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The Merchant and the Mathematician: Commerce and Accounting

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The Merchant and the Mathematician: Commerce and Accounting

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Abstract

In this article, we describe the invention of double-entry bookkeeping (or *partita doppia* as it was called in Italian) as a fertile intersection between mathematics and early commerce. We focus our attention on this seemingly simple technique that requires only minimal mathematical expertise, but whose discovery is clearly the result of a mathematical way of thinking, in order to make a conceptual point about the role of mathematics as the humus from which disciplines as different as operations research, computer science, and data science have evolved.

"When you make a deposit, be sure it is counted and weighed,
and when you give or receive, put it all in writing."
Sirach 42. 7, Old Testament

1. Introduction

The issue at hand is an apparently simple one, but one that kept merchants awake at night in the early part of the Renaissance (*senza alcun riposo la loro mente sempre staria in gran travaglio*) [25, page 35]. How do we keep track of the various transactions that a merchant makes, in a way that facilitates what Fra' Luca Pacioli called his *lecito e competente guadagno* (just reward)? In *Particularis de computis et scripturis* [24], Pacioli introduced, and possibly formalized for the first time, an idea that was known before him and that he attributed to the merchants in Venice. (We discuss the historical record in Section 2.)

The idea of double-entry bookkeeping is indeed a simple one that hides the complex thinking behind it. To set up the system of double-entry, first of all it was necessary to understand what the key variables were that the merchant had to keep in mind (*debito* and *credito*). Then it was necessary to figure out a scheme that would allow the bookkeeping to faithfully and practically represent only those variables in a way that naturally emphasized their role. Most importantly, Pacioli wanted to give a precise algorithm, namely a simple, unequivocal system or process that would allow any merchant to apply the system and thus to run a successful business.

Without pretense of completeness we would like to give a quick explanation of double-entry bookkeeping, so that the reader can follow our considerations even before the full explanation that will be given in Section 3. In traditional single-entry bookkeeping, each transaction was recorded only once. The simplest example the reader can think of, and that is used even today in low-volume businesses, is the cash-book journal.

In the double-entry system, on the other hand, every transaction is recorded twice, once as a debit (entry) and once as a credit (contra entry). Moreover, each entry in an account includes the precise reference to the book and page of the account in which the contra entry has been inserted. In conclusion, since a debit in one account offsets a credit in another account, the sum of all debits equals the sum of all credits and this fact facilitates the detection of possible mistakes.

In order to provide the necessary historical context, this paper begins with a short and necessarily incomplete discussion of the salient moments in the history of bookkeeping, up to the late Middle Ages and Renaissance, with particular attention to both resistance to and motivation for progress (Section 2).

We then dedicate Section 3 to an extended and detailed discussion of the use of double-entry bookkeeping as applied to a fictionalised transaction inspired by Tagliente's work.[1] This section is purposely of a didactic nature, so that the reader not familiar with this technique, will gradually develop the perspective needed to fully understand the claims we will make in the final section. In Section 4 we finally try to offer a conceptual understanding of this accounting methodology, by attempting to see double-entry bookkeeping as an example of a subtle and rarely understood way in which mathematics plays a role in the understanding of reality.

2. Fragments of the Western history of bookkeeping

The birth of bookkeeping

As noted by Eleanor Robson, the development of bookkeeping seems to have began in Mesopotamia in the third millennium before the Common Era [31]. While records of those distant times are hard to acquire, it is fascinating to realize that the hundreds of thousands of cuneiform tablets that have reached us describe an increasingly diversified economic system: acquisitions, sales, loans, interest rates, credits, debts, taxes, fees, and even, it has been suggested, the concept of futures, namely financial derivatives [30]. What emerges is a trade economy capable to envision conditions and clauses, profits and risks, loans and interest rates [4, 32].

Some scholars claim that the existence of certain characters on clay tablets seemed to indicate a double accounting, and specifically the relation between debt and credit (see, for example, [8, Chapter 1.3.1] and [14]), but it is not unreasonable to believe that the material nature itself of the clay tablets used by the Sumerians, together with the ways to utilize and store them, might have forced the scribes to a *single-entry* bookkeeping.

Not much progress seems to have been made by the ancient Greeks, who did not use a formalized language for the study of economic phenomena, while recognizing the importance of economic and financial relationships (see, e.g., [5, 27]). For example Plato well understood that the money lender

[1] Giovanni Antonio Tagliente was a calligrapher, printer and author of textbooks of various typologies, and in particular of abacus and computation. The textbook [38] of only 16 not numbered *folia*, published in 1525, probably in Venice, had very large diffusion and was used in the schools of arithmetic and in commerce.

and the currency exchanger, did not need any special capital, but rather a table ($\tau\rho\alpha\pi\epsilon\zeta\alpha$) in the marketplace ($\epsilon\nu\ \alpha\gamma o\rho\alpha$), where he could conduct his business [29], and exchange currency, weigh metals, and verify the weight and composition of pawns.[2]

In Roman times, records were kept on wax tablets, according to the system called tabular bookkeeping. Among the account books, an *adversaria*[3] (day book) was used to record all the debts, purchases, credits, and obligations of trade. Financial data were then posted from the day book to a *tabulae rationum*, or tabular ledger. In the *tabulae rationum*, each account was divided into two pages facing each other, one for debit and one for credit entries. But all the entries made were single-entry in form, and there was no second account.

During the centuries of the High Middle Ages (indicatively between the eighth and the tenth centuries), as the Roman empire gradually lost its grip on Europe, the courtly economy prevailed, and banks were not yet developed. Indeed, in this period, the Hellenic-type of commerce based on fairs — less common in the Roman world — was progressively resumed: circulation and exchanges of goods were conducted by *mercatores*, which wandered from one fair to the other, from a castle to the other, alone or in caravans, while selling different merchandise, from exotic objects to more common goods, but embracing also questionable kinds of exchanges. Doing so, they confirmed their image of wandering men, not fearful of God. This wandering commerce slowly disappeared in favour of merchants that, even if travellers, had also permanent seats in the towns of fairs and in harbours, were capable to read and write, and had employees who themselves were also initiated to the rules of business accounting (see, for example, [7, Chapter I] and [11]).

Towards double-entry bookkeeping

The pressures from the work of the merchants, together with the increasing cultural contact between the world of Islam and the remnants of the Greek world, formed the terrain on which Algebra eventually developed [10, 41].

[2] An interesting viewpoint on Greek numeracy and monetary practice can be found in [20].

[3] The name indicates the fact that debt and credit were placed in physical opposition to each other.

To see the emergence of double-entry bookkeeping in the Western world,[4] let us look more carefully to what was happening in the Italian peninsula. It was around the end of the thirteenth century that Venetian and Tuscan businessmen passed to the new methodologies of accounting bookkeeping that we have indicated, and that were necessary to maintain a picture of the economic situation and of the financial balance of the family company and of partner companies, often situated in different regions and countries. This was made possible by the diffusion of the knowledge of commercial arithmetic and by a larger and larger experience of mercantile exchanges.[5]

The accounting registrations that reached us from companies located in Lombardia, Liguria, Romagna and Toscana testify a progress in the experience of computation and in the accounting management, which will see its evolution completed at the beginning of the fifteenth century, with the introduction of the Journal, the Ledger and the double-entry bookkeeping.

The Journal and the Ledger employed for accounting were also used to chronologically and exactly register every single commercial movement; for this reason they became also essential documents to solve legal controversies, as it happened in the case of the accounting documents of a few Venetian families, including the Barbarigo and the Fraterna Soranzo. These documents are considered among the oldest examples of double-entry bookkeeping.[6]

These forms of accounting books with two sections (for debts and for credits), in double-entry and written in Venetian vernacular (in some other places the use of Latins remained), became known as "the Venetian Method" and, influenced significantly Luca Pacioli, who had lived in Venice for a long period, and likely led him to compose his *Tractatus XI (De computis et scripturis)* of his *Summa*.

[4] Here and throughout the paper we are limiting our discussion to the European history of double-entry bookkeeping, without seeking to describe the additional sources that show the development of similar ideas in India, China, and throughout Islam. See, for example, [21].

[5] *"C'est la naissance de la comptabilité appelée à ses débuts méthode italienne ou méthode vénitienne, jusqu'en 1755 où Pietro Paolo Scali la nommera en partie double."* (English: "This is the birth of accounting called at its beginnings the Italian method or the Venetian method, until 1755 when Pietro Paolo Scali will name it double entry.") [17, page 84]. See also [2, 10].

[6] For further information see [33].

Datini and the transition

Thus, progressively from the eleventh century on, the Italian merchants went beyond the practice of the simple chronological registration of commercial operations (the Journal) and developed a more elaborate form called *dual* registration, where entries were distinguished for typology and inserted in different columns indicating debts and credits, even if in general with no cross reference between *contra* entries, an element that was instead central to double-entry bookkeeping.

The case of Francesco Datini can be placed in this context, for at least two reasons: for the extraordinary richness of Datini's archives (about twelve hundred accounting books, huge epistolary and many other documents)[7] that arrived entirely to us, and because he was a merchant who, through the progressive growth of his commercial traveling and of the complexity of his partner companies, was in a position to naturally make the transition from single to dual, and then to double-entry bookkeeping. Indeed this transition can be traced out, since he kept accounts of his books for almost half of a century. His books show the use of single-entry bookkeeping from 1367 to 1372, a transition period after 1372, and double-entry starting from 1390 on (see [37]).

Datini exemplified the notion of a global citizen, and it is remarkable to see how in a rather short period he was able to connect his activities under what we should today call his own multinational holding company. Indeed, the activities of his original company in Avignon were intertwined first with those of three other companies he had built, in Firenze, in Prato, and in Pisa, and eventually we have a company system, a holding, encompassing eight branches (or *fondaci* as Datini calls them), namely Avignon, Prato, Pisa, Firenze, Genova, Barcelona, Valenza, and Majorca.

But the success of markets needed more than just good bookkeeping. It required reliability, and it required risk management. Both these innovations came through the standardization of the transfer of goods and funds. Reliability was made possible through the increased use of and adjustment of the

[7] See [39]. See also "Fondo Datini" at https://archiviodistatoprato.cultura.gov.it and http://museocasadatini.it/it/palazzo-francesco-datini-prato.php, both last accessed on January 31, 2023.

bills of exchange and checks. Once merchants reached the East, they would first handle their bills of exchange and bank transfers; these two instruments allowed the merchants to transfer funds, both within the same locality, as well as between distant cities [12]. This allowed the elimination of a major risk element in the process of goods transfer, creating a consistent standard for borrowing. Commissions paid for honouring the bills of exchange served as a way to skirt the Church's ban on usury, thereby allowing a means for gaining interest, which could not be openly registered because of the ban.[8]

Indeed the merchants of the early modern age faced the question of the Church on the legitimacy of wealth, and in any case the necessity to distinguish between excessive wealth accumulation and just compensation, between usurers and bankers. Consequently, entries in the accounts of debtors did not always reflect the real movements of cash. For example, a loan was recorded by a bank in the account of the borrower for the full amount that had to be repaid, including all the interest applied. As a result, a cash account could not be maintained, otherwise the two account entries for the transaction would not have been equal and any inspection in the books would have detected the bank's usurious activities. The credit entry in Figure 1 is taken by the book of Tagliente [38] (see also [34], [35, page 304]) and records that a payment was due from a debtor; the entry reflected that a payment was to be made, but no contra entry was posted in another account. This is consistent with the fact that Tagliente himself calls this "single-entry." In the case of dual entry, the other account would have been indicated, without its location in the book. Finally, if these were entries in a double-entry Ledger, then the Ledger had to contain accounts for cinnamon and for the bank of Antonio Cappello e Luca Vendramin, and each entry had to contain the name of the contra account and the reference to the page where the contra entries were recorded.

[8] Despite its less than positive judgement on merchants and trading, the Catholic religion was not opposed to honest merchants, especially if the profits are used for the common good. The *Decretum Gratiani* (1159), for example, emphasized that merchants were expelled from the temple, but it stressed that the most insidious sin is that of usury since money made in this way is not the fruit of labour. For further information, see [1].

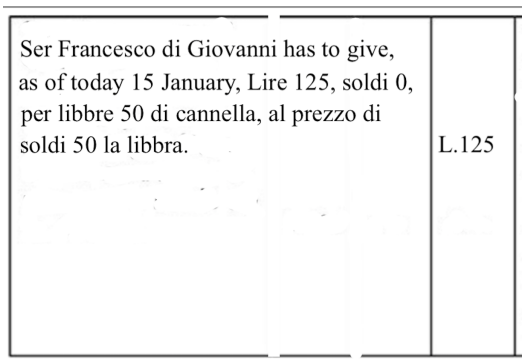


Figure 1: Example of single-entry bookkeeping. Entry does not even mention the account of the contra entry.

Note that this figure actually describes a single transactions managed with single-entry accounting. The treatment of this same transactions that Pacioli would have made is distinctly different as we will show in the next section in a somewhat more articulate situation.

Datini was again one of the first to adopt the check as a new form of payment in 1398 when he opened a bank with a partner in Firenze. Checks ensured safety because they offered a method to transfer funds without having to carry the actual cash in person. Because funds could be transferred in relatively greater safety, the risk of loss that came from carrying and transferring more capital decreased tremendously. Hence those with wealth had more incentive to use their capital in economic endeavours. The circulation of this capital alone was enough to inspire more market activity, and led to greater opportunity for investment.

Inside this process of transformation of the social conditions, the training of apprentices and the formation of accountants helped the development of commerce and accounting methodology. A central element became the presence of a class of employees — *abachieri*, *scrivani*, *chiavai*, and *segretari* — increasingly capable of high-level accountancy, also thanks to their school education. In fact the *abacus* schools provided young apprentices with the fundamental knowledge of arithmetic necessary to manage the various aspects of mercantile accounting, as we read explicitly in some of Datini's documents, when, for example, he explains how Simone Bellandi, one of his

employees, *stette da mesi* 2 *in* 3 *all'abaco*.[9] We know, as well, that Datini often encouraged his young employees to attend the abacus school: *Tomaso è mecho; è che io volglio ch'elgli inpari uno pocho d'abacho, e poi lo manderò a Mano a Pisa e ch'è mia intenzione di fare di lui chome al filgliuolo*.[10]

We give here explicit evidence of the transition operated by Datini in his accounting system, from the single-entry to the double-entry bookkeeping method. Figure 2, a screenshot taken from a book of Melis [16, page 213], represents the collection of all entries, relative to a single transaction, taken from the archives of Datini and dated 2 June 1403. Transcription "a)" comes from one entry of the Journal; transcription "b)" comes from the Ledgers. Transcription "b)" has a double-entry: entry "1)" recorded as a debt, and (the *contra)* entry "2)" recorded as a credit. It is important to notice that each of the entries in "b)" has precise references to the other (the *contra*) entry. The use of double-entry method is quite evident.

1403

a) nel giornale:

Francescho e Stoldo dè avere, a dì 2 di giugnio, f. tre d'oro; per loro da Jachopo di Piero selaio f. 3 (16).

b) nel mastro:

1) addebitamento nel libro dell'entrata e uscita:

Da Francesco e Stoldo, dì 2 di giungno. f. 3; per loro da Jacopo di Piero sellaio, recò Domenico suo figliolo; a libro nero A, c. 79, f. 3 (17).

2) accreditamento nel « libro nero A »:

Franciescho di Marcho e Stoldo di Lorenzo nostri deono avere

. . . . . .

. . . . . .

E deono avere, a dì 2 di giugno, f. tre d'oro; per loro da Jachopo di Piero selaio, rechò Domenicho suo figluolo, chome apare a entrata A, c. 2, f. 3 (18).

Figure 2:

[9] "He spent two-three months at the abacus school." Quoted in [15, Volume I, page 343, footnote 10]. See [3].

[10] "Tomaso is with me; the fact is that I want him to learn some abacus, and then I will send him soon to Pisa, since it is my intention to treat him like a son." (ASPo, Datini, Prato-Firenze, Francesco Datini, I. 7. 1392). Quoted in [22, page 82]. See also the English edition [23].

3. Learning double-entry bookkeeping: a more articulate example

The best description of double-entry bookkeeping is probably given by Luca Pacioli himself [25, Ch. XIV, page 63]:

> Per la qual cosa sappi che di tutte le partite che tu avrai poste nel Giornale, al Quaderno grande, te ne convien sempre fare due, cioè una in dare e l'altra in avere perchè lì si chiama lo debitore per lo *Per* e lo creditore per lo *A*, come di sopra dicemmo; chè dell'uno e dell'altro si deve da per sè fare una partita, quella del debitore ponere alla man sinistra e quella del creditore alla man destra, e in quella del debitore chiamare la carta dove sia quella del suo creditore, e così in quella del creditore chiamare la carta di quella dove sia il suo debitore; e in questo modo sempre vengono incatenate tutte le partite del ditto Quaderno grande, nel quale mai si deve mettere cosa in dare che quella ancora non si ponga in avere, e così mai si deve mettere cosa in avere che ancora quella medesima con suo ammontare non si metta in dare. E di qua nasce poi il bilancio, che del libro si fa nel suo saldo.

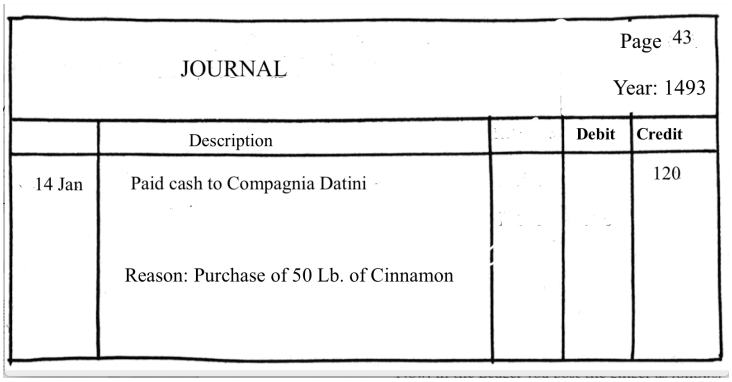
An English translation from [26, Chapter XIV, page 25]):

> As to each entry you have written in the Journal, you must always write up two entries in the Ledger. In the Journal, each debit entry is preceded by the word "to" and each credit entry by the word "by". In the Ledger, you write up a record for each of them. The debit entry must always be put on the left; the credit entry always on the right. By the debit entry, you write in the reference page of the corresponding credit entry. In this way, you cross-reference all the entries in the Ledger, and you must never make a credit entry without a balancing debit entry. The extract of a trial balance depends upon adherence to this rule.

In order to help the reader get a full grasp of this process, we now take the Tagliente's example but increase its complexity by adding one more transaction, the one that the merchant had to do to buy Cinnamon, which he then sold to Francesco. The reason for this additional complexity is to show to the reader how intertwined operations can be managed through the use of double-entry bookkeeping.

Suppose that on the 14th of January of the year 1493, the merchant owner of our Florentine shop, realizes that his shop is out of Cinnamon, and then he buys 50 *Lb.* of Cinnamon, by paying 120 Lire cash to the Compagnia Datini of Prato.

If the merchant uses the single-entry bookkeeping, then he will register this purchase using the following record:



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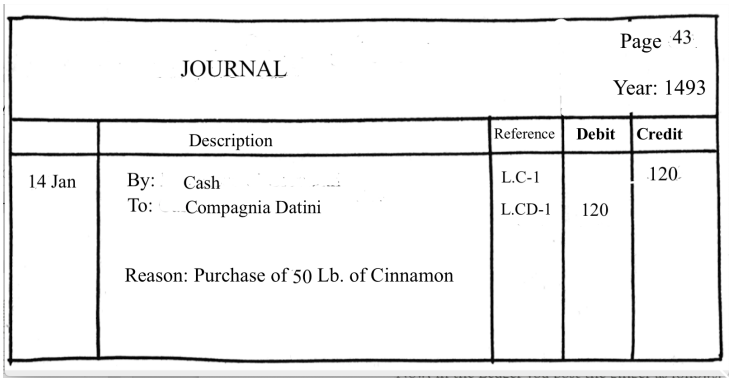
Year: 1493

	Description		Debit	Credit
14 Jan	Paid cash to Compagnia Datini			120
	Reason: Purchase of 50 Lb. of Cinnamon			

Figure 3: Single-entry bookkeeping example.

If instead the merchant uses the double-entry bookkeeping, he will have to work a bit more to register the purchase. First he has to insert in the Journal the following two entries. The first one to registers that he paid Lire 120 cash to the Compagnia Datini, and the reason why.



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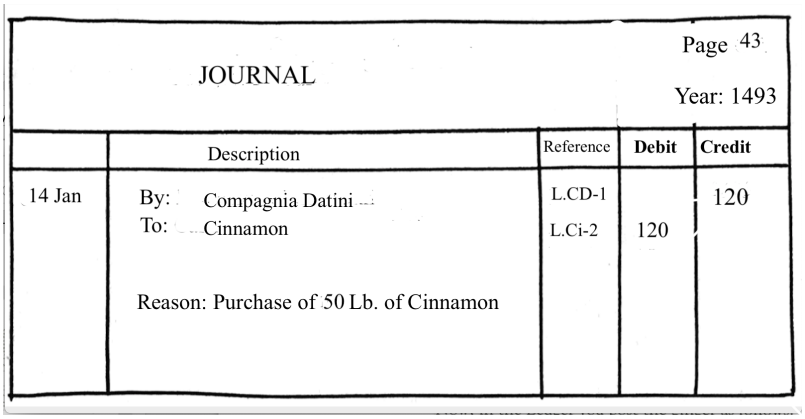
Page 43

Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Cash	L.C-1		120
	To: Compagnia Datini	L.CD-1	120	
	Reason: Purchase of 50 Lb. of Cinnamon			

Figure 4: Double-entry bookkeeping example: the first step.

Then he has to register the fact that he receives the Cinnamon from the Compagnia Datini.



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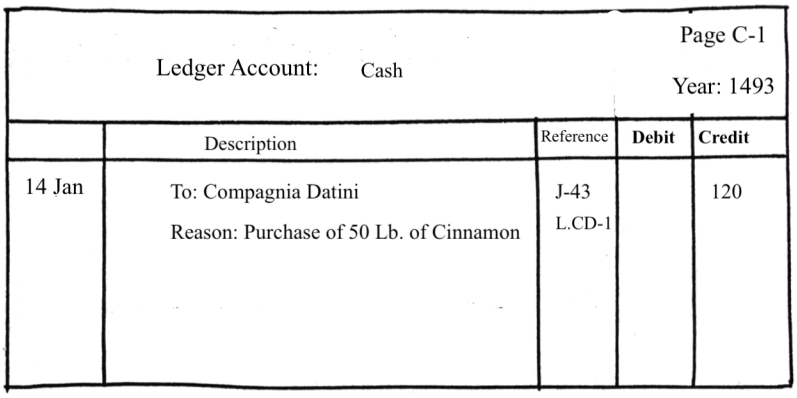
Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Compagnia Datini	L.CD-1		120
	To: Cinnamon	L.Ci-2	120	
	Reason: Purchase of 50 Lb. of Cinnamon			

Figure 5: Double-entry bookkeeping example: the second step.

The shop owner is still not yet ready. In fact, in the evening, after the shop closes, he has — for each record of the journal — to insert the pertinent double entries in the Ledger. For the entry of Figure 4 he writes in the Ledger the entry concerning the Account Cash (which includes the reference to the page of the Journal)



Ledger Account: Cash

Page C-1

Year: 1493

	Description	Reference	Debit	Credit
14 Jan	To: Compagnia Datini	J-43		120
	Reason: Purchase of 50 Lb. of Cinnamon	L.CD-1		

Figure 6: Double-entry bookkeeping example: the third step.

and the entry concerning the Account Compagnia Datini (again including the reference to the page of the Journal)

Ledger Account: Compagnia Datini — Page CD-1 — Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Cash Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.C-1	120	

Figure 7: Double-entry bookkeeping example: the fourth step.

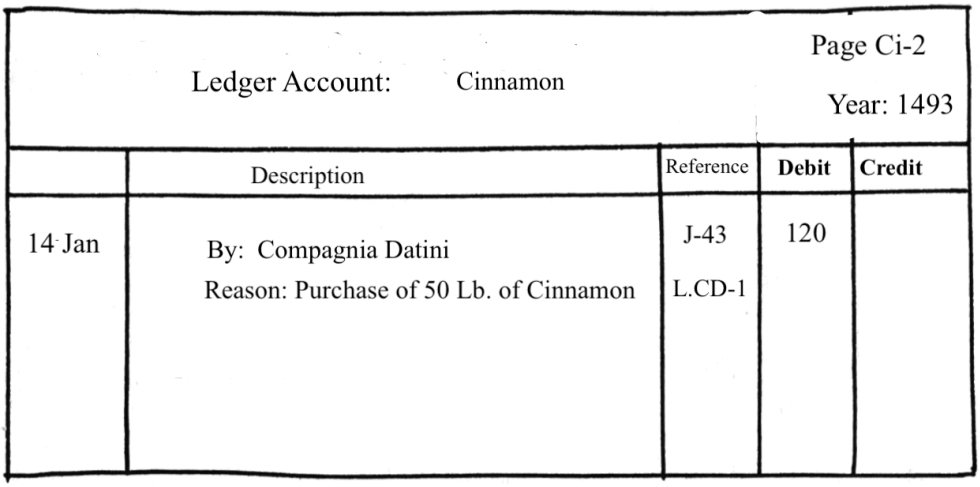
Now the owner has to do the analogue for the Journal record of Figure 5, by inserting the Ledger entry concerning the Account Compagnia Datini

Ledger Account: Compagnia Datini — Page CD-1 — Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Cash Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.C-1	120	
14 Jan	To: Cinnamon Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.Ci-2		120

Figure 8: Double-entry bookkeeping example: the fifth step.

and finally the Ledger entry at the concern of the Account Cinnamon.



Ledger Account: Cinnamon

Page Ci-2

Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Compagnia Datini Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.CD-1	120	

Figure 9: Double-entry bookkeeping example, completed.

Of course the reader has the right to be puzzled: so much work is needed for the double-entry system, so why use it? The single-entry method gives the shop-owner the only datum that on that 14 January 1493 he bought from Compagnia Datini 50 *Lb.* of Cinnamon, and paid Lire 120 cash. That might seem to capture the essence of the transaction. The double-entry method, however, gives a lot of additional information. To see this, we update the various Accounts that are involved, by summing up separately their columns of Debts and Credits. At the end of the day 14 January 1493 these are the daily variations of the Accounts involved:

Cinnamon:	Debt: +120 Lire	
Cash:		Credit: +120 Lire
Compagnia Datini:	Debt: +120 Lire	Credit: +120 Lire

Therefere the merchant can easily:

(i) see that the balance of the day (credits – debts) is zero, which suggests that the accounting was properly performed;
(ii) see that the value of Cash has diminished.
(iii) check the value of the Cinnamon which is in the shop, ready to be sold.
(iv) see that he is even with the Compagnia Datini (debts – credits = 0).

Sometimes, for the sake of brevity, a purchase could be registered using only one Journal record (with the corresponding two entries in the two appropriate Ledger accounts). In our case, the single Journal entry could register the overall transition of 120 Lire by the account Cash to the account Cinnamon, forgetting the role played by the Compagnia Datini. But this approach has precisely the disadvantage of ignoring the role played, and the documentation produced, by the seller company, and in any case it cannot be used if the payment and the transferring of goods are not happening in the same day.

Now, on the 15$^{\text{th}}$ of January, the day after, Messer Francesco di Giovanni enters the Florentine shop, and buys 50 *Lb.* of Cinnamon, paying the price of 125 Lire cash to the merchant.

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	Description		Debit	Credit
15 Jan	Francesco di Giovanni paid Cash Reason: Purchase of 50 Lb. of Cinnamon		125	

Figure 10: Single-entry bookkeeping example, continued.

But since our fictitious merchant uses the double-entry bookkeeping, he will have to insert in the Journal the following two entries. The first one to register that Francesco paid lire 125 cash and the reason why.

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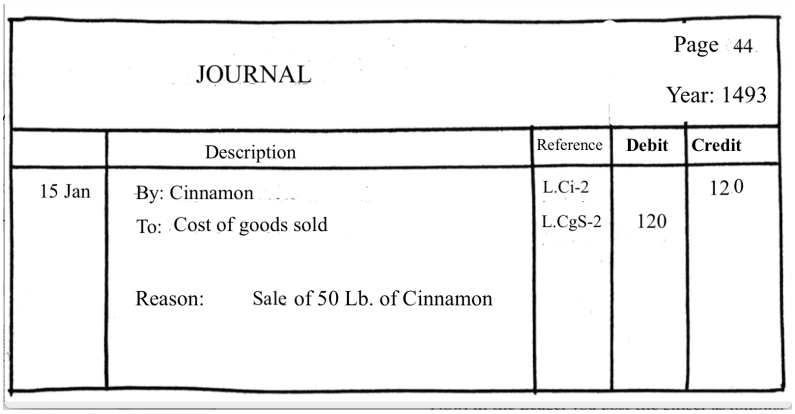
	Description	Reference	Debit	Credit
15 Jan	By: Francesco di Giovanni	L F-1		125
	To: Cash	L C-1	125	
	Reason: Sale of 50 Lb. of Cinnamon			

Figure 11: Double-entry bookkeeping example, continued.

and then he has to record the fact that the inventory (asset) of Cinnamon has diminished of the value of the sold Cinnamon, and to move this value to the Account Cost of goods sold.



JOURNAL

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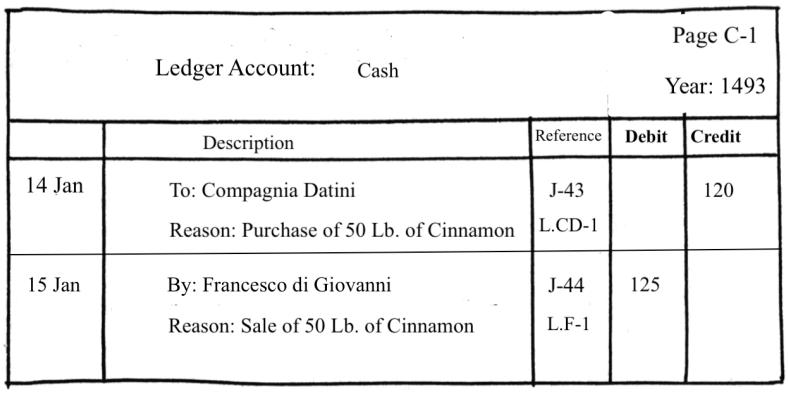
Year: 1493

	Description	Reference	Debit	Credit
15 Jan	By: Cinnamon	L.Ci-2		120
	To: Cost of goods sold	L.CgS-2	120	
	Reason: Sale of 50 Lb. of Cinnamon			

Figure 12: Double-entry bookkeeping example, continued: the second step.

This last record takes into account the fact that the inventory (asset) of Cinnamon has diminished of 120 Lire (and hence the "system" will automatically register a profit of 5 Lire).

Like every day, in the evening, after the shop closes, the merchant has — for each record of the journal — to insert the appropriate double entries in the Ledger. For the entry of Figure 11 he writes in the Ledger the entry concerning the Account Cash (with the reference to the page of the Journal)



Ledger Account: Cash

Page C-1

Year: 1493

	Description	Reference	Debit	Credit
14 Jan	To: Compagnia Datini	J-43		120
	Reason: Purchase of 50 Lb. of Cinnamon	L.CD-1		
15 Jan	By: Francesco di Giovanni	J-44	125	
	Reason: Sale of 50 Lb. of Cinnamon	L.F-1		

Figure 13: Double-entry bookkeeping example, continued: the third step.

and the entry concerning the Account Francesco di Giovanni (with the reference to the page of the Journal).

Ledger Account: Francesco di Giovanni — Page F-1, Year: 1493

	Description	Reference	Debit	Credit
15 Jan	To: Cash Reason: Sale of 50 Lb. of Cinnamon	J-44 L.C-1		125

Figure 14: Double-entry bookkeeping example, continued: the fourth step.

The merchant then has to do the analogue for the Journal record of Figure 12, by inserting the Ledger entry concerning the Account Cinnamon

Ledger Account: Cinnamon — Page Ci-2, Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Compagnia Datini Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.CD-1	120	
15 Jan	To: Cost of goods sold Reason: Sale of 50 Lb. of Cinnamom	J-44 L.CgS-2		120

Figure 15: Double-entry bookkeeping example, continued: the fifth step.

and the Ledger entry at the concern of the Account Cost of goods sold.

Ledger Account: Cost of goods sold — Page CgS-2 — Year: 1493

	Description	Reference	Debit	Credit
15 Jan	By: Cinnamon Reason: Sale of 50 Lb. of Cinnamon	J-44 L.Ci-2	120	

Figure 16: Double-entry bookkeeping example, continued: the sixth step.

and of course the Ledger Account Compagnia Datini remains unchanged on January 15.

Ledger Account: Compagnia Datini — Page CD-1 — Year: 1493

	Description	Reference	Debit	Credit
14 Jan	By: Cash Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.C-1	120	
14 Jan	To: Cinnamon Reason: Purchase of 50 Lb. of Cinnamon	J-43 L.Ci-2		120

Figure 17: Double-entry bookkeeping example, continued: the final step.

Here is the information given by the double-entry method. By updating the various Accounts (summing up separately their columns of Debts and Credits), we find that at the end of the day 15 January 1493, these are the daily variations of the Accounts involved:

Cash:	Debt: +125 Lire	
Cinnamon:		Credit: +120 Lire
Francesco di Giovanni:		Credit: +125 Lire
Cost of goods sold:	Debt: +120 Lire	

The merchant can now easily check:

(i) that the balance of the day (credits—debts) is zero, which suggests that the accounting was properly performed;
(ii) that there is a sale income of 125 Lire and that the cost of the Goods sold is 120 Lire, so there is a profit of 5 Lire;
(iii) the variation in the value of the Cinnamon which is in the shop

If we consider together the variations of the two days 14 and 15 January we get the following table

Cash:	Debt: +125 Lire	Credit: +120 Lire
Compagnia Datini:	Debt: +120 Lire	Credit: +120 Lire
Cinnamon:	Debt: +120 Lire	Credit: +120 Lire
Francesco di Giovanni:		Credit: +125 Lire
Cost of goods sold:	Debt: +120 Lire	

which makes the situation well clear, and in particular points out that after two days:

(i) the shop has a profit of 5 Lire,
(ii) there is return on investment of 5/120 Lire
(iii) there is again no Cinnamon in the shop.

Note that the double-entry bookkeeping introduces a lack of symmetry between purchase and sale. When the store purchases Cinnamon there are only three Ledger accounts that are affected (Cash, the seller and Cinnamon).

Despite the simplicity of this example, the power of the double-entry system is apparent, and if one imagines the situation of a complex business (the one of the Compagnia Datini, for instance), it is clear that the double-entry system can give an everyday clear and complete picture of the situation.

4. Mathematics and double-entry bookkeeping: further reflections

The permeability between different knowledges and competences was not unusual in the early Modern Age. The ways arithmetic and geometry — and more generally mathematics — showed up as essential elements of commerce during this time furnish a remarkable example.

Maybe the most important instance of mathematics commingling with commerce is found in Pacioli's *Su[m]ma de Arithmetica Geometria Proportioni [et] Proportionalita* (see Figure 18). To set the historical stage, Luca Pacioli was born in San Sepolcro around 1445, but soon moved to Venice with the role of tutor in the house of the merchant Antonio Rompiasi. While at his service, he accompanied the merchant in his business travels, thus enriching his knowledge in the practice of commercial accounting. Together with Rompiasi's son, Pacioli attended the lectures of commercial mathemat-

Figure 18: Frontispice of *Summa*, edition 1523.

ics given by Domenico Bragadin, lecturer of mathematics of the Venetian Republic in the Rialto school. Indeed it was a duty of mathematicians to teach the discipline of effective calculations. Pacioli himself, in 1468, became lecturer of mathematics and canon priest in San Marco; in 1470 he wrote his first book of mathematics (which is unfortunately lost), dedicated to the three sons of Rompiasi. Successively Pacioli taught mathematics in several towns, and had the opportunity to meet and know Leonardo da Vinci, Leon Battista Alberti, and Piero della Francesca.

Pacioli's *Summa* investigates the meaning and the rules for investments, sales, profits and losses, and habits of trades in other countries together with weights and currencies. It announces also the analyses of improvements and worsening of duties and taxes, as well as salaries of employees and workers: all aspects that will be treated in the subsequent parts of his work.

While the purpose of the *Summa* is very ambitious and it is written as a general textbook for merchants, the two parts that are of greatest interest to us are the second and the third section. In the second, Pacioli exemplifies, practically (*per via negotiaria*) and theoretically (*per via alghebratica*), the rules for the administration of mercantile companies. He does this by exposing with technical (but in vernacular language) terminology an ample spectrum of themes: from the lease of land, to the various forms of trade, to the mathematical notions, theoretical and practical, which regulate the letters of change, to currency, gold and silver.

The third section consists of the fifteen pages of the *Tractatus XI intitolato Particularis de computis et scripturis, della Distinctio IX*, [24]. This summary is a lucid synthesis of the meticulous, daily practice necessary for a correct double-entry bookkeeping. The importance of the role of the Journal, of the Ledger and of the cross references between entry and contra entry are pointed out, and the universality of the double-entry method is stressed.

The section of the *Summa Tractatus XI intitolato Particularis de computis et scripturis* had an extraordinary success, as a tool to circulate worldwide the procedures of the correct double-entry bookkeeping, together with the reasons of the legal and accounting efficacy of this method [6] (see also [13]). Pacioli's accurate attention to the economic phenomena and to the rules of behaviour of those who had a commercial enterprise is probably due to the activity in which his order, the Friars Minor, was involved at that time,

i.e., to trace the border between *just* interest and usury, between lawful and unlawful, moral and immoral for what concerned the transactions connected with loan and exchange. In other words, the growth of the economical and technical volume of trade, the daily practice and the social recognition of the role of merchants encouraged the friar to provide both a systematization of the mercantile accounting inside the mathematical knowledge, and a set of technical rules with the aim to legitimate the trade as a virtuous economical practice.

As we reach the end of this paper, we want to try to interpret double-entry bookkeeping within the framework of more modern mathematics, and to use this interpretation to draw a (maybe premature) consequence on the role of mathematics. As we already noticed in Section 3, a key feature of the double-entry method is the creation of a link between each entry of the Ledger and its contra-entry, in such a way that they remain connected for the entire accounting procedure (see, e.g., Figure 12). This fact recalls, in modern terms, the features of a spreadsheet (excel for example): some entries of an excel sheet can be coupled (or related), and each couple of entries can be forced to contain the same value (or related values), in such a way that it is automatically guaranteed that the computations performed will satisfy certain fundamental constraints. Of course in the past, accountants, or businessmen, had to perform the computations — with the care and the control techniques indicated in the medieval texts — instead of computers. Indeed the entire accounting method can be viewed as a possibly enormous, extremely versatile, human-run spreadsheet, which can be adapted to any kind of business and developed accordingly, and which has to be kept precisely up-to-date every day, so as to have the everyday control of the entire situation of the business, including the status of each account, the status of the inventory, as well as profits and losses (the discussion after Figure 17 highlights an instance of this phenomenon). A spreadsheet is of course also a dynamic database, which, to be effective and useful, should allow easy insertion and retrieving of data. And, maybe most importantly, it should allow an easy and transparent construction and development of its structure.

A different way of thinking about double-entry bookkeeping was proposed by Sangster [36], who suggested that one needs to think of double-entry bookkeeping as an axiomatic system, namely a set of procedures which can be implemented by following a set of axioms and the consequences that one

can derive from them. Specifically, he claims that the entire double-entry bookkeeping can be based on the following axioms (that can be traced back to the example in Section 3):

(i) All transactions involve two elements: an item exchanged and a form of settlement.
(ii) All forms of settlement can substitute for each other.
(iii) One element is debit and the other is credit.
(iv) Debit = credit.
(v) The entries in the money columns are to be in one currency only.

and the three postulates

(i) Cash received is a debit.
(ii) Capital given (by the businessman) is a credit.
(iii) Cash given is a credit.

From this point of view, Sangster suggests that double-entry bookkeeping can be formulated in the language of Euclid, with a set of rules established a-priori, and a set of permissible actions, that will allow the construction of an algorithm, for the dynamic preparation and development of the spreadsheet/database structure of the double-entry bookkeeping of a given business.

Finally a third way, which has been pointed out by David P. Ellerman [9], consists in realizing the fundamental isomorphism between the double-entry bookkeeping setup and the construction of integers (positive and negative) as equivalence classes of pairs of natural numbers. This idea had apparently already been expressed by mathematicians such as Cayley and Hamilton, as pointed out by Ellerman, in the aforementioned paper. We refer the reader to the cited articles for details, but the key observation here is the fact that profits and losses are described not through positive and negative numbers, but rather through pairs of positive numbers so that a profit of 5 liras is indicated as the pair $(125, 120)$, while a loss would be indicated by the pair $(120, 125)$. While this may appear to be an odd complexity, it is important to remember that negative numbers had not yet been accepted at the time of Pacioli. For example, we know that while in the tenth century Abū'l-Wafā' used negative numbers to represent a debt in his work on arithmetic for merchants, [40], this seems to be the only place where negative numbers have been found in medieval Arabic mathematics.

As for Europe, negative numbers did not begin to appear explicitly until the fifteenth century, when scholars began to study and translate the ancient texts that had been recovered from Islamic and Byzantine sources.[11] In this context, the ideas hidden in the double-entry bookkeeping acquire a particular significance, because the way in which the methodology is designed bypasses the lack of negative numbers, with the placement of numbers on different columns.

This last remark makes us realize that the way in which mathematics is embroiled in the development of double-entry bookkeeping is a very special and subtle one, less apparent and less well-known than the more standard ways mathematics shows up in other contexts to help us understand and model a great variety of phenomena and situations.[12]

Indeed, since antiquity, mathematics has developed both as an investigation of its own objects, dictated by curiosity and the desire to understand them better, as well as to explain and describe a variety of phenomena such as, for example, the complexity of harmony, that can only be understood through a careful use of fractions and ratios [28]. Along these lines, mathematical theories have also been inspired to help solve practical problems, an example being the creation of projective geometry as a method to represent faithfully pictorial scenes with the use of perspective.

More generally, however, mathematics applies to the real world in many different manners. We distinguish here four conceptually distinct ways.

Maybe the first, and most obvious, application of mathematics is exemplified by scientists who use mathematical techniques to solve concrete problems at hand. The rocket scientists who calculate the trajectory of the rocket use basic Newtonian physics and then solve the related differential equations that allow them to identify the quantity of fuel necessary to reach the orbit.

[11] Even then, they were often referred to as *absurd numbers* or *numeri absurdi* by the Italian mathematical school of Cardano.

[12] As we all know, double-entry bookkeeping has had a wide success. It is nevertheless worth mentioning that its method also passed to, and had an influence in, the sphere of literature. An emblematic case is that of Defoe's Robinson Crusoe who, on the desert island, once he had paper and ink, set up a "double-entry" journal of *Evil* and *Good*, "like debtor and creditor."

More complex is the role of mathematics in elaborating models for complex situations whose behavior is not fully understood. For example, a molecular biologist may be interested in understanding the ion transfer process that allows neurons to fire at the right time. The actual system is too complex for a precise description, but the biologist can select a few central variables and describe a model (usually a set of integro-differential equations) whose behaviour is then studied through the instruments of calculus, to offer a prediction of how the system works. This process is subjected to several iterations until one has a model that can be considered to be a good description of the phenomenon at hand.

Mathematics is also playing an increasing role in the analysis of big data, where algorithms are being developed to extract information from large databases, while acknowledging that scientists do not have at their disposal any coherent model for the problem at hand. This approach, whose theoretical underpinnings are being explored by data scientists as well as philosophers, aims to find a mathematical understanding of the process by which algorithms are developed and the reasons why they work (see, e.g., [18, 19]).

But, as it has hopefully become apparent in this paper, there is a less visible, and rarely mentioned fourth contribution that mathematics can offer to the progress of science. One of the qualities that mathematicians develop through their studies of mathematics is the ability to perceive the hidden structure of a phenomenon. In this particular case the phenomenon is a collection of very simple transactions, where a customer buys (and pays for) a product from a supplier. There is ostensibly no need for any real mathematics in this setting. But what Pacioli and his predecessors discovered is the fact that in order to properly describe the phenomenon, one needs to introduce a mechanism, a method, an algorithm if you want, that implicitly embeds negative numbers and the notion of subtraction into the context. This recognition unveils the hidden nature of the phenomenon that was not visible *a priori*.

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