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Existence of solutions to uncertain differential equations of nonlocal type via an extended Krasnosel'skii fixed point theorem

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Abstract In the present study, we investigate the existence of the solutions to a type of uncertain differential equations subject to nonlocal derivatives. The approach is based on the application of an extended Krasnosel'skii fixed point theorem valid on fuzzy metric spaces. With this theorem, we deduce that the problem of interest has a fuzzy solution, which is defined on a certain interval. Our approach includes the consideration of a related integral problem, to which the above-mentioned tools are applicable. We finish with some physical motivations.

1 Introduction

Uncertainty quantification is a quantitative characterization that tries to reduce the ambiguities in mathematical, computational and real-world applications.

Uncertainty appears in mathematical models and experimental measurements due to various sources. Uncertainty principle is important, for instance, in quantum theory [1].

In [2], the author studied the Dirac equation in the context of generalized uncertainty principle, and, in [3], it has been introduced a new generalized momentum operator taking into account, among others, the extended uncertainty principle.

On the other hand, the importance of nonlocal derivatives has been increasing in the last years as a consequence of the interest and power of its applications in the construction of models for complex phenomena in relation with different fields of science and engineering. Some well-known examples of proposed applications are, for instance, the study of electromagnetic or control processes, diffusion processes through heterogeneous media, biologic problems, or electrical circuits [4, 5].

In 2010, the concept of solution to uncertain differential equations with nonlocal derivatives of fractional type (which we denote as UFDEs) is presented by Agarwal et al. [6]. As we have already mentioned, the operators of arbitrary (non-necessarily integer) order are adequate for the study of phenomena under the influence of memory effects [7–11]. The type of equations proposed in [6] exhibits an interesting behavior, since they combine the introduction of uncertainty and nonlocal effects both together. This work has leaded to the establishment of a new kind of hybrid operators [4], that collect the particularities of both uncertain equations and equations of arbitrary order. It is in this framework that several researchers have been making contributions in the recent years. In particular, many researchers have investigated UFDEs and some related integro-differential problems by considering the notions of differentiability by Riemann–Liouville or Caputo [12–15]. To mention one of the contributions, Arshad and Lupulescu [16, 17] have deduced certain results concerning the existence and the uniqueness problems for the solutions to UFDEs by considering the Riemann–Liouville derivative.

One usual approach in the study of the solutions to differential problems consists on the transformation into an integral problem and the definition of a mapping to which fixed point theory is applicable. Therefore, different types of fixed point results have proved its usefulness for the research work in the field of arbitrary-order differential equations. For instance, [18] is devoted to the existence of solutions for implicit differential equations including Caputo-Fabrizio operators through the application of Banach and Krasnosel'skii fixed point results, analyzing also some stability properties. An analogous study for equations with Caputo-Hadamard derivative is made in [19]. On the other hand, systems of fractional functional differential equations under Caputo operators are analyzed in [20] under Krasnosel'skii's fixed point theorem, and [21] also deals with the existence of positive solutions for arbitrary-order differential equations of Caputo-type with the help of Guo-Krasnosel'skii fixed point theorem. The same result,

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Guo-Krasnosel'skii fixed point theorem, is also the main tool used in [22] to analyze fractional equations with p-Laplacian. In the very recent work [23], the authors apply Banach contraction principle and Leray–Schauder alternative to study equations under Riemann–Liouville perspective.

In the context of UFDEs, the development of new extensions of fixed point results is relevant, this fact is motivated by the special particularities found in the spaces of fuzzy sets and functions. An interesting example is the extension of Schauder fixed point theorem by Agarwal et al. [6]. In their research work, the classical fix point result is generalized to semilinear spaces with the cancellation property. This extension allows to study the existence of solutions to UFDEs. Using this theorem, the authors in [24] have considered UFDEs and analyzed the existence of solutions. Other generalized fixed point results have been given in [9], where the authors have proved Krasnosel'skii fixed point result in the context of semilinear Banach spaces of general type, and applied it in order to deduce some existence theorems for UFDEs.

In this contribution, we consider a family of nonlinear differential equations subject to uncertainty and nonlocal derivatives that constitutes a class of equations larger than that in [24]. The main mathematical tool for the study is the application of a fixed point result of Krasnosel'skii type available in the context of fuzzy metric spaces. With this result, we provide the existence of at least one solution to each of the equations of interest, provided that some sufficient conditions hold. Our aim is to present a result that extends the results in [25] to the case of uncertain differential equations. Hence, the original contribution of the paper is providing a new proof for a fixed point result of Krasnosel'skii type for fuzzy metric spaces by using the particular cone structure of the base space and H-differences. With this result, on the existence of at least a fixed point for mappings being the addition of a compact mapping and a contractive mapping, several results on the existence of solutions to arbitrary-order uncertain differential equations are deduced, covering more general situations than the ones in [24], which was based on a fixed point result in abstract spaces valid just for compact mappings.

2 Basic notions

To establish the problem of interest and the main results, we first need to introduce several concepts and useful notations. For details, we refer to [26, 27].

In this study, we denote by $\mathcal{E}$ the space of fuzzy intervals. The α-cuts of an element $x \in \mathcal{E}$ are defined as

$$[x]^\alpha := \{t \in \mathbb{R} \mid x(t) \geq \alpha\}, \quad \text{for } \alpha \in (0, 1],$$

and the support of x is

$$[x]^0 := cl\{t \in \mathbb{R} \mid x(t) > 0\}.$$

The notation for the α-cuts of x

$$[x]^\alpha = [\underline{x}^\alpha, \overline{x}^\alpha]$$

will be also useful.

Given two elements $x, y \in \mathcal{E}$ and a scalar $\lambda \in \mathbb{R}$, we define the sum $x + y$ and the product by scalar λx by using, respectively, the expressions

$$[x + y]^\alpha = [x]^\alpha + [y]^\alpha, \text{ and } [\lambda x]^\alpha = \lambda[x]^\alpha,$$

for all $\alpha \in [0, 1]$. As we have observed, these notions are based on the classical sum of real intervals and the product of a real interval by a scalar, respectively.

The space $\mathcal{E}$ is easily endowed with a metric $D : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_+ \cup \{0\}$, based on the Pompeiu–Hausdorff distance between the α-cuts, as follows:

$$D(x, y) := \sup_{\alpha \in [0,1]} d_H([x]^\alpha, [y]^\alpha) = \sup_{\alpha \in [0,1]} \max\{|\underline{x}^\alpha - \underline{y}^\alpha|, |\overline{x}^\alpha - \overline{y}^\alpha|\}, \quad x, y \in \mathcal{E}.$$

The above-mentioned distance satisfies some interesting properties, including the invariance by translations and the positive homogeneity:

$D(x + z, y + z) = D(x, y)$, for all $x, y, z \in \mathcal{E}$,

$D(kx, ky) = |k| D(x, y)$, for all $k \in \mathbb{R}$, and $x, y \in \mathcal{E}$,

$D(x + y, z + s) \leq D(x, z) + D(y, s)$, for all $x, y, z, s \in \mathcal{E}$, $D(\lambda x, \mu x) = |\lambda - \mu| D(x, \tilde{0})$, for all $\lambda, \mu \geq 0$, and $x \in \mathcal{E}$, and the metric space $(\mathcal{E}, D)$ is complete.

The space $\mathcal{E}$ is not a vector space, so traditional fixed point results on Banach spaces are not useful to tackle problems in this context. To this purpose, we recall the concept of semilinear Banach space, which formalizes a framework to include the collection of all fuzzy intervals. The notion of semilinear space received attention in previous research works, see, for instance, [28]. We say that U is a semilinear metric space if U is a semilinear space endowed with a metric $d : U \times U \to \mathbb{R}_+$ satisfying

- $d(\alpha, \beta) = d(\alpha + \gamma, \beta + \gamma)$, (translation invariant)
- $\lambda d(\alpha, \beta) = d(\lambda\alpha, \lambda\beta)$, being $\lambda \geq 0$, (positively homogeneous)

for every $\alpha, \beta, \gamma \in U$. Let $\tilde{0}$ be the zero element in U. A norm on U is defined by considering $||x|| := d(x, \tilde{0})$. Concerning the behavior of the operations on semilinear metric spaces, it is possible to prove that the sum and the multiplication by a fixed scalar are continuous mappings. If the semilinear metric space U is a complete metric space, then U is called a semilinear Banach space. For instance, since $\mathcal{E}$ is not a linear space, then it is impossible for $\mathcal{E}$ to be a Banach space. If $\alpha + \beta = \gamma + \beta$ implies that $\alpha = \gamma$ for every $\alpha, \beta, \gamma \in U$, then the semilinear space U is said to have the cancellation property.

Let $a > 0$ be fixed. We define $C([0, a], \mathcal{E})$ as the set of all continuous fuzzy-interval valued functions defined on the compact interval $[0, a]$. The space $C([0, a], \mathcal{E})$ is, in relation with the concepts exposed, a semilinear space and has the cancellation property. Besides, it is known that the metric

$$\rho_0(x, y) := \max_{t \in [0,a]} D(x(t), y(t)), \quad x, y \in C([0, a], \mathcal{E}),$$

gives the space a structure of complete space [27], so we have an example of semilinear Banach space.

The elements in $\mathcal{E}$ are not necessarily continuous with respect to the degree of membership α. If we consider the set of all the fuzzy intervals $x \in \mathcal{E}$ such that the mapping $\alpha \mapsto [x]^\alpha$ is continuous on $[0, 1]$ (considering the Pompeiu–Hausdorff metric on the set of compact intervals), then this space of continuous fuzzy intervals is denoted by $\mathcal{E}^c$. Taking the same metric D, $(\mathcal{E}^c, D)$ has also the structure of complete metric space [29].

Definition 2.1 [29] Let $M \subseteq \mathcal{E}^c$. We say that M is compact-supported provided that there exists a real compact subset N with $[x]^0 \subseteq N$, $\forall x \in M$.

Definition 2.2 [29] Let $M \subseteq \mathcal{E}^c$. If, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\alpha - \alpha_0| < \delta \text{ implies } D([x]^\alpha, [x]^{\alpha_0}) < \varepsilon, \forall x \in M,$$

then we say that M is level-equicontinuous at $\alpha_0 \in [0, 1]$.

We say that M is level-equicontinuous on $[0, 1]$ if the property explained in Definition 2.2 is valid at every $\alpha \in [0, 1]$.

Theorem 2.3 [29] *Suppose that M is a compact-supported subset of $\mathcal{E}^c$. The set M is relatively compact in $(\mathcal{E}^c, D)$ if and only if it is level-equicontinuous on [0, 1].*

Next, we recall the extension of the famous Schauder fixed point theorem to semilinear spaces as presented in [30].

Theorem 2.4 *Let U be a semilinear Banach space with the cancellation property. Suppose that S is a nonempty, bounded, closed, and convex subset of U. Let $P : S \to S$ be a compact mapping. Then P has (at least) a fixed point in S.*

In [9], Long et al. have presented an extension of the Krasnosel'skii fixed point result for a class of spaces called generalized semilinear Banach spaces. In the following, we present a similar result in the fuzzy context, with a slightly different proof by using H-differences.

Theorem 2.5 *(Fixed point result of Krasnosel'skii type for fuzzy metric spaces) Let $\mathcal{M}$ be a non-empty, closed and convex subset of $C(I, \mathcal{E}^c)$ and suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathcal{M}$ into S and*

(i) $\mathcal{A}$ is continuous and compact,
(ii) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$, for every $x, y \in \mathcal{M}$,
(iii) $\mathcal{B}$ is a contraction mapping.

Then, there exists a fixed point for $\mathcal{A} + \mathcal{B}$ in $\mathcal{M}$, that is, there is $y \in \mathcal{M}$ for which $\mathcal{A}y + \mathcal{B}y = y$.

Proof: For every arbitrarily fixed element $y \in \mathcal{M}$, we define a self-mapping $Q : \mathcal{M} \to \mathcal{M}$ as

$$x \to Q(x) := \mathcal{A}y + \mathcal{B}x.$$

We check that Q satisfies the condition of contraction mapping

$$\rho_0(Q(x), Q(z)) = \rho_0(\mathcal{A}y + \mathcal{B}x, \mathcal{A}y + \mathcal{B}z) = \rho_0(\mathcal{B}x, \mathcal{B}z) \leq \kappa\rho_0(x, z),$$

where $0 \leq \kappa < 1$, and we also know that $\mathcal{M}$ is a complete metric space. Then, considering the equation $z = \mathcal{A}y + \mathcal{B}z$, we can deduce the existence of a unique solution in $\mathcal{M}$. For y fixed, let $\psi(y) \in \mathcal{M}$ be the unique solution of this equation in $\mathcal{M}$, i.e., $\psi(y) = \mathcal{A}y + \mathcal{B}\psi(y)$. Then, $(I \ominus \mathcal{B})\psi(y) = \mathcal{A}y$. Thus, $I \ominus \mathcal{B} : \psi(\mathcal{M}) \to C(I, \mathcal{E}^c)$ exists and $\mathcal{A}(\mathcal{M}) \subseteq (I \ominus \mathcal{B})(\psi(\mathcal{M}))$. We prove that it is injective so that

$$(I \ominus \mathcal{B})^{-1} : (I \ominus \mathcal{B})(\psi(\mathcal{M})) \to \psi(\mathcal{M})$$

exists and, therefore, $\psi(y) = (I \ominus \mathcal{B})^{-1}\mathcal{A}y$. Now, we show that $I \ominus \mathcal{B} : \psi(\mathcal{M}) \to C(I, \mathcal{E}^c)$ is continuous. Let $\hat{x}, \bar{x} \in \mathcal{M}$ fixed, and consider $\psi(\hat{x}), \psi(\bar{x}) \in \psi(\mathcal{M})$. Then, we obtain

$$\begin{aligned}\rho_0(\psi(\hat{x}), \psi(\bar{x})) &= \rho_0((I \ominus \mathcal{B})\psi(\hat{x}) + \mathcal{B}\psi(\hat{x}), (I \ominus \mathcal{B})\psi(\bar{x}) + \mathcal{B}\psi(\bar{x}))\\ &\leq \rho_0((I \ominus \mathcal{B})\psi(\hat{x}), (I \ominus \mathcal{B})\psi(\bar{x})) + \rho_0(\mathcal{B}\psi(\hat{x}), \mathcal{B}\psi(\bar{x}))\\ &\leq \rho_0((I \ominus \mathcal{B})\psi(\hat{x}), (I \ominus \mathcal{B})\psi(\bar{x})) + \kappa\rho_0(\psi(\hat{x}), \psi(\bar{x})),\end{aligned}$$

where we have used that $\mathcal{B}$ is a contraction. Therefore,

$$\rho_0((I \ominus \mathcal{B})\psi(\hat{x}), (I \ominus \mathcal{B})\psi(\bar{x})) \geq (1-\kappa)\rho_0(\psi(\hat{x}), \psi(\bar{x})). \tag{2.1}$$

The inequality (2.1) proves that $I \ominus \mathcal{B}$ is injective on $\psi(\mathcal{M})$. Thus, $I \ominus \mathcal{B}$ is bijective and continuous from $\psi(\mathcal{M})$ to $(I \ominus \mathcal{B})(\psi(\mathcal{M}))$. This proves that $(I \ominus \mathcal{B})^{-1}$ exists from $(I \ominus \mathcal{B})(\psi(\mathcal{M}))$ to $\psi(\mathcal{M})$. Moreover, from the inequality (2.1), $(I \ominus \mathcal{B})^{-1}$ is continuous on $(I \ominus \mathcal{B})(\psi(\mathcal{M}))$. Indeed, we get

$$\begin{aligned}&\rho_0((I \ominus \mathcal{B})^{-1}(I \ominus \mathcal{B})\psi(\hat{x}), (I \ominus \mathcal{B})^{-1}(I \ominus \mathcal{B})\psi(\bar{x})) = \rho_0(\psi(\hat{x}), \psi(\bar{x}))\\ &\leq \frac{1}{1-\kappa}\rho_0((I \ominus \mathcal{B})\psi(\hat{x}), (I \ominus \mathcal{B})\psi(\bar{x})).\end{aligned}$$

Now, we use that $\psi(y) = (I \ominus \mathcal{B})^{-1}\mathcal{A}y$ and that, by hypothesis, $(I \ominus \mathcal{B})^{-1}\mathcal{A}$ is continuous and compact, then using Theorem 2.4, ψ has a fixed point y_0 in $\mathcal{M}$, which satisfies $\psi(y_0) = y_0$. Since, $\psi(y_0)$ is the unique solution to $z = \mathcal{A}y_0 + \mathcal{B}z$, then, we have $y_0 = \mathcal{A}y_0 + \mathcal{B}y_0$. □

In the following, we set $V := C((0, a], \mathcal{E}) \cap L^1((0, a], \mathcal{E})$.

Definition 2.6 Let $y \in V$. The fuzzy Riemann–Liouville fractional (arbitrary order) integral of order $\nu > 0$ of y, for $t \in (0, a)$, is defined by

$$I^\nu y(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}y(s)ds,$$

assuming that the integral at the right is well-defined on the interval $(0, a)$.

Definition 2.7 [30] Let $y \in V$ and $\nu \in (0, 1)$. Let $t \mapsto \int_0^t (t-s)^{-\nu}y(s)ds$ be a Hukuhara differentiable fuzzy function on the interval $(0, a]$. The fuzzy Riemann–Liouville fractional (arbitrary order) derivative of order ν of y at t is defined by

$$D^\nu y(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t (t-s)^{-\nu}y(s)ds,$$

which is, according to the assumption, a fuzzy interval $D^\nu y(t) \in \mathcal{E}$.

3 Uncertain differential equations of nonlocal type

Next, we consider a class of uncertain differential equations whose nonlocal behavior is given by fractional (arbitrary order) operators, as follows,

$$D^\nu x = f(t, x) + g(t, x), \tag{3.1}$$

where the order of derivation is $0 < \nu < 1$, the fuzzy function $f : [0, a] \times \mathcal{E}^c \to \mathcal{E}^c$ is continuous and compact, and $g : [0, a] \times \mathcal{E}^c \to \mathcal{E}^c$ satisfies

$$D(g(t, x), g(t, y)) \leq LD(x, y), \quad x, y \in \mathcal{E}^c, \text{ for some } L \geq 0. \tag{3.2}$$

Next, we recall the concept of solution to (3.1).

Definition 3.1 [24] We say that the fuzzy function $x \in V$ is a solution to (3.1) if:

- There exists the fractional derivative $D^\nu x$ of x on $(0, a]$ and it is continuous.
- $D^\nu x(t) = f(t, x(t)) + g(t, x(t)), \quad \forall t \in (0, a]$.

The next remark gives us a procedure to obtain solutions to the equation (3.1) through the solutions of a related arbitrary-order integral equation.

Remark 3.2 [24] Suppose that $x \in C([0, a], \mathcal{E}^c)$ is a solution to the following fuzzy arbitrary-order integral equation

$$x(t) = I^\nu(f(t, x(t)) + g(t, x(t))),$$

and that $f(\cdot, x(\cdot)),\ g(\cdot, x(\cdot)) \in V$. Then x is a solution to the uncertain differential equation of nonlocal type (3.1).

At this point, we present the essential elements that will allow to prove the main existence results.

Let $I_{\delta_0} := [0, \delta_0]$, where $\delta_0 \leq a$, and $r_0 > 0$ be such that

$$R := \sup\Big\{D(\tilde{0}, f(t, x)) \,|\, t \in I_{\delta_0}, D(x, \tilde{0}) \leq r_0\Big\} < +\infty.$$

Note that it is also possible to choose δ_0 small enough such that $\frac{R\delta_0^{\nu}}{\Gamma(\nu+1)} \leq r_0$. Define the set

$$\Phi := \Big\{x \in C(I_{\delta_0}, \mathcal{E}^c) \,|\, D(\tilde{0}, x(t)) \leq r_0, \text{ for all } t \in I_{\delta_0}, \text{ and } x(0) = \tilde{0}\Big\}.$$

We can easily check that $\Phi \subseteq C(I_{\delta_0}, \mathcal{E}^c)$ is bounded, closed, and convex. Besides, recall that $C(I_{\delta_0}, \mathcal{E}^c)$ is a semilinear Banach space. On the set Φ, we consider two mappings related to functions f and g, respectively. First, we consider the mapping $\mathcal{T} : \Phi \to C(I_{\delta_0}, \mathcal{E}^c)$ defined by

$$(\mathcal{T}x)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, x(s)) ds, \text{ for } t \in I_{\delta_0}. \tag{3.3}$$

To simplify the procedure, we fix some notations referring to some restrictions on the function f appearing in the nonlinearity of the equation.

We introduce the condition (H) for the function $f \in C([0, a] \times \mathcal{E}^c, \mathcal{E}^c)$ consisting on the restriction: for every pair of points (t, x), $(t, y) \in [0, a] \times C([0, a], \mathcal{E}^c)$, the following inequality holds

$$D(f(t, x(t)), f(t, y(t))) \leq p(t) w(\rho_0(x, y)), \tag{3.4}$$

where w is a real continuous function on $[0, \infty)$ and such that $w(0) = 0$, and $p : [0, a] \to \mathbb{R}^+$ satisfies $I^{\nu} p(t) < N$, for every $t \in [0, a]$.

Lemma 3.3 *Let f satisfy (H) on $[0, a] \times \mathcal{E}^c$. Then, the following assertions are satisfied:*

- *$\mathcal{T}$ is well-defined on $C(I_{\delta_0}, \mathcal{E}^c)$.*
- *$\mathcal{T}$ is a continuous mapping on $C(I_{\delta_0}, \mathcal{E}^c)$.*

Proof First, we prove that the mapping $\mathcal{T}$ is well-defined. By definition, it is obvious that $(\mathcal{T}x)(0) = \tilde{0}$, for every $x \in \Phi$.

Now, for fixed $x \in \Phi$, we show that $\mathcal{T}x \in C(I_{\delta_0}, \mathcal{E}^c)$. Moreover, we show that $\mathcal{T}x$ is a uniformly continuous function on the interval I_{δ_0}. We take fixed $t, t' \in I_{\delta_0}$, with $t < t'$. For these elements, we deduce

$$\begin{aligned}
&D((\mathcal{T}x)(t), (\mathcal{T}x)(t')) \\
&= \frac{1}{\Gamma(\nu)} D\left(\int_0^t (t-s)^{\nu-1} f(s, x(s)) ds, \int_0^{t'} (t'-s)^{\nu-1} f(s, x(s)) ds\right) \\
&= \frac{1}{\Gamma(\nu)} D\Big(\int_0^t (t-s)^{\nu-1} f(s, x(s)) ds, \int_0^{t} (t'-s)^{\nu-1} f(s, x(s)) ds \\
&+ \int_t^{t'} (t'-s)^{\nu-1} f(s, x(s)) ds\Big) \\
&\leq \frac{1}{\Gamma(\nu)} \left[D\left(\int_0^t (t-s)^{\nu-1} f(s, x(s)) ds, \int_0^{t} (t'-s)^{\nu-1} f(s, x(s)) ds\right)\right. \\
&\left. + D\left(\int_t^{t'} (t'-s)^{\nu-1} f(s, x(s)) ds, \tilde{0}\right)\right] \\
&\leq \frac{1}{\Gamma(\nu)} \left[\int_0^t D((t-s)^{\nu-1} f(s, x(s)), (t'-s)^{\nu-1} f(s, x(s))) ds\right. \\
&\left. + \int_t^{t'} (t'-s)^{\nu-1} D(f(s, x(s)), \tilde{0}) ds\right] \\
&\leq \frac{R}{\Gamma(\nu)} \left[\int_0^t |(t-s)^{\nu-1} - (t'-s)^{\nu-1}| ds + \int_t^{t'} (t'-s)^{\nu-1} ds\right] \\
&\leq \frac{R}{\Gamma(\nu+1)} (2(t'-t)^{\nu} + t^{\nu} - t'^{\nu}) \\
&\leq \frac{2R}{\Gamma(\nu+1)} (t'-t)^{\nu}.
\end{aligned}$$

Therefore, $D((\mathcal{T}x)(t), (\mathcal{T}x)(t')) \to 0$, when $|t - t'|$ tends to zero, so $\mathcal{T}x$ is a continuous function on the interval I_{δ_0}, for $x \in \Phi$. We also have

$$D((\mathcal{T}x)(t), \tilde{0}) \leq \frac{1}{\Gamma(\nu)} \int_0^t D((t-s)^{\nu-1} f(s, x(s)), \tilde{0})ds \leq \frac{R\delta_0^\nu}{\Gamma(\nu+1)} \leq r_0,$$

for every $t \in I_{\delta_0}$. Therefore, $\mathcal{T}$ is a self-mapping $\mathcal{T} : \Phi \to \Phi$.

Now, we show that $\mathcal{T}$ is a continuous mapping. Let $y_n, y \in \Phi, n = 1, 2, ...$ such that $y_n \underset{n\to\infty}{\longrightarrow} y$, with convergence in the space $C([0, a], \mathcal{E}^c)$, i.e, satisfying $\rho_0(y_n, y) \underset{n\to\infty}{\longrightarrow} 0$. In consequence, for every $t \in J := [0, a]$, we deduce that

$$\begin{aligned}
&\rho_0(\mathcal{T}y_n, \mathcal{T}y)\\
&= \sup_{t\in J} D((\mathcal{T}y_n)(t), (\mathcal{T}y)(t))\\
&= \sup_{t\in I_{\delta_0}} D\left(\frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, y_n(s))ds, \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, y(s))ds\right)\\
&\leq \frac{1}{\Gamma(\nu)} \sup_{t\in I_{\delta_0}} \int_0^t (t-s)^{\nu-1} D(f(s, y_n(s)), f(s, y(s)))ds\\
&\leq \frac{1}{\Gamma(\nu)} \sup_{t\in I_{\delta_0}} \int_0^t (t-s)^{\nu-1} p(s) w(\rho_0(y_n, y))ds\\
&\leq \sup_{t\in I_{\delta_0}} I^\nu p(t)\, w(\rho_0(y_n, y)).
\end{aligned}$$

Since, by the specifications in (H), $w(0) = 0$ and w is continuous on its domain $[0, \infty)$, it is clear that $w(r) \underset{r\to 0^+}{\longrightarrow} 0$. Due to $\rho_0(y_n, y) \underset{n\to\infty}{\longrightarrow} 0$ and the estimate $I^\nu p(t) < N$, we conclude that $\mathcal{T}y_n \to \mathcal{T}y$ as $n \to \infty$, i.e., $\mathcal{T}$ is continuous. □

Theorem 3.4 *Consider a fixed $\nu \in (0, 1)$ and suppose that the fuzzy function $f : [0, a] \times \mathcal{E}^c \to \mathcal{E}^c$ is continuous and compact, satisfying (H), and assume that $g : [0, a] \times \mathcal{E}^c \to \mathcal{E}^c$ satisfies (3.2). Under these assumptions, the uncertain differential equation of nonlocal type (3.1) has (at least) one solution defined on the interval $[0, \delta_0]$, which is a continuous function, being δ_0 a suitable positive number with $\delta_0 < a$.*

Proof By Remark 3.2, we are reduced to study the uncertain integral equation

$$x(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, x(s))ds + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, x(s))ds.$$

Lemma 3.3 provides the proper definition of the mapping $\mathcal{T}$, given by Eq. (3.3), and also its continuity on the space $C(I_{\delta_0}, \mathcal{E}^c)$.

In the following, we justify that $\mathcal{T}$ is a compact mapping. Take an arbitrarily fixed $x \in \Phi$, and let $t, t' \in I_{\delta_0}$ with $t \leq t'$, then

$$\begin{aligned}
&D((\mathcal{T}x)(t), (\mathcal{T}x)(t'))\\
&= \frac{1}{\Gamma(\nu)} D\left(\int_0^t (t-s)^{\nu-1} f(s, x(s))ds, \int_0^{t'} (t'-s)^{\nu-1} f(s, x(s))ds\right)\\
&\leq \frac{1}{\Gamma(\nu)} D\left(\int_0^t (t-s)^{\nu-1} f(s, x(s))ds, \int_0^t (t'-s)^{\nu-1} f(s, x(s))ds\right)\\
&\quad + \frac{1}{\Gamma(\nu)} D\left(\int_0^t (t'-s)^{\nu-1} f(s, x(s))ds, \int_0^{t'} (t'-s)^{\nu-1} f(s, x(s))ds\right)\\
&\leq \frac{R}{\Gamma(\nu+1)} (t^\nu - t'^\nu + 2(t'-t)^\nu).
\end{aligned}$$

This implies that $\mathcal{T}(\Phi)$ is equicontinuous in $C(I_{\delta_0}, \mathcal{E}^c)$.

Next, to show the relative compactness of $\mathcal{T}(\Phi)(t)$ in $\mathcal{E}^c$, we use Theorem 2.3, hence, equivalently, we show that $\mathcal{T}(\Phi)(t)$ is compact-supported and level-equicontinuous in $\mathcal{E}^c$.

Fix $t \in [0, \delta_0]$, we check that $\mathcal{T}(\Phi)(t) \subseteq \mathcal{E}^c$. For every $v \in \mathcal{T}(\Phi)(t)$, it is possible to write

$$v = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, y(s))ds, \quad \text{for some } y \in \Phi.$$

Since, by hypothesis,f is a compact mapping, then $f(I_{\delta_0} \times \Phi)$ is a relatively compact set in $\mathcal{E}^c$. By virtue of Theorem 2.3, $f(I_{\delta_0} \times \Phi)$ is a level-equicontinuous set. Therefore, for all $\varepsilon > 0$, we can affirm the existence of $\delta > 0$ such that

$$D([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}) < \frac{\Gamma(\nu+1)\varepsilon}{2\delta_0^\nu}, \quad \forall (s, y) \in I_{\delta_0} \times \Phi,$$

provided that $|\alpha_1 - \alpha_2| < \delta$. Therefore, similarly to [30], for $|\alpha_1 - \alpha_2| < \delta$, we deduce

$$\begin{aligned} D([v]^{\alpha_1}, [v]^{\alpha_2}) &= D([\mathcal{T}(y)(t)]^{\alpha_1}, [\mathcal{T}(y)(t)]^{\alpha_2}) \\ &\leq \frac{1}{\Gamma(\nu)} D\left(\left[\int_0^t (t-s)^{\nu-1} f(s, y(s))ds\right]^{\alpha_1}, \left[\int_0^t (t-s)^{\nu-1} f(s, y(s))ds\right]^{\alpha_2}\right) \\ &\leq \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} D([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}) ds \\ &\leq \varepsilon. \end{aligned}$$

So, $\mathcal{T}(\Phi)(t)$ is level-equicontinuous in $\mathcal{E}^c$. Now, according to the relative compactness of $f(I_{\delta_0} \times \Phi)$, we can guarantee the existence of a compact set $M \subset \mathbb{R}^n$ with $[f(s, y(s))]^0 \subseteq M$ for every $(s, y) \in I_{\delta_0} \times \Phi$. Therefore, we obtain

$$\begin{aligned} \left[\frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, y(s))ds\right]^0 &= \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} [f(s, y(s))]^0 ds \\ &\subseteq \frac{M}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds = \frac{Mt^\nu}{\Gamma(\nu+1)}. \end{aligned}$$

Therefore, as a subset of $\mathcal{E}^c$, $\mathcal{T}(\Phi)(t)$ is compact-supported.

Thus, using the Arzelà-Ascoli theorem, the relative compactness of $\mathcal{T}(\Phi)$ in $C([0, \delta_0], \mathcal{E}^c)$ is deduced.

On the other hand, we also consider the mapping $\widetilde{\mathcal{T}} : \Phi \to C(I_{\delta_0}, \mathcal{E}^c)$ defined by

$$(\widetilde{\mathcal{T}}x)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, x(s))ds, \quad t \in I_{\delta_0}. \tag{3.5}$$

Therefore,

$$\begin{aligned} D((\widetilde{\mathcal{T}}x)(t), (\widetilde{\mathcal{T}}y)(t)) &\leq \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} D(g(s, x(s)), g(s, y(s)))ds \\ &\leq \frac{L}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} D(x(s), y(s))ds \leq \frac{Lt^\nu}{\Gamma(\nu+1)} \rho_0(x, y) \leq \frac{L\delta_0^\nu}{\Gamma(\nu+1)} \rho_0(x, y). \end{aligned}$$

In consequence, if δ_0 is small enough such that $L\delta_0^\nu < \Gamma(\nu+1)$, then the mapping $\widetilde{\mathcal{T}}$ is contractive. Hence, using Theorem 2.5, the mapping $\mathcal{T} + \widetilde{\mathcal{T}}$ has at least a fixed point in Φ. Therefore, there exists (at least) one continuous fuzzy solution x for (3.1) on I_{δ_0}, for a certain $0 < \delta_0 \leq a$. □

Remark 3.5 In Theorem 3.4, we can relax the compactness condition of f. Let $f(I_{\delta_0} \times \Phi)$ be level-equicontinuous and p be bounded on I_{δ_0}, and assume the existence of $y_0 \in \mathcal{E}^c$ with $\cup_{t \in I_{\delta_0}} [f(t, y_0)]^0$ bounded. Consider a bounded set B in $I_{\delta_0} \times F$, where F is a bounded set in $\mathcal{E}^c$. Now, we check the relative compactness of $f(B) \subseteq f(I_{\delta_0} \times F)$.

If $t \in I_{\delta_0}$ and $y \in F$, then it is deduced

$$\begin{aligned} D([f(t, y)]^0, [f(t, y_0)]^0) &\leq \sup_{\alpha \in [0,1]} D([f(t, y)]^\alpha, [f(t, y_0)]^\alpha) \\ &= D(f(t, y), f(t, y_0)) \\ &\leq p(t) w(\rho_0(C_y, C_{y_0})), \end{aligned}$$

where we consider the auxiliary constant function associated with each $z \in \mathcal{E}^c$, given by

$$\begin{aligned} C_z : I_{\delta_0} &\to \mathcal{E}^c, \\ t &\to C_z(t) := z. \end{aligned}$$

Since F is bounded and $y \in F$, then $D(y, y_0)$ is bounded and, thus, $\rho_0(C_y, C_{y_0})$ is bounded. On the other hand, since w is continuous, then $w(\rho_0(C_y, C_{y_0}))$ is bounded. By hypothesis, since p is bounded on the interval I_{δ_0}, then $D([f(t, y)]^0, [f(t, y_0)]^0)$ is bounded for all $y \in F$. So, we can affirm the existence of $K > 0$ with

$$D([f(t, y)]^0, [f(t, y_0)]^0) \leq K, \ \forall t \in I_{\delta_0}, \ \forall y \in F. \tag{3.6}$$

Denote $f(t, y) =: u$, and $f(t, y_0) =: v$. Then, the assumptions allow to guarantee the existence of $N > 0$ with

$$[f(t, y_0)]^0 = [\underline{v}^0, \overline{v}^0] \subseteq [-N, N].$$

So, by (3.6), we have

$$D([u]^0, [v]^0) = \max\{|\underline{u}^0 - \underline{v}^0|, |\overline{u}^0 - \overline{v}^0|\} \le K.$$

Then, $|\underline{u}^0 - \underline{v}^0| \le K$ and $|\overline{u}^0 - \overline{v}^0| \le K$. Therefore, we deduce

$$\underline{u}^0 \in [\underline{v}^0 - K, \underline{v}^0 + K] \subseteq [-N - K, N + K],$$
$$\overline{u}^0 \in [\overline{v}^0 - K, \overline{v}^0 + K] \subseteq [-N - K, N + K].$$

Hence,

$$[f(t, y)]^0 = [u]^0 \subseteq [-N - K, N + K], \ \forall t \in I_{\delta_0}, \forall y \in F.$$

Therefore, there exists the compact set $K_0 := [-N - K, N + K]$ with the property $[f(I_{\delta_0} \times F)]^0 \subseteq K_0$.

Remark 3.6 If we consider the nondecreasing character of w and the inequality

$$D(f(t, x), f(t, y)) \le p(t)w(D(x, y)), \text{ for every } t \in [0, a], x, y \in \mathcal{E}^c, \tag{3.7}$$

the previous results also hold. As remarked in [25] for the classical case, if $p(t) = \theta$, with $\theta > 0$ a constant, then the inequality (3.7) is in fact the Osgood condition. In particular, if $p(t) = \theta$, with $\theta > 0$, and $w(D(x, y)) = D(x, y)$, that is, w is the identity, then the inequality (3.7) represents the Lipschitzian character of f in the second variable.

4 Some physical applications

The potential applications of arbitrary-order calculus in physical problems have been highlighted in the literature in a vast way.

In order to provide an example of application of the results of the paper, we will focus on the specific model proposed in [31]. In this reference, it is connected the phenomenon of nuclear magnetic resonance with the Bloch equation, used for modeling of materials through a simple first-order system of equations. With a fractional generalization of this model, heteorogeneous materials can be considered. This approach has shown to be appropriate due to the appearance of multiexponential processes in the analysis of the signal decay. In particular, in [31], a Caputo fractional derivative is used to model a single-spin system in a static magnetic field at resonance.

Here, we consider an analogous Riemann–Liouville-type equation subject to uncertainty of the form

$$\begin{cases} \dfrac{1}{\sigma_t^{1-\nu}} D^\nu M_z(t) = A_0 \ominus A_1 M_z(t), \\ \dfrac{1}{\sigma_t^{1-\nu}} D^\nu M_x(t) = B_0 M_y(t) \ominus B_1 M_x(t), \\ \dfrac{1}{\sigma_t^{1-\nu}} D^\nu M_y(t) = (-B_0) M_x(t) \ominus B_1 M_y(t), \end{cases} \tag{4.1}$$

where $M_x(t)$, $M_y(t)$, and $M_z(t)$ represent the coordinates of the system magnetization and A_0, A_1, B_0, B_1 are suitable (uncertain) constants related with the equilibrium magnetization, the spin-lattice relaxation time, the spin-spin relaxation time, and the resonant frequency. The parameter $\sigma_t > 0$ is introduced in order to provide consistency with dimensionality (see [32]).

The first equation in this system corresponds to the framework considered. On the other hand, a similar type of results can be obtained for the space $C([0, a], \mathcal{E} \times \mathcal{E} \times \mathcal{E})$ with obvious adaptations; hence, it would be possible to rewrite problem (4.1) as a higher dimension generalization of problem (3.1), by choosing $f \equiv 0$ and

$$g(t, x_1, x_2, x_3) = \sigma_t^{1-\nu}(B_0 x_2 \ominus B_1 x_1, (-B_0)x_1 \ominus B_1 x_2, A_0 \ominus A_1 x_3),$$

which satisfy the required conditions as long as g is well-defined.

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