The 21^{st} Century Systems: an updated vision of Discrete-Time Fractional Models

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***Abstract*—Two different approaches for describing discrete-time fractional linear systems are presented. The first is based on the nabla and delta discrete-time derivatives. In this case suitable exponentials are introduced and used to define discrete Laplace transforms. The second approach is based on the bilinear (Tustin) transformations. For both cases, appropriate algorithms for obtaining the impulse, step, and frequency responses are presented. The state-variable representation is also analysed.**

***Index Terms*—Fractional system, Fractional derivative, Nabla derivative, Nabla Laplace transform, Delta derivative, Delta Laplace transform, Tustin transformation**

I. Introduction

Discrete-Time Fractional Models must be truly regarded as representative of the 21^{st} century systems. Indeed, while the continuous-time fractional models exist since the 19^{th} century, the discrete-time versions have only recently been introduced. This state of affairs happens because it is not easy to "fractionalize" discrete integer order systems defined by difference equations. An attempt was rehearsed in [1] where a model based on a fractional delay equation was introduced. It was possible to define and compute the traditional tools like, impulse response (IR) and transfer function (TF). However, it could not be considered as a fractional discrete-time system, in the sense followed in [2], since the corresponding TF is not truly fraccional. This failure showed that we had to give a deeper thought to the problem and, the most natural way to do it, was to go back to the origins.

Discrete-time systems began as a mere set of numerical tecniques to approximate and solve continuous-time differential equations. Such procedure was based on the incremental ratia used to aproximate the derivatives and is currently known as Euler method [3]. This approach gave rise to the delta systems [4], but, with a slight modification of the perspective, originated the important class of the discrete-time systems based on the difference equations [5], [6]. Nonetheless, the original Euler procedure was not completely abandoned, even if there was no formal introduction of such systems. Besides remaining important as an intermediate step to obtain difference equations from continuous-time systems, they were used under the delta system format. Indeed, the procedure can be applied when approximating continuous-time systems for filter implementation and control [7], [8], [4], [9], [10], as well as in modeling [11], [12], [13], [14].

More recently, the delta approach was revisited soaked in the Hilger's formulation for a continuous/discrete unification [15], which is nowadays called *calculus on time scales* [16], [17]. This formulation considers a general domain, called time scale or more generally measure chain, that can be continuous, discrete or mixed [17], [18], [16], [19]. In this domain, Hilger defined two derivatives, named *delta and nabla*, that are the incremental ratio or their limit to zero when calculated at a non isolated point. With these derivatives we can devise the corresponding differential equations representative of some given systems. Using a current nomenclature we will call them *nabla and delta systems* in agreement with the adopted derivatives. The nabla derivative is causal, while the delta is anti-causal. Based on this framework, suitable formulations for fractional discrete-time system definitions had to be sought. They are based on discrete nabla and delta derivatives so that the resulting systems mimic the continuous-time that emerge from the discrete-time as a limit when the sampling rate, $r = 1/h$, increases without bound, meaning that the inter-sample interval, $h \in \mathbb{R}^+$ (also called *graininess*) goes to zero. In fact, it is like a return back to the origins when the discrete were mere approximations to the CT systems [20], [19], [21]. However, these approaches did not propose coherent formulations of the nabla and delta system theory, that was developed later in a more comprehensive way [22].

The above approach has two drawbacks.

- While in the traditional discrete-time systems the unit circle is the reference for stability, the nabla/delta approach uses a circle with center $+/-1/h$, passing at the origin [22].
- In the traditional discrete-time signal processing the Z transform (ZT) is the tool par excellence for working with systems defined by difference equations. In the case of nabla/delta systems, we need to define new transforms [22].

As it is well known, the stability domain of causal continuous-time system is the right half complex plane (HCP). We can establish a one to one correspondence between the left (right) HCP and the interior (exterior) of the unit disk. Such relation can be expressed by a particular case of the bilinear (or, Möbius) transformation that is commonly named the Tustin map [23], used for the discrete-time approximation of continuous-time linear systems [24], [5], [6], [25]. However,

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no discrete-time derivative was introduced. This idea was explored in [26]. In fact, the bilinear transformation allows us to formulate a general discrete-time fractional calculus that mimics the corresponding continuous-time version, while being fully autonomous. On the other hand, it leads to tools and concepts similar to those of continuous-time fractional signals and systems (see the companion paper [2]). Another important characteristic of the proposed derivatives and systems lies in the fact that the Z transform is the most appropriate one in this framework. As the unit circle recovers its traditional role, the usefulness of the Fast Fourier Transform (FFT) becomes clear from the numerical and calculation time perspectives.

These two formulations led to discrete-time differential equations describing input-output relations in a similar way to what occurs with the continuous-time fractional systems. However, there is a generalized framework that can be used to deal with continuous-time and discrete-time systems usually called *state space representation*. While the first approaches consider the system like a "black-box" relating merely the input and output, this one allows us to "see" what is inside the system, through the introduction of inner variables describing the so-called *state* of the system. In this case, the model is defined by two equations that aquire the same form for all systems: the dynamic and observation equations. To solve the dinamic equation we need to introduce the *state transition operator* verifying the semi-group properties. We present this general formulation together with a correct definition of state transition operator.

The paper outlines as follows. In Section II the nabla and delta linear time invariant systems (LTIS) are introduced and studied. We focus our attention in the nabla versions, because they correspond to the causal case. We introduce the nabla and delta discrete-time derivatives (subsection II-A) and study their eigenfunctions that will be called nabla and delta exponentials (subsection II-B). From these concepts, we can define two discrete Laplace transforms. We discuss the Nabla Laplace transform only (subsection II-C), as well as its properties and the backward compatibility with the continuous-time Laplace transform. The nabla linear systems are introduced and studied in subsection II-E where the transient and steady-state responses are obtained. The stability and initial-conditions are also considered.

The linear systems based on the bilinear transformation are studied in Section III. The formulation is essentially performed in the Z transform domain which makes easier the introduction and study of the forward/backward derivatives (subsection III-A). The time formulation of the derivatives is presented in subsection III-B. With these derivatives we define the bilinear differential discrete-time linear systems (subsection III-C). Some illustrative examples are presented. In Section IV we introduce the state-variable formulation in a general setup valid also for the continuous-time systems. Finally some conclusions are drawn in Section V.

A. Abbreviations

The following abbreviations are used in this manuscript:

ARMA	Autoregressive-Moving Average
CT	Continuous-Time
DT	Discrete-Time
FARMA	Fractional Autoregressive-Moving Average
FD	Fractional derivative
FIR	Finite Impulse Response
FT	Fourier transform
FFT	Fast Fourier transform
FR	Frequency response
IC	Initial-conditions
IR	Impulse Response
IIR	Infinite Impulse Response
GL	Grünwald-Letnikov
LTIS	Linear time-invariant system
LS	Linear system
LT	Laplace transform
MLF	Mittag-Leffler function
NLT	Nabla Laplace transform
ROC	Region of convergence
TF	Transfer function
ZT	Z transform

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II. On the nabla and delta linear time invariant systems

A. Fractional nabla and delta derivatives

Consider that our working domain is the *time scale*

$$\mathbb{T} = (h\mathbb{Z}) = \{\ldots, -3h, -2h, -h, 0, h, 2h, 3h, \ldots\},$$

with $h \in \mathbb{R}^+$. We can consider a slided time scale by a given value, $a < h$, but it does not bring any new relevant notion for this paper.

Set $t = nh$. We define the *nabla derivative* by:

$$f'_{\nabla}(t) = \frac{f(t) - f(t-h)}{h} \tag{1}$$

and the *delta derivative* by

$$f'_{\Delta}(t) = \frac{f(t+h) - f(t)}{h}. \tag{2}$$

As it can be seen, the first derivative is causal, while the second is anti-causal. The repeated application of these derivatives allows us to obtain the N^{th}, ($N \in \mathbb{N}$) order derivatives and from them the general non integer order (i. e., $\alpha \in \mathbb{R}$) formulations [22]:

$$f^{(\alpha)}_{\nabla}(t) = \frac{\sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t-nh)}{h^{\alpha}} \tag{3}$$

and

$$f^{(\alpha)}_{\Delta}(t) = e^{-j\alpha\pi} \frac{\sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t+nh)}{h^{\alpha}}, \tag{4}$$

obtained from the generalised Grünwald-Letnikov (GL) derivative (see [2]). The symbol $(-\alpha)_n$ stands for the

Pochhamer representation of the raising factorial: $(-\alpha)_0 = 1$, $(-\alpha)_n = \prod_{k=0}^{n-1}(-\alpha+k)$. We will call these derivatives *forward* and *backward* respectively (this terminology is the reverse of the one used in some mathematical literature). The exponential factor in (4) is frequently removed, mainly when the variable is not time. These formulations for the fractional derivatives state different forms from those we find in some current texts on discrete fractional calculus (see [17], [16]), but are suitable for generalizing classic tools like the impulse and frequency responses.

Example II.1. *Consider the Heaviside unit step:*

$$\varepsilon(nh) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases} \tag{5}$$

with $n \in \mathbb{Z}$. It is straightforward to show that the nabla derivative of the unit step is the discrete-time impulse

$$D_\nabla \varepsilon(nh) = \begin{cases} \frac{1}{h}, & n = 0 \\ \\ 0, & n \neq 0. \end{cases} \tag{6}$$

The anti-causal unit step is given by $\varepsilon(-nh)$. Using (2) we obtain

$$D_\Delta \varepsilon(-nh) = \begin{cases} -\frac{1}{h}, & n = 0 \\ \\ 0, & n \neq 0. \end{cases} \tag{7}$$

The above defined derivatives enjoy the same properties as the continuous-time (CT) derivatives [22], namely

- Linearity
- Causality
- Time reversal
  The substitution $t \Rightarrow -t$, converts the forward (nabla) derivative into the backward (delta) and vice-versa.
- Additivity and Commutativity of the orders
  Let α and β real numbers. Then,
  $D^\alpha\left[D^\beta f(t)\right] = D^\beta\left[D^\alpha f(t)\right] = D^{\alpha+\beta} f(t)$
- Neutral element
  The existence of neutral element is a consequence of the previous property. Letting $\beta = -\alpha$, then it results $D^\alpha\left[D^{-\alpha} f(t)\right] = D^0 f(t) = f(t)$.
- Inverse element
  From the last result we conclude that there is always an anti-derivative.

Example II.2. *Fractional derivatives of impulses*
It is straightforward to show that the derivative of any order of the impulse is essentially given by the binomial coefficients. In fact, from (3) and (4) we get

$$D_\nabla^\alpha \delta(n) = h^{-\alpha-1}\frac{(-\alpha)_n}{n!}\varepsilon(nh) \tag{8}$$

and

$$D_\Delta^\alpha \delta(n) = (-h)^{-\alpha-1}\frac{(-\alpha)_{-n}}{(-n)!}\varepsilon(-nh). \tag{9}$$

According to the above properties, it is easy to obtain the fractional derivative of the step functions. We only have to substitute $\alpha - 1$ for α and divide by h, so that:

$$D_\nabla^\alpha \varepsilon(nh) = h^{-\alpha}\frac{(-\alpha+1)_n}{n!}\varepsilon(nh)$$

and

$$D_\Delta^\alpha \varepsilon(nh) = (-h)^{-\alpha}\frac{(-\alpha+1)_{-n}}{(-n)!}\varepsilon(-nh).$$

Example II.3. *Fractional derivatives of the discrete power functions*
For negative values of α these expressions can be considered the definitions of fractional discrete "powers". Their derivatives are given by

$$D_\nabla^\beta\left[\frac{(a)_n}{n!}\varepsilon(nh)\right] = h^{-\beta+1}\frac{(a-\beta)_n}{n!}\varepsilon(nh), \tag{10}$$

which represents the analogue of the derivative of the power function. Similarly we obtain for the delta derivative

$$D_\Delta^\beta\left[\frac{(a)_{-n}}{(-n)!}\varepsilon(-nh)\right] = (-h)^{-\beta+1}\frac{(a-\beta)_n}{(-n)!}\varepsilon(-nh). \tag{11}$$

Remark II.1. *It can be shown that these "powers" tend to the causal CT powers when $h \to 0$ [27].*

Remark II.2. *The theory described in this and in the following sub-sections can be generalised for irregular time scales. However, such study goes beyond the objectives of this work [27].*

B. The nabla and delta exponentials

As we discussed for the CT LS [2], the usual (eternal) exponentials, $e^{st}, t \in \mathbb{R}$ and $s \in \mathbb{C}$, are the eigenfunctions of such systems. As it is well known, the discrete-time exponentials, z^n, $n \in \mathbb{Z}$, are the eigenfunctions of the discrete systems described by difference equations. Here, we introduce the exponentials suitable for dealing with the systems defined by the nabla and delta derivatives. These exponentials play a similar role to those discussed in the study of CT LS.
The *nabla exponential* is defined by:

$$e_\nabla(t,s) = [1-sh]^{-t/h}, \tag{12}$$

where $s \in \mathbb{C}$. Similarly, the *delta exponential* is given by:

$$e_\Delta(t,s) = [1+sh]^{t/h}. \tag{13}$$

The corresponding complex sinusoids are obtained when s is over the circles $[1-sh] = 1$ and $[1+sh] = 1$, $s \in \mathbb{C}$, respectively. They are called right and left Hilger circles [15], [16]. The main properties of the exponentials read [22]

1) Relation between the nabla and delta exponentials:
$$e_\Delta(t,s) = 1/e_\nabla(t,-s) = e_\Delta(-t,-s). \tag{14}$$
2) As $h \to 0$ both exponentials converge to e^{st}.
3) Eigenfunctions
$$D_\nabla^\alpha e_\nabla(t,s) = s^\alpha e_\nabla(t,s) \tag{15}$$
and
$$D_\Delta^\alpha e_\Delta(t,s) = s^\alpha e_\Delta(t,s). \tag{16}$$
4) Behaviour on $\mathbb{C}$
The nabla exponential increases/decreases, as follows (for the delta exponential the results are similar) [15], [16].

- It is real for real s.
- It is positive for $s = x < \frac{1}{h},\ x \in \mathbb{R}$.
- It oscillates for $s = x > \frac{1}{h},\ x \in \mathbb{R}$.
- For values of s inside the right Hilger circle it is bounded and goes to zero as $s \to \frac{1}{h}$.
- On the Hilger circle it has absolute value equal to 1 and it degenerates into a sisoid.
- Outside the Hilger circle its absolute value increases as $|s|$ increases and goes to infinite as $|s| \to \infty$.

5) Delayed exponentials
Let $n_0 \in \mathbb{Z}^+$. The delayed exponentials verify the following relations

$$e_\nabla(t \mp n_0 h, s) = e_\nabla(t, s) \cdot e_\Delta(\pm n_0 h, -s) \tag{17}$$

and

$$e_\Delta(t \mp n_0 h, s) = e_\Delta(t, s) \cdot e_\nabla(\pm n_0 h, -s). \tag{18}$$

6) *Product of exponentials*

- *Different types at the same time*

$$e_\nabla(t, s) \cdot e_\Delta(t, v) = e_\nabla(t, \tfrac{s+v}{1+vh}) = e_\Delta(t, \tfrac{s+v}{1-sh}), \tag{19}$$

with $v \in \mathbb{C}$

- *Different types at distincts times*

$$\begin{aligned} e_\nabla(t, s) \cdot e_\Delta(\tau, -s) &= e_\nabla(t - \tau, s) \\ e_\nabla(t, -s) \cdot e_\Delta(\tau, s) &= e_\Delta(\tau - t, s), \end{aligned} \tag{20}$$

with $\tau \in \mathbb{T}$.

- *Same type*

$$e_\nabla(t, s) \cdot e_\nabla(t, v) = e_\nabla(t, s + v - svh)$$

and

$$e_\Delta(t, s) \cdot e_\Delta(t, v) = e_\Delta(t, s + v + svh).$$

7) *Cross derivatives*

$$D_\Delta e_\nabla(t, s) = sh \cdot e_\nabla(t + h, s)$$

$$D_\nabla e_\Delta(t, s) = sh \cdot e_\Delta(t - h, s)$$

C. Nabla Laplace transform

The nabla and delta exponentials allow us to define two transforms [22]. Let $f(nh)$ be a signal and assume that it has a Nabla Laplace transform (NLT), $F_\nabla(s)$. The analysis equation for the NLT is given by

$$\mathcal{N}\left[f(nh)\right] = F_\nabla(s) = h \sum_{n=-\infty}^{+\infty} f(nh) e_\Delta\left(nh, -s\right). \tag{21}$$

Its inverse transform (synthesis equation) is given by

$$f(nh) = \mathcal{N}^{-1}\left[F_\nabla(s)\right] = -\frac{1}{2\pi j} \oint_\gamma F_\nabla(s) e_\nabla\left((n+1)h, s\right) ds, \tag{22}$$

where the integration path, γ, is any simple closed contour in a region of analyticity of the integrand that includes the point $s = \frac{1}{h}$. The simplest path is a circle with centre at $s = \frac{1}{h}$. Attending to the properties of the exponential that we listed before, the limit as $h \to 0$ in (21) leads to the usual two-sided LT.

We assume that s is inside the ROC of the transform. The NLT enjoys the following properties

- Linearity
- Transform of the derivative

$$\mathcal{N}\left[f_\nabla^{(\alpha)}(nh)\right] = s^\alpha F_\nabla(s), \tag{23}$$

reproducing a well known property of the CT Laplace transform. The ROC is the disk inside the Hilger circle.

- Time shift
The NLT of $f(nh - n_0 h)$, with $n_0 \in \mathbb{Z}$, is given by:

$$\mathcal{N}\left[f(nh - n_0 h)\right] = e_\Delta(n_0 h, -s) F_\nabla(s). \tag{24}$$

- Convolution in time

$$\mathcal{N}\left[h \sum_{k=-\infty}^{+\infty} f(kh) g(nh - kh)\right] = F_\nabla(s) \cdot G_\nabla(s). \tag{25}$$

From (8) we obtain immediately the impulse response (IR) of the “differintegrator” s^α. In particular, when $\alpha = -1$, we get $\mathcal{N}^{-1} s^{-1} = \varepsilon(n)$, which leads to a classic result: *the step response is the accumulation (“integral”) of the IR*

$$r_\varepsilon(nh) = h \sum_{k=0}^{n} f(kh).$$

Remark II.3. *As it can be verified by computation, the NLT of the correlation requires the introduction of the Delta LT that is defined by [22]*

$$F_\Delta(s) = h \sum_{n=-\infty}^{+\infty} f(nh) e_\nabla(nh, -s). \tag{26}$$

The corresponding inverse is given by

$$f(nh) = \frac{1}{2\pi j} \oint F_\Delta(s) e_\Delta((n-1)h, s) \mathrm{d}s. \tag{27}$$

Example II.4 (Causal and anti-causal exponentials)**.** *A nabla causal exponential is defined by*

$$e_{c\nabla}(t, p) = e_\nabla(t + h, p) \cdot \varepsilon(t).$$

for any $p \in \mathbb{C}$. Its NLT reads [22]:

$$\mathcal{N}\left[e_\nabla(t + h, p) \cdot \varepsilon(t)\right] = \frac{1}{s - p}. \tag{28}$$

The ROC is defined by all points distancing from $1/h$ less than $\left|p - \frac{1}{h}\right|$. A simple criterion imposes that $\left|s - \frac{1}{h}\right| < 1$ and $\left|p - \frac{1}{h}\right| > 1$. The pole must stay outside the Hilger circle.
The anti-causal exponential is defined by

$$e_{a\nabla}(t, p) = -e_\nabla(t + h, p) \cdot \varepsilon(-t - h)$$

$$NLT\left[-e_\nabla(t + h, p) \cdot \varepsilon(-t - h)\right] = \frac{1}{s - p}. \tag{29}$$

The ROC of (29) *is the inverse of the one for the causal: the pole must be inside the Hilger circle.*
We can generalise the above results for multiple poles by

considering $\frac{1}{s-p}$ as a function of p and computing sucessive integer order derivatives.

Letting $p = 0$ in the expressions (28) and (29) we obtain immediately the NLT of the unit step and powers. For the causal case, we obtain

$$\mathcal{N}\left[h^{\alpha}\frac{(\alpha+1)_n}{n!}\varepsilon(nh)\right] = \frac{1}{s^{\alpha+1}}. \tag{30}$$

The ROC is the disk inside the Hilger circle.

D. The Discrete-Time Fourier transform

Similarly to the CT case, the exponential degenerates into a sisoid when its absolute value is equal to 1. From (12), $|1 - sh| = 1$ defines a circle centred at $1/h$ and has radius equal to $1/h$ (right hand Hilger circle). Similarly, the left Hilger circle comes from $|1 + sh| = 1$ (13). With the change of variable $s = \frac{1-e^{-j\omega h}}{h}$ in (12), (21), and (22), we obtain

$$e_{\nabla}(nh,\omega) = e^{j\omega hn}, \;\; n \in \mathbb{Z},$$

$$F(e^{j\omega}) = h\sum_{-\infty}^{\infty} f(nh)e^{-j\omega hn},$$

$$f(nh) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} F(e^{j\omega})e^{j\omega hn}\mathrm{d}\omega.$$

With the substitution $\omega h = \Omega$ we obtain expressions that are independent of the sampling interval (graininess) h

$$e_{\nabla}(n,\Omega) = e^{j\Omega n},$$

$$F(e^{j\Omega}) = h\sum_{-\infty}^{\infty} f(nh)e^{-j\Omega n} \tag{31}$$

and

$$f(nh) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\Omega})e^{j\Omega n}\mathrm{d}\Omega. \tag{32}$$

Expressions (31) and (32), define the discrete-time Fourier transform pair (DTFT), namely the direct and inverse transforms, respectively [28], [24]. The presence of the factor h in (31) ensures a back compatibility with the CT Fourier transform.

E. The nabla linear systems

1) Steady-state response: We define a nabla linear system through the following differential equation [22]

$$\sum_{k=0}^{N} a_k D_{\nabla}^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k D_{\nabla}^{\beta_k} x(t) \tag{33}$$

where a_k and b_k ($k = 0, 1, \cdots,$) with $a_N = 1$ are real numbers. The operator D_{∇} is the nabla derivative defined previously. The orders N and M are any positive integers. The positive real numbers α_k and β_k with $k = 0, 1, \cdots,$ form strictly increasing sequences. It is interesting to remark that when $\alpha_k = k$, relation (33) can be transformed into a difference equation, by using (1) and the the binomial decomposition.

Let $g(t)$ be the IR of the system defined by (33) with $x(t) = \delta(nh)$. The output is the convolution (25) of the input and the IR

$$y(t) = g(t) * x(t). \tag{34}$$

If $x(t) = e_{\nabla}(nh, s)$, then the output is given by:

$$y(t) = e_{\nabla}(nh,s)\left[h\sum_{n=-\infty}^{\infty} g(nh)e_{\Delta}(nh,-s)\right].$$

The summation expression is the *transfer function* as usually. We have

$$G_{\nabla}(s) = h\sum_{n=-\infty}^{\infty} g(nh)e_{\Delta}(nh,-s), \tag{35}$$

showing that the TF is the NLT of the IR [22]. With these results we can write the TF as

$$G_{\nabla}(s) = \frac{\sum\limits_{k=0}^{M} b_k s^{\beta_k}}{\sum\limits_{k=0}^{N} a_k s^{\alpha_k}}. \tag{36}$$

As in the CT case, we conclude that:

- The exponentials are the eigenfunctions of the linear systems (33)
- The eigenvalues are the transfer function values.

Let us analyse the following example.

Example II.5. *Let $h = 1$ and consider the differential equation*

$$y'''(t) + y''(t) - 4y'(t) + 2y(t) = x(t).$$

If $x(n) = 2^{-n}$, that corresponds to set $s = -1$ in (12)*, then the solution is given by:*

$$y(n) = \frac{1}{(-1)^3 + (-1)^2 + 4 + 2}2^{-n} = \frac{1}{6}2^{-n}.$$

Now we are in conditions of defining the *frequency response* (FR) of the system (33). We only need to make the substitution of variable $s = \frac{1-e^{-j\omega h}}{h}$ in (36). This involves the transformation of the parameters of the TF using the binomial coefficients. We conclude that the TF (36) gives rise to the following FR

$$G(e^{j\omega}) = \frac{\sum\limits_{k=0}^{M_0} B_k e^{-j\omega k}}{\sum\limits_{k=0}^{N_0} A_k e^{-j\omega k}}, \tag{37}$$

where the coefficients B_k are given by

$$B_k = \sum_{l=1}^{M} b_l h^{-\alpha l-1}\frac{(-\alpha l)_k}{k!}, \tag{38}$$

for $k = 0, 1, 2, \ldots, M_0$. For the A_k coefficients, the computation is similar.

Computing the inverse FT of $G(e^{j\omega})$ we obtain a difference equation equivalent to the differential equation (33). As observed, for practical purposes, we have to truncate the sequence.

2) Transient responses: In general, the systems described by (33) or (36) are IIR systems, difficult to study due to the problems in computing the poles and zeros. We only have FIR systems when $N = 0$ and all the derivative orders are positive integers.

As for the CT we will consider the commensurate case. The TF assumes the form

$$G_\nabla(s) = \frac{\sum\limits_{k=0}^{M} b_k s^{\alpha k}}{\sum\limits_{k=0}^{N} a_k s^{\alpha k}}. \quad (39)$$

The fraction in (39) can be decomposed into a sum of a polynomial (only zeros) and a proper fraction (pole-zero). We are going to study the two cases separately.

1) Polynomial case

Let us consider a transfer function with the form

$$G_\nabla(s) = \sum_{k=0}^{M} b_k s^{\alpha k}. \quad (40)$$

To invert this expression, we recall the previous statement regarding s^α that is the TF of the differentiator. This is a system with IR given by the binomial coefficients in agreement with (8). The IR corresponding to (40) is

$$g(nh) = b_0 \delta(nh) + \sum_{k=1}^{M} b_k h^{-k\alpha-1} \frac{(-\alpha k)_n}{n!} \varepsilon(nh). \quad (41)$$

Theoretically they are IIR systems, but the IR may go to zero sufficiently fast to exhibit an FIR behaviour.

2) Proper fraction case

Herein, we consider that we have M zeroes and N poles in (39) with $N > M$. For the sake of simplicity we assume that the poles have multiplicity one. In this case we can write $G_\nabla(s)$ as

$$G_\nabla(s) = \sum_{k=1}^{N} \frac{A_k}{s^\alpha - p_k}, \quad (42)$$

where A_k and p_k, $k = 1, 2, \ldots, N$, are the residues and poles obtained by substituting w for s^α in (39), respectively. The IR can be obtained by inverting a combination of partial fractions such:

$$F_\nabla(s) = \frac{A}{s^\alpha - p}. \quad (43)$$

To invert (43) we can insert it into the inversion integral, (22). The residue theorem tells us that the inversion is obtained by computing the $(n+1)^{th}$ derivative of $F(s)$ and substituting $1/h$ for s. A simpler alternative is to use the properties of the geometric series. We have two possibilities corresponding to the regions:

a) *Intersection of the Hilger circle with the disk* $|s| < |p|^{\frac{1}{\alpha}}$

In this case we have

$$F_\nabla(s) = -\frac{A}{p}\frac{1}{1-\frac{s^\alpha}{p}} = -\frac{A}{p}\left[1 + \sum_{k=1}^{\infty} p^{-k} s^{\alpha k}\right]. \quad (44)$$

From the above expression it results that the IR $f(nh)$, corresponding to a partial fraction of order one, is given by

$$f(nh) = -A\left[\sum_{k=0}^{\infty} p^{-k-1} h^{-\alpha k-1} \frac{(-\alpha k)_n}{n!} \varepsilon(nh)\right]. \quad (45)$$

However, it will not be an interesting system in applications, since the values of p must be large for small values of h.

b) *Intersection of the Hilger circle with the disk* $|s| > |p|^{\frac{1}{\alpha}}$

For this case, we can write

$$F_\nabla(s) = A\frac{s^{-\alpha}}{1 - ps^{-\alpha}} = A\left[\sum_{k=1}^{\infty} p^{k-1} s^{-\alpha k}\right]. \quad (46)$$

The corresponding IR, $f(nh)$, is given by

$$f(nh) = A\left[\sum_{k=1}^{\infty} p^{k-1} h^{\alpha k-1} \frac{(\alpha k)_n}{n!} \varepsilon(nh)\right]. \quad (47)$$

This expression is the discrete-time version of the α-exponential [29] that we find in CT systems and that is related to the Mittag-Leffler function (MLF). Therefore, this case is interesting in practice, since it allows approaching CT systems closely using small values of h.

For a complex conjugate pair we obtain:

$$v(nh) = 2\sum_{k=1}^{\infty} Re\left\{Ap^{k-1}\right\} h^{\alpha k-1} \frac{(\alpha k)_n}{n!} \varepsilon(nh). \quad (48)$$

The properties of the nabla exponential and the sequence of operations that we followed to compute the NLT of the causal transform showed that if the poles are outside the right Hilger circle then the system is stable. In parallel, the partial fraction inversions computed previously showed that the series defining the time functions are convergent if $|p|h^\alpha > 1$, and $|p|h^\alpha < 1$, in the first (44) and second (46) cases, respectively. This means that if p is outside the Hilger circle, then the system is stable. Moreover, the system can be stable even if the pole is located inside the Hilger circle provided that it is outside the circle $|s| = |p|^{\frac{1}{\alpha}}$. This represents the discrete counterpart of the stability criterion for CT systems [30]. For the integer order systems we can study the pole distribution by a Routh-Hurwitz like criterion [31]. It is important to remark again that the exterior (interior) of the right Hilger circle degenerates in the left (right) half complex plane when $h \to 0$.

F. On the initial conditions

The initial condition problem was discussed in the companion paper [2]. Therefore, everything argued in that paper remains valid here as long as we consider the required adaptation. For the commensurate case, this can be found in [22] and reads

$$\mathcal{N}\left[g_{\nabla}^{(N\alpha)}(t)\varepsilon(t)\right] = s^{N\alpha}\mathcal{N}\left[f(t)\varepsilon(t)\right] - \sum_{k=0}^{N-1} g_{\nabla}^{(k\alpha)}(-h)s^{(N-k)\alpha-1} \tag{49}$$

With this expression we can insert the initial conditions in any system as proposed in [32] and [33].

III. The bilinear transformation based linear systems

The systems described in the previous section are useful for introducing the discrete-time systems defined by fractional discrete-differential equations. We could define the standard tools, such us the IR and TF. However, these systems are somehow different from the standard discrete-time descriptions, since they have stability domains that are defined by the Hilger circles, instead of the unit circle that is the reference in the theory of classic discrete-time systems. On the other hand, they require new LT instead of the ZT. This observation motivated the search for an alternative for defining fractional linear systems.

In the DT approximation of CT systems, the Tustin (bilinear) transformation [23] is of current use. In fact, the definition of DT LS having as base such transformation was proposed just recently [26]. The Tustin transformation is a particular case of the conformal Möbius mapping. As it is well-known, it establishes a bijection between the left (right) half complex plane and the interior (exterior) of the unit disk. This property allows us to consider such transformation as the base for defining alternative discrete-time derivatives and the fractional systems that mimic the analogous CT versions. Such strategy enable us to adopt the tools and the results available in the DT domain for the CT fractional systems introduced in the companion paper [2]. Moreover, the proposed derivatives and systems have the important feature of being suitable to be implemented through the FFT with the corresponding advantages, from the numerical and calculation time perspectives.

A. Forward and backward derivatives based on the bilinear transformation

The Tustin transformation is usually expressed by [6], [5]

$$s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}, \tag{50}$$

where h is the sampling interval, s is the derivative operator associated with the (continuous-time) LT and z^{-1} the delay operator tied with the Z transform.

Let $x(nh)$ be a discrete-time function, we define the order 1 *forward bilinear derivative* $Dx(nh)$ of $x(nh)$ as the solution of the difference equation

$$Dx(nh) + Dx(nh-h) = \frac{2}{h}\left[x(nh) - x(nh-h)\right]. \tag{51}$$

Similarly, we define the order 1 *backward bilinear derivative* $Dx(nh)$ of $x(nh)$ as the solution of

$$Dx(nh+h) + Dx(nh) = \frac{2}{h}\left[x(nh+h) - x(nh)\right]. \tag{52}$$

The bilinear exponential $e_s(nh)$ is the eigenfunction of the equations (51) or (52). If we set $x(nh) = e_s(nh)$, $y(nh) = se_s(nh)$, $s \in \mathbb{C}$, with $e_s(0) = 1$, then

$$e_s(nh) = \left(\frac{2+hs}{2-hs}\right)^n, \quad n \in \mathbb{Z},\ s \in \mathbb{C}. \tag{53}$$

The properties of the bilinear exponential $e_s(nh)$ are

- When $n \to \infty$, this exponential
  - Increases, if $Re(s) > 0$,
  - Decreases, if $Re(s) < 0$,
  - Is sinusoidal, if $Re(s) = 0$, with $s \neq 0$,
  - Is constant equal to 1, if $s = 0$,
- It is real for real s,
- It is positive for $s = |x| < \frac{2}{h}$, $x \in \mathbb{R}$,
- It oscillates for $s = |x| > \frac{2}{h}$, $x \in \mathbb{R}$.

Following the procedure in the previous section, we could use this exponential to construct a bilinear discrete-time LT. However, formula (53) suggests that $z = \frac{2+hs}{2-hs}$ leads to the ZT, since such transformation sets the unit circle $|z| = 1$ as the image of the imaginary axis in s, independently of the value of h. Accordingly, the bilinear exponential has the usual properties:

- When $n \to \infty$, this exponential
  - Increases, if $|z| > 1$,
  - Decreases, if $|z| < 1$,
  - Is sinusoidal, if $|z| = 1$, with $z \neq 1$,
  - Is constant equal to 1, if $z = 1$,
- It is real for real z,
- It is positive for $z = x > 0$, $x \in \mathbb{R}$,
- It oscillates for $z \neq \mathbb{R}_0^+$.

Therefore, we do not need to introduce a new transform, since the ZT is suitable.

In what concerns derivative definitions, instead of considering (51) or (52), we start from the ZT formulations. Let $z \in \mathbb{C}$ and $h \in \mathbb{R}^+$. Consider the discrete-time exponential function, z^n, $n \in \mathbb{Z}$. We define the forward bilinear derivative (D_f) as an elemental DT system such that

$$D_f z^n = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}z^n. \tag{54}$$

The forward TF of such derivative, $H_f(z)$, is defined by

$$H_f(z) = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}, \qquad |z| > 1. \tag{55}$$

The backward bilinear derivative (D_b) is defined as the system verifying

$$D_b z^n = \frac{2}{h}\frac{z-1}{z+1}z^n, \tag{56}$$

with backward TF, $H_b(z)$, given by

$$H_b(z) = \frac{2}{h}\frac{z-1}{z+1}, \qquad |z| < 1. \tag{57}$$

The repeated application of the above operators lead to the forward and backward derivatives for any positive integer order that can be generalized for any real order. Let $\alpha \in \mathbb{R}$. The $\alpha-$order forward bilinear FD is a DT LS with TF

$$H_f(z) = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha, \qquad |z| > 1, \tag{58}$$

such that

$$D_f^\alpha z^n = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha z^n, \qquad |z|>1. \tag{59}$$

Identically, the backward bilinear FD has TF

$$H_b(z) = \left(\frac{2}{h}\frac{z-1}{z+1}\right)^\alpha, \qquad |z|<1, \tag{60}$$

such that

$$D_b^\alpha z^n = \left(\frac{2}{h}\frac{z-1}{z+1}\right)^\alpha z^n, \qquad |z|<1. \tag{61}$$

Once we defined the derivative of an exponential we are in conditions of obtaining the derivative of any signal having ZT. We only have to use the inversion integral of the ZT

$$x(n) = \frac{1}{2\pi j}\oint_\gamma X(z)z^{n-1}\mathrm{d}z. \tag{62}$$

From (62) and (59) we conclude that, if $x(n)$ is a function with ZT $X(z)$, analytic in the ROC defined by $z\in\mathbb{C}:|z|>a,\ \ a<1$, then

$$D_f^\alpha x(n) = \frac{1}{2\pi ij}\oint_\gamma \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha X(z)z^{n-1}\mathrm{d}z, \tag{63}$$

with the integration path outside the unit disk. This implies that

$$\mathcal{Z}\left[D_f^\alpha x(n)\right] = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha X(z), \qquad |z|>1. \tag{64}$$

Let $x(n)$ be a function with ZT $X(z)$, analytic in the ROC defined by $z\in\mathbb{C}:|z|<a,\ \ a>1$. We define

$$D_b^\alpha x(n) = \frac{1}{2\pi j}\oint_\gamma \left(\frac{2}{h}\frac{z-1}{z+1}\right)^\alpha X(z)z^{n-1}\mathrm{d}z, \tag{65}$$

with the integration path inside the unit disk and the branchcut line is a segment joining the points $z=\pm1$. This implies that

$$\mathcal{Z}\left[D_b^\alpha x(n)\right] = \left(\frac{2}{h}\frac{z-1}{z+1}\right)^\alpha X(z), \qquad |z|<1. \tag{66}$$

Remark III.1. *We must note that:*

1) *In (58) and (59) we have two branchcut points at $z=\pm1$. The corresponding branchcut line is any line connecting these values and being located in the unit disk. The simplest is a straight line segment (see figure 1).*
2) *In (60) and (61) we have the same branchcut points, but with branchcut line(s) lying outside the unit disk. For simplifying, we can use two half-straight lines starting at $z=\pm1$ on the real negative and positive half lines, respectively (see figure 1).*
3) *In both of the previous cases, we can extend the domain of validity to include the unit circumference, $z=e^{j\omega n},\ \ |\omega|\in(0,\pi)$, with exception of the points $z=\pm1$. In these cases, the integration path in (63) must be deformed around such points. This deformation is very important when using the FFT. In such situation, a small numerical trick can be used: push the branchcut points slightly inside (outside) the unit circle, that is, to $z=-1+\epsilon$ and $z=1-\epsilon$ $(-1-\epsilon, 1+\epsilon)$, with ϵ being a small positive real number.*

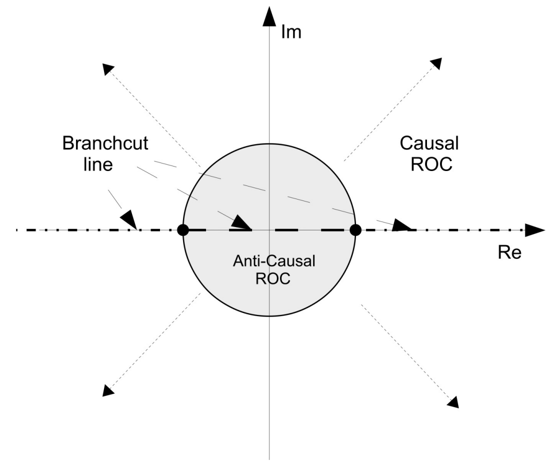

Fig. 1. ROC for causal and anti-causal derivatives and branchcut points and lines.

4) *The ROC is independent on the scale graininess, h, and consequently we can establish a one to one correspondence between the unit disk, in z, and the left half-plane, in s given by $s=\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}$.*

According to what we just wrote, we can extend the above definitions to include sinusoids. We define the derivative of $x(n)=e^{j\omega n},\ \ n\in\mathbb{Z}$, through

$$D_{f,b}\ e^{j\omega n} = \left[\frac{2}{h}\tan\left(\frac{\omega}{2}\right)\right]^\alpha e^{j\omega n}, \qquad |\omega|<\pi, \tag{67}$$

independently of considering the forward or backward derivatives. With this result, we can obtain de derivative of any function having discrete-time FT, that is expressed as:

$$D_{f,b}\ x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\left[\frac{2}{h}\tan\left(\frac{\omega}{2}\right)\right]^\alpha e^{j\omega n}\mathrm{d}\omega. \tag{68}$$

In accordance with the existence conditions of the FT, we can say that, if $x(n)$ is absolutely summable, then the derivative (68) exists.

These derivatives enjoy the same properties of the nabla and delta derivatives [26].

B. Time formulations

In the previous subsection we introduced the derivatives using a formulation based on the ZT. However, we can obtain the corresponding time framework, getting formulae similar to the GL derivatives [34]. From the binomial series [35]

$$(1\pm w)^a = \sum_{k=0}^\infty \frac{(\mp1)^k(-a)_k}{k!}w^k, \qquad |w|<1,$$

we conclude that the TF in (58) and (60) can be expressed as power series,

$$\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha = \sum_{k=0}^\infty \psi_k^\alpha z^{-k},\ |z|>1,$$

where $\psi_k^\alpha,\ k=0,1,\cdots$, is the inverse ZT of $\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha$ and represents the IR (see Appendix).

In agreement with the meaning attributed to the sequence $\psi_k^\alpha, \quad k=0,1,\cdots$, we define the $\alpha-$order bilinear forward and backward derivatives as

$$D_f^{(\alpha)}x(n) = \left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(n-k) \tag{69}$$

and

$$D_b^{(\alpha)}x(n) = e^{i\alpha\pi}\left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(n+k). \tag{70}$$

As before, we can remove the exponential factor, $e^{i\alpha\pi}$, in (70). In the following we consider the causal derivative (69) represented by the simplified notation D^α and with ZT given by (64).

Some interesting results are [26]

- The first is causal while the second is anti-causal.
- *Fractional derivative of the impulse*
  Let us introduce the Kroneckker impulse, $\delta(n),\ n \in \mathbb{Z}$, by
  $$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}.$$
  The Heaviside discrete unit step is usually defined by
  $$\varepsilon(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$
  and its ZT is given by
  $$\mathcal{Z}\left[\varepsilon(n)\right] = \frac{1}{1-z^{-1}}, \ \ |z| > 1.$$
  As we can see, the derivative of any order of the Kroneckker impulse is essentially given by the coefficients ψ_n^α defined in the Appendix. In fact, from (69) we get
  $$D^\alpha \delta(n) = \left(\frac{2}{h}\right)^\alpha \psi_n^\alpha \varepsilon(n), \tag{71}$$
  where $\varepsilon(n)$ is used to express the right behaviour of the derivative of the delta, stating the causality of the operator.
- *Fractional derivative of the unit step*
  The function ψ_n^{-1}, introduced in the example A.1, is a modified version of the unit step. It is straightforward to confirm that
  $$\psi_n^{-1} = 2\varepsilon(n) - \delta(n),$$
  with ZT given by $\mathcal{Z}\left[\psi_n^{-1}\right] = \frac{h}{2}\frac{1+z^{-1}}{1-z^{-1}}, \quad |z| > 1,$ as expected. According to the above properties, we can obtain the FD of the unit step function. We have
  $$\varepsilon(n) = \frac{1}{2}\psi_n^{-1} + \frac{1}{2}\delta(n).$$
  Consequently
  $$D^\alpha \varepsilon(n) = \frac{1}{2}\left(\frac{2}{h}\right)^{\alpha-1}\psi_n^{\alpha-1} + \frac{1}{2}\left(\frac{2}{h}\right)^\alpha \psi_n^\alpha.$$
- *Fractional derivative of the ψ function*
  We are interested in computing the derivative of ψ_n^α, for any α with $n \in \mathbb{Z}$. From (71) and the additivity property, we can write
  $$D^\beta D^\alpha \delta(n) = D^\beta \left[\left(\frac{2}{h}\right)^\alpha \psi_n^\alpha\right] = \left(\frac{2}{h}\right)^{\alpha+\beta} \psi_n^{\alpha+\beta}\varepsilon(n)$$
  which leads to
  $$D^\beta\left[\psi_n^\alpha\right] = \left(\frac{2}{h}\right)^\beta \psi_n^{\alpha+\beta}\varepsilon(n). \tag{72}$$

1) Backward compatibility: Usually, DT systems are considered as mere approximations of their CT counterparts. Nevertheless, and as shown above, the DT systems exist by themselves and have properties that, although similar to, are independent from the CT analogues. However, this observation does not prevent us from establishing a continuous path from each other. In fact, we can go from the discrete into the continuous domain by reducing the graininess. To see it, let us return to (69) and rewrite it as

$$D_f^{(\alpha)}x(nh) = \left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(nh-kh).$$

Assume that $x(nh)$ resulted from a CT function $x(t)$ and define a new function, $y(t)$, by

$$y(t) = \left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(t-kh). \tag{73}$$

The LT of expression (73) is

$$Y(s) = \left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha e^{-khs}X(s) = \left(\frac{2}{h}\frac{1-e^{-hs}}{1+e^{-hs}}\right)^\alpha X(s), \tag{74}$$

where $Y(s) = \mathcal{L}\left[y(t)\right]$ and $X(s) = \mathcal{L}\left[x(t)\right]$. Knowing that $\lim\limits_{h\to 0} \frac{1-e^{-hs}}{h} = s$, we can write

$$Y(s) = s^\alpha X(s), \ \ Re(s) > 0,$$

meaning that $Y(s)$ is the LT of the (continuous-time) derivative of $x(t)$. This relation expresses the compatibility between the new formulation described above and the well known results from the continuous-time derivative formulation [34] (see the companion paper [2]). With the backward formulation, we would obtain the same result, but with a ROC valid for $Re(s) < 0$. The above equations together with (69), lead to the conclusion that, for $t \in \mathbb{R}$, we can write:

$$D_f^{(\alpha)}x(t) = \lim_{h\to 0}\left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(t-kh). \tag{75}$$

Similarly, from the backward formulation, mainly (70), we obtain

$$D_b^{(\alpha)}x(t) = e^{i\alpha\pi}\lim_{h\to 0}\left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(t+kh). \tag{76}$$

Relations (75) and (76) state two new ways of computing the continuous-time FD that are similar to the Grünwald-Letnikov derivatives. However, it may be interesting to remark that we can compute derivatives with (73) instead of (65).

C. The bilinear discrete-time linear systems

The above derivatives lead us to consider systems defined by constant coefficient differential equations with the general form (33), where the operator D_∇ is substituted by the forward (or backward) derivative previously defined.
Let $g(n)$ be its IR. The output is $y(n) = g(n) * v(n)$. With the definition of forward derivative and mainly formula (64) we write the TF

$$G(z) = \frac{\sum\limits_{k=0}^{M} b_k \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\beta_k}}{\sum\limits_{k=0}^{N} a_k \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha_k}}, \quad |z| > 1, \tag{77}$$

for the causal case, and

$$G(z) = \frac{\sum\limits_{k=0}^{M} b_k \left(\frac{2}{h}\frac{z-1}{z+1}\right)^{\beta_k}}{\sum\limits_{k=0}^{N} a_k \left(\frac{2}{h}\frac{z-1}{z+1}\right)^{\alpha_k}}, \quad |z| < 1, \tag{78}$$

for the anti-causal case. We can give to expressions (77) and (78) a form that states their similarity with the classic fractional LS [33], [36]. For example, for the first, let $v = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)$. We have

$$G(v) = \frac{\sum\limits_{k=0}^{M} b_k v^{\beta_k}}{\sum\limits_{k=0}^{N} a_k v^{\alpha_k}}. \tag{79}$$

Remark III.2. *It is important to note that the factors* $\left(\frac{2}{h}\right)^{\alpha_k}$, $k = 1, 2, \cdots$, *do not have any important role in the computations. Therefore, they can be merged with the coefficients* a_k *and* b_k *.*

The procedure to invert (79) is identical to the one we followed in the nabla system (42). For simplicity we assume that $M < N$ and all the roots, $p_k, k = 1, 2, \cdots$, of $\sum\limits_{k=0}^{N} a_k z^k$ are simple which allows us to write

$$G(v) = \sum_{k=1}^{N} \frac{A_k}{v^\alpha - p_k}, \tag{80}$$

where the A_k and p_k, $k = 1, 2, \ldots, N$, are the residues and pseudo-poles obtained by substituting w for s^α in (39). The IR results from the inversion of a combination of partial fractions such as:

$$F_\nabla(s) = \frac{A}{s^\alpha - p}. \tag{81}$$

Example III.1. *Consider the simple system with TF*

$$G(v) = \frac{1}{v^\alpha + 2}. \tag{82}$$

In figure 2 we represent the step responses for several values of the order, $\alpha = 0.5k$, $k = 1, 2, 3$, *obtained with nabla and bilinear formulations.*

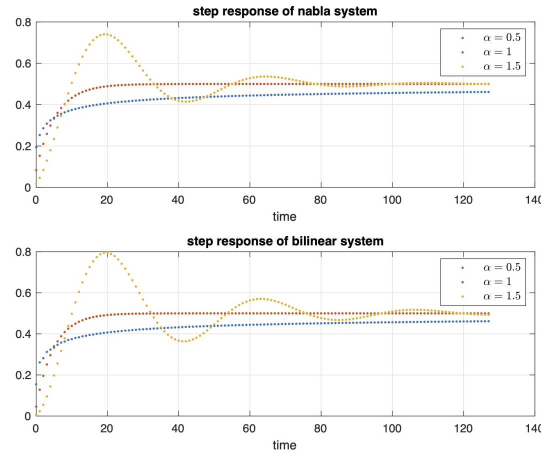

Fig. 2. Step responses of the system (82) for $\alpha = \{0.5, 1, 1.5\}$. (from below) for nabla (above) and bilinear (below) cases, with $h = 0.1$.

IV. State-Variable representation

A. A brief introduction

Three equivalent representations of linear invariant systems were studied in the previous sections: the differential equation, IR, and TF. In the time-variant case, the differential equation remains a valid representation and we can still introduce the notion of IR, but we cannot define a TF. On the other hand, these representations consider the system like a "black box" relating an input with an output, while forgetting what happens "inside" the system. This limitation can be avoided by the representation in *state variables*, also called internal representation. This approach has several advantages, namely, a simple matrix formulation and the ability to formulate, in an identical form, a large number of different cases, such as non-linear, time-varying, or multivariate systems [37], [38], [39], [40], [41], [30]. Besides the input and output signals, this representation introduces another one, the state, that may be vectorial or matricial. Here, we present such a description valid for the fractional LS, not only DT, but also CT.

B. Standard form of the equations

Let $\mathbf{v}(t), \mathbf{y}(t)$, and $\mathbf{x}(t)$ be the input, output, and state of a system respectively described by the following set of equations:

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = \mathbf{\Phi}[t, \mathbf{x}(t), \mathbf{v}(t)] \tag{83}$$
$$y(t) = \mathbf{\Psi}[t, \mathbf{x}(t), \mathbf{v}(t)], \tag{84}$$

where $\boldsymbol{\alpha} = [\alpha_1 \ \alpha_2 \ \ldots \alpha_N]^T$ is a vector with (positive) differentiation orders. Moreover, the derivative $D^{\boldsymbol{\alpha}}$ of vectorial order $\boldsymbol{\alpha}$ is defined as

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = [D^{\alpha_1}x_1(t) \ D^{\alpha_2}x_2(t) \ \ldots \ D^{\alpha_n}x_n(t)]^T \tag{85}$$

and represents any of the previously defined causal derivatives, CT and DT. The expressions (83) and (84) are called *state* (or *dynamic*), and *output* (or *observation*) equations, respectively. We will assume that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are linear functions, so that

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{v}(t), \tag{86}$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{v}(t), \tag{87}$$

where $\mathbf{x}(t)$ is a $N \times 1$ vector and $\mathbf{A}(t)$ is an $N \times N$ matrix. The dimensions of the other matrices are chosen in agreement with the type of system:

- SISO system:

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = \overbrace{\mathbf{A}(t)}^{N \times N \text{ matrix}} \overbrace{\mathbf{x}(t)}^{N \times 1 \text{ vector}} + \overbrace{\mathbf{B}(t)}^{N \times 1 \text{ vector}} \overbrace{\mathbf{v}(t)}^{\text{scalar}} \tag{88}$$

$$\mathbf{y}(t) = \underbrace{\mathbf{C}(t)}_{1 \times N \text{ vector}} \underbrace{\mathbf{x}(t)}_{N \times 1 \text{ vector}} + \underbrace{\mathbf{D}(t)}_{\text{scalar}} \underbrace{\mathbf{v}(t)}_{\text{scalar}} \tag{89}$$

- MIMO system with n_v inputs and n_y outputs:

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = \overbrace{\mathbf{A}(t)}^{N \times N \text{ matrix}} \overbrace{\mathbf{x}(t)}^{N \times 1 \text{ vector}} + \overbrace{\mathbf{B}(t)}^{N \times n_v \text{ matrix}} \overbrace{\mathbf{v}(t)}^{n_v \times 1 \text{ vector}} \tag{90}$$

$$\mathbf{y}(t) = \underbrace{\mathbf{C}(t)}_{n_y \times N \text{ matrix}} \underbrace{\mathbf{x}(t)}_{N \times 1 \text{ vector}} + \underbrace{\mathbf{D}(t)}_{n_y \times n_v \text{ matrix}} \underbrace{\mathbf{v}(t)}_{n_v \times 1 \text{ vector}} \tag{91}$$

The above equations represent the standard form of expressing a linear system by means of *state variables*, which are the elements of the state $\mathbf{x}(t)$. A system is *time invariant* if matrices $\mathbf{A}, \mathbf{B}, \mathbf{C},$ and $\mathbf{D}$ are constant. This is the situation assumed in the following. Therefore, a given LTIS has a state-space representation in the form

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{v}(t) \tag{92}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{v}(t). \tag{93}$$

Define the diagonal matrix diag $(s^{\boldsymbol{\alpha}})$ with diagonal elements $s^{\alpha_k}, \;\; k = 1, 2, \cdots, N$. The LT of (92)-(93) is

$$\text{diag}\,(s^{\boldsymbol{\alpha}})\,\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{V}(s) \tag{94}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{V}(s). \tag{95}$$

From (94),

$$\mathbf{X}(s) = (\text{diag}\,(s^{\boldsymbol{\alpha}}) - \mathbf{A})^{-1}\mathbf{B}\mathbf{V}(s) \tag{96}$$

and replacing this in (95) yields

$$\frac{\mathbf{Y}(s)}{\mathbf{V}(s)} = \mathbf{C}\left[\text{diag}\,(s^{\boldsymbol{\alpha}}) - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}, \tag{97}$$

which is the TF matrix. If the derivative orders are equal, $\alpha_k = \alpha, \; k = 1, 2 \ldots n, \; 0 < \alpha \leq 1$, we obtain the commensurable state-space representation in which

$$\text{diag}\,(s^{\boldsymbol{\alpha}}) = s^{\alpha}\mathbf{I}, \tag{98}$$

where $\mathbf{I}$ is the identity matrix.

Remark IV.1. *To simplify, we use the generic designation LT, without making a clear distinction between the different transforms, namely in the bilinear case, where* $s = \frac{1-z^{-1}}{1+z^{-1}}$ *is implicit.*

C. State transition operator

Let us come back to the dynamic equation

$$D^{\boldsymbol{\alpha}}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{99}$$

and assume that the input is null for $t > 0$. Using the results introduced in (49), we can write

$$\text{diag}\,(s^{\boldsymbol{\alpha}})\,\mathbf{X}(s) - \text{diag}\left(s^{\alpha-\mathbf{1}}\right)\mathbf{x}(0) = \mathbf{A}\mathbf{X}(s). \tag{100}$$

Therefore, there exists a *state transition operator*, $\Phi(0,t)$, verifying

$$\mathbf{x}(t) = \Phi(0,t) \cdot \mathbf{x}(0), \tag{101}$$

that is expressed by the inverse LT

$$\Phi(0,t) = \mathcal{L}^{-1}\left\{\left[\text{diag}\,(s^{\boldsymbol{\alpha}}) - \mathbf{A}\right]^{-1} \cdot \text{diag}\left(s^{\boldsymbol{\alpha}-\mathbf{1}}\right)\right\}. \tag{102}$$

There is a closed form for such inverse, but we do not present it, since it is a bit involved [42], [40] and it is not necessary in the follow-up. In the commensurate case, we can write

$$\Phi(0,t) = \mathcal{L}^{-1}\left\{\left[s^{\alpha}\mathbf{I} - \mathbf{A}\right]^{-1} \cdot s^{\alpha-1}\mathbf{I}\right\} \tag{103}$$

and

$$\Phi(0,t) = \mathcal{L}^{-1}\left\{\sum_{0}^{\infty} \mathbf{A}^n s^{-n\alpha-1}\right\} \tag{104}$$

that assume different forms according to the used LT. In the NLT case, we obtain from (30) the result

$$\Phi(0,nh) = \sum_{k=0}^{\infty} \mathbf{A}^k h^{k\alpha} \frac{(k\alpha+1)_n}{n!}\varepsilon(nh), \tag{105}$$

while in the bilinear case, we get

$$\Phi(0,nh) = \sum_{k=0}^{\infty} \mathbf{A}^k \left(\frac{h}{2}\right)^{k\alpha+1} \psi_n^{-k\alpha-1}\varepsilon(nh). \tag{106}$$

If we use a causal CT system, the inverse LT is given by

$$\Phi(0,t) = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\varepsilon(t), \tag{107}$$

that is the multidimensional MLF [43]. This is the result we obtain from (105) or (106) when $h \to 0$. For the first, with $t = nh$, we can write

$$\begin{aligned} h^{k\alpha}\frac{(k\alpha+1)_n}{n!} &= h^{k\alpha}\frac{\Gamma(k\alpha+1+n)}{\Gamma(n+1)} \\ &= \frac{\Gamma(k\alpha+1+t/h)h^{k\alpha}}{\Gamma(k\alpha+1)\Gamma(t/h+1)} \approx \frac{h^{k\alpha}}{\Gamma(k\alpha+1)}\left[\frac{t}{h}\right]^{k\alpha} \end{aligned}$$

and

$$\lim_{h\to 0} h^{k\alpha}\frac{(k\alpha+1)_n}{n!} = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

The case of expression (106) is somehow more involved, but yields an indentical result.

With $\Phi(0,t)$, we can define the general state transition operator, $\Phi(\tau,t)$ relating the states at two instants τ and t:

$$\mathbf{x}(t) = \Phi(\tau,t) \cdot \mathbf{x}(\tau). \tag{108}$$

The state transition operator is defined by

$$\Phi(\tau,t) = \Phi(0,t) \cdot \Phi^{-1}(0,\tau), \tag{109}$$

with $\Phi(0,0) = \mathbf{I}$. It can be shown that this operator verifies the usual properties, namely the semi-group property. In applications, we need to compute its inverse, $\Phi^{-1}(0,\tau)$. We compute it for the nabla, bilinear and CT cases. To compute the inverse of (105) we set

$$\Phi^{-1}(0,nh) = \sum_{k=0}^{\infty} \phi_k \mathbf{A}^k h^{k\alpha} \frac{(k\alpha+1)_n}{n!}\varepsilon(nh), \tag{110}$$

where the coefficients $\phi_k, k = 0, 1, 2\cdots$, need to be determined. The product of (105) and (110) gives the indentity matrix. With some manipulation, we obtain then $\phi_0 = 1$ and

$$\phi_k = -\frac{\alpha^n}{n!}\sum_{m=0}^{k-1}\phi_m\frac{(1/\alpha - m)_n(1/\alpha - l + m)_n}{(1/\alpha - l)_n}, \quad k = 1, 2, \cdots$$

Similarly, (106) suggests that we write

$$\varPhi^{-1}(0, nh) = \sum_{k=0}^{\infty}\phi_k\mathbf{A}^k\left(\frac{h}{2}\right)^{k\alpha+1}\psi_n^{-k\alpha-1}\varepsilon(nh). \quad (111)$$

The values $\phi_k, \ k = 0, 1, \cdots$, are given by:

$$\phi_k = \sum_{m=0}^{k-1}\phi_m\frac{\psi_n^{-m\alpha-1}\psi_n^{-(k-m)\alpha-1}}{\psi_n^{-k\alpha-1}\psi_n^{-1}}, \quad k = 1, 2, \cdots,$$

with $\phi_0 = 1$.

For the CT case, we proceed similarly. Letting

$$\varPhi^{-1}(0, \tau) = \sum_{k=0}^{\infty}\phi_k\mathbf{A}^k\frac{\tau^{k\alpha}}{\Gamma(k\alpha+1)}\varepsilon(\tau) \quad (112)$$

we can show that $\phi_0 = 1$ and

$$\phi_k = -\sum_{m=0}^{k-1}\phi_m\binom{k\alpha}{m\alpha}, \quad k = 1, 2, \cdots$$

With the state transition operator defined by (103) and (109) we can obtain the output of the system using the standard procedure [42], [40], [30].

V. Conclusions

In this paper we presented two different approaches for describing discrete-time fractional linear systems. The first is based on the nabla and delta discrete-time derivatives. We found their eigenfunctions, nabla and delta exponentials, that we used to define discrete LT. The second approach is based on the bilinear (Tustin) transformations. For both cases, appropriate algorithms for obtaining the impulse, step, and frequency responses were presented. Finally, the state-variable representation was also introduced. This one is very general in the sense that it can be used for continuous-time systems too.

Appendix
The ψ_k^α sequence

The sequence $\psi_k^\alpha, \ k = 0, 1, \cdots$, is obtained as the discrete convolution of two binomial sequences:

$$\psi_k^\alpha = \frac{(-\alpha)_k}{k!} * \frac{(-1)^k(\alpha)_k}{k!}k \in \mathbb{Z}_0^+.$$

In previous works [44], [45], [46], ARMA approximations to these sequences were also proposed.

Performing this discrete convolution we obtain the following results [26]. If $\alpha \in \mathbb{R}$ but $\alpha \notin \mathbb{Z}^-$, then

$$\psi_k^\alpha = (-1)^k\frac{(\alpha)_k}{k!}\sum_{m=0}^{k}\frac{(-\alpha)_m(-k)_m}{(-\alpha-k+1)_m}\frac{(-1)^m}{m!}, \quad k \in \mathbb{Z}_0^+ \quad (113)$$

and

$$\psi_k^{-N} = \frac{(N)_k}{k!}\sum_{m=0}^{min(k,N)}\frac{(-N)_m(-k)_m}{(-N-k+1)_m}\frac{(-1)^m}{m!}, \quad k \in \mathbb{Z}_0^+. \quad (114)$$

when $\alpha = -N, \ N \in \mathbb{Z}^+$. The sequence ψ_k^α verifies the properties:

1) The sequence $\psi_k^\alpha, \ k = 0, 1, \cdots$, is causal and, therefore, is null for $k < 0$.
2) For any $\alpha \in \mathbb{R}$, we have
$$\psi_k^{-\alpha} = (-1)^k\psi_k^\alpha \quad k \in \mathbb{Z}_0^+ \quad (115)$$
3) Initial value
From the initial value theorem of the ZT, it is immediate that $\psi_0^\alpha = 1$, independently of the order.
4) Final value
Let $\alpha \leq 0$. From the final value of the ZT,
$$\psi_\infty^\alpha = \lim_{z\to 1}(z-1)\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha,$$
then $\psi_\infty^\alpha = 0$, if $-1 \leq \alpha \leq 0$, and $\psi_\infty^\alpha = 2$, if $\alpha = -1$. For $\alpha < -1$ the sequence increases to ∞, otherwise, for $\alpha > 0$ we apply (115).
5) A recursion
The IR sequence verifies the recursion
$$\psi_k^\alpha = -\frac{2\alpha}{k}\psi_{k-1}^\alpha + \left(1 - \frac{2}{k}\right)\psi_{k-2}^\alpha, \qquad k \geq 2, \quad (116)$$
with $\psi_0^\alpha = 1$ and $\psi_1^\alpha = -2\alpha$.
This recursion shows that, if $\alpha < 0$, then ψ_k^α is a positive sequence. As consequence, attending to (115), the sequence corresponding to positive orders is always oscillating: successive values have alternating sign.
6) Relation with the Hypergeometric function
The second factor in (113) is a sequence drawn from the Gauss Hypergeometric function
$$\sum_{m=0}^{k}\frac{(-\alpha)_m(-k)_m}{(-\alpha-k+1)_m}\frac{(-1)^m}{m!} = {}_2F_1(-\alpha, -k; 1-k-\alpha; -1), \quad (117)$$
for $n \in \mathbb{Z}_0^+$.
7) For a fixed $k \in \mathbb{Z}$, ψ_k^α is a polynomial in α of degree k with the coefficient of α^k decreasing with increasing k.

Example A.1. *We present $\psi_k^{\pm N}$ for some values of $N \in \mathbb{Z}^+$ and for any real order, obtained by recursive computation*

1) $N = 1$

- $\psi_k^1 = \begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ 2(-1)^k & k > 0 \end{cases}$
- $\psi_k^{-1} = \begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ 2 & k > 0 \end{cases}$

2) $N = 2$

- $\psi_k^2 = \begin{cases} 0 & k<0 \\ 1 & k=0 \\ (-1)^k 4k & k>0 \end{cases}$
- $\psi_k^{-2} = \begin{cases} 0 & k<0 \\ 1 & k=0 \\ 4k & k>0 \end{cases}$

3) *For any negative order $-\alpha$, with $\alpha > 0$*
Using the recursion (116) with $\psi_0^{-\alpha} = 1$ and $\psi_1^{-\alpha} = 2\alpha$, we obtain successively:

$$\begin{aligned}
&\psi_2^{-\alpha} = 2\alpha^2 \\
&\psi_3^{-\alpha} = \frac{4}{3}\alpha^3 + \frac{2}{3}\alpha \\
&\psi_4^{-\alpha} = \frac{2}{3}\alpha^4 + \frac{4}{3}\alpha^2 \\
&\psi_5^{-\alpha} = \frac{4}{15}\alpha^5 + \frac{20}{15}\alpha^3 + \frac{6}{15}\alpha \\
&\psi_6^{-\alpha} = \frac{4}{45}\alpha^6 + \frac{40}{45}\alpha^4 + \frac{46}{45}\alpha^2 \\
&\psi_7^{-\alpha} = \frac{8}{315}\alpha^7 + \frac{140}{315}\alpha^5 + \frac{392}{315}\alpha^3 + \frac{90}{315}\alpha \\
&\psi_8^{-\alpha} = \frac{2}{315}\alpha^8 + \frac{56}{315}\alpha^6 + \frac{308}{315}\alpha^4 + \frac{264}{315}\alpha^2 \\
&\cdots \qquad \cdots
\end{aligned} \tag{118}$$

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