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On the order-up-to policy with intermittent integer demand and logically consistent forecasts ☆

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ABSTRACT

We measure the impact of a first-order integer auto-regressive, INAR(1), demand process on order-up-to (OUT) replenishment policy dynamics. We obtain a unique understanding of the bullwhip behaviour for slow moving integer demand. We forecast the integer demand in two ways; with a *conditional mean* and a *conditional median*. We investigate the impact of the two forecasting methods on the bullwhip effect and inventory variance generated by the OUT replenishment policy. While the conditional mean forecasts result in the tightest inventory control, they result in real-valued orders and inventory levels which is inconsistent with the integer demand. However, the conditional median forecasts are integer-valued and produce logically consistent integer order and inventory levels. The conditional median forecasts minimise the expected absolute forecasting error, but it is not possible to obtain closed form variance expressions. Numerical experiments reveal existing results remain valid with high volume correlated demand. However, for low volume demand, the impact of the integer demand on the bullwhip effect is often significant. Bullwhip with conditional median forecasts can be both lower and higher than with conditional mean forecasts; indeed it can even be higher than a known conditional mean upper bound (that is valid for all lead times under real-valued, first-order auto-regressive, AR(1), demand), depending on the auto-regressive parameter. Numerical experiments confirm the conditional mean inventory variance is a lower bound for the conditional median inventory variance.

1. Introduction

The bullwhip effect refers to the tendency for supply chain replenishment decisions to amplify the variability of the demand when releasing production orders onto the factory shop floor or placing replenishment orders onto suppliers (Lee et al., 2000). A rich literature has been developed in the last 25 years on the bullwhip effect since the seminal work of Lee et al. (1997). Many of the existing studies on the bullwhip effect (Chen et al., 2000a; Dejonckheere et al., 2003) have assumed that real-valued demand exists. That is, demand can take on any number, even fractional values. For some products sold by volume or weight (for example, powders, granules, or liquids) this may be appropriate. However, for other products, only integer (or batch) demand makes sense. For example, you cannot buy half a bicycle or half a laptop. In these situations, demand and replenishment orders must be integers. For some situations, with high volume demand, replenishment calculations can ignore integer effects as rounding becomes negligible. However, low volume demand settings may be more susceptible to integer (rounding) effects. Consider, for example, the daily demand for a single product in a single grocery store in Fig. 1. These products have low volume, integer, occasionally zero, demands.

Continuous-valued time series can be modelled using auto-regressive integrated moving average (ARIMA) type processes (Box et al., 2015). First order ARIMA demand processes are popular assumptions in bullwhip studies due to their mathematical simplicity and their relevance to practical settings. These demand processes are often forecasted with minimum mean squared error (MMSE) forecasting methods (Graves, 1999; Chen et al., 2000a,b; Hosoda and Disney, 2006; Duc et al., 2008). However, standard ARIMA techniques are not suitable for modelling non-negative integer-valued series (Silva and Oliveira, 2004). Therefore, another family of stationary models, integer auto-regressive moving average processes (INARMA), has been proposed, Al-Osh and Alzaid (1988).

Wang and Disney (2016) provide a comprehensive review of bullwhip research categorising it into empirical, experimental, and analytical approaches. They note integer demand processes have not

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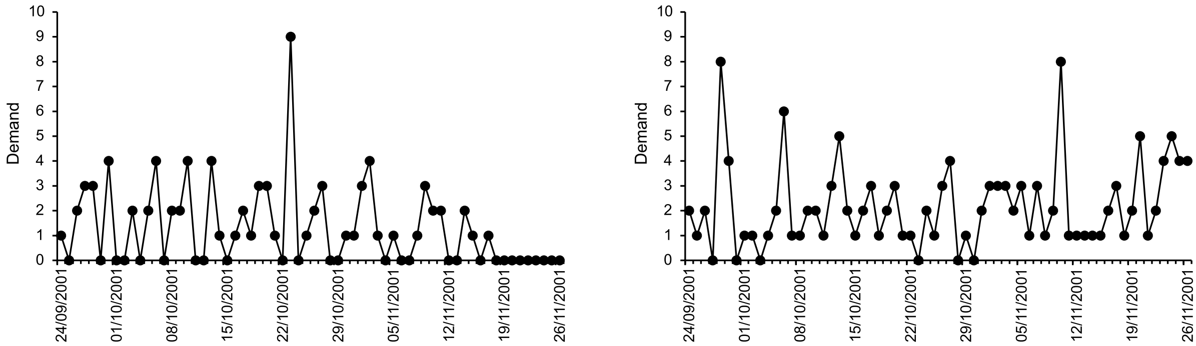

Fig. 1. Low volume, occasionally zero, integer demand for two products from a single store of a UK grocery retailer.

been extensively covered. The bullwhip effect can be measured in different ways, however it is often convenient to quantify it via the ratio of the variance of orders divided by the variance of demand. For normally distributed errors in the demand process and when a linear replenishment system exists, the capacity costs are known to be a linear function of the standard deviation of the orders, Boute et al. (2022). The standard deviation of the orders is closely related to the bullwhip measure. The other side of the bullwhip problem is the variance of the *inventory* levels. In a linear system (where a positive net stock represents inventory holding and a negative net stock represents unmet demand, or a backlog) it is convenient to use the term *net stock amplification* (*NSAmp*) as a moniker for the variance of the net stock divided by the variance of the demand. *NSAmp* is an important measure as it is closely related to the expected, per period, inventory holding and backlog costs when normally distributed error terms are present in the demand process and a linear replenishment system exists. Later in this paper we will explore the economic consequences of Poisson distributed errors and non-normal orders and net stock levels.

Our contribution is to investigate the consequences of integer demand on the performance of the OUT replenishment policy when two different forecasting approaches are used. The first forecasting method is based on the–frequently assumed–conditional mean of future demand (Wang and Disney, 2016). While, this forecasting method creates a MMSE forecast of future demands, it results in non-integer, forecasts, orders, and inventory levels. This is logically inconsistent with the integer demand assumption. The second forecasting method is based on the conditional median forecast which produces–conceptually consistent–integer forecasts, orders, and inventory levels. We find the conditional mean forecasts for INAR(1) demand results in exactly the same variance ratios as the conditional mean forecasts under real-valued AR(1) demand. The conditional median forecasts produce different order and inventory dynamics. Numerical experiments show existing results for real-valued demand can be used with confidence when we have high volume demand integer demand. However, under low volume, possibly intermittent, integer demand there can be a significant difference between the real- and integer-valued *Bullwhip* and *NSAmp* predictions.

The structure of the paper is as follows. Section 3 provides background information on the INAR(1) process and the OUT replenishment policy. Section 4 is devoted to creating conditional mean forecasts. Section 5 is devoted to creating conditional median forecasts. Section 6 characterises the *Bullwhip* and *NSAmp* performance of the OUT policy with the two different forecasts. We also compare the conditional mean and median forecasts to two empirical forecasting methods for intermittent demand. Integer, independent and identically distributed (i.i.d.) demands are considered in more detail in Section 7. Section 8 summarises the paper and concludes.

2. Literature review

Forecasts are often used as an input to planning, policy, or decision-making processes. For instance, in supply chains, a forecast is used to determine inventory replenishment quantities. However, classical time series models such as exponential smoothing state–space (ETS), ARIMA, and multiple linear regression do not produce integer forecasts required by the decision-making process and may poorly describe the nature of count series (Davis et al., 2021). Count time series can be modelled and forecasted using classical marginal count distributions (for example, Poisson, negative binomial) that are often described using generalised linear models (GLM), (Dobson and Barnett, 2018). GLM extends linear regression and can accommodate both continuous or discrete series. In line with the concept of ARIMA for real-valued discrete time series, several models are proposed for count series including integer-valued ARMA (INARMA) (Al-Osh and Alzaid, 1988; Silva and Oliveira, 2004), discrete ARMA (Biswas and Song, 2009) and generalised ARMA (Zheng et al., 2015).

If a time series contains many zeros (often called intermittent time series), classical count distributions such as Poisson and Negative Binomial may not be able to describe the series. Intermittent demand occurs when a product experiences frequent periods of zero demand. Often in these situations, when demand occurs, it is small and sometimes highly variable in size. Various approaches have been proposed to forecast intermittent demand time series. The first method was proposed by Croston (1972), followed by some refinements known as the Syntetos–Boylan Approximation (SBA), (Syntetos and Boylan, 2005) and the Teunter–Syntetos–Babai (TSB) approximation (Teunter et al., 2011). These approaches compensate for the bias present in Croston's method introduced by exponential smoothing, however they do not generate integer forecasts. A different approach to model intermittent demand time series is to use statistical models such as INARMA, zero-inflated and hurdle models (Hu et al., 2011). Lolli et al. (2017) compares the intermittent demand forecasting performance of single hidden layer neural networks for forecasting intermittent demand, one-, three- and five-periods ahead. Two approaches were taken to train the neural networks; a slower back-propagation algorithm resulted in a smaller mean average percentage error (MAPE) than a quicker extreme learning algorithm. The neural networks were also shown to outperform the popular exponential smoothing models for predicting intermittent demand: Croston's Method, (Croston, 1972) and the SBA (Syntetos and Boylan, 2005). Other methods to forecast intermittent demand have been proposed in the literature, including temporal aggregation (Rostami-Tabar et al., 2013, 2014, 2019), temporal hierarchies (Nikolopoulos et al., 2011) and bootstrapping (Hasni et al., 2019).

Goltsos et al. (2022) conduct a literature review of forecasting for inventory management finding the forecasting literature largely assumes the forecast to be an end in itself, without due consideration of the how those forecasts will be used in business decisions and their economic consequences. Conversely, much of the inventory/supply chain literature assume parameters describing the demand process are known beforehand. Petropoulos et al. (2019) study the utility of using exponential smoothing, ARIMA models, the Theta method of Assimakopoulos and Nikolopoulos (2000), and forecasting approaches

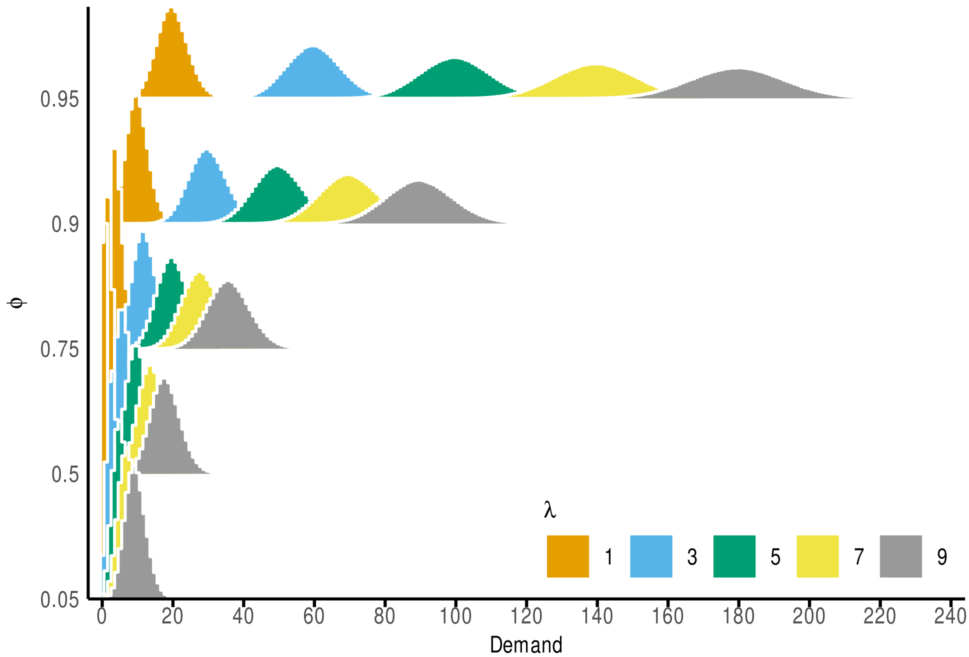

Fig. 2. Probability mass function of the demand of an INAR(1) process for different ϕ and λ.

based on multiple temporal aggregation in the OUT policy with different lead-times. The monthly demand patterns from the M3 competition were used in a study by Petropoulos et al. (2019) to evaluate forecasting performance. They found combinations of forecasts resulted in superior replenishment performance.

Lee et al. (2000) showed the OUT policy under AR(1) demand process with positive auto-correlation ($\phi > 0$) induces bullwhip. Luong (2007) also considered AR(1) demand, deriving important results including an upper bound for the bullwhip effect that is tight for small positive ϕ and a lower and upper bound for the ϕ that produces the maximum bullwhip. Chen and Lee (2016) show the bullwhip effect is an increasing function of the lead time under real-valued integrated moving average, IMA(0,1,1), demand. Gaalman et al. (2022) revealed when, and when not, the bullwhip effect was always increasing in the lead time under ARMA(p,q) demand. They showed that higher order ARMA(p,q) demand has a complex lead time-bullwhip behaviour and that bullwhip is not always an increasing function in the lead time. Hosoda and Disney (2009) showed that mis-specifying ARMA demand did not always increase total supply chain costs when the forecasts were judged on their utility.

Alwan and Weiß (2017) study integer-valued correlated newsvendor models using the INAR(1) process. For a blood supply chain they show how to estimate the demand model from real data. Mohammadipour and Boylan (2012) study the conditional mean k-step ahead forecast of INARMA demand. Babai et al. (2011) investigated a continuous time version of the OUT policy reacting to an i.i.d. demand compound Poisson demand using queuing theory. They find optimal safety stock targets and order-up-to levels in their model with a stochastic lead time. They note there is a significant difference between their results and standard newsvendor approaches to setting the order-up-to levels when the demand is low volume and intermittent. Integer demand and intermittent demand is rarely discussed in the bullwhip literature, Wang and Disney (2016). To the best of our knowledge, there is no prior literature on correlated intermittent demand on the bullwhip problem apart from early conference versions of this paper.

3. Model development

In this section, we first present the INAR(1) demand process and the OUT policy.

3.1. The INAR(1) demand process

We assume the demand follows a first-order integer auto-regressive process, INAR(1), where demand d in period t is given by

$$d_t = \phi \circ d_{t-1} + z_t. \tag{1}$$

Here, d_t is the demand in period t, $0 \leq \phi \leq 1$ is the auto-regressive parameter, and z_t is a sequence of i.i.d. non-negative integer-valued Poisson distributed random variables, with mean λ and finite variance λ (Silva et al., 2009). Notice, in the INAR(1) model, the auto-regressive parameter is valid in the range $0 \leq \phi < 1$ rather than the range $-1 < \phi < 1$ that is relevant in the real-valued AR(1) process. The atomic expression $\phi \circ d_{t-1}$ is the binomial thinning operation,

$$\phi \circ d_{t-1} = \sum_{i=1}^{d_{t-1}} X_i. \tag{2}$$

Here, X_i is a sequence of i.i.d. Bernoulli indicators with parameter ϕ (i.e. with $\mathbb{P}(X_i = 1) = \phi$ for $i = \{1, 2, \dots, d_{t-1}\}$). A natural interpretation of (1) is that d_t is the total number of guests in a hotel at time t, z_t is the number of new guests that arrived today, and $\phi \circ d_{t-1}$ is the number of guests that remained in the hotel from the day before Ristić and Nastić (2012). Silva and Oliveira (2004) provide a number of useful relations and properties of the INAR(1) model. Notably, the relations

$$\mathbb{E}[\phi \circ d_t] = \phi \mathbb{E}[d_t], \tag{3}$$

$$\mathbb{E}[\phi \circ d_t]^2 = \phi^2 \mathbb{E}[d_t^2] + \phi(1-\phi)\mathbb{E}[d_t], \tag{4}$$

and

$$\mathbb{V}[\phi \circ d_t] = \mathbb{E}[\phi \circ d_t]^2 - (\mathbb{E}[\phi \circ d_t])^2 = \phi \mathbb{V}[d_t], \tag{5}$$

are important. Here, $\mathbb{E}[\cdot]$ is the expectation operator and $\mathbb{V}[\cdot]$ is the variance operator.

Lemma 1. *The INAR(1) demand process has mean μ_d, and an auto-covariance with lag j, γ_j, of*

$$\mu_d = \frac{\lambda}{1-\phi} \quad \text{and} \quad \gamma_j = \begin{cases} \frac{\lambda}{1-\phi}, & j = 0, \\ \phi^j \gamma_0, & j \geq 1. \end{cases} \tag{6}$$

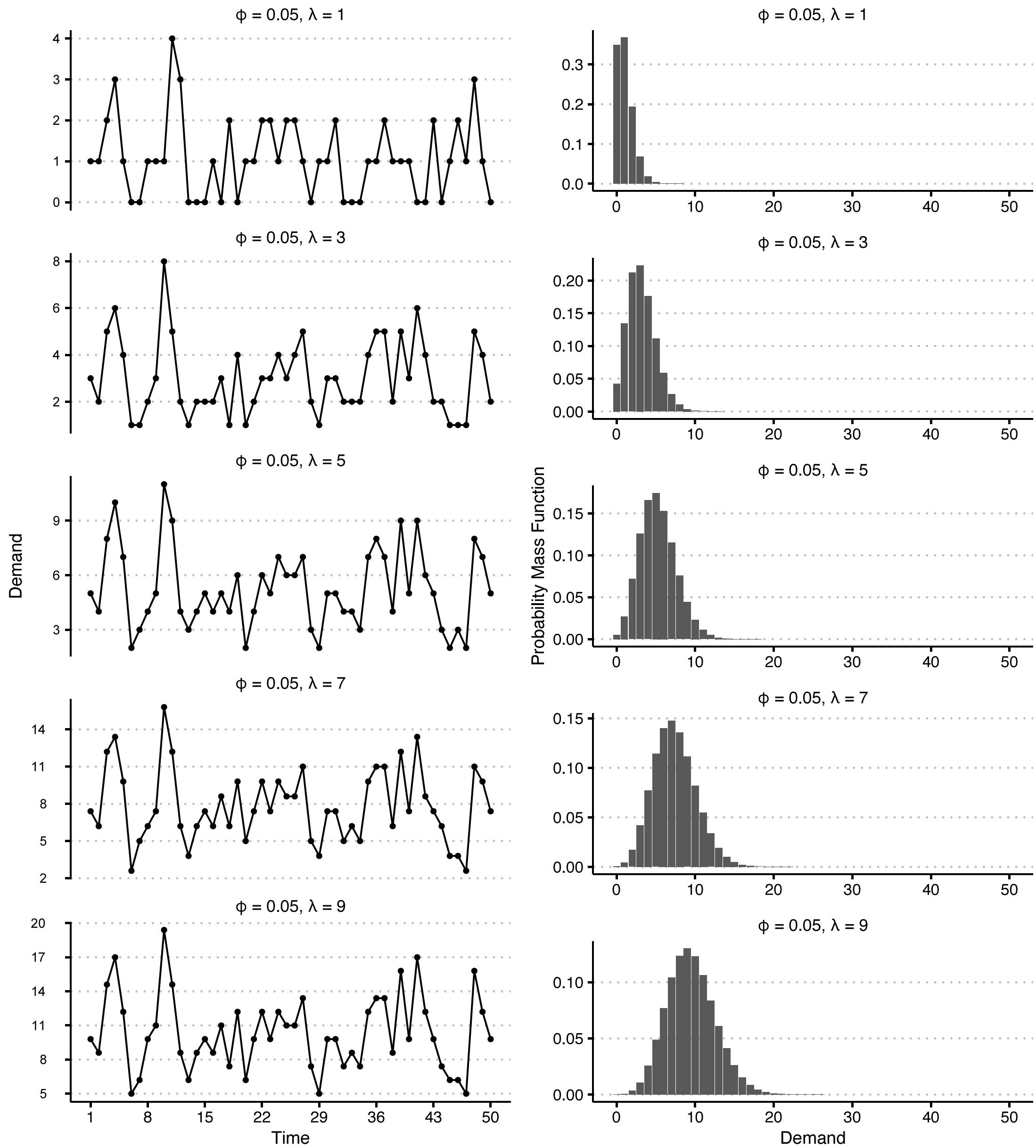

Fig. 3. Demand distribution and time series: Small λ and small ϕ produce intermittent demands. Note, the random seed is identical for all time series (left column). The exact pmf (right column) is produced from (7).

***Proof*.** The relations in (6) are provided by Al-Osh and Alzaid (1988). To make our paper self-contained, an alternative proof is provided in Appendix A. □

The auto-covariance function at lag $j = 0$ gives the demand variance. Silva et al. (2009) show the INAR(1) demand has a Poisson distribution with a shape parameter of $\lambda_d = \lambda/(1-\phi)$:

$$\mathbb{P}[d_t = x] = \frac{\lambda_d^x e^{-\lambda_d}}{x!}. \tag{7}$$

Fig. 2 illustrates the probability mass function (pmf) of the demand for different ϕ and λ. The probability of a zero demand increases as ϕ and λ get smaller, demonstrating the power of the INAR(1) model for representing low volume, intermittent, integer demand processes. Figs. 3 and 4 illustrates example INAR(1) time series with different ϕ and λ in the left column of panels. When ϕ and λ are small, the time series is intermittent; increasing ϕ or λ increases the average demand (see (6)). The right column of panels in Figs. 3 and 4 provides an exact pmf (from (7)) of the corresponding time series. The probability of zero demand, $e^{-\lambda/(1-\phi)} \to 1$, when $\{\phi, \lambda\} \to 0$.

3.2. The periodic order-up-to replenishment policy

We assume the OUT policy operates in discrete time and define and analysis the system via difference equations. At the end of period t, the retailer uses the periodic OUT policy to order q_t items from the manufacturer:

$$q_t = s_t - s_{t-1} + d_t. \tag{8}$$

The order-up-to level, s_t, in time period t is determined by

$$s_t = \hat{d}_{t,L} + i^{\star}, \tag{9}$$

where $\hat{d}_{t,L} = \sum_{i=1}^{L} \hat{d}_{t+i}$ is the forecast of the demand over the lead-time L. $i^{\star}$ is a constant target net stock (safety stock) set to achieve a given

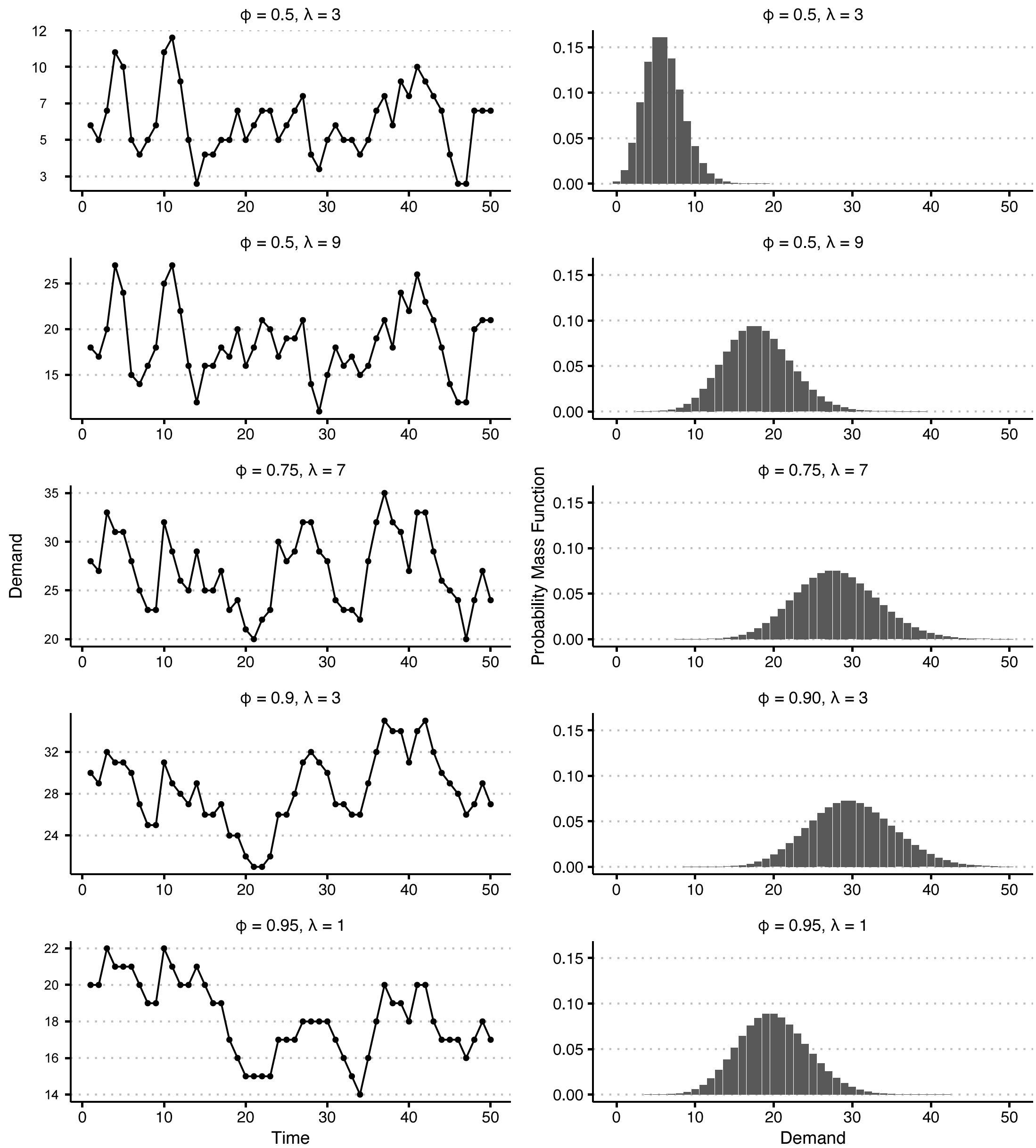

Fig. 4. Demand distribution and time series: Increasing ϕ produces demand processes with trends and higher means. Note, the random seed is identical for all time series (left column). The exact pmf (right column) is produced from (7).

target level of inventory availability. Appendix B shows the commonly assumed, per period inventory holding (h) and backlog (b) costs,

$$C_{i,t} = h[i_t]^+ + b[-i_t]^+, \tag{10}$$

are minimised when $i^\star$ is set equal to the smallest $\bar{i}$ that satisfies

$$\sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n] \geq \frac{b}{b+h}, \tag{11}$$

where $\mathbb{P}[n]$ is the probability that the inventory $i_t = n$ when $\bar{i} = 0$. As the net stock levels are stationary and $i^\star$ is a constant it has no influence on the order and net stock variances (it does however have an influence on the economics of the system, a factor we study later in Section 7). The inventory balance equation is given by

$$i_t = i_{t-1} + q_{t-L} - d_t, \tag{12}$$

where i_t is the inventory level at time t, q_{t-L} is the replenishment order placed in period $t-L$, and the integer lead-time $L \geq 1$ includes a sequence of events delay. Note, an order with zero lead time is not deemed to have been received until the next order quantity is determined. Also note, (12) allows the net stock to become negative. Positive net stock indicates there is inventory on-hand (at the end of the period); negative net stock indicates a backlog has occurred (that is, demand cannot be immediately satisfied from stock and the demand is backlogged to be satisfied in a later period).

Eq. (12) implies the following sequence of events is present in the periodic OUT policy: 1) Orders placed in period $t-L$ are received at the start of the period t. 2) Throughout period t, demand is observed and satisfied from inventory. 3) The order-up-to level, s_t, is updated and 4) inventory levels are observed and replenishment orders placed at the end of the period. This sequence of events is illustrated in Fig. 5. The inventory variance calculation requires the determination of: the variance of demand over the lead time, the variance of forecast, and the co-variance between demand and its forecast over the lead time. In the following sections we develop these expressions which depends upon the forecasting method present. We will consider two forecasting methods:

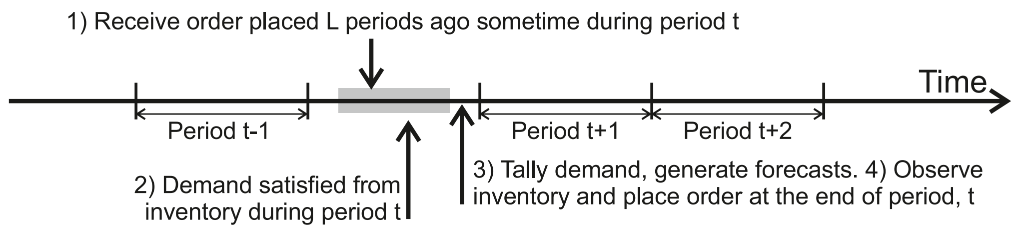

Fig. 5. The sequence of events in the order-up-to policy.

- Real-valued conditional mean forecasts, denoted $\bar{d}$, that originate from an MMSE forecast.
- Integer-valued conditional median forecasts, denoted $\tilde{d}$, that minimise the expected absolute forecast error.

In the next section, we define and explore these two approaches for forecasting INAR(1) demand.

4. Forecasting demand over the lead time with the conditional mean

Here we assume the demand forecast minimises the mean square forecast error over the lead-time and review period, conditional upon d_t. That is, $\hat{d}_{t+k} = \bar{d}_{t+k} = \mathbb{E}[d_{t+k}|d_t]$.

Lemma 2. *Under INAR(1) demand, the conditional mean forecast of demand over the lead time is*

$$\bar{d}_{t,L} = \frac{\phi(1-\phi^L)}{1-\phi}d_t + \frac{L\lambda(1+L)}{2}. \tag{13}$$

Proof. We start by deriving an exact expression of demand k periods ahead, d_{t+k}. We note

$$d_{t+k} = \phi \circ d_{t+k-1} + z_{t+k} \tag{14}$$

and

$$d_{t+k-1} = \phi \circ d_{t+k-2} + z_{t+k-1}. \tag{15}$$

Substituting (15) into (14) recursively (for $d_{t+k-2}, d_{t+k-3}, \dots$), and collecting ϕ terms (3) yields,

$$d_{t+k} = \phi^k \circ d_t + z_{t+1} + z_{t+2} + \cdots + z_{t+k}. \tag{16}$$

Replacing the future values of z_{t+i} in (16) with their expectation, $\mathbb{E}[z_{t+i}] = \lambda$, yields,

$$\begin{aligned}\bar{d}_{t+k} &= \mathbb{E}[\phi^k \circ d_t + z_{t+1} + z_{t+2} + \cdots + z_{t+k}|d_t] \\ &= \phi^k d_t + k\lambda.\end{aligned} \tag{17}$$

As $\bar{d}_{t,L} = \sum_{k=1}^{L} \bar{d}_{t+k}$, closing the $\sum_{k=1}^{L}\left(\phi^k d_t + k\lambda\right)$ sum provides (13), the stated relation for $\bar{d}_{t,L}$. □

Note $\bar{d}_{t,L} \in \mathbb{R}$ is increasing in d_t. The first term is increasing in L as $0 \le \phi \le 1$, and is independent of λ. The second term is also a constant and simplifies out in subsequent analysis.

4.1. Consequences of forecasting INAR(1) demand with the conditional mean

Under i.i.d. integer demand, the demand forecasts based on the conditional mean are constant over time, $\bar{d}_{t,L} = \bar{d}_{t-1,L} = L\mu_d = L\lambda$, and from (8), orders equal demand, $q_t = d_t$. Furthermore, as the variance of the demand is λ when $\phi = 0$, $\mathbb{V}[q_t] = \lambda$, and $\mathbb{V}[i_t] = L\lambda$. As both the demand and the orders are integer processes, so are the inventory levels. However, for correlated INAR(1) demand, the forecast is dynamic ($\bar{d}_{t,L} \neq \bar{d}_{t-1,L}$) and is also real-valued ($\bar{d}_{t,L} \in \mathbb{R}$). This means real-valued orders and inventory levels are present under correlated demand. These non-integer orders and inventory levels are inconsistent with the integer demand assumption. Hence, we now seek integer forecasts, via conditional median forecasts, which lead to integer orders and inventory levels.

5. Forecasting INAR(1) demand over the lead time with the conditional median

Let $\tilde{d}_{t+k|t}$ be an integer forecast of the demand k periods ahead, conditional upon d_t. The median of the pmf provides logically consistent integer forecasts which Freeland and McCabe (2004) claim minimises the expected absolute error conditional upon d_t, $\mathbb{E}[|d_{t+k} - \tilde{d}_{t+k} \parallel d_t]$. The k periods ahead median forecast, $\tilde{d}_{t+k} = X$, where X is the smallest X such that

$$\sum_{x=0}^{X} \mathbb{P}[\tilde{d}_{t+k} = x|d_t] > 1/2. \tag{18}$$

Lemma 3. *The pmf of $\tilde{d}_{t+k}$, given d_t is*

$$\begin{aligned}&\mathbb{P}[\tilde{d}_{t+k} = x|d_t] \\ &= \frac{\phi^{xk}}{x!}\left(\phi^k - 1\right)^{-x}\left(1-\phi^k\right)^{d_t} e^{-\frac{\lambda\left(\phi^k-1\right)}{\phi-1}}\, U\left[-x, d_t - x + 1, \frac{\lambda\phi^{-k}\left(\phi^k-1\right)^2}{\phi-1}\right].\end{aligned} \tag{19}$$

where, x is a non-negative integer and $U[a,b,z]$ is the confluent hypergeometric function, Mathworld (2022).

Proof. Given d_t, Freeland and McCabe (2004) and Silva et al. (2009) provide the following expression for the pmf of $\tilde{d}_{t+k}$,

$$\begin{aligned}&\mathbb{P}[\tilde{d}_{t+k} = x|d_t] \\ &= e^{\frac{\lambda(1-\phi^k)}{\phi-1}} \sum_{i=0}^{M_k} \frac{1}{(x-i)!}\binom{d_t}{i}\left(\phi^k\right)^i\left(1-\phi^k\right)^{d_t-i}\left(\frac{\lambda\left(1-\phi^k\right)}{1-\phi}\right)^{x-i}.\end{aligned} \tag{20}$$

where $M_k = (\tilde{d}_{t+k} \wedge d_t)$ is the minimum of $\tilde{d}_{t+k}$ and d_t. Algebra allows one to close the sum, producing (19). □

Further analytical work with correlated INAR(1) demand cannot be done with this forecasting technique, but (18) and (19) can be easily implemented and studied numerically in software such as Excel, Mathematica, and R. However, some progress can be made for i.i.d. integer demand that we explore in the next section.

5.1. Forecasting i.i.d. integer demand with the conditional median

An i.i.d. integer demand can be modelled within the INAR(1) framework by setting $\phi = 0$. However, the pmf of $\tilde{d}_{t+k}$ when $\phi = 0$ given by (19) is indeterminate and another approach to determine the median forecasts must be taken. When $\phi = 0$, the INAR(1) demand degenerates into an i.i.d. random variable drawn from a Poisson distribution with mean and variance λ those pmf is

$$\mathbb{P}[d_t = x] = \frac{\lambda^x e^{-\lambda}}{x!}. \tag{21}$$

Substituting (21) into (18) produces an implicit expression for the median forecast of the Poisson demand; the median forecasts concur with the smallest X that ensures

$$\sum_{x=0}^{X} \frac{\lambda^x e^{-\lambda}}{x!} = \frac{\Gamma[1+X,\lambda]}{\Gamma[1+X]} > 1/2. \tag{22}$$

Here $\Gamma[\cdot,\cdot]$ is the incomplete Gamma function. Note, when λ is a positive integer, the median of the Poisson demand is equal to its mean λ. When λ is not an integer, (22) implies $\tilde{d}_{t+k} = \lceil \lambda \rceil$ or $\tilde{d}_{t+k} = \lfloor \lambda \rfloor$ depending on the value of λ. Notice, there are no time dependent variables in (22); the median forecast (and s_t) remains constant over time. The consequences of this ensure the orders always equal the demand under i.i.d. Poisson demand (that is, $q_t = d_t$). This is exactly how the order-up-to policy responds to real-valued i.i.d. demands. Further note, $\mathbb{V}[q_t] = \lambda$ and $\mathbb{V}[i_t] = L\lambda$, concurring with the order and inventory variances of the OUT policy with conditional mean forecasts.

6. Analysis of the OUT variances with INAR(1) demand

In this section we compare how the OUT policy responds to the INAR(1) demand with the two different forecasting methods. The analysis will focus on the *Bullwhip* and *NSAmp* ratios as they are often used to assess dynamic supply chain performance, Wang and Disney (2016):

$$Bullwhip = \mathbb{V}[q_t]/\mathbb{V}[d_t] \quad \text{and} \quad NSAmp = \mathbb{V}[i_t]/\mathbb{V}[d_t]. \tag{23}$$

When $Bullwhip > 1$ we say a bullwhip effect is present. The demand variance was given in (6). In Section 6.1, we determine the order and inventory variance when the conditional mean is used to forecast demand in order to investigate the $Bullwhip$ and $NSAmp$ measures. Section 6.2 does the same for the OUT policy with conditional median forecasting.

6.1. Order and inventory variance with conditional mean forecasting

Vassian (1955) shows $\mathbb{V}[i_t]$ is given by the variance of forecast error over lead time:

$$\mathbb{V}[i_t] = \mathbb{V}\left[d_{t,L} - \hat{d}_{t,L}|d_t\right] = \mathbb{V}[d_{t,L}] + \mathbb{V}[\hat{d}_{t,L}|d_t] - 2\mathrm{cov}[d_{t,L}, \hat{d}_{t,L}|d_t]. \tag{24}$$

We need three components: the variance of the demand over the lead time $\mathbb{V}[d_{t,L}]$, the variance of the forecast of the demand over the lead time $\mathbb{V}[\hat{d}_{t,L}]$, and the co-variance between the demand over the lead-time and the forecast of the demand over the lead time $\mathrm{cov}[d_{t,L}, \hat{d}_{t,L}|d_t]$. These are provided in the following three Lemmas.

Lemma 4 (*The Variance of Demand Over Lead Time L*). *The variance of the lead-time demand is*

$$\mathbb{V}\left[d_{t,L}\right] = \frac{\lambda\left(\left(\phi^2-1\right)L - 2\phi\left(\phi^L-1\right)\right)}{(\phi-1)^3}. \tag{25}$$

Proof. The lead time demand is given by

$$d_{t,L} = \sum_{i=1}^{L} d_{t+i} = d_{t+1} + d_{t+2} + \cdots + d_{t+L}. \tag{26}$$

The variance of demand over the lead time is calculated from

$$\begin{aligned}\mathbb{V}\left[d_{t,L}\right] &= \mathbb{V}\left[d_{t+1} + d_{t+2} + \cdots + d_{t+L}\right] \\ &= \sum_{i=1}^{L}(\mathbb{V}\left[d_{t+i}\right] + 2\mathrm{cov}[d_{t+1}, d_{t+2}] + \cdots + 2\mathrm{cov}[d_{t+1}, d_{t+L}] \\ &\quad + 2\mathrm{cov}[d_{t+2}, d_{t+3}] + \cdots \\ &\quad + 2\mathrm{cov}[d_{t+2}, d_{t+L}) + \cdots + 2\mathrm{cov}[d_{t+L-1}, d_{t+L}]).\end{aligned} \tag{27}$$

Using (6) in (27) yields the variance of the demand over the lead time,

$$\begin{aligned}\mathbb{V}\left[d_{t,L}\right] &= L\gamma_0 + 2\left(\gamma_1 + \gamma_2 + \cdots \gamma_{L-1}\right) + 2\left(\gamma_1 + \gamma_2 + \cdots \gamma_{L-2}\right) + \cdots + 2\gamma_1 \\ &= L\gamma_0 + 2\sum_{j=1}^{L-1}\sum_{i=1}^{j}\gamma_0\phi^i.\end{aligned} \tag{28}$$

Using (6) and the telescoping method, the nested sum in (28) becomes the stated relation (25). □

Lemma 5 (*Variance of the Forecast Over the Lead Time*). *The variance of the forecast over the lead time is*

$$\mathbb{V}\left[\hat{d}_{t,L}|d_t\right] = \frac{\lambda}{1-\phi}\left(\frac{\phi(1-\phi^L)}{1-\phi}\right)^2. \tag{29}$$

Proof. First note, with conditional mean forecasting, $\hat{d}_{t,L} = \bar{d}_{t,L}$. From (13), the variance of the forecast over the lead time is

$$\mathbb{V}\left[\hat{d}_{t,L}|d_t\right] = \mathbb{V}\left[d_t\frac{\phi(1-\phi^L)}{1-\phi} + \frac{L\lambda(1+L)}{2}\right]. \tag{30}$$

Note, as $L\lambda(1+L)/2$ is a constant it has no impact on the variance,

$$\mathbb{V}\left[\hat{d}_{t,L}|d_t\right] = \mathbb{V}\left[d_t\frac{\phi(1-\phi^L)}{1-\phi}\right]. \tag{31}$$

As $\mathbb{V}[d_t] = \lambda/(1-\phi)$ from (6), the variance operator provides (29). □

Remark. The variance of the forecast of demand over the lead time (29), is increasing in L as $0 \le \phi \le 1$; it is also increasing in λ.

Lemma 6 (*Covariance of the Demand and Its Forecast Over L*). *The covariance of demand over lead time and its forecast is given by*

$$cov\left[d_{t,L}, \hat{d}_{t,L}|d_t\right] = \gamma_0\left(\frac{\phi\left(1-\phi^L\right)}{1-\phi}\right)^2. \tag{32}$$

Proof. The co-variance between the demand over the lead time and it forecast is calculated as,

$$\mathrm{cov}\left[d_{t,L}, \hat{d}_{t,L}|d_t\right] = \mathrm{cov}\left[d_{t+1} + d_{t+2} + \cdots + d_{t+L}, \hat{d}_{t,L}|d_t\right]. \tag{33}$$

By substituting (13) into (33), using the additive law of covariance (the covariance of a random variable with a sum of random variables is the sum of the covariances with each of the random variables) we have:

$$\begin{aligned}\mathrm{cov}\left[d_{t,L}, \hat{d}_{t,L}|d_t\right] &= \mathrm{cov}\left[d_{t+1} + \cdots + d_{t+L}, d_t\frac{\phi(1-\phi^L)}{1-\phi} + \frac{L\lambda(1+L)}{2}\right] \\ &= \frac{\phi(1-\phi^L)}{1-\phi}\left(\gamma_1 + \gamma_2 + \cdots + \gamma_L\right).\end{aligned} \tag{34}$$

From (6), the sum $\sum_{i=1}^{L}\lambda_i = \lambda_0\frac{\phi(1-\phi^L)}{1-\phi}$, which when substituted into (34) produces (32). □

Finally, we can now provide the inventory variance expression in Proposition 1:

Proposition 1 (*Variance of the Inventory Levels*). *The inventory variance is given by*

$$\mathbb{V}[i_t] = \frac{\lambda}{1-\phi}\left(L + 2\phi\left(\frac{\phi^L + L(1-\phi) - 1}{(\phi-1)^2}\right) - \left(\frac{\phi(1-\phi^L)}{1-\phi}\right)^2\right). \tag{35}$$

Proof. Eq. (24) highlighted the variance of the inventory levels is given by the variance of the forecast error over the lead time. Substituting (6), (25), (29), and (32) into (24) yields (35). □

Remark. Dividing $\mathbb{V}[i_t]$ by the $\mathbb{V}[d_t]$ yields the net stock amplification ratio $NSAmp$:

$$NSAmp = \frac{\mathbb{V}[i_t]}{\mathbb{V}[d_t]} = L + 2\phi\left(\frac{\phi^L + L(1-\phi) - 1}{(\phi-1)^2}\right) - \left(\frac{\phi(1-\phi^L)}{1-\phi}\right)^2. \tag{36}$$

Notice, the Poisson parameter λ has no influence on $NSAmp$. The $NSAmp$ measure is plotted in Fig. 6. Notice, $NSAmp$ under INAR(1)

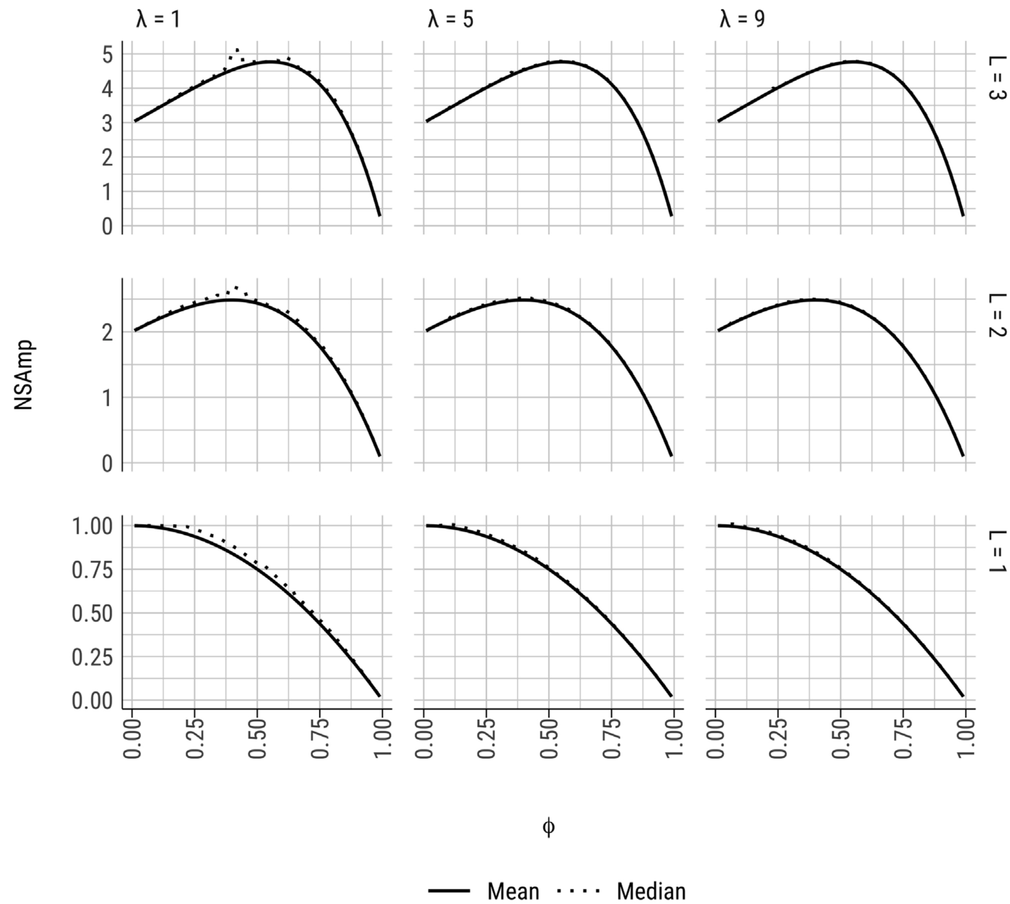

Fig. 6. *NSAmp* maintained by the OUT policy under INAR(1) demand with conditional mean and conditional median forecasting.

demand has exactly the same form as the same as the $NSAmp$ measure under AR(1) demand, Disney and Lambrecht (2008). Also notice, as $\phi \to 1$, the demand variance $\mathbb{V}[d_t] \to \infty$. Together with a finite inventory variance, this means $NSAmp \to 0$ as $\phi \to 1$. Furthermore, $NSAmp \to L$ as $\phi \to 0$.

Proposition 2 (*Variance of the Orders*). *The variance of the replenishment orders is given by*

$$\mathbb{V}\left[q_t\right] = \frac{\lambda}{1-\phi}\left(1 + 2\phi\left(1-\phi^L\right)\left(1 + \frac{\phi\left(1-\phi^L\right)}{1-\phi}\right)\right). \tag{37}$$

Proof. First note, substituting s_t and s_{t-1} from (9) into (8) yields

$$q_t = \hat{d}_{t,L} - \hat{d}_{t-1,L} + d_t. \tag{38}$$

Using $\hat{d}_{t,L}$ and $\hat{d}_{t-1,L}$ from (13) in (38) and collecting together like terms we obtain:

$$\begin{aligned} q_t =& \frac{L\lambda}{1-\phi} + \phi\left(d_t - \frac{\lambda}{1-\phi}\right)\left(\frac{1-\phi^L}{1-\phi}\right) - \frac{L\lambda}{1-\phi} \\ &- \phi\left(d_{t-1} - \frac{\lambda}{1-\phi}\right)\left(\frac{1-\phi^L}{1-\phi}\right) + d_t \\ =&(d_t - d_{t-1})\phi\left(\frac{1-\phi^L}{1-\phi}\right) + d_t. \end{aligned} \tag{39}$$

The variance of order quantity is calculated as follows:

$$\begin{aligned} \mathbb{V}\left[q_t\right] =& \mathbb{V}\left[\frac{\phi d_t(1-\phi^L)}{1-\phi} - \frac{\phi d_{t-1}(1-\phi^L)}{1-\phi} + d_t\right] \\ =& \mathbb{V}\left[\frac{\phi d_t(1-\phi^L)}{1-\phi}\right] + \mathbb{V}\left[\frac{\phi d_{t-1}(1-\phi^L)}{1-\phi}\right] \\ &+ \mathbb{V}\left[d_t\right] + 2\text{cov}\left[\frac{\phi d_t(1-\phi^L)}{1-\phi}, d_t\right] - \\ &2\text{cov}\left[\frac{\phi d_t(1-\phi^L)}{1-\phi}, \frac{\phi d_{t-1}(1-\phi^L)}{1-\phi}\right] - 2\text{cov}\left[\frac{\phi d_{t-1}(1-\phi^L)}{1-\phi}, d_t\right]. \end{aligned} \tag{40}$$

As $\mathbb{V}[d_{t-k}] = \gamma_0$ and $\forall k \geq 1$, $\text{cov}[d_t, d_{t-k}] = \gamma_k$, (40) reduces to (37). □

By substituting (6) and (37) into (38) we obtain the following expression for the bullwhip effect,

$$Bullwhip = 1 + 2\phi\left(1-\phi^L\right)\left(1 + \frac{\phi(1-\phi^L)}{1-\phi}\right). \tag{41}$$

Eq. (41) shows the Poisson distribution λ has no influence on the bullwhip effect. Also note, (41) has the same structural form as the *Bullwhip* generated by the OUT policy with MMSE forecasting under AR(1) demand, Disney and Lambrecht (2008). That the $Bullwhip$ and $NSAmp$ are so similar should not be a surprise; the autocorrelation function (ACF) of the demand process ACF=ϕ^k, given in (6), has exactly the same form as the ACF for the real-valued AR(1) process. Furthermore, as $0 \leq \phi \leq 1$, (41) is increasing in L and always $Bullwhip > 1$ indicating bullwhip always exists under an INAR(1) demand process with conditional mean forecasts, regardless of ϕ and L.

Using the *Bullwhip* measure in (41) we can investigate the existence of the bullwhip effect and the impact of ϕ and L on its magnitude.

Corollary 1. *The bullwhip effect exists (that is,* $Bullwhip > 1$*) regardless of the values of the auto-regressive parameter,* ϕ *and the lead time,* L*.*

Proof. As $0 < \phi < 1$, it is clear from (41) that $Bullwhip > 1$ as the second addend of (41) is always positive. □

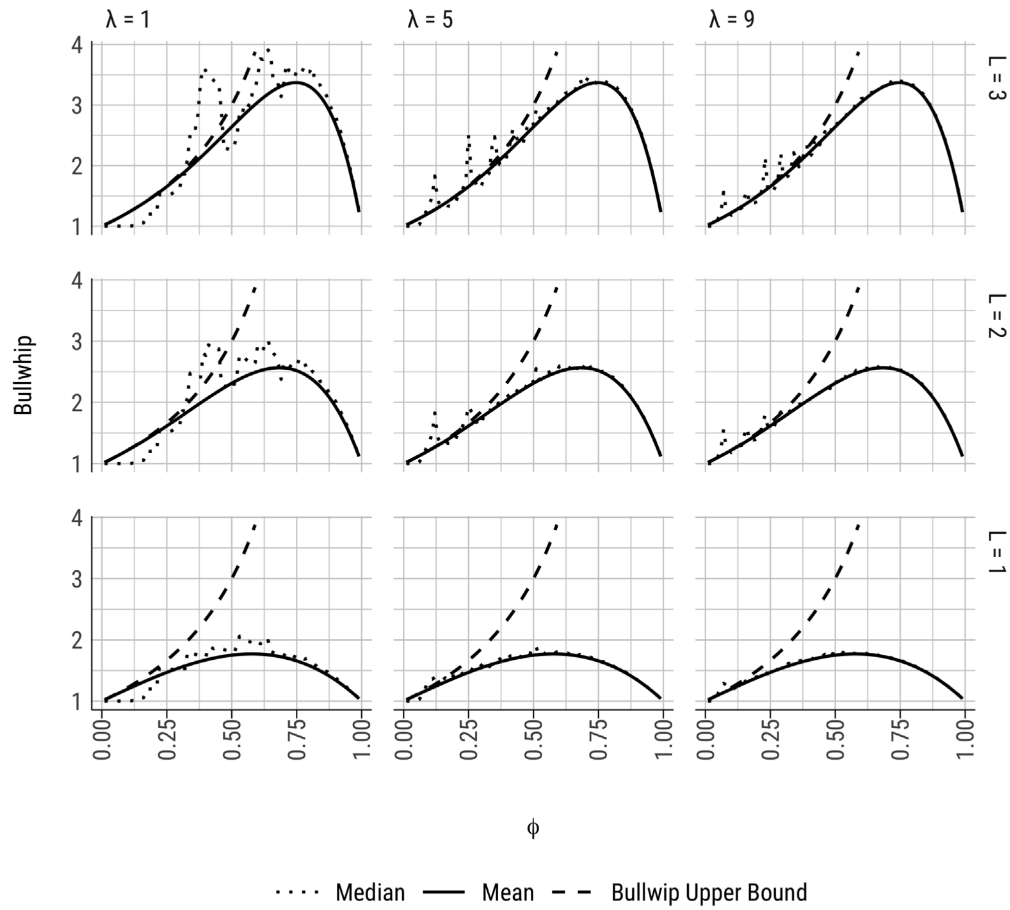

Fig. 7. The bullwhip effect in the OUT policy under INAR(1) demand with conditional mean and conditional median forecasting.

From the above Corollary 1, we see that always $Bullwhip > 1$ highlighting that bullwhip always exists under INAR(1) demand. Fig. 7 illustrates the impact of the demand process parameter and lead time on the bullwhip effect confirming it is always present and increases in the lead time L.

Corollary 2. *An upper bound for the bullwhip effect, valid for all lead times, and is given by Bullwhip* $= (1+\phi)/(1-\phi)$.

Proof. The proof of Corollary 2 is provided by Luong (2007). An alternative proof is given in Appendix C. □

Remark. Luong (2007) also provide an upper and lower bound for the $\phi_{\max}$ that corresponds to the maximum bullwhip, between which a bisection search is efficient for finding the ϕ that produces the maximum bullwhip, $\phi_{\max}$:

$$\left(\frac{1}{L+2}\right)^{1/(L+1)} \leq \phi_{\max} \leq \left(\frac{L}{2(L+1)}\right)^{1/(L+1)}. \tag{42}$$

Note, $\phi_{\max}$ tends to unity when $L \to \infty$.

Fig. 7 illustrates the impact of the demand process parameter and lead time on the bullwhip effect confirming it is always present and increases in the lead time L. The bullwhip upper bound of Luong (2007), valid for all lead times and real-valued conditional mean forecasts, is also plotted in Fig. 7. The upper bound is tight when ϕ is small.

6.2. Order and inventory variance with conditional median forecasting

As no further analytical work on the conditional median forecasts is possible, we resort to a numerical investigation of the orders and inventory variance. We built a simulation in the R software that ran on a HAWK High-Performance Computing Cluster. To generate the demands in each period t, we first generated random Poisson distributed error terms, z_t and constructed demand series using (1). Then, k-period ahead conditional median forecasts were generated using (18); these were summed to create integer forecasts of demand over the lead time. Following that, inventory levels and orders are calculated using (8) and (12), respectively. Finally, the *Bullwhip* and *NSAmp* ratios are calculated using (23). The parameter values used in the simulation are $\lambda = \{1,5,9\}$, $L = \{1,2,3\}$ and $0 < \phi < 1$ in steps of 0.01. For all combinations of λ and ϕ, a time series of one million observations was generated. We have plotted the results of this exercise in Fig. 6 for the *NSAmp* ratio and Fig. 7 for the *Bullwhip* ratio.

Fig. 6 revealed the $NSAmp$ expressions under conditional mean forecasting is a good predictor of the $NSAmp$ measure when conditional median forecasting is present, especially with high volume integer demand (i.e. when $\lambda \gg 1$). As the conditional mean forecast minimises the mean squared error forecasts over the lead-time and review period, the conditional mean $NSAmp$ represents a *lower bound* of conditional median $NSAmp$; lower demand volumes (small λ and ϕ near 0.5) increase the conditional median $NSAmp$.

The conditional median $Bullwhip$ curves in Fig. 7 were somewhat more complex. When ϕ is near zero, the $Bullwhip$ curves remain unchanged from the i.i.d. case, close to unity, possibly as a result of the forecast remaining unchanged from the i.i.d. case. In the region $\phi \approx 0.1$ to $\phi \approx 0.8$ there is a period of seemingly erratic $Bullwhip$ which may be either above or below the conditional mean $Bullwhip$ curve; it may also be above the AR(1) bullwhip upper bound from Luong (2007). When ϕ is close to unity, above $\phi \approx 0.8$, the conditional median $Bullwhip$ closely resembles the conditional mean $Bullwhip$ curve. We do not

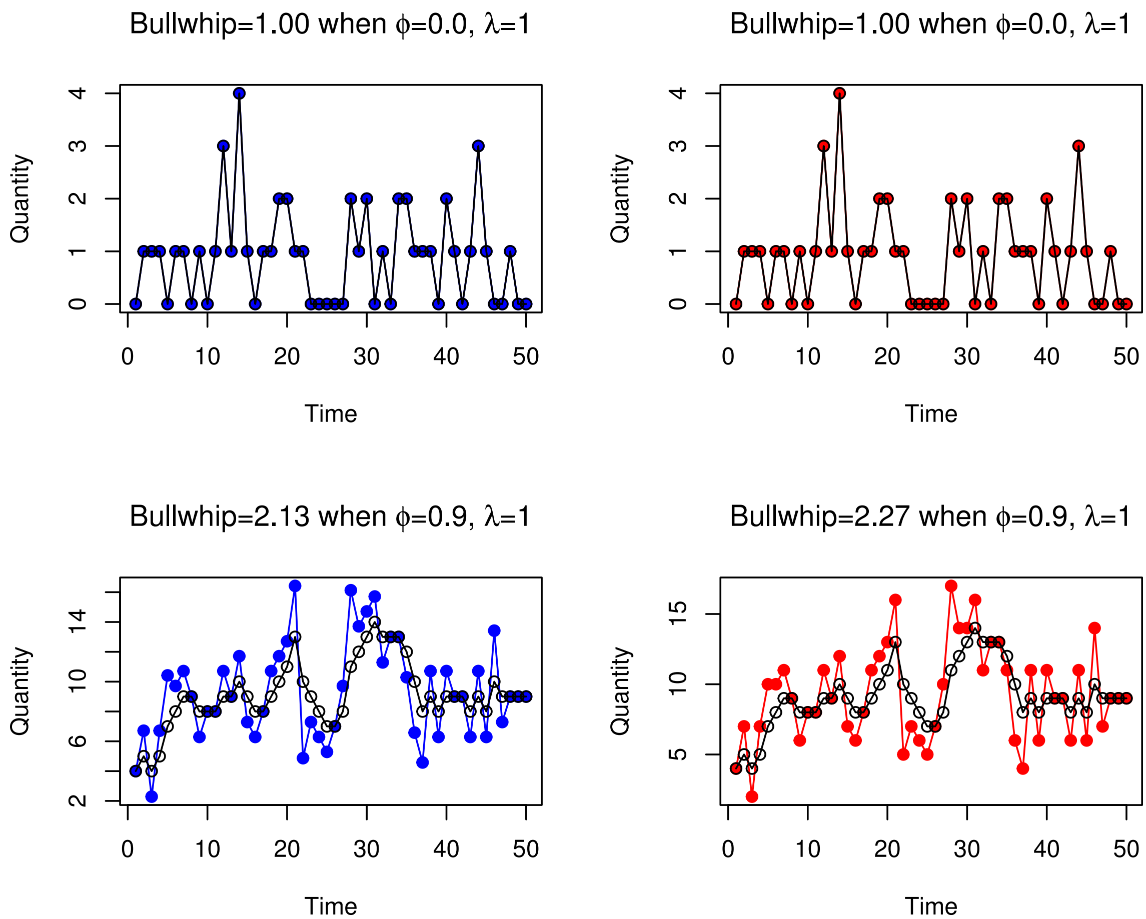

Fig. 8. Time series of demand and orders. Conditional mean forecasts result in real orders (left column); conditional median forecasts result in integer orders (right column). The top row contains i.i.d. demand; the bottom row contains correlated demand. Key: Black — demand, blue — orders with conditional mean forecasts, red — orders with condition median forecasts. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

know whether a conditional median ever produces $Bullwhip < 1$ (but we have not found a convincing numerical example when $Bullwhip < 1$).

Fig. 8 provides example time series of the demand and orders for both the conditional mean forecasts (left column of panels) and the conditional median forecasts (right column of panels). The Poisson noise driving the demand is the same in all panels. The top row of panels correspond to i.i.d. integer Poisson demand and both the conditional mean (top left panel) and conditional median (top right panel) forecasts produce integer orders that equal demand. The bottom row of panels illustrate correlated INAR(1) demand. The conditional mean forecasts produce real-valued orders (bottom left panel). The conditional median forecasts produce integer-valued orders, panel d). All panels concur with our previous discussions.

6.3. NSAmp and Bullwhip comparison to Croston and SBA forecasts

In this section we will compare the OUT performance of conditional mean and median forecasts with two empirical forecasting methods: Croston's method (Croston, 1972) and the SBA, (Syntetos and Boylan, 2005). Croston's method and the SBA are compound forecasts based on exponential smoothing forecasts of levels and intervals. As the forecasts are only updated when a demand occurs these are essentially non-linear in nature and are thus difficult to analyse analytically. However, we have conducted a simulation study to understand their *NSAmp* and *Bullwhip* performance. Table 1 details the *NSAmp* maintained by the OUT policy with the four different forecasts; Table 2 details the *Bullwhip* generated. The simulation used to create each data point was one million time periods in duration. We have compared the real-valued Croston, SBA and conditional mean forecasts (which produce real-valued orders and inventory levels) and the integer-valued conditional median forecasts (which produce integer-valued orders and inventory levels). The simulation results confirm the conditional mean forecasts result in the smallest *NSAmp* measures. This result is to be expected as the conditional mean forecasts produce MMSE forecasts of demand over the lead time—a condition that leads the OUT policy to become the inventory optimal policy (as the inventory variance is equal to the variance of the sum of the forecasts errors over the lead-time and review period, Vassian (1955)). The conditional median forecasts come a close second in terms of *NSAmp* performance, and also closely track the *Bullwhip* performance of the conditional mean forecasts. The *Bullwhip* performance of Croston's method the SBA forecasts is superior to the conditional mean and median forecasts, with the SBA forecasts generally besting Croston's method. However, this *Bullwhip* performance comes at the cost of significantly increased inventory variance.

7. Economic performance of the OUT policy under i.i.d. integer demands

When $\phi = 0$, we have i.i.d. Poisson distributed demands, and all future demand forecasts are a constant. In this case it is easy to verify the inventory pmf is a simple reflection and translation of the pmf of the sum of demand over the lead-time:

$$\mathbb{P}[i_t = x] = \frac{(L\lambda)^{L\lambda+\bar{i}-x}e^{-L\lambda}}{(L\lambda+\bar{i}-x)!}. \tag{43}$$

Here, $\bar{i}$ is the target net stock. We can determine the mean and variance of the inventory levels directly from the mean and variance of the sum

Table 1
NSAmp generated by the OUT policy under AR(1) with different forecasting methods.

Forecasting method	$\phi = 0$	$\phi = 0.1$	$\phi = 0.2$	$\phi = 0.3$	$\phi = 0.4$	$\phi = 0.5$	$\phi = 0.6$	$\phi = 0.7$	$\phi = 0.8$	$\phi = 0.9$
Conditional mean	**1**	**0.990**	**0.960**	**0.911**	**0.841**	**0.751**	**0.642**	**0.512**	**0.362**	**0.192**
Conditional median	**1**	1.000	0.992	0.954	0.883	0.782	0.678	0.536	0.380	0.199
Croston's method	1.071	1.074	1.183	1.059	1.031	0.982	0.905	0.786	0.630	0.400
SBA	1.057	1.059	1.054	1.041	1.013	0.964	0.889	0.774	0.619	0.599

Note: Always $\lambda = 1$, $L = 1$, and $\alpha = \beta = 0.2$ in Croston's and SBA forecasts. Bold font indicates minimum *NSAmp* for a given demand process.

Table 2
Bullwhip generated by the OUT policy under AR(1) with different forecasting methods.

Forecasting method	$\phi = 0$	$\phi = 0.1$	$\phi = 0.2$	$\phi = 0.3$	$\phi = 0.4$	$\phi = 0.5$	$\phi = 0.6$	$\phi = 0.7$	$\phi = 0.8$	$\phi = 0.9$
Conditional mean	**1**	1.198	1.384	1.547	1.674	1.751	1.771	1.718	1.580	1.345
Conditional median	**1**	**1.002**	1.203	1.531	1.717	1.794	1.962	1.737	1.664	1.382
Croston's method	1.127	1.154	1.183	1.214	1.246	1.273	1.290	1.285	1.246	1.160
SBA	1.112	1.136	**1.162**	**1.190**	**1.218**	**1.242**	**1.258**	**1.253**	**1.219**	**1.142**

Note: Always $\lambda = 1$, $L = 1$, and $\alpha = \beta = 0.2$ in Croston's and SBA forecasts. Bold font indicates minimum *Bullwhip* for a given demand process.

of L Poisson distributed random variables:

$$\mathbb{E}[i_t] = \sum_{x=-\infty}^{L\lambda+\bar{i}} \frac{(L\lambda)^{L\lambda+\bar{i}-x} e^{-L\lambda}}{(L\lambda+\bar{i}-x)!} = \bar{i}, \text{ and}$$

$$\mathbb{V}[i_t] = \sum_{x=-\infty}^{L\lambda+\bar{i}} \frac{(L\lambda)^{L\lambda+\bar{i}-x} e^{-L\lambda}}{(L\lambda+\bar{i}-x)!} (x - \mathbb{E}[i_t])^2 = L\lambda. \tag{44}$$

The expected per period inventory holding and backlog costs (see (B.1)) can be then obtained from (43) as follows:

$$\begin{aligned}\mathbb{E}[C_t^i] = &\sum_{x=-\infty}^{0} \frac{b(-x)\left(e^{L\lambda}(L\lambda)^{\bar{i}+L\lambda-x}\right)}{(\bar{i}+L\lambda-x)!} \\ &+ \sum_{x=1}^{L\lambda+\bar{i}-1} \frac{hx\left(e^{-L\lambda}(L\lambda)^{\bar{i}+L\lambda-x}\right)}{(\bar{i}+L\lambda-x)!} + he^{-L\lambda}(\bar{i}r + L\lambda) \\ = &\frac{(b+h)e^{-L\lambda}\left((L\lambda)^{\bar{i}+L\lambda+1} + \bar{i}e^{L\lambda}\Gamma[\bar{i}+L\lambda+1, L\lambda]\right)}{\Gamma[\bar{i}+L\lambda+1]} - b\bar{i}.\end{aligned} \tag{45}$$

Appendix B shows the optimal $i^\star$, the $\bar{i}$ that minimises the expected inventory holding and backlog cost (45) can be found as the smallest $\bar{i}$ that ensures

$$\sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n] \geq \frac{b}{b+h} \tag{46}$$

where $\mathbb{P}[n] = \mathbb{P}[i_t = n | \bar{i} = 0] = \frac{(L\lambda)^{L\lambda-n} e^{-L\lambda}}{(L\lambda-n)!}$. In a like manner, due to the constant forecasts under i.i.d. INAR(1) demand, (8) shows the order pmf is equal to the demand pmf and $Bullwhip = 1$. The per period production cost C_t^q, with a nominal hours unit cost of u within a nominal capacity of K and flexible per unit overtime cost of um, where m is the overtime multiplier, Boute et al. (2022), can be formulated as:

$$C_t^q = uK + um[q_t - K]^+. \tag{47}$$

Using (21), the expected per period capacity costs are given by

$$\mathbb{E}[C_t^q] = uK + \sum_{x=k}^{\infty} \frac{mu(x-K)e^{-\lambda}\lambda^x}{x!} \tag{48}$$

$$= u\left(\frac{e^{-\lambda}\lambda m\left(\lambda^k - e^{\lambda}\Gamma[k+1,\lambda]\right)}{\Gamma[k+1]} + \frac{m\Gamma[k+1,\lambda]}{\Gamma[k]} + K + m(\lambda - K)\right). \tag{49}$$

Appendix D shows $K^\star$, the minimiser of (49), is the smallest K such that

$$\sum_{n=0}^{K} \mathbb{P}[n] \geq \frac{m-1}{m} \tag{50}$$

where $\mathbb{P}[n] = \mathbb{P}[q_t = n] = \mathbb{P}[d_t = n] = \lambda^n e^{-\lambda}/(n!)$. Note, $\mathbb{P}[q_t = n] = \mathbb{P}[d_t = n]$ comes from the fact $q_t = d_t$ under i.i.d. demand and MMSE forecasts. The last relation comes from (21).

Fig. 9 (panel b) shows the $K^\star$ is both increasing in λ and m. Fig. 9 also illustrates the minimised inventory holding and backlog costs in panel c and the minimised capacity costs in panel d. Total costs are equal to the sum of these two costs. The minimised capacity (inventory holding and backlog) costs are independent of $K^\star$ ($i^\star$) respectively. Minimising λ or L reduces the inventory and backlog related costs; minimising m or L reduces the capacity costs.

8. Conclusions

We have examined the *Bullwhip* and *NSAmp* behaviour of the OUT policy under an integer-valued INAR(1) demand series with two different forecasting methodologies; one based on the conditional mean, the other forecast on the conditional median. The variance ratios derived for conditional mean forecasts under integer demand were found to be the same as those for the corresponding real-valued demand. This should be expected as the results for the real demand variance are *distribution free*. We conjecture the variance ratios maintained by OUT policy under INARMA demand with conditional mean forecasts will remain the same (as those already obtained in the literature) for ARMA demand. The consequences of the conditional median forecasts of INARMA demand remains unknown; to the best of our knowledge the conditional median forecasts have not yet been found.

The auto-regressive parameter ϕ and the lead time affects the variance ratios under INAR(1) demand. The Poisson distribution parameter λ has no impact on the variance ratios. However, the real order and inventory levels produced are inconsistent with the integer demand. The bullwhip effect always exists regardless of the auto-regressive parameter and the lead time. There exists a lower bound, $Bullwhip > 1$, and an upper bound which is a function of the auto-regressive parameter, ϕ. For a given value of ϕ, the upper bound represents the maximum value of the bullwhip effect regardless of the lead time L. The upper bound is tight when ϕ is small. The *NSAmp* maintained by the conditional mean forecasts of the INAR(1) demand behaves exactly as the *NSAmp* measure for AR(1) demand. For i.i.d. demand $NSAmp = L$; as $\phi \to 1$, $NSAmp \to 0$.

When conditional median forecasts are used, *Bullwhip* is somewhat more erratic and can deviate significantly (positively and negatively) from both the conditional mean *Bullwhip* and the upper bound. With large λ (or when both λ and ϕ are large), the demand distribution becomes more *normal* and the existing bullwhip knowledge based on real-valued demand becomes more relevant. The $Bullwhip$ and $NSAmp$ expressions can be used with confidence in high volume settings. However, low values of ϕ and λ lead to intermittent demand series that contain a high proportion of zeros and to time series where the

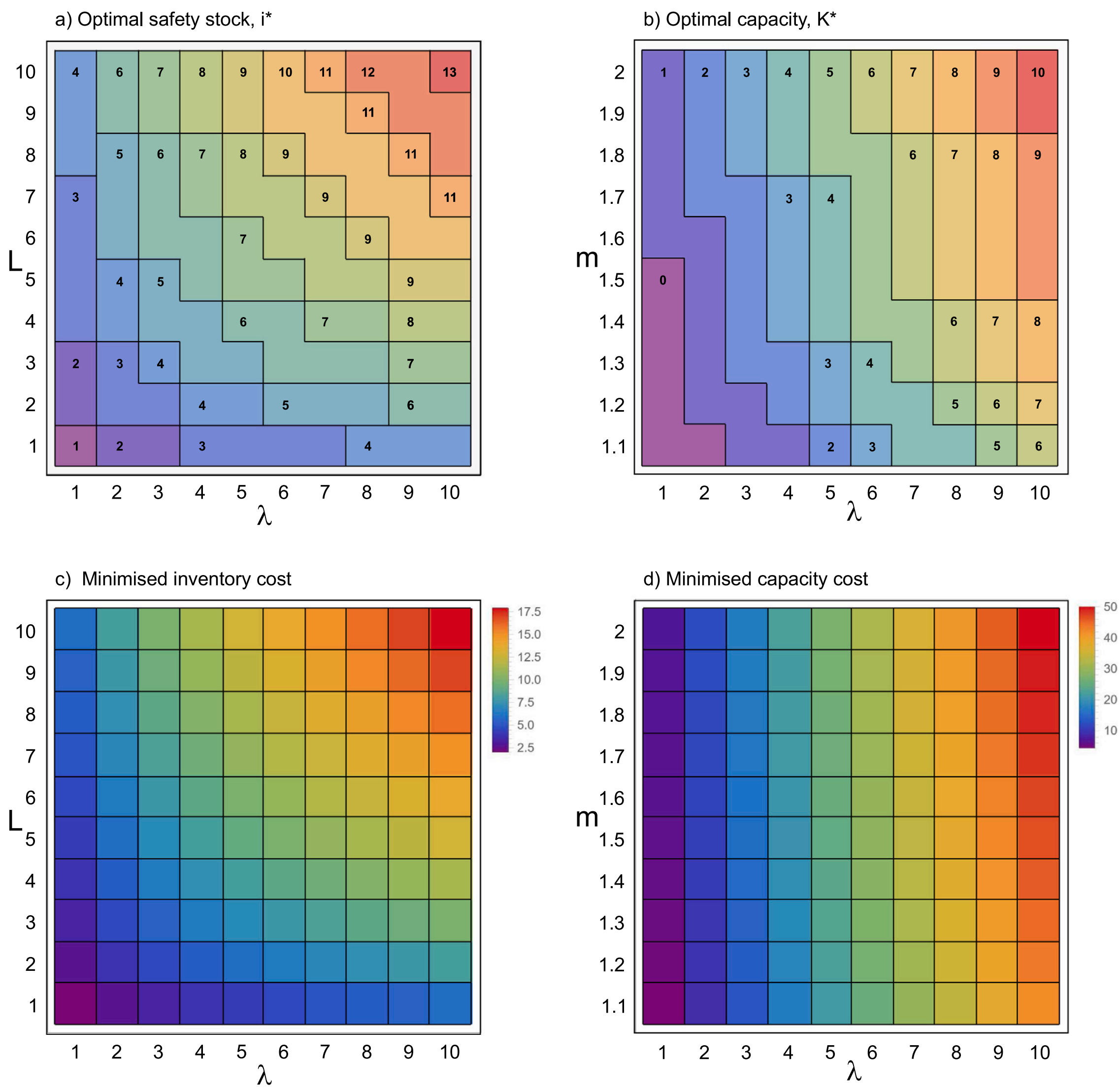

Fig. 9. Optimal OUT policy settings. Panel (a) The safety stock, $i^{\star}$, required to minimise the inventory costs when $h = 1, b = 9$ for different demand λ and lead times L. Panel (b) The optimal capacity, $K^{\star}$, required to minimise the capacity costs when $u = 4$ for different demand λ and over-time multiplier m. Panel (c) The minimised inventory cost when $i^{\star}$ is present. Panel (d) The minimised capacity cost when $K^{\star}$ is present.

integer effects become more significant. Small ϕ and small λ lead the demand distribution (which is always non-negative) to become skewed to the right. In the extreme case, when $\phi = 0$, the demand is Poisson distributed. In these cases the existing knowledge will be less valuable. Bullwhip seems to always exists for INAR(1) demand; the inventory variance (and the *NSAmp* measure) for conditional mean forecasts is a lower bound for the inventory variance under conditional median forecasts. When the demand is i.i.d., a constant forecast is produced by both forecasting methods that means $q_t = d_t$. For this case we were able to extend our variance ratio analysis to include an economic study of the system which searched for the target safety stock to minimise the expected inventory holding and backlog costs and the target capacity level to minimise the regular labour and overtime costs. We found that the optimal capacity requirements, $K^{\star}$, increases in both λ and m; Capacity costs are reduced by minimising λ and m. Inventory costs are minimised by decreasing L and λ.

CRediT authorship contribution statement

Bahman Rostami-Tabar: Conceptualization, Programming, Formal analysis, Model development, Writing – original draft. **Stephen M. Disney:** Conceptualization, Programming, Formal analysis, Model development, Writing – original draft.

Data availability

No data was used for the research described in the article.

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Appendix A. Auto-covariance function of the demand

We can show that the mean of d_t is given by,

$$\mathbb{E}\left[d_t\right] = \mathbb{E}\left[\phi \circ d_{t-1} + z_t\right] = \mathbb{E}\left[\phi \circ d_{t-1}\right] + \mathbb{E}\left[z_t\right] = \phi \mathbb{E}\left[d_{t-1}\right] + \lambda. \tag{A.1}$$

INAR(1) processes are stationary, $\mathbb{E}[d_t] = \mathbb{E}[d_{t-1}]$ and (A.1) reduces to $\mathbb{E}\left[d_t\right] = \lambda/(1-\phi)$. As the variance of a sum is the sum of the variances of the addends and twice the covariance between the addends, the variance of demand at period t is

$$\mathbb{V}\left[d_t\right] = \mathbb{V}\left[\phi \circ d_{t-1} + z_t\right) = \mathbb{V}\left[\phi \circ d_{t-1}\right] + \mathbb{V}\left[z_t\right] + 2\mathrm{cov}\left[d_{t-1}, z_t\right]. \quad (A.2)$$

Algebra then leads to

$$\begin{aligned}\mathbb{V}\left[d_t\right] &= \mathbb{V}\left[\phi \circ d_{t-1}\right] + \lambda && (\text{As } \mathrm{cov}[d_{t-1}, z_t] = 0)\\ &= \phi\mathbb{V}\left[d_{t-1}\right] + \lambda && (\mathbb{V}[\phi \circ d_{t-1}] = \phi\mathbb{V}[d_{t-1}])\\ &= \phi\mathbb{V}\left[d_t\right] + \lambda && (\text{As demand is stationary, } \mathbb{V}\left[d_{t-1}\right] = \mathbb{V}\left[d_t\right])\\ &= \lambda/(1-\phi). && (\text{After collecting together terms})\end{aligned}$$

By recursive substitutions of d_{t-k} for $k \geq 1$, (1) can be written as

$$d_t = \phi^k \circ d_{t-k} + \sum_{j=0}^{k-1} \phi^j z_{t-j}. \quad (A.3)$$

The auto-covariance of lag $k \geq 1$ can then be calculated as

$$\begin{aligned}\gamma_k = \mathrm{cov}\left[d_t, d_{t-k}\right] &= \mathrm{cov}\left[\phi^k \circ d_{t-k} + \sum_{j=0}^{k-1} \phi^j \circ z_{t-j}, d_{t-k}\right]\\ &= \phi^k \mathrm{cov}\left[d_{t-k}, d_{t-k}\right] + \mathrm{cov}\left[\sum_{j=0}^{k-1} \phi^j \circ z_{t-j}, d_{t-k}\right]. \quad (A.4)\end{aligned}$$

As the correlation between d_{t-k} and z_{t-j} for all $j \leq k-1$ is equal to zero, the covariance term $\mathrm{cov}\left[d_{t-k}, \sum_{j=0}^{k-1} \phi^j \circ z_{t-j}\right] = 0$. The auto-covariance function of lag $k \geq 1$ for INAR(1) demand is

$$\gamma_k = \phi^k \mathrm{cov}\left[d_{t-k}, d_{t-k}\right] = \phi^k \gamma_0. \quad (A.5)$$

Appendix B. Determining the target net stock in the OUT policy under discrete demand

Hill (2011) outlined a procedure to find the optimal ordering quantity for the discrete newsvendor problem; we adopt his approach for the target net stock levels in the OUT policy. Let $\mathbb{P}[n]$ be the probability that the net stock level $i_t = n$ when the target net stock $\bar{i}$ is zero (i.e. $\mathbb{P}[n] = \mathbb{P}[i_t = n | \bar{i} = 0]$). When per unit, per period inventory holding (of h) and backlog (of b) costs are present the following costs are incurred in each time period,

$$C_{i,t} = h[i_t]^+ + b[-i_t]^+. \quad (B.1)$$

Taking expectations, the long run average costs per period, with an arbitrary target net stock can be expressed

$$\mathbb{E}[C_{\bar{i}}] = h \sum_{n=-\bar{i}}^{\infty} \mathbb{P}[n](n+\bar{i}) - b \sum_{n=-\infty}^{-\bar{i}-1} \mathbb{P}[n](n+\bar{i}). \quad (B.2)$$

To facilitate taking the inverse of the inventory distribution later we make the $\bar{i}$ in the index of the first sum positive and recast the expected inventory costs as

$$\mathbb{E}[C_{\bar{i}}] = h \sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n](\bar{i}-n) - b \sum_{n=\bar{i}+1}^{\infty} \mathbb{P}[-n](\bar{i}-n). \quad (B.3)$$

Notice the backlog costs are non-increasing in $\bar{i}$ and the holding costs are non-decreasing in $\bar{i}$. This means there is only one minimum (or at most two consecutive minimums) in the inventory costs. When the expected costs with a target net stock of $\bar{i}$ units is approximately the same as with a target net stock of $\bar{i}+1$ units we will be near a cost minimum. Setting $\mathbb{E}[C_{\bar{i}}] = \mathbb{E}[C_{\bar{i}+1}]$ and noting that $\sum_{n=-\infty}^{\bar{i}+1} \mathbb{P}[-n](\bar{i}+1-n) = \sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n](\bar{i}+1-n)$ and $\sum_{n=\bar{i}+2}^{\infty} \mathbb{P}[-n](\bar{i}+1-n) = \sum_{n=\bar{i}+1}^{\infty} \mathbb{P}[-n](\bar{i}+1-n)$ leads to

$$\begin{aligned}&h \sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n](\bar{i}-n) - b \sum_{n=\bar{i}+1}^{\infty} \mathbb{P}[-n](\bar{i}-n)\\ &= h \sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n](\bar{i}+1-n) - b \sum_{n=\bar{i}+1}^{\infty} \mathbb{P}[-n](\bar{i}+1-n). \quad (B.4)\end{aligned}$$

Collecting together both sides yields

$$h \sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n]\left((\bar{i}+1-n)-(\bar{i}-n)\right) - b \sum_{n=\bar{i}+1}^{\infty} \mathbb{P}[-n]\left((\bar{i}+1-n)-(\bar{i}-n)\right) = 0 \quad (B.5)$$

Simplifying further provides

$$h \sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n] - b \sum_{n=\bar{i}+1}^{\infty} \mathbb{P}[-n] = 0. \quad (B.6)$$

Defining $\sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n] = F[\bar{i}]$ as the cumulative distribution function of the net stock levels (with $\bar{i} = 0$) and simplifying then leads to

$$\begin{aligned}&hF[\bar{i}] - b(1 - F[\bar{i}]) = 0 && (B.7)\\ &F[\bar{i}](h-b) - b = 0 && (B.8)\\ &F[\bar{i}] = \frac{b}{b+h} && (B.9)\\ &i^\star = F^{-1}\left[\frac{b}{b+h}\right]. && (B.10)\end{aligned}$$

Thus, the optimal target net stock $i^\star$ can then be found as the smallest $\bar{i}$ that ensures

$$\sum_{n=-\infty}^{\bar{i}} \mathbb{P}[-n] \geq \frac{b}{b+h}. \quad (B.11)$$

Note, there are no assumptions made about the distribution, or autocorrelation of the discrete net stock levels, only that it is stationary (that is, the net stock level has a finite and constant variance and mean). We find it interesting that the *critical fractile* (of the inventory distribution) for OUT policy's target net stock has the same form as the *critical fractile* (of the demand distribution) for the newsvendor problem.

Appendix C. Bounds on the bullwhip measure

The bullwhip ratio was given in (41). Using the limit function to avoid the divide by zero issues in the final addend of (41), $0 \leq \phi \leq 1$ and $L \geq 1$, implies

$$0 \leq \phi^L \leq 1. \quad (C.1)$$

Multiplying (C.1) by minus one (-1) and adding plus one ($+1$) to all sides, we have

$$0 \leq 1 - \phi^L \leq 1. \quad (C.2)$$

Given $\phi > 0$ and $1 - \phi > 0$ are positive, multiplying by 2ϕ yields

$$0 \leq 2\phi(1-\phi^L) \leq 2\phi. \quad (C.3)$$

Squaring (C.3) and dividing by 2 leads to

$$0 \leq 2 \times 2\left(\phi(1-\phi^L)\right)^2 \leq 2 \times 2\phi^2 = 0 \leq 2\left(\phi(1-\phi^L)\right)^2 \leq 2\phi^2. \quad (C.4)$$

Knowing that $\frac{1}{1-\phi} > 1$, multiplying by $\frac{1}{1-\phi}$ yields

$$0 \leq \frac{2\left(\phi(1-\phi^L)\right)^2}{1-\phi} \leq \frac{2\phi^2}{1-\phi}. \quad (C.5)$$

By adding (C.3) to (C.5)

$$0 \leq 2\phi(1-\phi^L) + \frac{2\left(\phi(1-\phi^L)\right)^2}{1-\phi} \leq 2\phi + \frac{2\phi^2}{1-\phi} \quad (C.6)$$

and simplifying (C.6)

$$0 \leq 2\phi(1-\phi^L) + \frac{2\left(\phi(1-\phi^L)\right)^2}{1-\phi} \leq \frac{2\phi}{1-\phi}. \quad (C.7)$$

To obtain the bullwhip expression, we add plus one (+1) to (C.7)

$$1 \leq 1 + \left(2\phi(1-\phi^L) + \frac{2\left(\phi(1-\phi^L)\right)^2}{1-\phi}\right) \leq 1 + \frac{2\phi}{1-\phi}. \quad (C.8)$$

Therefore, the upper and lower bound of the bullwhip effect measure of an INAR(1) is

$$1 \leq Bullwhip \leq \frac{1+\phi}{1-\phi}. \tag{C.9}$$

This means that bullwhip always exists but is less than $(1+\phi)/(1-\phi)$, regardless the lead time and the auto-regressive parameter.

Appendix D. Determining the optimal production capacity under discrete demand

The per period production cost C_t^q, with a nominal hours unit cost of u within a nominal capacity of K and flexible per unit overtime cost of um, where m is the overtime multiplier, Boute et al. (2022), can be expressed as:

$$C_t^q = uK + um[q_t - K]^+. \tag{D.1}$$

The expected capacity costs per period can be found by taking the expectation of (D.1),

$$\mathbb{E}[C_t^q] = uK + um\sum_{n=K}^{\infty}(\mathbb{P}[n](n-K)), \tag{D.2}$$

where $\mathbb{P}[n] = \mathbb{P}[q_t = n]$ is the probability of the orders q_t equalling n. Notice, in (48), the first addend uK is increasing in K, and the second addend, the sum, is decreasing in K. Thus, there is a unique K (or at most two consecutive Ks) that minimises the expected capacity costs.

At the minimum the expected capacity costs with capacity K will be approximately equal to the expected capacity cost with capacity $K+1$,

$$uK + um\sum_{n=k}^{\infty}(\mathbb{P}[n](n-k)) = u(K+1) + um\sum_{n=K+1}^{\infty}(\mathbb{P}[n](n-(k+1))). \tag{D.3}$$

Noting that $\sum_{n=k}^{\infty}(\mathbb{P}[n](n-k)) = \sum_{n=k+1}^{\infty}(\mathbb{P}[n](n-k))$, we can rearrange the above to yield

$$u((K+1)-K) + um\sum_{n=K+1}^{\infty}(\mathbb{P}[n](n-(k+1)) - (n-k)) = 0 \tag{D.4}$$

$$u + um\sum_{n=K+1}^{\infty}(\mathbb{P}[n](-1)) = 0 \tag{D.5}$$

$$1 - m\sum_{n=K+1}^{\infty}\mathbb{P}[n] = 0 \tag{D.6}$$

As $\sum_{n=K+1}^{\infty}\mathbb{P}[n] = 1 - F[K]$, where $F[K]$ is the cumulative distribution function the order distribution we have

$$1 - m(1 - F[K]) = 0 \tag{D.7}$$

$$-m + mF[K] = -1 \tag{D.8}$$

$$F[K] = \frac{m-1}{m} \tag{D.9}$$

$$K^{\star} = F^{-1}\left[\frac{m-1}{m}\right]. \tag{D.10}$$

The optimal production capacity $K^{\star}$ can then be found as the smallest K that ensures

$$\sum_{n=0}^{K}\mathbb{P}[n] \geq \frac{m-1}{m}. \tag{D.11}$$

Note, in our derivation, we have not made any assumptions about the distribution or auto-correlation of the production orders, only that the orders are stationary (with a finite and constant mean and variance).

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