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Decentralized revenue sharing from broadcasting sports

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Abstract
We study the problem of sharing the revenues from broadcasting sports leagues among participating clubs. First, we characterize the set of rules satisfying two basic axioms: *anonymity* and *additivity*. Then, we decentralize the problem by letting clubs vote for rules. No majority equilibrium exists when they are allowed to vote for any rule within the characterized set. However, if the set is restricted in a meaningful and plausible way (just replacing *anonymity* by *equal treatment of equals*), majority equilibrium does exist.

Keywords Resource allocation · Broadcasting problems · Voting · Majority · Anonymity

JEL Classification D63 · C71 · Z20.

1 Introduction

Sports organizations crucially rely on the revenues they obtain from broadcasting their events. In 2019, the National Football League (NFL—the professional American football league) generated more than US$ 4.5 billion in broadcasting rights worldwide. The English Premier League (EPL—the top English football league), the National Basketball Association (NBA—the major professional basketball league in North America), and La Liga (the top professional football division of the Spanish football league system) followed in the ranking, generating US$ 3.83, US$ 3.12 and US$ 2.27 billion, respectively.[1] These massive amounts are typically obtained via collective bargaining. That is, each sports league organizes itself via a governing body, which negotiates with TV broadcasters selling broadcasting rights for all games played in the league. An interesting problem emerges once the deal is closed; namely, how to share those revenues among participating clubs in the league. This problem can be solved in two ways. One way is by means of a social planner (ideally, but not necessarily, considering some sort of rule that is singled out from an axiomatic

[1] The data are provided by Statista: https://www.statista.com/statistics/1120170/broadcasting-rights-sports-by-league/. Last accessed February 2022.

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characterization). We refer to this as a *centralized* way. The other way involves no social planner, and the involved parties (clubs in this case) themselves decide via some voting protocol. We refer to this as a *decentralized* way. We will mostly focus in this paper on the latter (although we also include a characterization result in our analysis).

One might argue that in a one-club/one-vote environment, uniform sharing is more or less guaranteed because weak-drawing clubs may block unequal sharing by refusing to permit broadcasting games involving them and strong-drawing clubs (Fort & Quirk, 1995). This is less realistic nowadays. In recent years, most leagues have expanded timetables, so that very few games overlap. We also have more accurate data about audiences in each game. Consequently, it is easier to estimate the (broadcasting) strength of each participating club in a league, and the uniform sharing might easily be dismissed on grounds of fairness.

Our analysis will take as a starting point the *audience matrix* associated with each league—that is, the collection of the (TV) audiences of all games in the league. More precisely, if all clubs involved in a league are listed as columns and rows of a matrix, then each entry of the matrix will collect the audience (number of TV viewers) of the game involving the team at the corresponding row and the team at the corresponding column, played at the former's stadium.[2] If we assume a pay-per-view system, in which each viewer pays the same fee to watch a game, then the overall number of viewers can be interpreted as the overall amount of revenue to be allocated (normalizing the fee to one). Then, we will consider rules that fully allocate such an amount among all participating clubs. The uniform rule (outlined above) is obviously an option. But others exist too. We concentrate on those that satisfy two basic principles: *anonymity* and *additivity*. The first states that a permutation of the set of agents equally permutes the allocation. Thus, the name of the agents does not matter. The second one is standard in axiomatic work and it can be traced back to Shapley (1953). It states that revenues should be additive on the audience matrix, which precludes some externalities, while conveying a form of simplicity.

Our first result (Theorem 1) characterizes all the rules satisfying *anonymity* and *additivity*. They require that the amount received by each club is dependent on three characteristics: its overall home audience, its overall away audience, and the overall audience in the whole tournament (league). We then let clubs vote among these rules and obtain a first negative result (Theorem 2): no majority equilibrium exists. That is, for each problem within a large class we identify, and each solution for it (obtained from one of the rules characterized), there exists another solution (obtained from another rule, among those characterized) that is strictly preferred by a majority of clubs.

The above negative result is another instance of Condorcet's paradox of voting, which is perhaps best exemplified by the problem of determining the division of a cake by majority rule (Hamada, 1973). Such a result might lead one to despair of ever achieving a voting equilibrium in our setting. Nevertheless, as Campbell (1975) puts it, majority voting is never allowed to operate by itself without restraints imposed by constitution and convention. We actually show that we can avoid majority cycles in our setting with a simple move in that direction. More precisely, instead of *anonymity*, we could consider another axiom, dubbed *equal treatment of equals*, stating that if two clubs have the same audiences, then each time they play a third, they should receive the same amount. Thus, both axioms are

2 Thus, for a league with a standard double round-robin format, the matrix would be square and the diagonal entries would be empty.

related to the principle of *impartiality*.[3] It turns out that the combination of this axiom with *additivity* characterizes another family of rules, which happens to be included in the family of rules characterized in Theorem 1. We refer to them as *general compromise rules*, because they offer a general (not necessarily convex) compromise between two focal rules: on the one hand, the *uniform rule* already mentioned; on the other hand, a rule dubbed *concede-and-divide*, which allocates revenues comparing individual and average (broadcasting) performance in a meaningful way. We then obtain a positive result (Theorem 3). That is, we show that if we allow clubs to vote among rules within any bounded subset of the family of general compromise rules, majority equilibrium does exist.

To prove the previous result, we exploit an interesting feature of the family of general compromise rules. Rules within the family satisfy the so-called *single-crossing property*, which allows one to separate those clubs that benefit from the choice of one or the other rule, depending on the rank of their overall audiences. This is a sufficient condition for the existence of a majority voting equilibrium (Gans & Smart, 1996). And it also has some other implications, referring to the distributive impact of the rules within the family, as well as the identification of the majority voting equilibrium (Hemming & Keen, 1983).

Our work relates to several branches of the literature. On the one hand there is the literature on voting regarding taxes, pioneered by Foley (1967), who analyzed the problem of voting on taxes in an endowment economy (and showed that there always exists a majority voting equilibrium for the class of flat taxes).[4] In our setting, voting refers to allocation rules, rather than tax methods. As such, we are closer to the literature on voting for resource allocation (Birnberg et al., 1970; Barzel & Sass, 1990). Thus, our work also touches the sizable body of literature on fair allocation, with a special emphasis on its well-developed component dealing with rationing problems (O'Neill, 1982; Kaminski, 2000; Thomson, 2019). On the other hand, this paper is a new stage in our research agenda on sharing the revenues raised from the collective sale of broadcasting rights for sports leagues (Bergantiños & Moreno-Ternero, 2020a, 2020b, 2021, 2022a, 2022b, 2022c, 2022d, 2023). As such, it connects to literature dealing with broadcasting and revenue sharing in sports (Cave & Crandall, 2001; Szymanski & Késenne, 2004).

We conclude this introduction mentioning that our analysis may have potential applications that go beyond the four big leagues mentioned at the beginning of this section. Two of those referred to (European) football, also known as soccer. Table 1 below lists the most important leagues for that sport, ranked according to their value per game (based on the contracts in place as of September 2022). This allows us to obtain a more accurate picture for soccer numbers worldwide.

Note that most of these leagues have a standard round-robin tournament from which we can obtain an audience matrix, thus satisfying the informational requirements of our model. This is not the case in some other major sports with a different format, such as F1, golf, or tennis. In the case of F1 (and MotoGP and similar racing sports), participants likely compete in several races during the season. We can therefore only have combined

[3] Impartiality is a basic requirement in the theory of justice (Young, 1994; Moreno-Ternero & Roemer 2006). In the context of fair allocation, it essentially translates as imposing the rule that ethically irrelevant aspects be excluded from the allocation process. The two axioms mentioned above (*anonymity* and *equal treatment of equals*) are indeed motivated by the desire for impartiality, although many other axioms are also motivated by that objective.

[4] Foley's work mostly relies on verbal discussion. A more formal treatment of his model (and some of his results) is provided by Gouveia and Oliver (1996). See also Gouveia (1997) or Moreno-Ternero (2011) for similar models and results.

Table 1 Top 10 domestic soccer leagues in value per game[a]

League	Country	Value per game	Currency
Premier League	England	8 million	GBP
Bundesliga	Germany	3.59 million	EUR
La Liga	Spain	2.60 million	EUR
Serie A	Italy	2.44 million	EUR
Ligue 1	France	1.53 million	EUR
English Football League	England	0.708 million	GBP
Scottish Premiership	Scotland	0.555 million	GBP
Primeira liga	Portugal	0.588 million	EUR
Brasileirao	Brazil	2.734 million	BRL
Major League Soccer	US	0.525 million	USD

[a]The source is https://en.wikipedia.org/wiki/List_of_domestic_football_league_broadcast_deals_by_country. Last accessed, September 30, 2022

audiences for each race, without disentangling the audience for each pair or participants. Similarly, in the case of golf, participants compete at different tournaments, and even though some participants do not make the cut by the last days of each tournament, we can only obtain combined audiences. Finally, both tennis and international football competitions (such as the FIFA World Cup) rely on knock-out tournament formats. In that case, we would easily obtain an audience matrix as well (although it would have more empty entries than an audience matrix from a round-robin tournament), and thus our model could also be applied.

The rest of the paper is organized as follows. We introduce the model, basic rules and axioms in Sect. 2, and we provide the main characterization result in Sect. 3. Section 4 is devoted to the decentralization process. We first show that in letting clubs vote among the rules within the family characterized in Sect. 3, we cannot guarantee the existence of a majority voting equilibrium. We then show that such an existence is guaranteed when the set of voting alternatives is restricted to another focal family of rules. We then explore additional features of such a family and its (associated) majority voting equilibrium. Section 5 concludes. For smooth flow, some proofs have been deferred to an appendix.

2 The model

We consider the model introduced in Bergantiños and Moreno-Ternero (2020a). Let N be a finite set of clubs. Its cardinality is denoted by n. We assume that $n \geq 3$. For each pair of clubs $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i's stadium. We use the notational convention that $a_{ii} = 0$ for each $i \in N$.[5] Let $A \in \mathcal{A}_{n\times n}$ denote the resulting matrix of broadcasting audiences generated in the whole tournament involving the clubs within N. As the set N will be fixed throughout our analysis, we will not explicitly consider it in the description of each problem. Each matrix $A \in \mathcal{A}_{n\times n}$

[5] Alternatively, we could write $a_{ii} = \emptyset$ to specify that a team i cannot play against itself.

with zero entries in the diagonal will thus represent a *problem*, and we will refer to the set of problems as $\mathcal{P}$.

Let $\alpha_i(A)$ denote the overall audience achieved by club i, i.e.,

$$\alpha_i(A) = \sum_{j \in N} (a_{ij} + a_{ji}).$$

Without loss of generality, we normalize the revenue generated from each viewer to 1 (to be interpreted as the "pay-per-view" fee). Thus, we sometimes refer to $\alpha_i(A)$ as the *claim* of club i. When no confusion arises, we write α_i instead of $\alpha_i(A)$. We then denote each club's (overall) home audience by h_i and its (overall) away audience by w_i. Formally, for each $i \in N$,

$$h_i = \sum_{j \in N \setminus \{i\}} a_{ij}, \text{ and}$$
$$w_i = \sum_{j \in N \setminus \{i\}} a_{ji}.$$

Note that $\alpha_i = h_i + w_i$ for each $i \in N$.

We denote by $\overline{\alpha}$ the average audience of all clubs.; namely,

$$\overline{\alpha} = \frac{\sum_{i \in N} \alpha_i}{n}.$$

For each $A \in \mathcal{A}_{n \times n}$, let $||A||$ denote the overall audience of the tournament; namely,

$$||A|| = \sum_{i,j \in N} a_{ij} = \frac{1}{2} \sum_{i \in N} \alpha_i = \frac{n\overline{\alpha}}{2}.$$

A **rule** is a mapping that associates with each problem the list of the amounts clubs get from the overall revenue. Formally, $R : \mathcal{P} \to \mathbb{R}^N$ is such that, for each $A \in \mathcal{P}$,

$$\sum_{i \in N} R_i(A) = ||A||.$$

Rules can generally be structured in the following way. Assume the amount received by each club i has three parts: one depending on its (overall) home audience, another depending on its (overall) away audience, and the third depending on the overall audience in the whole tournament. Formally:

General rules $\{G^{xyz}\}_{x+y+nz=1}$. For each trio $x, y, z \in \mathbb{R}$ with $x + y + nz = 1$, each $A \in \mathcal{P}$, and each $i \in N$,

$$G_i^{xyz}(A) = xh_i + yw_i + z||A|| = x\left(h_i - \frac{\bar{\alpha}}{2}\right) + y\left(w_i - \frac{\bar{\alpha}}{2}\right) + \frac{\bar{\alpha}}{2}.$$

General rules have been studied in Bergantiños and Moreno-Ternero (2022c, 2022d). Note that if $(x, y, z) = (0, 0, \frac{1}{n})$, we obtain the so-called *uniform rule*, which divides equally among all clubs the overall audience of the whole tournament.[6] If $(x, y, z) = (\frac{n-1}{n-2}, \frac{n-1}{n-2}, \frac{-1}{n-2})$, we

[6] This rule was first studied in Bergantiños and Moreno-Ternero (2020b). It also appears in Bergantiños and Moreno-Ternero (2022a, b, c, d).

obtain another focal rule (the so-called *concede-and-divide*), which is based on a comparison of the performance of a club with the average performance of the other clubs.[7] Formally:

Uniform, U: for each $A \in \mathcal{P}$, and each $i \in N$,

$$U_i(A) = \frac{||A||}{n} = \frac{\overline{\alpha}}{2}.$$

Concede-and-divide, CD: for each $A \in \mathcal{P}$, and each $i \in N$,

$$CD_i(A) = \alpha_i - \frac{\sum_{j,k \in N \setminus \{i\}} \left(a_{jk} + a_{kj}\right)}{n-2} = \frac{(n-1)\alpha_i - ||A||}{n-2} = \frac{2(n-1)\alpha_i - n\overline{\alpha}}{2(n-2)}.$$

The linear combinations of the above two rules give rise to a new family of (compromise) rules, which is fully included within the family of *general rules*.[8] Formally:

General compromise rules $\left\{UC^{\lambda}\right\}_{\lambda \in \mathbb{R}}$: for each $\lambda \in \mathbb{R}$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UC_i^{\lambda}(A) = (1-\lambda)U_i(A) + \lambda CD_i(A).$$

Equivalently,

$$UC_i^{\lambda}(A) = (1-\lambda)\frac{||A||}{n} + \lambda\frac{(n-1)\alpha_i - ||A||}{n-2} = \frac{\overline{\alpha}}{2} + \lambda\frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right).$$

We conclude this section introducing three basic axioms of rules.

First, the axiom that revenues should be additive on A. Formally:

Additivity: For each pair A and $A' \in \mathcal{P}$,

$$R\left(A + A'\right) = R(A) + R\left(A'\right).$$

Second, an axiom indicating that the name of the agents does not matter. Formally, let σ be a permutation of the set of agents. Thus, $\sigma : N \to N$ such that $\sigma(i) \neq \sigma(j)$ when $i \neq j$. Given a permutation σ and $A \in \mathcal{P}$, we define the problem A^{σ}, where for each pair. $i,j \in N$, $a_{ij}^{\sigma} = a_{\sigma(i)\sigma(j)}$.

Anonymity: For each $A \in \mathcal{P}$, each permutation σ, and each $i \in N$,

$$R_i(A) = R_{\sigma(i)}(A^{\sigma}).$$

Third, the axiom that if two clubs have the same audiences, then each time they play a third, they should receive the same amoun.

Equal treatment of equals: For each $A \in \mathcal{P}$, and each pair $i,j \in N$ such that $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i,j\}$,

[7] This rule was first studied in Bergantiños and Moreno-Ternero (2020a). It also appears in Bergantiños and Moreno-Ternero (2021, 2022a, 2022b, 2022c, 2022d)

[8] These rules have been considered in Bergantiños and Moreno-Ternero (2022b, c, d).

$$R_i(A) = R_j(A).$$

3 A characterization result

Our first result characterizes the family of rules satisfying the first two axioms introduced above.

Theorem 1 *A rule satisfies additivity and anonymit if and only if it is a general rule.*

Proof It is not difficult to show that each general rule satisfies the two axioms at the statement. Conversely, let R be a rule satisfying the two axioms. Let $A \in \mathcal{P}$. For each pair $i,j \in N$, with $i \neq j$, let $\mathbf{1}^{ij}$ denote the matrix with the following entries:

$$\mathbf{1}^{ij}_{kl} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

Let $i \in N$. By *additivity*,

$$R_i(A) = \sum_{j,k \in N : j \neq k} a_{jk} R_i\left(\mathbf{1}^{jk}\right). \tag{1}$$

Let $i,j,k \in N$ be three different clubs. By *anonymity*,

$$\begin{aligned} R_k\left(\mathbf{1}^{ij}\right) = & R_k\left(\mathbf{1}^{ji}\right) = R_i\left(\mathbf{1}^{kj}\right) = R_i\left(\mathbf{1}^{jk}\right) = R_j\left(\mathbf{1}^{ki}\right) = R_j\left(\mathbf{1}^{ik}\right), \\ R_i\left(\mathbf{1}^{ij}\right) = & R_i\left(\mathbf{1}^{ik}\right) = R_j\left(\mathbf{1}^{ji}\right) = R_j\left(\mathbf{1}^{jk}\right) = R_k\left(\mathbf{1}^{ki}\right) = R_k\left(\mathbf{1}^{kj}\right), \text{ and} \\ R_i\left(\mathbf{1}^{ji}\right) = & R_i\left(\mathbf{1}^{ki}\right) = R_j\left(\mathbf{1}^{ij}\right) = R_j\left(\mathbf{1}^{kj}\right) = R_k\left(\mathbf{1}^{ik}\right) = R_k\left(\mathbf{1}^{jk}\right). \end{aligned}$$

Then, there exist $\hat{x}, \hat{y}, \hat{z} \in \mathbb{R}$ such that for each $i,j \in N$ with $i \neq j$ and $k \in N \backslash \{i,j\}$ we have that

$$\hat{x} = R_i\left(\mathbf{1}^{ij}\right),\ y' = R_j\left(\mathbf{1}^{ij}\right) \text{ and } \hat{z} = R_k\left(\mathbf{1}^{ij}\right). \tag{2}$$

As $\sum_{k \in N} R_k\left(\mathbf{1}^{ij}\right) = 1$, it follows that $\hat{x} + \hat{y} + (n-2)\hat{z} = 1$.

We define $x = \hat{x} - \hat{z}$, $y = \hat{y} - \hat{z}$, and $z = \hat{z}$. Then, $x + y + nz = 1$ and

$$R_k\left(\mathbf{1}^{ij}\right) = \begin{cases} x + z & \text{if } k = i \\ y + z & \text{if } k = j \\ z & \text{otherwise.} \end{cases}$$

By (1),

$$\begin{aligned} R_i(A) = & \sum_{j,k \in N : j \neq k} a_{jk} R_i\left(\mathbf{1}^{jk}\right) = (x+z) \sum_{j \in N \backslash \{i\}} a_{ij} + (y+z) \sum_{j \in N \backslash \{i\}} a_{ji} + z \sum_{j,k \in N \backslash \{i\} : j \neq k} a_{jk} \\ = & \ x h_i + y w_i + z||A|| = \ G_i^{xyz}(A). \end{aligned}$$

□

If, instead of *anonymity*, we add *equal treatment of equals* to *additivity*, then we characterize the family of *general compromise* rules (Bergantiños & Moreno-Ternero, 2022c). This implies that, under the presence of *additivity*, *equal treatment of equals* is a stronger axiom than *anonymity*.

4 Decentralization

In the previous section we provided normative foundations for a family of rules to share revenues raised from broadcasting. Two basic axioms (*anonymity* and *additivity*) characterize the family of *general rules*. Such an axiomatic analysis is, nevertheless, silent regarding the specific rule to choose within the family. We explore such a problem in this section, taking a decentralized approach. More precisely, we study whether the choice of a rule within the family could be made by means of simple majority voting, letting each club vote for a rule within the family. Due to the overwhelming existence of majority cycles (Greenberg, 1979; Balasko & Crés, 1997), one should normally not expect a positive answer to this question. This is indeed what the next result confirms.

Some formal definitions first. Given a problem $A \in \mathcal{P}$, we say that $R(A)$ is a *majority winner* (within the set of rules $\mathcal{R}$) for A if there is no other rule $R' \in \mathcal{R}$ such that $R'_i(A) > R_i(A)$ for a majority of clubs. We say that the family of rules $\mathcal{R}$ has a *majority voting equilibrium* if there is at least one majority winner (within $\mathcal{R}$) for each problem $A \in \mathcal{P}$.

Theorem 2 *There is no majority voting equilibrium for the family of general rules.*

Even though the technical proof of the previous result (see the appendix) might be cumbersome, its logic should be clear. It all amounts to realizing that, given a general rule, one can construct another general rule which, at a certain problem, increases the amount obtained by a majority of the clubs involved, while reducing the amount obtained by all the others. The argument is similar to others used in related models (Klingaman, 1969; Marhuenda & Ortuño-Ortin, 1998).

Given the previous result, our aim now shifts to prove the existence of a majority voting equilibrium for a sufficiently large family of rules. Surprisingly, as the next result shows, we can do so for any bounded family of general compromise rules. Recall that, as mentioned in the previous section, such a family is characterized by the combination of *equal treatment of equals* and *additivity*. We conclude from here that in the presence of *additivity*, *anonymity* characterizes a family that is *too* large to guarantee the existence of a majority voting equilibrium, whereas *equal treatment of equals* characterizes another family (large, but not *too large*) that does guarantee the existence of a majority voting equilibrium. Note that we need to impose the *bounded* requirement, which restricts the domain of the parameter describing general compromise rules to a bounded interval. Otherwise, the majority equilibrium might not exist, as it may correspond to the rule obtained with one of the bounds of the interval.

To prove the result, we show first that all rules within the family of general compromise rules satisfy the *single-crossing* property. That is, for each pair of rules within the family, and each problem $A \in \mathcal{P}$, there exists a club $i^* \in N$ separating those clubs benefitting from the choice of one rule and those benefitting from the choice of the other. It turns out that i^* is precisely the club whose overall audience is closest (from below) to the average overall audience.

Proposition 1 *All rules within the family of general compromise rules satisfy the single-crossing property.*

Proof Let $A \in \mathcal{P}$. Let $\underline{\lambda}, \overline{\lambda}, \lambda_1, \lambda_2 \in \mathbb{R}$, with $\underline{\lambda} \leq \lambda_1 \leq \lambda_2 \leq \overline{\lambda}$ (one of them strict), and $i \in N$. We distinguish two cases:

If $\alpha_i \leq \overline{\alpha}$, then

$$UC_i^{\lambda_1}(A) = \frac{\overline{\alpha}}{2} + \lambda_1 \frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right) \geq \frac{\overline{\alpha}}{2} + \lambda_2 \frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right) = UC_i^{\lambda_2}(A).$$

If $\alpha_i > \overline{\alpha}$, then

$$UC_i^{\lambda_1}(A) = \frac{\overline{\alpha}}{2} + \lambda_1 \frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right) \leq \frac{\overline{\alpha}}{2} + \lambda_2 \frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right) = UC_i^{\lambda_2}(A).$$

Let i^* be the agent whose claim is closest to $\overline{\alpha}$ from below. And assume, without loss of generality, that $N = \{1, \ldots, n\}$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then, we have the following:

(i) $UC_i^{\lambda_1}(A) \leq UC_i^{\lambda_2}(A)$ for each $i = 1, ..., i^*$ and
(ii) $UC_i^{\lambda_1}(A) \geq UC_i^{\lambda_2}(A)$ for each $i = i^* + 1, ..., n$.

□

It is well known that the single-crossing property of preferences is a sufficient condition for the existence of a majority voting equilibrium (Gans & Smart, 1996). Thus, the next result follows.

Theorem 3 *There is a majority voting equilibrium for each bounded family of general compromise rules* $\left\{UC^{\lambda}\right\}_{\lambda \in [\underline{\lambda}, \overline{\lambda}]}$.

Theorem 3 states that if we let clubs vote for a rule within any bounded family of general compromise rules, then there will be a majority winner for each problem. The identity of this winner will be problem-specific and will depend on the characteristics of the problem at stake, as stated in the next result (whose proof appears in the appendix). First, some notation. For each $A \in \mathcal{P}$, we consider the following partition of N, with respect to the average claim ($\bar{\alpha}$): $N_l(A) = \{i \in N : \alpha_i < \bar{\alpha}\}$, $N_u(A) = \{i \in N : \alpha_i > \bar{\alpha}\}$, and $N_e(A) = \{i \in N : \alpha_i = \bar{\alpha}\}$. That is, taking the average claim (within the tournament) as the benchmark threshold, we consider three groups referring to clubs with claims below, above, or exactly at, the threshold.[9]

Proposition 2 *Let* $\left\{UC^{\lambda}\right\}_{\lambda \in [\underline{\lambda}, \overline{\lambda}]}$ *be the domain of rules for voting and* $A \in \mathcal{P}$. *The following statements hold*:

(i) *If* $2|N_l| > n$, *then* $UC^{\underline{\lambda}}(A)$ *is the unique majority winner.*
(ii) *If* $2|N_u| > n$, *then* $UC^{\overline{\lambda}}(A)$ *is the unique majority winner.*
(iii) *Otherwise, each* $UC^{\lambda}(A)$ *is the majority winner.*

[9] When no confusion arises, we simply write N_l, N_u, and N_e. Note that $n = |N_l| + |N_u| + |N_e|$.

The single-crossing property exhibited at the proof of Theorem 3 also guarantees that the social preference relationship obtained under majority voting is transitive, and corresponds to that of the median voter. In our setting, the median voter, which we denote by m, corresponds to the club with the median overall audience (claim). Thus, depending on whether this median claim is below or above the average claim, the majority winner for each problem A will be either the rule $UC^{\underline{\lambda}}(A)$ or the rule $UC^{\overline{\lambda}}(A)$. In other words, a tournament with a small number of very strong clubs (i.e., with very high claims) will proclaim the allocation $UC^{\underline{\lambda}}(A)$ (the one favoring weaker clubs more within the family) as the majority winner, whereas a tournament with a small number of very weak clubs (i.e., with very small claims) will proclaim the allocation $UC^{\overline{\lambda}}(A)$ (the one favoring stronger clubs more within the family). The proofs of the next corollaries appear in the appendix.

Corollary 1 *Let* $\left\{UC^{\lambda}\right\}_{\lambda\in[\underline{\lambda},\overline{\lambda}]}$ *be the domain of rules for voting and let* $A \in \mathcal{P}$ *be such that* n *is odd. The following statements hold*:

(i) *If* $\alpha_m < \bar{\alpha}$, *then* $UC^{\underline{\lambda}}(A)$ *is the unique majority winner.*
(ii) *If* $\alpha_m > \bar{\alpha}$, *then* $UC^{\overline{\lambda}}(A)$ *is the unique majority winner.*
(iii) *If* $\alpha_m = \bar{\alpha}$, *then each* $UC^{\lambda}(A)$ *is the majority winner.*

Corollary 2 *Let* $\left\{UC^{\lambda}\right\}_{\lambda\in[\underline{\lambda},\overline{\lambda}]}$ *be the domain of rules for voting and let* $A \in \mathcal{P}$ *be such that* n *is even. The following statements hold*:

(i) *If* $\alpha_{\frac{n+2}{2}} < \bar{\alpha}$, *then* $UC^{\underline{\lambda}}(A)$ *is the unique majority winner.*
(ii) *If* $\alpha_{\frac{n}{2}} > \bar{\alpha}$, *then* $UC^{\overline{\lambda}}(A)$ *is the unique majority winner.*
(iii) *If* $\alpha_{\frac{n}{2}} \leq \bar{\alpha} \leq \alpha_{\frac{n+2}{2}}$, *then each* $UC^{\lambda}(A)$ *is the majority winner.*

If the parameter describing general compromise rules ranges from zero to 1, then it yields precisely convex combinations between the *uniform rule* and *concede-and-divide*. Proposition 2 translates into the following.

Corollary 3 *Let* $\left\{UC^{\lambda}\right\}_{\lambda\in[0,1]}$ *be the domain of rules for voting and* $A \in \mathcal{P}$. *The following statements hold*:

(i) *If* $2|N_l| > n$, *then* $U(A)$ *is the unique majority winner.*
(ii) *If* $2|N_u| > n$, *then* $CD(A)$ *is the unique majority winner.*
(iii) *Otherwise*, *each* $UC^{\lambda}(A)$ *is the majority winner.*

Corollary 3 implies that if the distribution of claims is skewed to the left (i.e., the median claim is below the mean claim), then the *uniform* allocation (the most equal allocation within the family) is the majority winner, whereas if it is skewed to the right (i.e., the median claim is above the mean claim), then the *concede-and-divide* allocation (the most unequal allocation within the family, as proved below) is the majority winner. If it is not skewed, then any *compromise* allocation can be the majority winner.

Another consequence of the single-crossing property is that it guarantees progressivity comparisons of schedules (Jakobsson, 1976; Hemming & Keen, 1983). Thus, we can also obtain an interesting result, referring to the distributive power of the rules within the family of general compromise rules. Formally, given $x, y \in \mathbb{R}^n$ satisfying $x_1 \leq x_2 \leq \ldots \leq x_n$, $y_1 \leq y_2 \leq \ldots \leq y_n$, and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, we say that x *is greater than* y *in the Lorenz ordering* if $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$, for each $k = 1, \ldots, n-1$, with at least one strict inequality. When x is greater than y in the Lorenz ordering, one can state (see, for instance, Dasgupta et al., 1973) that x is unambiguously "more egalitarian" than y. In our setting, we say that a rule R *Lorenz dominates* another rule R' if for each $A \in \mathcal{P}$, $R(A)$ is greater than $R'(A)$ in the Lorenz ordering. As the Lorenz criterion is a partial ordering, one might not expect to be able to perform many comparisons of vectors. It turns out, however, that the general compromise rules are fully ranked according to this criterion. This is stated in the next result.

Proposition 3 *The following statements hold*:

- If $0 \leq \lambda_1 \leq \lambda_2$ then UC^{λ_1} Lorenz dominates UC^{λ_2}.
- If $\lambda_1 \leq \lambda_2 \leq 0$ then UC^{λ_2} Lorenz dominates UC^{λ_1}.

Proof Assume, without loss of generality, that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. It is not difficult to show that, for each $k = 1, \ldots, n-1$, $\sum_{i=1}^{k} \alpha_i \leq k\overline{\alpha}$, and $\sum_{i=1}^{k} \alpha_{n-i+1} \geq k\overline{\alpha}$.

In order to prove the first statement of the proposition, let $0 \leq \lambda_1 \leq \lambda_2$. Then, the corresponding general compromise rules preserve the order of claims. Formally,

$$\begin{aligned} &UC_1^{\lambda_1}(A) \leq UC_2^{\lambda_1}(N, A) \leq \ldots \leq UC_n^{\lambda_1}(A), \text{ and} \\ &UC_1^{\lambda_2}(A) \leq UC_2^{\lambda_2}(N, A) \leq \ldots \leq UC_n^{\lambda_2}(A). \end{aligned}$$

Thus, it suffices to show that, for each $k = 1, \ldots, n-1$,

$$\sum_{i=1}^{k} UC_i^{\lambda_1}(A) \geq \sum_{i=1}^{k} UC_i^{\lambda_2}(A).$$

Equivalently,

$$\sum_{i=1}^{k} \left[\frac{||A||}{n} + \lambda_1 \frac{n-1}{n-2} \left(\alpha_i - \overline{\alpha} \right) \right] \geq \sum_{i=1}^{k} \left[\frac{||A||}{n} + \lambda_2 \frac{n-1}{n-2} \left(\alpha_i - \overline{\alpha} \right) \right].$$

Or,

$$\lambda_1 \sum_{i=1}^{k} \left(\alpha_i - \overline{\alpha} \right) \geq \lambda_2 \sum_{i=1}^{k} \left(\alpha_i - \overline{\alpha} \right),$$

which follows from the fact that $\sum_{i=1}^{k} \left(\alpha_i - \overline{\alpha} \right) \leq 0$ and $\lambda_1 \leq \lambda_2$.

In order to prove the second statement, let $\lambda_1 \leq \lambda_2 \leq 0$. In this case, the corresponding general compromise rules reserve the order of claims. Formally,

$$\begin{aligned} &UC_1^{\lambda_1}(A) \geq UC_2^{\lambda_1}(N, A) \geq \ldots \geq UC_n^{\lambda_1}(A), \text{ and} \\ &UC_1^{\lambda_2}(A) \geq UC_2^{\lambda_2}(N, A) \geq \ldots \geq UC_n^{\lambda_2}(A). \end{aligned}$$

Thus, it suffices to show that, for each $k = 1, \ldots, n-1$,

$$\sum_{i=1}^{k} UC^{\lambda_2}_{n-i+1}(A) \geq \sum_{i=1}^{k} UC^{\lambda_1}_{n-i+1}(A).$$

Equivalently,

$$\sum_{i=1}^{k}\left[\frac{||A||}{n}+\lambda_2\frac{n-1}{n-2}\left(\alpha_{n-i+1}-\overline{\alpha}\right)\right] \geq \sum_{i=1}^{k}\left[\frac{||A||}{n}+\lambda_1\frac{n-1}{n-2}\left(\alpha_{n-i+1}-\overline{\alpha}\right)\right].$$

Or,

$$\lambda_2\sum_{i=1}^{k}\left(\alpha_{n-i+1}-\overline{\alpha}\right) \geq \lambda_1\sum_{i=1}^{k}\left(\alpha_{n-i+1}-\overline{\alpha}\right),$$

which follows from the fact that $\sum_{i=1}^{k}\left(\alpha_{n-i+1}-\overline{\alpha}\right) \geq 0$ and $\lambda_1 \leq \lambda_2$. □

Proposition 3 implies that the parameter defining the family can actually be interpreted as an index of the distributive power of the rules within the family. The *uniform rule* is the center element of the family, obtained when $\lambda = 0$. It also happens to be the *maximal* element of the Lorenz ordering, as it generates fully egalitarian allocations. It is then obvious that all other rules within the family are Lorenz-dominated by it. The remarkable feature, which is stated by Proposition 3, is that, departing from the *uniform rule* in both directions (either with positive parameters or with negative parameters), we obtain rules that yield progressively less egalitarian allocations. That is, the more we depart from the center element in the family, the less egalitarian the rules become. And we can establish those comparisons for each pair of rules within each of the two sides of the family. When the pair of rules is composed of rules in different sides of the family (i.e., one corresponding to a negative parameter and the other corresponding to a positive parameter), then we cannot establish Lorenz comparisons for such a pair of rules.

To conclude this section, note that the rules considered above might impose negative amounts to some clubs. It might be interesting to restrict our focus to general compromise rules that allocate non-negative amounts to all players. It turns out that the family of rules satisfying *additivity*, *equal treatment of equals*, and *non-negativity* (for each problem A and each $i \in N$, $R_i(A) \geq 0$) is characterized at Proposition 2.2 in Bergantiños and Moreno-Ternero (2022c). This family of rules is precisely composed of the general compromise rules UC^{λ} where $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$. Note that, by Theorem 3 above, this family of rules has a majority voting equilibrium.

5 Discussion

We have studied in this paper the problem of sharing the revenues raised from the collective sale of broadcasting rights for sports leagues, from a decentralized perspective. Our starting point is the characterization of a large family of rules by means of two basic axioms: *additivity* and *anonymity*. We then explore majority voting for such a family and obtain a negative result. Although the two axioms allow us to narrow the class of possible rules to a workable size (*additivity* is the critical axiom for that), the resulting family is too large to bypass (majority) cycles.

We then consider a subfamily of general compromise rules (which are precisely characterized by replacing *anonymity* with *equal treatment of equals*) and obtain a positive result for it; namely, if clubs are allowed to vote among general compromise rules, majority equilibrium does exist. Furthermore, rules within the family of bounded general compromise rules satisfy the *single-crossing* property. This permits us to obtain further information about the majority voting equilibrium, as well as the distributional properties of the rules within the family. In particular, we show that the skewness of the distribution of clubs' claims (overall number of viewers during the whole tournament) determines the equilibrium. In cases such as that of La Liga, a small number of clubs have very high claims.[10] This would proclaim the allocation provided by the general compromise rule with the lowest parameter the majority equilibrium.

It is left for further research to explore alternative forms of decentralization (via voting). For instance, there exists a growing interest in considering *approval voting* (Brams & Fishburn, 1978) as an alternative to majority voting in many instances. This method allows each voter to cast her vote for as many candidates as she wishes; each positive vote is counted in favor of the candidate. The votes are then added by candidate, and the winner is the one who receives the largest number of votes. All other candidates can also be ranked according to the number of votes they obtain. An alternative to *approval voting* is *cumulative voting* (Glasser, 1959; Sawyer & MacRae, 1962). It allows voters to distribute points among candidates in any arbitrary way.[11] An interesting case is the one in which every agent is endowed with a fixed number of votes that are evenly divided among all candidates for whom she votes. This corresponds to the notion of *Shapley ranking* introduced by Ginsburgh and Zang (2003) for the so-called museum pass game and recently characterized by Dehez and Ginsburgh (2020).[12] The *Shapley ranking* can be rationalized as the Shapley value of an associated cooperative game with transferable utility. In the context of this paper, we can also consider a game-theoretical approach to associate a cooperative game with transferable utility to each (broadcasting) problem. A natural way to do this is to take an optimistic stance on the revenue a coalition can generate on its own. If so, the Shapley value of the game yields a rule that happens to be a member of the family of compromise rules (Bergantiños & Moreno-Ternero, 2020a). It seems plausible to conjecture that such a rule would arise as the equilibrium in a decentralized process with *Shapley ranking* as an alternative to majority voting.

To conclude, as mentioned above, *additivity* is the critical axiom in our work to narrow the class of possible rules to a workable size. One could think of weakening this axiom by means of *transfer*, as introduced by Dubey (1975). In this context, a rule R satisfies *transfer* if, for each pair A and $A' \in \mathcal{P}$,

$$R\left(\max\{A, A'\}\right) + R\left(\min\{A, A'\}\right) = R(A) + R\left(A'\right),$$

where if $B = \max\{A, A'\}$ and $C = \min\{A, A'\}$, then $b_{ij} = \max\{a_{ij}, a'_{ij}\}$, and $c_{ij} = \min\{a_{ij}, a'_{ij}\}$, for each pair $i, j \in N$.

[10] Typically, only 20% of the clubs playing La Liga have claims above the average (Bergantiños & Moreno-Ternero, 2020a, 2021).

[11] Both *approval voting* and *cumulative voting* can be seen as members of a family of voting procedures dubbed *size approval voting*, which are characterized by Alcalde-Unzu and Vorsatz (2009).

[12] See also Bergantiños and Moreno-Ternero (2015).

If we replace *additivity* by *transfer* in Theorem 1, then the idea of our proof cannot be easily adapted. Our proof first "computes" the rules in the elementary problems $\mathbf{1}^{ij}$. Later, by *additivity*, the rules are extended to all problems. Without *additivity*, we would therefore need a different route.

We have identified at least a family of rules satisfying *transfer* and *anonymity*, but violating *additivity*. Specifically, a rule within this family divides the audience of each game between the two teams playing the game, taking into account the audience of the game. Formally, for each map $f : \mathbb{R}_+ \to \mathbb{R}$, we define the rule R^f as follows. For each problem A and each $i \in N$,

$$R_i(a) = \sum_{j \in N \setminus \{i\}} \left[f\left(a_{ij}\right) a_{ij} + \left(1 - f\left(a_{ji}\right)\right) a_{ji} \right].$$

Appendix

Proof of Theorem 2

Let $A \in \mathcal{P}$. Let $N_l^h(A) = \{i \in N : h_i < \frac{\bar{\alpha}}{2}\}$, $N_u^h(A) = \{i \in N : h_i > \frac{\bar{\alpha}}{2}\}$, and $N_e^h(A) = \{i \in N : h_i = \frac{\bar{\alpha}}{2}\}$. Similarly, let $N_l^w(A) = \{i \in N : w_i < \frac{\bar{\alpha}}{2}\}$, $N_u^w(A) = \{i \in N : w_i > \frac{\bar{\alpha}}{2}\}$, and $N_e^w(A) = \{i \in N : w_i = \frac{\bar{\alpha}}{2}\}$.

Now, let G^{xyz} be a general rule. Then,

$$G_i^{xyz}(A) = x\left(h_i - \frac{\bar{\alpha}}{2}\right) + y\left(w_i - \frac{\bar{\alpha}}{2}\right) + \frac{\bar{\alpha}}{2}.$$

We consider several cases.

Case 1. $|N_l^h(A)| > \frac{n}{2}$.

Let $\varepsilon > 0$, $\hat{x} = x - \varepsilon$, $\hat{y} = y$, and $\hat{z} = \frac{1 - \hat{x} - \hat{y}}{n}$. Then, $G^{\hat{x}\hat{y}\hat{z}}$ is also a general rule. Furthermore, for each $i \in N_l^h(A)$, $G_i^{\hat{x}\hat{y}\hat{z}}(A) > G_i^{xyz}(A)$. Thus, more than half of the voters would prefer $G^{\hat{x}\hat{y}\hat{z}}(A)$ to $G^{xyz}(A)$. Hence, $G^{xyz}(A)$ is not the majority winner.

Case 2. $|N_u^h(A)| > \frac{n}{2}$.

Let $\varepsilon > 0$, $\hat{x} = x + \varepsilon$, $\hat{y} = y$, and $\hat{z} = \frac{1 - \hat{x} - \hat{y}}{n}$. Then, $G^{\hat{x}\hat{y}\hat{z}}$ is also a general rule. Furthermore, for each $i \in N_u^h(A)$, $G_i^{\hat{x}\hat{y}\hat{z}}(A) > G_i^{xyz}(A)$. Thus, more than half of the voters would prefer $G^{\hat{x}\hat{y}\hat{z}}(A)$ to $G^{xyz}(A)$. Hence, $G^{xyz}(A)$ is not the majority winner.

Case 3. $|N_l^w(A)| > \frac{n}{2}$.

Let $\varepsilon > 0$, $\hat{x} = x$, $\hat{y} = y - \varepsilon$, and $\hat{z} = \frac{1 - \hat{x} - \hat{y}}{n}$. Then, $G^{\hat{x}\hat{y}\hat{z}}$ is also a general rule. Furthermore, for each $i \in N_l^w(A)$, $G_i^{\hat{x}\hat{y}\hat{z}}(A) > G_i^{xyz}(A)$. Thus, more than half of the voters would prefer $G^{\hat{x}\hat{y}\hat{z}}(A)$ to $G^{xyz}(A)$. Hence, $G^{xyz}(A)$ is not the majority winner.

Case 4. $|N_u^w(A)| > \frac{n}{2}$.

Let $\varepsilon > 0$, $\hat{x} = x$, $\hat{y} = y + \varepsilon$, and $\hat{z} = \frac{1 - \hat{x} - \hat{y}}{n}$. Then, $G^{\hat{x}\hat{y}\hat{z}}$ is also a general rule. Furthermore, for each $i \in N_h^w(A)$, $G_i^{\hat{x}\hat{y}\hat{z}}(A) > G_i^{xyz}(A)$. Thus, more than half of the voters would prefer $G^{\hat{x}\hat{y}\hat{z}}(A)$ to $G^{xyz}(A)$. Hence, $G^{xyz}(A)$ is not the majority winner.

Case 5. $\max\{|N_l^h(A)|, |N_u^h(A)|, |N_l^w(A)|, |N_u^w(A)|\} \leq \frac{n}{2}$.

In contrast with the previous cases, we cannot guarantee here that $G^{xyz}(A)$ is not the majority winner. For instance, if we take a problem A in which all games have the same audience (thus, $|N_l^h(A)| = |N_u^h(A)| = |N_l^w(A)| = |N_u^w(A)| = 0$), then all general rules would yield the same (uniform) solution. Hence, $G^{xyz}(A)$ is the majority winner for each (x, y, z) such that $x + y + nz = 1$.

Similarly, assume we take a fully polarized and symmetric problem, namely, A is such that n is even, $a_{ij} = a_{ji} = s$ for each pair $i,j \in \{1, \dots, \frac{n}{2}\}$ such that $i \neq j$, $a_{ij} = a_{ji} = t$ for each pair $i,j \in \{\frac{n}{2} + 1, \dots, n\}$ such that $i \neq j$, $a_{ij} = a_{ji} = 0$ when $i \in \{1, \dots, \frac{n}{2}\}$ and $j \in \{\frac{n}{2} + 1, \dots, n\}$, and $s < t$. Note that for each $i \in \{1, \dots, \frac{n}{2}\}$ $h_i = w_i = \frac{n-2}{2}s$ and $\alpha_i = (n-2)s$, whereas for each $i \in \{\frac{n}{2} + 1, \dots, n\}$ $h_i = w_i = \frac{n-2}{2}t$ and $\alpha_i = (n-2)t$. Thus, $\bar{\alpha} = \frac{n-2}{2}(s + t)$ and $|N_l^h(A)| = |N_u^h(A)| = |N_l^w(A)| = |N_u^w(A)| = \frac{n}{2}$. For this problem, all general rules would yield the same amounts to each club within the same group, and symmetric amounts to both groups. More precisely, for each $i \in \{1, \dots, \frac{n}{2}\}$, $G_i^{xyz}(A) = \frac{n-2}{4}(s - t)(x + y) + \frac{n-2}{4}(s + t)$ and for each $i \in \{\frac{n}{2} + 1, \dots, n\}$, $G_i^{xyz}(A) = \frac{n-2}{4}(t - s)(x + y) + \frac{n-2}{4}(s + t)$. Thus, again, $G^{xyz}(A)$ is a majority winner for each (x, y, z) such that $x + y + nz = 1$.

Nevertheless, the above does not mean that we can always find a majority winner in this case. Consider, for instance, the following problem:

$$A = \begin{pmatrix} 0 & 5 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As $\frac{\bar{\alpha}}{2} = 3$, it follows that $N_l^h(A) = \{5\}$, $N_l^w(A) = \{3, 5\}$, $N_u^h(A) = \{1, 2\} = N_u^w(A)$, and therefore, $\max\{|N_l^h(A)|, |N_u^h(A)|, |N_l^w(A)|, |N_u^w(A)|\} = 2 \leq \frac{n}{2} = \frac{5}{2}$. We have the following:

i	h_i	w_i	α_i	$G_i^{xyz}(A)$
1	5	7	12	$2x + 4y + 3$
2	4	5	9	$x + 2y + 3$
3	3	3	6	3
4	3	0	3	$-3y + 3$
5	0	0	0	$-3x - 3y + 3$

Let $\varepsilon > 0$, $\hat{x} = x + \varepsilon$, $\hat{y} = y - \frac{\varepsilon}{3}$, and $\hat{z} = \frac{1-\hat{x}-\hat{y}}{n}$. Then, $G^{\hat{x}\hat{y}\hat{z}}$ is also a general rule. Furthermore, for each $i \in \{1, 2, 4\}$, $G_i^{\hat{x}\hat{y}\hat{z}}(A) > G_i^{xyz}(A)$. Thus, more than half of the voters would prefer $G^{\hat{x}\hat{y}\hat{z}}(A)$ to $G^{xyz}(A)$. Hence, $G^{xyz}(A)$ is not a majority winner. □

Proof of Proposition 2

Let $\underline{\lambda}, \overline{\lambda}, \lambda \in \mathbb{R}_+$, with $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$, and $A \in \mathcal{P}$. For each $i \in N$,

$$UC_i^{\lambda}(A) = \frac{||A||}{n} + \lambda \frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right).$$

If $\alpha_i > \bar{\alpha}$, then $UC_i^{\lambda}(A)$ is an increasing function of λ, thus maximized at $\lambda = \overline{\lambda}$. This implies that, for each $i \in N_u$, $UC^{\overline{\lambda}}(A)$ is the most preferred outcome.

If $\alpha_i < \bar{\alpha}$, then $UC_i^{\lambda}(A)$ is a decreasing function of λ, thus maximized at $\lambda = \underline{\lambda}$. This implies that, for each $i \in N_l$, $UC^{\underline{\lambda}}(A)$ is the most preferred outcome.

If $\alpha_i = \bar{\alpha}$, then $UC_i^{\lambda}(A) = \frac{||A||}{n}$ for each $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. This implies that, for each $i \in N_e$, all rules within the family $\left\{UC^{\lambda}\right\}^n_{\lambda \in [\underline{\lambda}, \overline{\lambda}]}$ yield the same outcome.

From the above, statements (i) and (ii) follow trivially.

Assume, by contradiction, that statement (iii) does not hold. Then, there exist $A \in \mathcal{P}$ and $\lambda \in [0, 1]$ such that UC^{λ} is not the majority winner for A. Thus, we can find $\lambda' \in [\underline{\lambda}, \overline{\lambda}]$ such that $UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)$ holds for the majority of the clubs. We then consider two cases:

Case $\lambda' > \lambda$.

In this case, $UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)$ if and only if $i \in N_l$. Now,

$$\begin{aligned} \left|N_l\right| &= \left|\left\{i \in N : UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)\right\}\right| \\ &> \left|\left\{i \in N : UC_i^{\lambda'}(A) \leq UC_i^{\lambda}(A)\right\}\right| \\ &= |N_u| + \left|N_e\right|, \end{aligned}$$

which is a contradiction.

Case $\lambda' < \lambda$.

In this case, $UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)$ if and only if $i \in N_u$. Now,

$$\begin{aligned} \left|N_u\right| &= \left|\left\{i \in N : UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)\right\}\right| \\ &> \left|\left\{i \in N : UC_i^{\lambda'}(A) \leq UC_i^{\lambda}(A)\right\}\right| \\ &= |N_l| + \left|N_e\right|, \end{aligned}$$

which is a contradiction. □

Proof of Corollary 1

If $\alpha_m < \bar{\alpha}$, then $\left|N_l\right| \geq m > \frac{n}{2}$. Hence, $2|N_l| > n$. By Proposition 2, statement (*i*) holds.

If $\alpha_m > \bar{\alpha}$, then $\left|N_u\right| \geq m > \frac{n}{2}$. Hence, $2|N_u| > n$. By Proposition 2, statement (*ii*) holds.

If $\alpha_m = \bar{\alpha}$, then $\left|N_l\right| < m, \left|N_u\right| < m$, and $\left|N_e\right| > 0$. Hence, we are in case (*iii*) of the statement of Proposition 2, which concludes the proof. □

Proof of Corollary 2

If $\alpha_{\frac{n+2}{2}} < \bar{\alpha}$, then $\left|N_l\right| \geq \frac{n+2}{2}$. Hence, $2|N_l| > n$. By Proposition 2, statement (*i*) holds.

If $\alpha_{\frac{n}{2}} > \bar{\alpha}$, then $\left|N_u\right| \geq \frac{n+2}{2}$. Hence, $2|N_u| > n$. By Proposition 2, statement (*ii*) holds.

Suppose now that $\alpha_{\frac{n}{2}} \leq \bar{\alpha} \leq \alpha_{\frac{n+2}{2}}$. Then, it is enough to prove that we are in case (*iii*) of the statement of Proposition 2.

If $\bar{\alpha} = \alpha_{\frac{n}{2}}$, then $\left|N_l\right| < \frac{n}{2}$ and $\left|N_u\right| \leq \frac{n}{2}$.

If $\alpha_{\frac{n}{2}} < \bar{\alpha} < \alpha_{\frac{n+2}{2}}$, then $\left|N_l\right| = \frac{n}{2}$ and $\left|N_u\right| = \frac{n}{2}$.

If $\bar{\alpha} = \alpha_{\frac{n+2}{2}}$, then $\left|N_l\right| \leq \frac{n}{2}$ and $\left|N_u\right| < \frac{n}{2}$.

In either case, the desired conclusion holds. □

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Conflict of interest Both authors declare that they have no conflict of interest.

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