15	C6	1046.50	52.64		-16	F3	174.61	80.02
13	B♭5	932.33	73.50		-17	E3	164.81	82.93
12	A5	880	74.50		-19	D3	146.83	88.76
10	G5	783.99	69.40		-21	C3	130.81	94.82
8	F5	698.46	61.62		-23	B♭2	116.54	99.03
7	E5	659.25	82.51		-24	A2	110	134.38
5	D5	587.33	74.67		-26	G2	98.00	130.95
3	C5	523.25	65.05		-28	F2	87.31	128.13
1	B♭4	466.16	57.41		-29	E2	82.41	142.34
0	A4	440	57.82		-31	D2	73.42	135.13
-2	G4	392.00	64.53		-33	C2	65.41	124.34
-4	F4	349.23	58.00		-35	B♭1	58.27	105.04

Table 5: Tension data (in Newtons) for all the strings of the harp.

We also performed a linear curve fitting on this data and the result is shown in Figure 2.

The equation of the function which models the tension in the strings is therefore

$$T(x) = -1.39x + 75.21. \tag{4.1}$$

Equation (4.1) describes the tension T for all of the strings of the modeled harp as a function of x, the distance in semitones from the reference note A4.

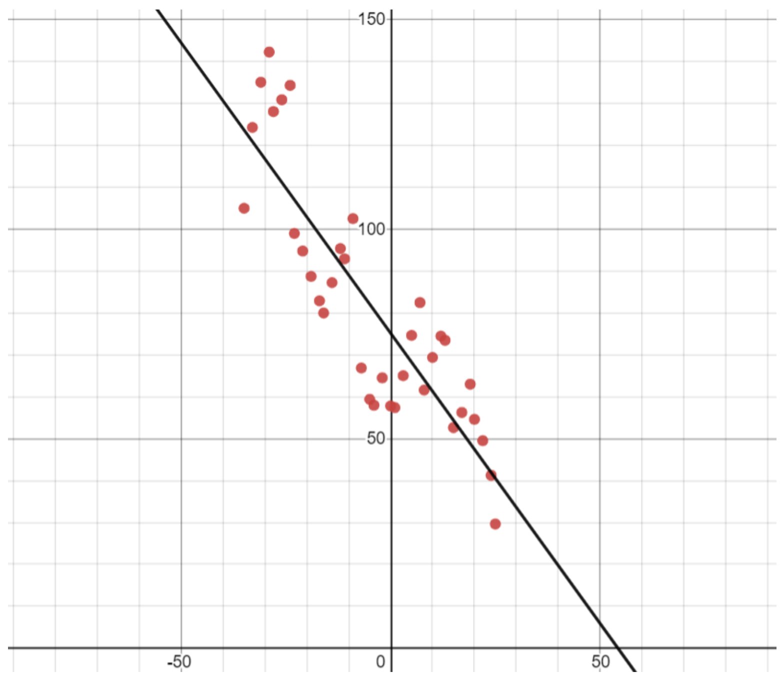


Figure 2: Tension data points from Table 5 and linear curve fitting for the 36-string harp modeled in this paper.

5. Linear density of a harp string

We move forward with finding a formula for $\rho(x)$, the linear density of a harp string as a function of x described above. For simplicity, we assume that all strings are made of nylon, which has a volume density of $P = 1.14 \times 10^3 kg/m^3$.

The linear density $\rho(x)$ is defined as

$$\rho(x) = P \cdot \pi \left(\frac{D}{2}\right)^2$$

where P is the volume density and $\pi \left(\frac{D}{2}\right)^2$ is the area of the cross section of the string with D being the diameter of the cross section. In Table 6 we list the values of D for all strings of the modeled harp and calculate the linear density using the above equation. We then performed a hyperbolic curve fitting on this data and plotted the results in Figure 3.

x	Note	$\nu(x)$	D	$\rho(x)$		x	Note	$\nu(x)$	D	$\rho(x)$
25	B♭6	1864.66	0.025	0.0003610		-5	E4	329.63	0.036	0.0007486
24	A6	1760	0.025	0.0003610		-7	D4	293.66	0.04	0.0009242
22	G6	1567.98	0.025	0.0003610		-9	C4	261.63	0.052	0.0015620
20	F6	1396.91	0.025	0.0003610		-11	B♭3	233.08	0.052	0.0015620
19	E6	1318.51	0.025	0.0003610		-12	A3	220	0.052	0.0015620
17	D6	1174.66	0.025	0.0003610		-14	G3	196.00	0.052	0.0015620
15	C6	1046.50	0.025	0.0003610		-16	F3	174.61	0.052	0.0015620
13	B♭5	932.33	0.028	0.0004529		-17	E3	164.81	0.052	0.0015620
12	A5	880	0.028	0.0004529		-19	D3	146.83	0.056	0.0018115
10	G5	783.99	0.028	0.0004529		-21	C3	130.81	0.06	0.0020795
8	F5	698.46	0.028	0.0004529		-23	B♭2	116.54	0.065	0.0024406
7	E5	659.25	0.032	0.0005915		-24	A2	110	0.076	0.0033365
5	D5	587.33	0.032	0.0005915		-26	G2	98.00	0.08	0.0036969
3	C5	523.25	0.032	0.0005915		-28	F2	87.31	0.085	0.0041735
1	B♭4	466.16	0.032	0.0005915		-29	E2	82.41	0.091	0.0047835
0	A4	440	0.032	0.0005915		-31	D2	73.42	0.096	0.0053236
-2	G4	392.00	0.036	0.0007486		-33	C2	65.41	0.1	0.0057765
-4	F4	349.23	0.036	0.0007486		-35	B♭1	58.27	0.1	0.0057765

Table 6: Linear density data (in kg/m) for all the strings of the harp.

The particular equation of the function that models the linear density of the strings for our harp is

$$\rho(x) = \frac{1}{21.77x + 906.04}. \tag{5.1}$$

Equation (5.1) describes the linear density of all the strings of the harp as a function of x, the distance in semitones from the reference note A4.

6. Model for the harmonic curve

Substituting the formulas for frequency, tension and linear density from Equations (3.1), (4.1) and (5.1) respectively into Equation (2.1) we find

$$L(x) = \frac{1}{2 \cdot 440 \cdot 2^{x/12}} \sqrt{\frac{-1.39x + 75.21}{1/(21.77x + 906.04)}}. \tag{6.1}$$

Equation (6.1) describes the length of all of the strings of the harp as a function x, the distance in semitones from the reference note A4.

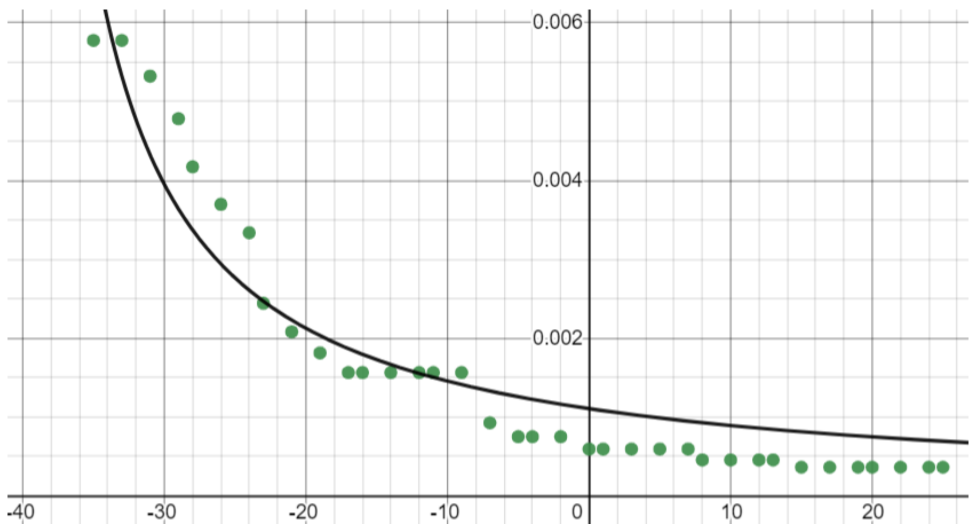


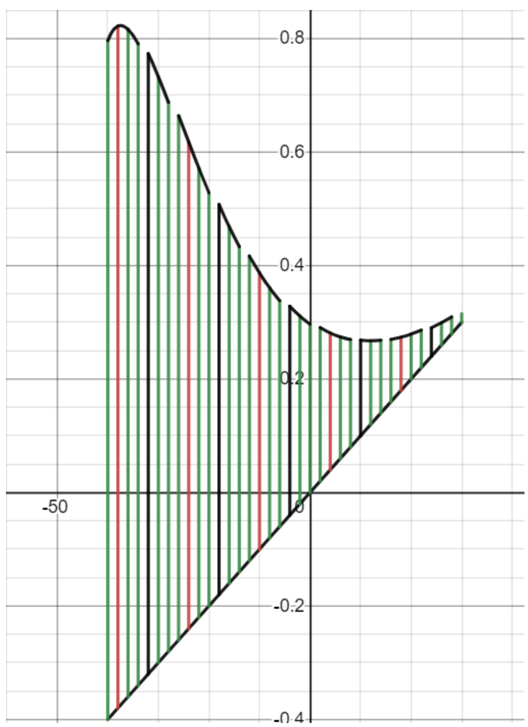
Figure 3: Linear density data points from Table 6 and hyperbolic curve fitting for the 36-string harp modeled in this paper.

We plot this function L in Figure 4(a) to represent the top curve of our modeled harp. This curve shows up as a piece-wise function due to adjustments necessary to account for the location of semitones in a major diatonic scale (and therefore the different semitone distances between the strings). In fact, the pattern of the pieces of the curve at the top of the modeled harp is similar to the pattern of the black and white keys on a piano. In Figure 4, we also plot vertical lines (strings) at every value of x from Table 1, each string having the length $L(x)$. To make the plot similar to a real harp, we start each vertical line from the line

$$S(x) = 0.01x \tag{6.2}$$

(representing the soundboard of the harp) instead of starting it from the x-axis.

The resulting model is very similar to the real classical harp pictured in Figure 4(b). The small differences can be explained by the different x-axis scale and a slight forward rotation in the column and strings of the real harp.



(a) Modeled harp

(b) Classical harp

Figure 4: The harp produced by our model versus a picture of a classical harp.

7. Futuristic harp designs

The harp model described by Equation 6.1 and plotted in Figure 4(a) relies on several parameters discussed in the previous sections: musical scales, specific regression models for tension, linear density of the strings and the shape of the soundboard (or the harp's body). Adjusting these parameters can lead to a dramatic change in the shape of the harp, which in turn could have a significant impact on the playability and structural integrity of the instrument.

As an exercise in mathematical creativity, we design new shapes for the harp (and name them based on their appearance) by adjusting only one of these parameters, the shape of the soundboard. The results are plotted in Figure 5 and were obtained by plotting the vertical strings of length $L(x)$ in Equation (6.1) starting from the following curves:

(i) for the harp model in Figure 5(a) (Elephant)]

$$S_1(x) = 0.01\sin(0.13x) + 0.01x$$

(ii) for the harp model in Figure 5(b) (Horn)

$$S_2(x) = \frac{1}{2 \cdot 440 \cdot 2^{x/12}} \sqrt{\frac{-1.39x + 75.21}{1/(21.77x + 906.04)}}$$

(iii) for the harp model in Figure 5(c) (Upside-down V)

$$S_3(x) = -0.01|x + 5|x$$

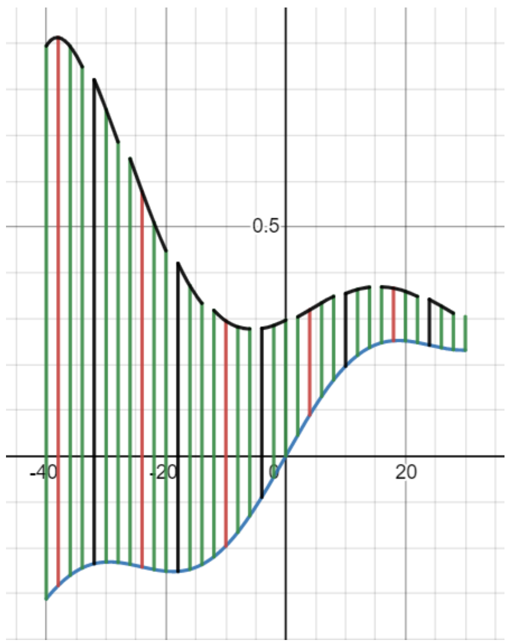
(iv) for the harp model in Figure 5(d) (Snail)

$$S_4(x) = 0.35\frac{\sin(0.35(x + 42))}{0.35(x + 42)}$$

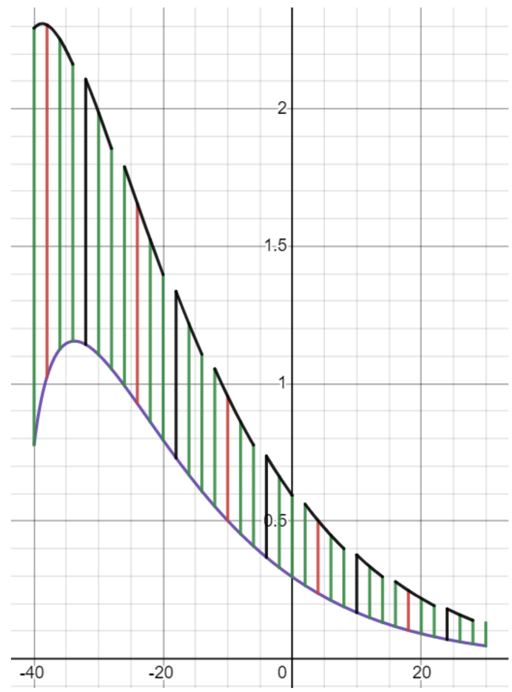
8. Conclusion

In this paper, we studied the mathematics used to model the classical harp and developed a formula for the shape of its neck. Our model was based on a harp that is widely produced and played today, a 36-string harp tuned in the key of F major. Interestingly, the same shape also appears in other instruments that use a multitude of strings to produce their sound, such as a grand piano, where the shape of the soundboard and ultimately the shape of the body follows a similar pattern. We then used the model of a classical harp to explore the potential to design new harp shapes by modifying the body or soundboard but keeping with the other restrictions brought by the types of strings, their tension and their specific densities.

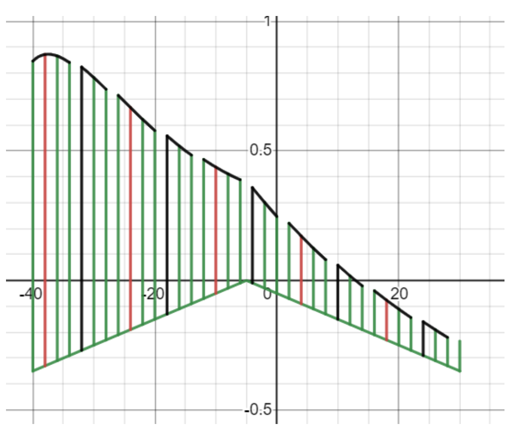
The possibilities of designing new shapes for the harp goes beyond aesthetics and mathematical creativity. Studies suggest that the soundboard and the soundbox of the harp are key factors in the creation and propagation of the sound and that even a slight modification of any component could have major impact on the acoustic radiation of the musical instrument [20], [15].



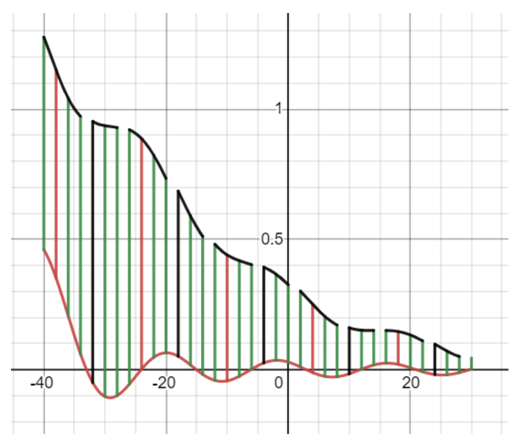
(a) The Elephant Harp



(b) The Horn Harp



(c) The Upside-down V Harp



(d) The Snail Harp

Figure 5: New harp designs using different shapes for the soundboard or the body of the harp.

Allowing for more shapes and, in general, for more freedom in the construction of a harp, could help find the optimal shape for improved sound quality and could inspire new possibilities for musicians and composers by giving them a chance to pioneer new sounds from an old instrument. However, more research is needed to understand how the new shapes presented here affect the sound of a harp.

Building a harp is an intricate process with numerous factors to consider: structural integrity of the frame, string tension, bridge and tuning pins, resonance of the soundboard and so on. These factors are not considered in this paper. The simple analysis and models uncovered by this work are a way of stimulating creativity through emphasizing existing connections between mathematics and music.

Acknowledgements

Cristina Carr gratefully acknowledges receiving a research stipend from the National Science Foundation through the Garden State LSAMP (NSF Award 1909824).

References

[1] Barbour, J.M., *Tuning and Temperament: A Historical Survey*, Michigan State College Press, E. Lansing, 1951.

[2] Bell A.J., "Acoustical effects of wood veneering on the soundboards of harps," *Acustica*, Volume **66** (1988), pages 114–116.

[3] Bell A.J., Firth I.M., "The directivity of the concert harp," *Acustica*, Volume **69** (1989), pages 26–30.

[4] Bell A.J., "The Helmholtz resonance and higher air modes of the harp soundbox," *Catgut Acoustical Society Journal*, Volume **3** (1997), pages 2–8.

[5] Benson, D., *Music: A Mathematical Offering*, Cambridge University Press, 2006.

[6] Bosanquet, R., *Musical Intervals and Temperaments*, Macmillan and Co., 1876.

[7] Boyce, W.E.; DiPrima, R.C.; Meade, D.B., *Elementary Differential Equations and Boundary Value Problems*, 11th Edition. Wiley, 2017.

[8] Christensen, T.S., *The Cambridge History of Western Music Theory*, Cambridge University Press, 2002.

[9] Feynman, R., "Sound. The wave equation," Chapter 47 in Volume 1 of *Lectures in Physics*, Caltech, 1963, 2006, 2013. Available at https://www.feynmanlectures.caltech.edu/I_47.html, last accessed on January 31, 2023.

[10] Firth, I.M., "Acoustics of the harp," *Acustica*, Volume **37** (1977), pages 139–147.

[11] Firth I.M., Bain R., Gallaher A., "String design equations," *Catgut Acoustical Society Journal*, Volume **46** (1986), pages 3–6.

[12] Galpin, F. W., "The Sumerian Harp of Ur, c. 3500 B.C.E.," *Oxford Journal of Music and Letters*, Volume X Issue 2 (1929),pages 108–123.

[13] Gautier F., Dauchez N., "Acoustic intensity measurement of the sound field radiated by a concert harp," *Applied Acoustics*, Volume **65** (2004), pages 1221–1231.

[14] Haas, R., "Cello Tangents of Quartic Polynomials," *Journal of Humanistic Mathematics*, Volume **8** Issue 1 (January 2018), pages 329–331. doi:10.5642/jhummath.201801.15.

[15] Le Carrou J.L., Leclere Q., Gautier F., "Some characteristics of the concert harp's acoustic radiation," *The Journal of the Acoustical Society of America*, Volume **127** Issue 5 (2010), pages 3203–3211.

[16] Lynch-Aird N.J., Woodhouse J., "Mechanical properties of nylon harp strings," *Materials*, Volume **10** Issue 5 (2017), page 497.

[17] Lynch-Aird N.J., Woodhouse J., "Comparison of mechanical properties of natural gut and synthetic polymer harp strings," *Materials*, Volume **11** Issue 11 (2018), page 2160

[18] Sethares, W.A., *Tuning, Timbre, Spectrum, Scale*, Springer 2005.

[19] Waltham, C., "Harp," Chapter 9 (pages 145–166) in *The Science of String Instruments*, edited by Thomas D. Rossing (Springer, New York, 2010).

[20] Waltham C., Kotlicki A. Vibrational characteristics of harp soundboards J. Acoust. Soc. Am., 124 (2008), pages 1774-1780.

[21] Welham, G., *Harp string Calculator*, https://www.harpsatsang.com/harp_design/data/stringcalculator.html, 2006.

[22] Woodhouse, J., Lynch-Aird, N.J., "Choosing strings for plucked musical instruments," *Acta Acustica united with Acustica*, Volume **105** Issue 3 (2019), pages 516–529.

[23] Woodhouse J., "The acoustics of a plucked harp string," *Journal of Sound and Vibration*, Volume **523**, 14 April 2022, pages 116669.