The 21^{st} Century Systems: an updated vision of Continuous-Time Fractional Models

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***Abstract*—This paper presents the continuous-time fractional linear systems and their main properties. Two particular classes of models are introduced: the fractional autoregresive-moving average type and the tempered linear system. For both classes, the computations of the impulse response, transfer function, and frequency response are discussed. It is shown that such systems can have integer and fractional components. From the integer component we deduce the stability. The fractional order component is always stable. The initial-condition problem is analysed and it is verified that it depends on the structure of the system. For a correct definition and backward compatibility with classic systems, suitable fractional derivatives are also introduced. The Grünwald-Letnikov and Liouville derivatives, as well as the corresponding tempered versions, are formulated.**

***Index Terms*—Fractional system, Fractional derivative, Differintegrator, Tempered derivative, Tempered linear system**

I. Introduction

A. On the fractional systems

Fractional systems are becoming increasingly adopted in the 21^{st} century for modelling many natural and man-made phenomena [1], [2], [3], [4], [5], [6], [7], [8]. In fact, we are presently dealing with phenomena that require a modeling beyond traditional tools. Signals obtained when observing many types of systems have spectra that exhibit slopes in the amplitude Bode diagrams that are not multiples of 20 dB per decade. This behaviour can be found, for example, in ECG, speech, music, electronic noise in junctions, network trafic, electrochemistry, and other applications [9], [10], [11]. Designations like $1/f$ noise, long range dependence [12], fractional Gaussian noise, and fractional Brownian motion (fBm), are ubiquitous in the scientific literature [13], [14], [15]. Other applications include the viscoelasticity [4], the J. Curie phenomenon [16], the Schrödinger fractional equation [17], [18], and the fractional Maxwell equations [19], [20], [21].

In a previous paper [22] (2008), and following Prof. Nishimoto, we forecasted that fractional systems would be the 21^{st} Century Systems and advanced an educated prediction about their evolution, both in the theoretical and practical perspectives. However, the following years revealed an unexpected unfolding that resulted in the appearance of formulations that were not only very inaccurate, but also far from those required when having in mind the solid background usual in engineering. This fact has certainly contributed to some lack of interest by many researchers in engineering. In fact, the confusion prevailing in the fractional world is an obstacle to its adoption by other applied sciences. This sequence of two articles intends to contribute for settling the dust that overshadows Fractional Calculus (FC), by highlighting concepts and tools that allow a correct fit between the traditional Signals & Systems and the FC. This backwards-compatibility is very important because it simplifies the adoption of FC by professionals with a more practical-oriented experience.

Instead of starting from the notion of fractional derivative and corresponding differential equations, we begin by introducing the concept of transfer function (TF) and some different forms it can assume, namely those involving irrational functions. From the TF, we define the corresponding differential equations with a considerable generality. In particular, we will call continuous-time fractional autoregressive-moving average (FARMA) those that are defined by an equation that results from the traditional ARMA model using a mere substitution of integer by fractional-order derivatives. This is a first type of "fractionalization" that consists in substituting the s Laplace variable by s^{α}. This approach corresponds to replacing the derivatives of orders $1, 2, 3, \cdots$ by the fractional ones with orders $\alpha, 2\alpha, 3\alpha, \cdots, \alpha \in \mathbb{R}$, yielding the so-called *commensurate systems*. This type of continuous-time fractional autoregressive-moving average models is the most interesting and is used in practical applications [23]. Nonetheless, we go further and use other real orders $\alpha_1 < \alpha_2 < \alpha_3, \cdots$ that yield *non-commensurate* systems.

It is important to compare the transient responses of both types of systems. While the integer order models are characterized by having exponential responses, the fractional ones have power functions as responses. For this reason, it is said that the first class has short memory while, on the contrary, the second has a long one (also called long range). However, we have applications, namely, in physics, where we find phenomena that are neither of short, nor of long range. These phenomena exhibit an intermediate behavior which requires a distinct approach. In the modelling of such kind of systems, we need to look for some form of embeding the two types of responses. One solution comes from the multiplication of the derivative kernels by an exponential that gives the required *medium range* systems. Several denominations were proposed in the literature for such systems. Similarly to what was considered in [24] we will use the designation *tempered systems*. Again, we have also commensurate and non-commensurate tempered systems.

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In terms of the TF and in a simple interpretation the tempering consists in substituting $s^\beta,\ \beta \in \mathbb{R}$, by $(s+\lambda)^\beta,\ \beta, \lambda \in \mathbb{R}$.

B. A short history

The FC is a generalization of the conventional calculus that was raised in 1695 from a Leibniz' idea, during the exchange of letters between him and J. Bernoulli [25]. The FC is not as accessible as the standard calculus, but leads to similar concepts and tools and enjoys a wider generality and applicability. The FC allows derivative operations of arbitrary order and represents an advance similar to the generalization from integer to real or complex numbers. During almost 200 years, namely since the works of Liouville (1832), the fractional derivative was considered as a curious and interesting topic, but merely an abstract mathematical concept. The main developments of the FC were accomplished by mathematicians without having in mind real world applications. However, the Liouville's proposals of fractional derivative had physical problems as their main motivation [26], [27], [28]. It is important to refer that Fourier proposed a general analysis/synthesis of the Fourier transform (FT) contemplating the fractional case. Liouville based his formulations on decompositions of the functions in terms of exponentials. However, he had a limitation, since, at that time, the Bromwich integral inverse of the Laplace transform (LT) was unknown. Therefore, he was not sufficiently convincing. The appearence of Riemann's proposal [29] allowed the formulation of a synthesis called *Riemann-Liouville derivative* (RL) [30], [31] that remained almost like a "standard" untill the end of the 20th century, when it was substitued by the (Dzherbashian-) Caputo (C) derivative, a particular case of second Liouville's formula. Heaviside considered some problems where fractional behaviour appears and applied his operational approach [1], [32]. Since the first Liouville's paper, several other definitions of derivative and integral operators were formulated, not necessarily compatible in the sense of giving always the same results. This state of affairs created difficulties when trying to extend specific tools based on the traditional integer order to the more general arbitrary order context [30], [31].

Since the early 1990's, some scientists and engineers have been working with those different forms, having in mind the perspective of practical applications [33], [34], [35], [36]. It is important to refer the pioneering works of Oustaloup's group in control and identification [37], [38], [39], [40], [41], [42]. The fractional electrical circuits started being investigated [43], [44], [45], [46], [47], [48]. In the last 10 years a number of papers were published describing electronic realizations of fractional systems [49]. However, in which concerns fractional System Theory we cannot say that we had a clear formulation generalizing the classic tools, namely impulse response (IR), TF, and frequency response (FR), keeping a backward compatibility. Aparently, this problem was examined, for the first time, in [50] and revisited in [22], proposing that FC would be the tool for system modelling in the 21^{th} century systems. A more general vision of fractional systems and their applications was presented in [51]. In the last decade, many applications appeared and important themes like, analysis, modelling, and synthesis, have been considered [52]. However, a closer look reveals that there are many integer order tools that need to be extended to the fractional framework, while keeping a backward compatibility [53]. Not all proposed formulations for the fractional operators are suitable for doing this task.

In this paper, the state-of-the-art of the compatible fractional system theory is described. We start by introducing the continuous-time ARMA like systems with commensurate and non commensurate orders. The traditional tools are introduced and computed with generality. The stability is also considered. The important initial condition problem is discussed.

C. Remarks

We assume that

- We work always on $\mathbb{R}$.
- We use the bilateral LT:
$$\mathcal{L}\left[f(t)\right] = F(s) = \int_{\mathbb{R}} f(t)e^{-st}\mathrm{d}t, \tag{1}$$
where $f(t)$ is any real or complex function defined on $\mathbb{R}$ and $F(s)$ is its transform, provided it has a non void region of convergence (ROC)
- The inverse LT is given by the Bromwich integral
$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} F(s)e^{st}\,\mathrm{d}s,\ t \in \mathbb{R}, \tag{2}$$
where $a \in \mathbb{R}$ is inside the region of convergence of the LT and $j = \sqrt{-1}$.
- The FT is obtained from the LT through the substitution $s = j\omega$ with $\omega \in \mathbb{R}$.
- The functions and distributions have LT and/or FT.
- Current properties of the Dirac delta distribution, $\delta(t)$, and its derivatives will be used.
- The standard convolution is given by
$$f(t) * g(t) = \int_{\mathbb{R}} f(\tau)g(t-\tau)\mathrm{d}\tau. \tag{3}$$
- The order of the fractional derivative is assumed to be any real number.
- The multi-valued expressions s^α and $(-s)^\alpha$ are used. To obtain functions from them we will fix for branch-cut lines the negative real half axis for the first and the positive real half axis for the second. For both expressions, the first Riemann surface is selected.
- The "floor" of a real number α is denoted as the integer $N = \lfloor \alpha \rfloor$ verifying $N \leq \alpha < N+1$.
- The Heaviside unit step and the signum function are represented by $\varepsilon(t)$ and $\mathrm{sgn}(t)$, respectively. These functions are related through $\mathrm{sgn}(t) = 2\varepsilon(t) - 1$.
- Let α and β two complex numbers with positive real parts. The two-parameter Mittag-Leffler function is defined by [54]
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}. \tag{4}$$

In applications, the causal function $t^{\beta-1}E_{\alpha,\beta}(pt^{\alpha})\varepsilon(t)$ is used, for it has LT

$$\mathcal{L}E_{\alpha,\beta}(-pt^{\alpha}) = \frac{s^{\alpha-\beta}}{s^{\alpha}+1}, \quad Re(s) > 0, \tag{5}$$

where $Re(\cdot)$ is the real part of a complex number.

- The general binomial series reads

$$(1-z)^a = \sum_{k=0}^{\infty} \frac{(-a)_k}{k!} z^k, \tag{6}$$

where $(-a)_n$ is the Pochhamer representation for the raising factorial:

$$(-a)_n = \prod_{k=0}^{n-1} (-a+k),$$

with $(-a)_0 = 1$. If a is not a negative integer, then

$$\frac{(-a)_k}{k!} = \binom{a}{k}$$

that represents the well-known binomial coefficients.

D. Abbreviations

The following abbreviations are used in this manuscript:

ARMA	Autoregressive-Moving Average
CT	Continuous-Time
DT	Discrete-Time
FARMA	Fractional Autoregressive-Moving Average
FD	Fractional derivative
FP	Feller Potential
FT	Fourier transform
FR	Frequency response
IC	Initial-conditions
IR	Impulse Response
GL	Grünwald-Letnikov
L	Liouville
LTIS	linear time-invariant system
LS	Linear system
LT	(Bilateral or two-sided) Laplace transform
MLF	Mittag-Leffler function
NLT	Nabla Laplace transform
RL	Riemann-Liouville
RP	Riesz Potential
RD	Riesz Derivative
RFD	Riesz-Feller Derivative
TF	Transfer function
TFD	Tempered Fractional Derivative
ULT	Unilateral (one-sided) Laplace transform

E. Outline of the paper

We start by recalling some important concepts regarding the study of linear systems in Section II. We define the IR and TF with great generality (II-A), and give examples of several types of TF. For defining such TF, we introduce the main basic tool for expressing the fractional systems: the *differintegrator* (II-C). The previously referred systems are presented progressively in Section III, from the simplest commensurate (III-A), to the non-commensurate (III-C). It is shown that fractional systems can be decomposed into two components: integer and fractional. This decomposition has consequences for the stability studied in subsection III-B. The very important problem of the IC is also analysed in subsection III-F. The IC depend on the structure of the system not on the used transform. A general formula for dealing with them is presented. The study uses the TF for starting point, without introducing any differential equation. A type ARMA fractional order differential equation together with the appropriate definitions of derivatives are presented in III-D. The FR is also studied in III-E. This set of topics cover the most important systems, namely the electric circuits. However, there are other interesting subjects that we found useful and that are studied briefly in sub-section III-G: the variable order derivatives and systems (III-G1), an introduction to fractional order stochastic processes (III-G2), and the particular case of the fractional Brownian motion (fBm) treated in III-G3. Most of the tools introduced in these sections are useful for dealing with another system generalization: the tempered fractional LS studied in Section IV. Therefore, we introduce the tempered fractional derivatives (IV-A) suitable for expressing the differential equations used to describe the tempered fractional LS (IV-B). Finally, some conclusions are presented in Section V.

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II. On the continuous-time linear time-invariant systems (LTIS)

A. Impulse response and transfer function

The linear systems (LS) are of primordial importance in Science and Engineering. In many situations we deal with nonlinear systems, but the linear are very useful and deserve study. Traditionally, linear systems are based on integer order differential or difference equations, if they are continuous- or discrete-time.

Let us consider continuous-time systems for which the input-ouput relation assumes the general form:

$$y(t) = \int_{-\infty}^{+\infty} g(t,\tau)x(\tau)\mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{7}$$

where $g(t,\tau)$ is the IR of the system [55]. The IR characterizes completely the system. In the *time-invariant* case $g(t,\tau) = g(t-\tau)$ and the input-output relation assumes a convolutional form [55], [56], [57]

$$y(t) = \int_{-\infty}^{+\infty} g(t-\tau)x(\tau)\mathrm{d}\tau. \tag{8}$$

Remark II.1. *If $g(t,\tau) = g(t/\tau)/\tau$, $t,\tau \in \mathbb{R}_0^+$, then (7) becomes a multiplicative (Mellin) convolution and the corresponding systems are not time-invariant, but they are essentially scale invariant. Among these types of systems we can consider the LS based on the Hadamard [31] and quantum derivatives [58], [59].*

Returning to (8) , if $x(t) = e^{st}$, $s \in \mathbb{C}, \in \mathbb{R}$, then

$$y(t) = G(s)e^{st}, \quad t \in \mathbb{R}, \tag{9}$$

with $G(s) = \mathcal{L}[g(t)]$ the LT of the IR that is the TF. Therefore,

1) The exponentials are the eigenfunctions of the LTIS and the eigenvalues, $G(s)$ are the LT of their IR [60].
2) If the system is causal, then its IR is a right function and $h_c(t) = 0$, for $t < 0$, implying that $H_c(s)$ has a ROC defined by $Re(s) > a \in \mathbb{R}$.
3) Similarly, if the system is anti-causal, then $h_a(t) = 0$, for $t > 0$, implying that $H_a(s)$ has a ROC defined by $Re(s) < b \in \mathbb{R}$.
4) Consider a bilateral system that is the sum of a causal and an anti-causal, $G(s) = G_c(s) + G_a(s)$. If its ROC is non void,$a < Re(s) < b$, the corresponding IR is two-sided, $g(t) = g_c(t) + g_a(t)$.
5) If the region of convergence (ROC) of $G(s)$ contains the imaginary axis, then we can set $s = j\omega$, $\omega \in \mathbb{R}$, so that *the response of a LTIS to a sisoid is also a sisoid with the same frequency.* In this case, the LT becomes the FT and originates the FR, denoted by $G(j\omega)$. It is frequently represented by the Bode diagrams that are logarithm plots of the amplitude, $A(\omega) = |G(j\omega)|$ and phase, $\phi(\omega) = \arg G(j\omega)$.

Therefore, we require that the IR, $g(t)$, [23]

- is continuous almost everywhere,
- has bounded variation,
- is of exponential order.

Concerning the stability of a system, we prefer usually the BIBO (bounded input-bounded output) stability criterion. This implies that the IR is absolutely integrable (AI). Therefore, any stable LS with TF given by $G(s)$ has frequency response, $G(j\omega)$. *For a given stable (unstable) causal there is always a unstable (stable) anti-causal with the same $G(s)$ as TF, but different ROC*. These considerations created a framework for the definition of LS through the IR and corresponding TF.

B. Some examples of TF

The above considerations do not suggest any particular form that the TF (or FR) can assume. The practical problem at hand can give insights into the one we must choose. In Engineering, it is frequent to use the information depicted by means of the Bode diagrams. Some interesting models are well-known and were suggested in different studies of electromagnetic media or electrical circuits (with suitable ROC):

1) Oscillator
$$G(s) = \frac{1}{s^2 + p}, \quad p \in \mathbb{R}^+;$$
2) Cole-Cole dielectric model
$$G(s) = \frac{1}{s^\alpha + p}, \quad p \in \mathbb{R}^+;$$
3) Causal highpass filter
$$G(s) = \frac{s^\alpha}{s^\alpha + p}, \quad p \in \mathbb{R}^+;$$
4) Cole-Davidson dielectric model [61]
$$G(s) = \frac{1}{(s + p)^\alpha}, \quad p \in \mathbb{R}^+;$$
5) Havriliak–Negami dielectric model [62]
$$G(s) = \frac{1}{[1 + (s\tau)^\alpha]^\beta};$$
6) Phase lead/lag compensator [24]
$$G(s) = \left(\frac{\tau s + a}{s + a}\right)^\alpha, \ \alpha, a \in \mathbb{R}^+, \ \tau > 1.$$

These examples suggest we can introduce some non common TF. Let N and M be positive integers and the polynomial coefficients a_k and b_k, $k = 0, 1, \cdots$, be real numbers. We introduce also the parameters α_k and β_k, $k = 0, 1, \cdots$, that, without loss of generality, we can assume to form positive real increasing sequences.

1) The most used model is the rational TF corresponding to a continuous-time autoregresive-moving average (CT-ARMA) model:
$$G(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}, \tag{10}$$
where the N and M are the orders of the model that correspond to the degrees of denominator and numerator polynomials, respectively.
2) The fractional continuous-time autoregresive-moving average (CT-FARMA) [63].
It is a direct generalization of expression (10) that gives
$$G(s) = \frac{\sum_{k=0}^{M} b_k s^{k\alpha}}{\sum_{k=0}^{N} a_k s^{k\alpha}}, \tag{11}$$
where $0 < \alpha \leq 1$. This is fractional commensurate LS.
3) Tempered CT-FARMA [24]
$$G(s) = \frac{\sum_{k=0}^{M} b_k (s + \lambda)^{k\alpha}}{\sum_{k=0}^{N} a_k (s + \lambda)^{k\alpha}}, \tag{12}$$
with $\lambda \in \mathbb{R}$.
4) Fractional non commensurate LS [64].
It generalizes (11)
$$G(s) = \frac{\sum_{k=0}^{M} b_k s^{\beta_k}}{\sum_{k=0}^{N} a_k s^{\alpha_k}}. \tag{13}$$
5) Tempered Fractional non commensurate LS [24]
$$G(s) = \frac{\sum_{k=0}^{M} b_k (s + \gamma_k)^{\beta_k}}{\sum_{k=0}^{N} a_k (s + \lambda_k)^{\alpha_k}}, \tag{14}$$
where λ_k, γ_k, $k = 0, 1, \cdots$ are real parameters.
6) Other generalizations can be found as $G(s) = H^\sigma(s)$ with $\sigma > 0$.

It is important to remark that:

- The complex variable functions introduced above may represent, at least, two different TF, a causal and an anti-causal, depending on the selected region of convergence [56], [57], [65];
- For stability reasons and without loosing generality, we will assume $M < N$.

C. The differintegrator

Classically, the sequence

$$\dots s^{-n} \dots s^{-2} \;\; s^{-1} \;\; 1 \;\; s^{1} \;\; s^{2} \dots s^{n} \dots \qquad n \in \mathbb{N}, \tag{15}$$

has a clear meaning due to the relation between integrals (negative exponents) or derivatives (positive exponents) and the LT. We note that:

- As we said previously each term with negative exponent represents two TF corresponding to two disjoint regions of convergence, namely $Re(s) > 0$ (causal system) and $Re(s) < 0$ (anti-causal system), with inverse LT given by

$$\dots \pm \frac{t^{n-1}}{(n-1)!} u(\pm t) \dots \pm \frac{t^2}{2!} u(\pm t) \quad \pm \frac{t^1}{1!} u(\pm t) \quad \pm u(\pm t) \tag{16}$$

- The terms with positive or null exponents are analytic on the whole complex plane. The corresponding inverse LT are

$$\begin{aligned} &\delta(t) \qquad \delta'(t) \qquad \dots \delta^{(n)}(t) \dots = \\ =&\frac{t^{-1}}{(-1)!} u(t) \;\; \frac{t^{-2}}{(-2)!} u(t) \;\; \dots \frac{t^{-n-1}}{(-n-1)!} u(t) \dots \end{aligned} \tag{17}$$

  where we recall a relation derived by Gel'fand and Shilov [66]:

$$\delta^{(n)}(t) = \frac{t^{-n-1}}{(-n-1)!} u(t), \quad n \geq 0. \tag{18}$$

The fractionalization of (18) allows us to fill in the gaps in sequence (15) yielding

$$\dots s^{-\pi} \dots s^{-2} \dots s^{-\sqrt{2}} \dots s^{-1} \dots s^{1/2} \dots s^{1} \dots s^{\frac{\pi}{2}} \dots s^{e} \dots \tag{19}$$

and also in (16) and (17) through

$$\mathcal{L}^{-1} s^{\alpha} = \pm \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} u(\pm t), \tag{20}$$

for any real order (we can consider complex orders, but the resulting systems are not Hermitian). The elemental system with TF $G(s) = s^{\alpha}$, $\alpha \in \mathbb{R}$, is called *differintegrator*. If $Re(s) > 0$, it will be called forward, otherwise if $Re(s) < 0$ it will be denoted backward.
The impedance of a circuit with a differintegrator, called *constant phase element* (CPE) [67], assumes the form

$$Z(j\omega) = K_{\alpha} (j\omega)^{\alpha}.$$

This impedance, called *fractance*, is complex for non integer α. An ideal fractional coil ($\alpha > 0$) has fractance [68], [69]

$$Z_L = L_{\alpha} (j\omega)^{\alpha}$$

with the indutance, L_{α}, with units $[H \cdot s^{\alpha}]$. Similarly, an ideal fractional capacitor has fractance [16]

$$Z_C = \frac{1}{C_{\alpha} (j\omega)^{\alpha}}$$

where the capacitance C_{α} has units $\left[F \cdot s^{1-\alpha}\right]$. With $\alpha = 1$ we obtain the classic coil and capacitor reactances. An interesting case is the frequency-dependent negative resistor (FDNR) [67], [70], [71] that corresponds to

$$Z_{FDNR} = \frac{K_{-2}}{(j\omega)^2}$$

Remark II.2. *The question of the unities in fractional elements like coils and capacitors is an open subject [72], [73].*

There are interesting realizations of differintegrators and applications to circuits [74], [75], [44], [48], [76], [77]. An interesting generalization of the Kramers-Krönig relations was presented in [21].

Example II.1. *A device with fractional characteristics is the electrochemical capacitor for which several fractional models were proposed for its impedance. This subject was studied in [78], [79] and the following model was obtained and validated experimentally*

$$Z(s) = R_0 + \frac{1}{s^{\gamma_1} \cdot C_{\gamma_1}} + \frac{1}{s^{\gamma_2} \cdot C_{\gamma_2}} + \frac{1}{s^{\gamma_1+\gamma_2} \cdot C_{\gamma_3}}, \tag{21}$$

where R_0 represents a series resistance, the symbols $C_{\gamma_1}, C_{\gamma_2}$, and C_{γ_3} stand for capacitances of the electrochemical capacitor model, and γ_1 and γ_2 are the fractional orders. It is interesting to remark that

1) *the model is a series or a resistor and 3 ideal capacitors;*
2) *the order of one ideal capacitor is the sum of the other two, assuming a value slightly greater than* 1.

III. Study of continuous-time FARMA linear systems

A. The commensurate case

The CT-FARMA LS are the most important class of fractional systems, since they constitute the base for the study of fractional electric circuits. This type of system was introduced in (11) by means of its TF that we reproduce here together with its partial fraction decomposition

$$\begin{aligned} G(s) = \frac{B(s^{\alpha})}{A(s^{\alpha})} &= \frac{\sum\limits_{k=0}^{M} b_k s^{\alpha k}}{\sum\limits_{k=0}^{N} a_k s^{\alpha k}} \\ &= \sum_{k=1}^{N_p} \frac{R_k}{(s^{\alpha} - p_k)^{n_k}}. \end{aligned} \tag{22}$$

The parameters $p_k,\ k = 1, 2, \cdots N_p$, are called *pseudo-poles* and are the roots of the characteristic polynomial $A(z) = \sum_{k=0}^{N} a_k z^k$. The roots of $B(z) = \sum_{k=0}^{M} b_k z^k$ are the *pseudo-zeroes* and R_k stands for the residues of the partial fraction decomposition of $B(z)/A(z)$.

Remark III.1. *Any system defined by (13), with rational orders can be converted into the form (11), that is to commensurate. We only may have to join additional null coefficients. The same happens with (14) that gives (12).*

Remark III.2. *Let* p *be a complex number,* $0 < \alpha < 1$*, and the equation* $s^\alpha - p = 0$ *that has infinite roots. However, there is only one solution for this equation in the first Riemann surface, if* $-\alpha\pi < \arg(p) \leq \alpha\pi$*, and it is given by* $s_0 = |p|^{\frac{1}{\alpha}} e^{j\frac{\arg(p)}{\alpha}}$*. For example, if* $\arg(p) = \pi$*, then the root of the equation is in the second Riemann surface. Therefore, it does not lead to a pole or zero of a system.*

The case $\alpha = 1$ corresponds to the classic model (10). As it is well-known [56], [57] the IR corresponding to this TF has the form

$$g(t) = \pm \sum_{k=1}^{N_p} R_k \frac{t^{n_k-1}}{(n_k-1)!} e^{p_k t} u(\pm t), \tag{23}$$

where $p_k,\ k = 1, 2, \cdots$, are the poles, n_k denotes their multiplicity, and R_k represents the corresponding residues in the partial fraction decomposition of (10). The function $\pm u(\pm t)$ is the causal/anti-causal Heaviside unit step used to make a clear distinction between the causal (right) and anti-causal (left) solutions. This is important if the variable t does not stand for time, but, for example, it represents space.
For $0 < \alpha < 1$, we need to invert a generic simple fraction $\frac{1}{(s^\alpha - p)^n}$. Since

$$\frac{(n-1)!}{(s^\alpha - p)^n} = \frac{d^{n-1} \frac{1}{(s^\alpha - p)}}{d^{n-1} p} \tag{24}$$

we only have to pay attention to the case:

$$G(s) = \frac{1}{s^\alpha - p}. \tag{25}$$

We will treat the causal case only. There are two different ways of inverting (25).

1) Using the Mittag-Leffler function: Using the geometric series, we can write successively

$$F(s) = \frac{1}{s^\alpha - p} = s^{-\alpha} \sum_{n=0}^{\infty} p^n s^{-\alpha n} = \sum_{n=1}^{\infty} p^{n-1} s^{-\alpha n}, \tag{26}$$

valid for $|ps^{-\alpha}| < 1$. Choosing the ROC $Re(s) > |ps^{-\alpha}|$, we will arrive at the causal inverse of $F(s)$:

$$f(t) = \sum_{n=1}^{\infty} p^{n-1} \frac{t^{n\alpha-1}}{\Gamma(n\alpha)} \varepsilon(t). \tag{27}$$

This function also called *alpha-exponential* [31] is normally expressed in terms of the MLF as

$$f(t) = t^{\alpha-1} E_{\alpha,\alpha}(pt^\alpha) \varepsilon(t) \tag{28}$$

and gives the IR corresponding to partial fraction (26). If $\alpha = 1$, then we have the classic result $f(t) = e^{pt}\varepsilon(t)$. The step response $r_\varepsilon(t)$ of (26) can be obtained from

$$\mathcal{L}\left[r_\varepsilon(t)\right] = \frac{1}{s^\alpha - p} \cdot \frac{1}{s} = \sum_{1}^{\infty} p^{n-1} s^{-\alpha n - 1}, \tag{29}$$

leading easily to

$$r_\varepsilon(t) = t^\alpha E_{\alpha,\alpha+1}(pt^\alpha) \cdot \varepsilon(t) = \frac{1}{p}\left[E_{\alpha,1}(pt^\alpha) - 1\right] \cdot \varepsilon(t), \tag{30}$$

that generalises the expression corresponding to $\alpha = 1$. Knowing relations (28) and (30) and attending to (24) we are able to compute the impulse and step responses of any LTIS defined by the TF (22).

2) Using the inverse LT: The inversion of a TF based on the MLF has an important drawback: the solution relies on one, or several, series, that create severe computational problems. Furthermore, such solution masks the underlying structure of the system, in the sense that it does not highlight the presence of two different terms

- One component of integer order that inherits the classical behaviour, mainly oscillations and (un)stability;
- Another component of fractional order responsible for the long range behavior of the fractional linear systems, that is intrinsically stable as we will demonstrate in the sequel.

We start from the Bromwich inversion integral (2) and decompose it in two parcells according to each path section [80]. We get [81]

$$\begin{aligned} g(t) &= \sum_{k=1}^{K_0} A_k e^{p^{1/\alpha} t} \varepsilon(t) \\ &+ \frac{1}{2\pi j} \int_0^\infty \left[G(e^{-j\pi}u) - G(e^{j\pi}u)\right] e^{-\sigma t} \mathrm{d}u \cdot \varepsilon(t), \end{aligned} \tag{31}$$

where the constants A_k, $k = 1, 2, \ldots, K_0$, are the residues of $G(s)$ at $p^{1/\alpha}$. The LT of both sides in (31) leads to

$$G(s) = G_i(s) + G_f(s), \tag{32}$$

where the integer order part is

$$G_i(s) = \sum_{k=1}^{K_0} \frac{A_k}{s - p^{1/\alpha}}, \qquad Re(s) > \max(Re(p^{1/\alpha})) \tag{33}$$

and the fractional part is

$$G_f(s) = \frac{1}{2\pi j} \int_0^\infty \left[G(e^{-j\pi}u) - G(e^{j\pi}u)\right] \frac{1}{s+u} \mathrm{d}u \tag{34}$$

which is valid for $Re(s) > 0$. The integer order part of the impulse response (33) is the classical sum of exponentials (or sinusoids), eventually multiplied by integer powers. The possible sinusoidal behaviour comes from this term. The numerical computation of this part does not put any significant problem. The fractional part is expressed as an integral of the product of an exponential, with negative real exponent, by a bounded function, that is zero at the origin and at infinite. This means that the integral is easily computed by means of a simple numerical procedure. Using a uniform sampling interval ν, we can write

$$h_f(t) = \frac{1}{2\pi i} \sum_{n=0}^{L} \left[G(e^{-j\pi}u_n) - G(e^{j\pi}u_n)\right] e^{-u_n t} \nu \tag{35}$$

with $u_n = n\nu,\ \ n = 0, 1, \ldots, L$. The sampling interval ν and the positive integer L are chosen to guarantee that the fraction in (35) is small for $u_L = L\nu$.

Remark III.3. *Consider the TF in (22) with $n_k = 1,\ k = 1, 2, \ldots$*

$$G(s) = \sum_{k=1}^{N} \frac{R_k}{s^\alpha - p_k}, \tag{36}$$

where R_k and $p_k,\ k = 1, 2, \ldots, N$, are the residues and pseudo-poles. Define the set
$Q = \{p_k : -\pi\alpha < \arg(p_k) \leq \pi\alpha,\ \ k = 1, 2, \ldots, N\}$.
The integer part $h_i(t)$ of the impulse response is given by:

$$h_i(t) = \sum_{k=1; p_k \in Q}^{N} R_k \cdot B_k e^{p_k^{1/\alpha} t} \epsilon(t), \tag{37}$$

where $R_k,\quad k = 1, 2, \ldots, N$, represent the coefficients obtained from the partial fraction decomposition and $B_K = \frac{1}{\alpha p_k^{1-1/\alpha}}$ are the residues of $\frac{1}{(s^\alpha - p_k)}$ at $p_k^{1/\alpha}$. The corresponding LT becomes [82]

$$H_i(s) = \sum_{k=1; p_k \in Q}^{N} \frac{R_k \cdot B_k}{s - p_k^{1/\alpha}} \tag{38}$$

for $Re(s) > \max(Re(p_k^{1/\alpha}))$, $p_k \in Q$, as defined above.

With generality, consider the situation where we have K pseudo-poles. The result in (35) allows us to state that:

- For $\alpha = 1,\ \ k = 1, 2, \ldots, K$, we have no fractional component. The TF is a sum of partial fractions and each one has an exponential for inverse LT. The corresponding TF is the quotient of two polynomials in s.
- For $\alpha < 1,\ \ k = 1, 2, \ldots, K$, we may have two components depending on the location of p_k in the complex plane:
  - If $|\arg(p_k)| > \pi\alpha,\ k = 1, 2, \ldots, K$, then we do not have the integer order component: it is a pure fractional system;
  - If $|\arg(p_k)| \leq \pi\alpha$, for some $k = 1, 2, \ldots, K$, then the system is mixed, in the sense that we have both components;
  - If $|\arg(p_k)| = \frac{\pi}{2}\alpha$, for some $k = 1, 2, \ldots, K$, then the integer order component is sinusoidal, but the fractional component exists as well.

Remark III.4. *From the above considerations we conclude that we can have purely fractional systems. We only have to choose the pseudo-poles with arguments in the region defined by $|\arg(p_k)| > \pi\alpha$.*

Example III.1. *Consider the fractional RC circuit with TF expressed by:*

$$G(s) = \frac{1}{1 + RC_\alpha s^\alpha},$$

where C_α is the capacitance in $\left[Fs^{1-\alpha}\right]$ and $\alpha < 1$. The IR of this system has only fractional component. It is interesting to note that when $\alpha = 1$ the situation is reversed. In figure 1 we depict the step responses of a RC circuit with $RC_\alpha = 1s^{-\alpha}$.

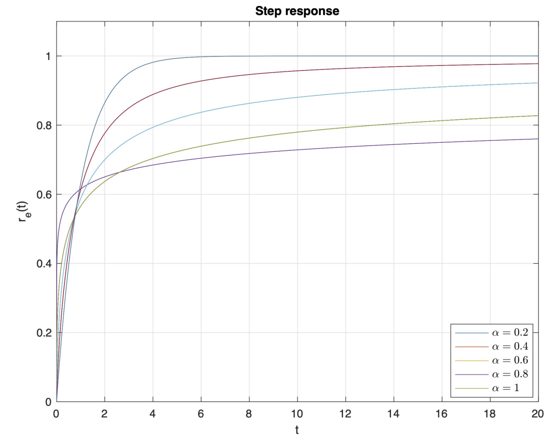

Fig. 1. Step responses of the RC circuit (example III.1) for $\alpha = 0.2k,\ k = 1, 2, \cdots 5$ (from below).

B. Stability

Consider a given term of (31) corresponding to one pseudo-pole p:

$$g(t) = \frac{1}{\alpha} p^{1/\alpha - 1} e^{p^{1/\alpha} t} \varepsilon(t) + \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{u^\alpha}{u^{2\alpha} - 2p\cos(\alpha\pi) + p^2} e^{-ut} \mathrm{d}u \cdot \varepsilon(t). \tag{39}$$

We can extract some conclusions [81]:

1) The fractional part is always bounded, for $\alpha > 0$ and any $p \in \mathbb{C}$. In fact, it is a simple matter to verify that function $\frac{u^\alpha}{u^{2\alpha} - 2p\cos(\alpha\pi) + p^2}$ is bounded. Therefore,

$$\int_0^\infty \left| \frac{u^\alpha}{u^{2\alpha} - 2p\cos(\alpha\pi) + p^2} \right| e^{-ut} \mathrm{d}u < \frac{A}{t},\ \ t > 0 \tag{40}$$

This term exists whenever $\alpha \neq 1$ and does not contribute to unstability.

2) As mentioned previously, the integer part only exists if $-\pi\alpha < \arg(p) \leq \pi\alpha$. In this case, we have three situations [81]:
   - $|\arg(p)| < \alpha\frac{\pi}{2}$ – the exponential increases without bound - unstable system;
   - $|\arg(p)| > \alpha\frac{\pi}{2}$ – the exponential decreases to zero - stable system;
   - $|\arg(p)| = \alpha\frac{\pi}{2}$ – the exponential oscillates sinusoidally - critically stable system.

The above considerations allow us to conclude that the behaviour of stable systems can be integer, fractional, or mixed:

- Classical integer order systems have impulse responses corresponding to linear combinations of exponentials that, in general, go to zero very fast. They are short memory systems.
- In fractional systems without poles, the exponential component disappears. These are long memory systems. All pseudo-poles have arguments with absolute values greater than π/α, where $\alpha < 1$.

- Mixed systems have both components. Some pseudo-poles have arguments with absolute values larger than $\frac{\pi}{2}\alpha$.

Remark III.5. *The three rules above are usually known as* Matignon's theorem*, that can be put in the following way: TF (10) is stable iff all pseudo-poles p_k verify $|\arg(p_k)| > \alpha\frac{\pi}{2}$.*

The above procedure for studying the stability demands knowing the pseudo-poles. In the integer order case, there are several criteria to evaluate the stability of a given linear system without the knowledge of the poles. One of the most important is the Routh-Hurwitz criterion that gives information on the number of poles on the right hand half complex plane [83]. The generalisation of this criterion for the fractional case was proposed in [84] and is very similar to the integer order case [23]. A very interesting alternative is the Mikhailov criterion [85] that is formulated in the frequency domain.

C. General non commensurate case

The non commensurate case cannot be dealt as we discussed above, unless we know the pseudo-poles. In this case, we can use (31). In the general case it is possible to obtain the IR in the form of a fractional McLaurin series through the recursive application of the general *initial value theorem* [86], [87]. This theorem relates the asymptotic behaviour of a causal signal, $g(t)$, as $t \to 0^+$ to the asymptotic behaviour of its LT, $G(\sigma) = \mathcal{L}[g(t)]$, as $\sigma = Re(s) \to \infty$.

Theorem III.1 (The initial-value theorem)**.** *Assume that $g(t)$ is a causal signal such that in some neighbourhood of the origin is a regular distribution corresponding to an integrable function and its LT is $G(s)$ with region of convergence defined by $Re(s) > 0$. Also, assume that there is a real number $\beta > -1$ such that $\lim_{t\to 0^+} g(t)t^\beta$ exists and is a finite complex value. Then*

$$\lim_{t\to 0^+} \frac{g(t)}{t^\beta} = \lim_{\sigma\to\infty} \frac{\sigma^{\beta+1}G(\sigma)}{\Gamma(\beta+1)}. \quad (41)$$

For proof see [86] (section 8.6, pages 243-248).
The repetitive use of this theorem allows us to express a given TF as a sum of negative power functions plus an error term. We are going to describe the procedure for obtaining the IR from the TF [87]. Let $G(s)$ be a TF and its associated ROC $Re(s) > 0$. We do the following steps:

1) Define $R_0(s) = G(s)$ with inverse LT given by $r_0(t)$
2) Let γ_0 be the real value such that
$$\lim_{\sigma\to\infty} \sigma^{\gamma_0} H(\sigma) = A_0,$$
where A_0 is finite and non null. Then, let
$$R_1(s) = G(s) - A_0 s^{-\gamma_0}.$$
It is clear that $\lim_{\sigma\to\infty} \sigma^{\gamma_0} R_1(\sigma) = 0$ and $A_0 = \lim_{t\to 0^+} r_0(t) = h^{(\gamma_0-1)}(0^+)$.
3) Repeat the process. Let γ_1 be the real value such that
$$\lim_{\sigma\to\infty} \sigma^{\gamma_1} R_1(\sigma) = A_1,$$
where A_1 is finite and non null. Again introduce
$$R_2(s) = G(s) - A_0 s^{-\gamma_0} - A_1 s^{-\gamma_1}$$
with $\lim_{\sigma\to\infty} \sigma^{\gamma_1} R_2(\sigma) = 0$ and $A_1 = \lim_{t\to 0^+} r_1^{(\gamma_1-1)}(t)$, having $r_1(t)$ as the inverse of $R_1(s)$.
4) In general, let γ_n be the real value such that
$$\lim_{\sigma\to\infty} \sigma^{\gamma_n} R_n(\sigma) = A_n,$$
where A_n is finite and non null. We arrive at the function:
$$R_n(s) = G(s) - \sum_{k=0}^{n-1} A_k s^{-\gamma_k} \quad (42)$$
and $\lim_{\sigma\to\infty} \sigma^{\gamma_n} R_n(\sigma) = 0$.
As above $A_n = \lim_{t\to 0^+} r_{n-1}^{(\gamma_n-1)}(t) = r_{n-1}^{(\alpha_n)}(0^+)$, to be coherent with the initial value theorem and $\gamma_n = \alpha_n + 1$, for $n \in \mathbb{Z}_0^+$.

We can write:

$$G(s) = \sum_{k=0}^{n-1} A_k s^{-\gamma_k} + R_n(s) \quad (43)$$

leading to the conclusion that $G(s)$ can be expanded in a generalized Laurent series [87]

$$G(s) = \sum_{k=0}^{\infty} r_k^{(\alpha_k)}(0^+) s^{-\gamma_k}. \quad (44)$$

The inverse LT of each $s^{-\gamma_k}$ allows us to obtain $g(t)$.
If we know the pseudo-poles, we can use the algorithm described in [64]. However, it must be emphasized that there are non-factorizable pseudo-polynomials .

D. From the TF to the differential equation. Fractional derivatives

Consider the causal case (we will omit the ROC). From (8), using the LT, we obtain $Y(s) = G(s)X(s)$ that leads to

$$\sum_{k=0}^{N} a_k s^{\alpha_k} Y(s) = \sum_{k=0}^{M} b_k s^{\beta_k} X(s). \quad (45)$$

Introduce an operator D^α (differintegrator) such that

$$\mathcal{L}^{-1}\left[s^\alpha F(s)\right] = D^\alpha f(t), \quad (46)$$

where $F(s) = \mathcal{L}\left[f(t)\right]$ and $\pm Re(s) > 0$. We obtain from (45)

$$\sum_{k=0}^{N} a_k D^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k} x(t), \quad (47)$$

that is an equation defining a system with TF

$$G(s) = \frac{\sum\limits_{k=0}^{M} b_k s^{\beta_k}}{\sum\limits_{k=0}^{N} a_k s^{\alpha_k}}.$$

We introduced the differintegrator, verifying the property (46). For positive order, we will call it *fractional derivative*.
FD is the name assigned to several mathematical operators,

namely the Grünwald-Letnikov (GL), Liouville (L), Riemann-Liouville, Caputo, Marchaud, Hadamard, Riesz, and others [30], [31], [63], [8], [23]. Without going into the discussion of a methodology for deciding if a given operator is a FD [88], we are interested in selecting those that are suitable for generalising well-known tools of Signals, Circuits and Systems and other Applied Sciences. In particular, we must consider the formulations that can be useful for the introduction of fractional linear systems and their characterisation in the perspective of compatibility with integer order definitions, as we assumed in the previous sub-sections..

Remark III.6. *We must note that*

1) *The traditional and more frequently used Riemann-Liouvile [30], [31] and Caputo [89], [31] derivatives do not verify (9);*
2) *The Caputo-Fabrizio and Atangana-Baleanu operators [90] are in fact highpass filters;*
3) *The so-called local derivatives [90] are essentially integer order derivatives;*

Therefore, these operators are incompatible with the traditional formulations we use in Circuits and Systems.

The main aim in this section is to present a coherent basis for establishing fractional operators compatible with the corresponding classic integer order. In particular, the formulation developed in the sequel allows currently used tools like, IR, TF and FR, and includes the standard derivatives obtained when the orders become integers. To look forward FD formulations consistent with the laws of Physics we recall the most important results from the classic calculus. The standard definition of derivative is

$$D_f f(t) = f'(t) = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}, \tag{48}$$

or

$$D_b f(t) = f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}. \tag{49}$$

Substituting $-h$ for h interchanges the definitions, meaning that we only have to consider $h > 0$. In this case, expression (48) uses the present and past values, while (49) uses the present and future values. In the following, we will distinguish the two cases by using the subscripts f (forward – in the sense that we go from past into future, a direct time flow) and b (backward – meaning a reverse time flow).

It is straightforward to invert the above equations to obtain

$$D_f^{-1} f(t) = \lim_{h\to 0} \sum_{n=0}^{\infty} f(t-nh)\cdot h \tag{50}$$

and

$$D_b^{-1} f(t) = \lim_{h\to 0} \sum_{n=0}^{\infty} f(t+nh)\cdot h. \tag{51}$$

These relations motivate the following comments:

- The different time flow shows its influence: the causality (anti-causality) is clearly stated,
- We have $D_f^{-1} D_f f(t) = D_f D_f^{-1} f(t) = f(t)$ and $D_b^{-1} D_b f(t) = D_b D_b^{-1} f(t) = f(t)$. We will call $D_{f,b}^{-1}$ *anti-derivative* [63].

We generalise the derivatives to the fractional case

$$\begin{aligned} D_f^{\alpha} f(t) &= \lim_{h\to 0^+} h^{-\alpha} \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} f(t-nh) \\ D_b^{\alpha} f(t) &= e^{-j\alpha\pi} \lim_{h\to 0^+} h^{-\alpha} \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} f(t+nh) \end{aligned} \tag{52}$$

that are at present called Grünwald-Letnikov (forward and backward) derivative, in spite of their first proposal having been done by Liouville [26]. The symbol $(-\alpha)_n$ is the Pochhamer representation of the raising factorial: $(-\alpha)_0 = 1$, $(-\alpha)_n = \prod_{k=0}^{n-1} (-\alpha + k)$. In applications where the variable t is not a time the constant factor $e^{-j\alpha\pi}$, in the backward case, can be removed [63]. In such situations, the derivatives can be called *left and right* respectively and we can show that they verify

$$\mathcal{L}\left[D_{l,r}^{\alpha} f(t)\right] = (\pm s)^{\alpha} F(s), \quad \pm Re(s) \geq 0, \tag{53}$$

in agreement with the requirement (46). In the rest of this section we consider merely the forward case. It must be highlighted an important fact: equation (52) is valid for any real (or complex) order. The relation (53) suggests another way of expressing the FD. We only have to remember that in (20) we obtained the impulse response of the causal differintegrator. So, the output for a given function, $f(t)$, is given by the convolution

$$D_{f,b}^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \tau^{-\alpha-1} f(t \mp \tau) d\tau. \tag{54}$$

Relation (54) is an integral formulation of the FD. However, this expression is not as handy as (52) due to the singularity of $\tau^{-\alpha-1}$ at the origin when $\alpha > 0$. Therefore, we adopt it only for negative orders (anti-derivative) and, for the positive (derivative) case we proceed with the regularization of the integral (we will consider the causal case). The regularised Liouville derivative is given by [91]

$$D_f^{\alpha} f(t) = \int_0^{\infty} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} \left[f(t-\tau) - \sum_0^{N} \frac{(-1)^m f^{(m)}(t)}{m!} \tau^m \right] d\tau, \tag{55}$$

where $N \in \mathbb{Z}_0^+$ is the greatest integer less than or equal to α, so that $\alpha - 1 < N \leq \alpha$. However, we have two alternatives for applying the convolution, avoiding the singularity. Let $\alpha \leq M \in \mathbb{Z}_0^+$. We can write

$$s^{\alpha} = s^{\alpha-M} s^M = s^M s^{\alpha-M}$$

which gives us two ways to solve the problem. The first reads [26]

$$D_f^{\alpha} f(t) = \int_0^{\infty} \frac{\tau^{M-\alpha-1}}{\Gamma(-\alpha+M)} f^{(M)}(t-\tau) d\tau. \tag{56}$$

This is called *Liouville-Caputo derivative* (LC) [26], [8]. The second decomposition, $s^{\alpha} = s^M s^{\alpha-M}$, gives

$$D_f^{\alpha} f(t) = D_f^M \left[\int_0^{\infty} \frac{\tau^{M-\alpha-1}}{\Gamma(-\alpha+M)} f(t-\tau) d\tau \right], \tag{57}$$

that constitues a derivative of the Riemann-Liouville type, that is also called Liouville derivative [30].

In conclusion, from the IR of the differintegrator, 3 different integral formulations were obtained from where current expressions can be derived, (55), (56), and (57). A fair comparison of the 3 derivatives lead us to conclude that:

- If $f(t)$ has LT with a nondegenerate region of convergence, then the 3 derivatives give the same result,
- The Liouville-Caputo derivative demands too much from analytical point of view, since it needs the unnecessary existence of the M^{th} order derivative,
- If $f(t) = 1,\ t \in \mathbb{R}$, then the Liouville derivative does not exist, since the integral is divergent.

There are several properties revealed by these derivatives. The most important are [92]:

1) Linearity
2) Additivity and Commutativity of the orders

$$D_f^\alpha D_f^\beta f(t) = D_f^\beta D_f^\alpha f(t) = D_f^{\alpha+\beta} f(t). \tag{58}$$

This result comes immediately from (53).

3) Neutral element

$$D_f^\alpha D_f^{-\alpha} f(t) = D_f^0 f(t) = f(t). \tag{59}$$

From (59) we conclude that there is always an inverse operator, that is, for every α there is always the $-\alpha$ order derivative that we call anti-derivative given by the same formula and so it is not needed to join any primitivation constant.

4) Backward compatibility ($n \in \mathbb{N}$)

If $\alpha = n$, then:

$$D_f^n f(t) = \lim_{h\to 0} \frac{\sum_{k=0}^{n}(-1)^k \binom{n}{k} f(t-kh)}{h^n}$$

We obtain this expression repeating the first order derivative.

If $\alpha = -n$, then:

$$D_f^{-n} f(t) = \lim_{h\to 0} \sum_{k=0}^{n} \frac{(n)_k}{k!} f(t-kh)\cdot h^n,$$

that corresponds to a n-th repeated summation [63].

E. The frequency response

The property (53) of the derivatives we are using can be extended to the imaginary axis yielding the following important result

$$D^\alpha e^{j\omega_0 t} = (j\omega_0)^\alpha e^{j\omega_0 t}, \quad t, \omega_0 \in \mathbb{R}. \tag{60}$$

In general, and for a system defined by (8), if $x(t) = e^{j\omega_0 t}$, then we obtain immediately from (9)

$$y(t) = G(j\omega_0) e^{j\omega_0 t}. \tag{61}$$

When the input is $x(t) = \cos(\omega_0 t)$, we have

$$y(t) = |H(\omega_0)| \cos\left(\omega_0 t + \varphi(\omega_0)\right), \tag{62}$$

where

1) $A(\omega) = |H(\omega)|$ is the *amplitude spectrum*, or *gain*, and is an even function,
2) $\phi(\omega) = \varphi(\omega)$ is the *phase spectrum*, or simply *phase*, and is an odd function.

For our objectives, it is preferable to give another form to the TF:

$$G(s) = K_0 \frac{\prod_{k=1}^{M_z} \left(\left(\frac{s}{\zeta_k}\right)^\alpha + 1\right)^{m_k}}{\prod_{k=1}^{N_p} \left(\left(\frac{s}{\theta_k}\right)^\alpha + 1\right)^{m_k}}, \tag{63}$$

where K_0 is called *static gain* and the integers $m_k = 1, 2, \cdots$, represent the multiplicity of the pseudo-poles/zeroes. As it is easy to verify, $\theta_k = (-p_k)^{1/\alpha}$ and $\zeta_k = (-z_k)^{1/\alpha}$. With these changes, (22) assumes a more classical form. Supposing that all the coefficients a_k and b_k in (22) are real, then all values of z_k and p_k are either real or, being complex, appear in conjugate pairs. In this case, its is usual to join the corresponding terms:

$$\left(\left(\frac{s}{\theta}\right)^\alpha + 1\right)\left(\left(\frac{s}{\theta^*}\right)^\alpha + 1\right) = \left(\frac{s^\alpha}{|\theta|}\right)^2 + \left(\frac{s}{|\theta|}\right)^\alpha \frac{2Re(\theta^\alpha)}{|\theta|^\alpha} + 1. \tag{64}$$

In the following we will assume multiplicities equal to 1. For a TF given by (22) the gain and phase are given by

$$\begin{aligned} A(\omega) = 20\log_{10}(K_0) &+ \sum_{k=1}^{M_z} m_k 20\log_{10}\left|\left(\frac{j\omega}{\zeta_k}\right)^\alpha + 1\right| \\ &- \sum_{k=1}^{N_p} n_k 20\log_{10}\left|\left(\frac{j\omega}{\theta_k}\right)^\alpha + 1\right|, \end{aligned} \tag{65}$$

and

$$\varphi(\omega) = \arg(K_0) + \sum_{k=1}^{M_z} m_k \arg\left[\left(\frac{j\omega}{\zeta_k}\right)^\alpha + 1\right] \tag{66}$$

$$- \sum_{k=1}^{N_p} n_k \arg\left[\left(\frac{j\omega}{\theta_k}\right)^\alpha + 1\right]. \tag{67}$$

As it is easy to verify a simple pseudo-pole/zero originates a decrease/increase of the amplitude by less than 20 dB per decade. In figure 2 we show the Bode plots corresponding to the fractional RC circuit of example III.1.

Remark III.7. *Frequently, we find papers that, assuming the Riemann-Liouville or Caputo derivatives, use the results we just presented. Nonetheless, this is not consistent with relations (60) to (62), since they are invalid for such derivatives.*

F. The Initial-Condition Problem

The initial condition (IC) problem is one of the most discussed topics in linear systems. The IC are a set of values that determine the output of a system when the input is null, i.e. the free response.

Since the thirties in the last century, this problem has been solved with the ULT. However, this approach uses values taken at $t = 0^+$, instead at $t = 0^-$, since the IC depend on the past, not on the future. This led to a modification of the

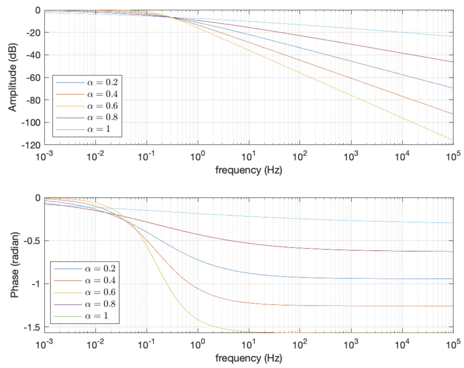

Fig. 2. Bode diagrams of the RC circuit for $\alpha = 0.2k,\ k = 1, 2, \cdots 5$ (from above).

ULT [93] that has been used in the study of integer order LS. Nonetheless, we shall avoid this approach and recall that the designation IC refers only to inputs and outputs before the reference instant.

Concerning the fractional systems the first approach was based in the RL derivative and the corresponding IC it induces [89], [94]. This procedure has been superseded by the one coming from the C derivative, since it uses initial conditions based on integer order derivatives [89]. These solutions were called into question differently by Lorenzo and Hartley [95] and Ortigueira [96] suggesting that the IC would be dependent on the structure of the system in question and not on the derivative used. This means that the RL or C derivatives have "their" own IC, not necessarily those posed by a system. Later Trigeassou et al. [97] and Sabatier et al. [42] proposed a new approach based on the "infinite state approach" following the so-called diffusive representation [98]. However, these approaches are not attractive in Engineering, because they do not show backwards compatibility with classic results.

The idea of having the IC as dependent on the structure of the system was recalled by Ortigueira and Coito in [99]. In particular, it was shown how we can choose suitable IC in a state space formulation, using an expansion of the solution in fractional Taylor series. However, a general approach based on a reinterpretation of the role of IC was proposed. To understand it, let us consider the simple example of a one-pole lowpass filter

$$Df(t) + af(t) = g(t),$$

where a is any real. Assume that $g(t) = 0$ for $t > t_0$ and that we want to compute the output, $f(t)$. This is equivalent to say that we observe the system by means of a unit step window $\varepsilon(t - t_0)$:

$$Df(t)\varepsilon(t - t_0) + af(t)\varepsilon(t - t_0) = 0.$$

Without loosing generality, we set $t_0 = 0$. As $Df(t)\varepsilon(t) = D\left[f(t)\varepsilon(t)\right] - f(t_0)\delta(t)$, we obtain

$$D\left[f(t)\varepsilon(t)\right] - f(0)\delta(t) + af(t)\varepsilon(t) = 0 \tag{68}$$

that has the well known solution $f(t)\varepsilon(t) = e^{-at}\varepsilon(t)$. Now, note that we can write

$$D\left[f(t)\varepsilon(t) - f(0)\varepsilon(t)\right] + af(t)\varepsilon(t) = 0 \tag{69}$$

that gives a new interpretation of the IC: *it is the amplitude of the subtracting step to make the function continuous.* In terms of the LT, we can write

$$s\mathcal{L}\left[f(t)\varepsilon(t) - f(0)\varepsilon(t)\right] + a\mathcal{L}\left[f(t)\right] = 0$$

or

$$sF(s) - f(0) + aF(s) = 0$$

with $F(s) = \mathcal{L}\left[f(t)\varepsilon(t)\right]$. This means that we made the substitution $sF(s) \to sF(s) - f(0)$. For the second order derivative, we have

$$\mathcal{L}\left[f''(t)\varepsilon(t)\right] = s\left[sF(s) - f(0)\right] - f'(0) = s^2F(s) - sf(0) - f'(0).$$

This procedure can be replicated for higher order derivatives leading to

$$\mathcal{L}\left[D^N f(t) \cdot \varepsilon(t)\right] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0)s^{(N-k-1)}. \tag{70}$$

that coincides with the classic formula obtained with the ULT. The way into the fractional case is similar to the described above with the substitutions $D \to D^\alpha$, with $\alpha \geq 0$. From (69), we have

$$D^\alpha\left[f(t)\varepsilon(t) - f(0)\varepsilon(t)\right] + af(t)\varepsilon(t) = 0 \tag{71}$$

and

$$s^\alpha\mathcal{L}\left[f(t)\varepsilon(t) - f(0)\varepsilon(t)\right] + a\mathcal{L}\left[f(t)\right] = 0$$

that gives

$$s^\alpha\mathcal{L}\left[f(t)\varepsilon(t)\right] - f(0)s^\alpha\frac{1}{s} + a\mathcal{L}\left[f(t)\right]. \tag{72}$$

Therefore, the substitution $s^\alpha F(s) \to s^\alpha F(s) - f(0)s^{\alpha-1}$ is used. The repetition of the process gives

$$\mathcal{L}\left[f^{2\alpha}(t)\varepsilon(t)\right] = s^\alpha\left[s^\alpha F(s) - f(0)s^{\alpha-1}\right] - f^\alpha(0)s^{\alpha-1} = s^{2\alpha}F(s) - s^{2\alpha-1}f(0) - f^\alpha(0)s^{\alpha-1}$$

that can be generalized to

$$\mathcal{L}\left[D^{N\alpha} f(t) \cdot \varepsilon(t)\right] = s^{N\alpha}F(s) - \sum_{m=0}^{N-1} f^{(m\alpha)}(0)s^{(N-m)\alpha-1}. \tag{73}$$

For the noncommensurate case, defined by a sequence of increasing orders, $\gamma_m,\ m = 0, 1, \cdots$, the same procedure led us to obtain the IC theorem of the LT [65]

$$\mathcal{L}\left[D^{\gamma_N} f(t) \cdot \varepsilon(t)\right] = s^{\gamma_N}F(s) - \sum_{m=0}^{N-1} f^{(\gamma_m)}(0)s^{\gamma_N-\gamma_m-1}, \tag{74}$$

that can be used in (47) to obtain the LT of the free response. We have

$$\sum_{k=0}^{N} a_k s^{\alpha_k} Y(s) - \sum_{k=0}^{N-1} y^{(\alpha_m)}(0)s^{\alpha_N-\alpha_m-1} = \sum_{k=0}^{M} b_k s^{\alpha_k} X(s) - \sum_{k=0}^{M-1} x^{(\alpha_m)}(0)s^{\alpha_N-\alpha_m-1}$$

with $F(s) = 0$, and the LT of the free response, $Y_f(s)$, is given by

$$Y_f(s) = \frac{\sum_{k=0}^{N-1} y^{(\alpha_m)}(0)s^{\alpha_N-\alpha_m-1} - \sum_{k=0}^{M-1} x^{(\alpha_m)}(0)s^{\alpha_M-\alpha_m-1}}{\sum_{k=0}^{N} a_k s^{\alpha_k}} \tag{75}$$

that can be inverted by the methods introduced in section III. Expression (75) recovers the classic formula when the derivative orders become positive integers.

G. Other generalizations

1) Variable order derivatives and systems: In the previous sections we assumed that the orders of derivatives were constant. However, most definitions and tools keep their validity when the orders become variable, provided that suitable derivative definitions are used. Several definitions with variable orders are known [100], [101], [102], [103], [104], but most are incompatible with our system framework. Suitable definitions were introduced in [105], [23] that recover the constant order definitions above introduced. We define the variable order (VO) forward GL derivative as

$$D_f^{\alpha(t)} f(t) = \lim_{h\to 0^+} h^{-\alpha(t)} \sum_{k=0}^{\infty} \frac{(-\alpha(t))_k}{k!} f(t-kh). \tag{76}$$

This definition preserves most important properties of the constant order GL FD. Let $f(t) = e^{st}, \ \ s \in \mathbb{C}$,

$$\begin{aligned} D_f^{\alpha(t)} e^{st} &= \lim_{h\to 0^+} h^{-\alpha(t)} \sum_{k=0}^{\infty} \frac{(-\alpha(t))_k}{k!} e^{s(t-kh)} \\ &= s^{\alpha(t)} e^{st}, \qquad Re(s) > 0. \end{aligned} \tag{77}$$

In particular, if $f(t) = e^{j\omega t}$, then $D_f^{\alpha(t)} f(t) = (j\omega)^{\alpha(t)} e^{j\omega t}$, and the derivative of a co-sine (or sine) is an amplitude-phase modulated co-sine (or sine), then

$$D_f^{\alpha(t)} \cos(\omega t) = \omega^{\alpha(t)} \cos\left[\omega t + \alpha(t)\frac{\pi}{2}\right], \ \ \omega > 0.$$

Example III.2. *The VOFD of the unit step is*

$$\begin{aligned} D_f^{\alpha(t)} \varepsilon(t) &= \frac{1}{\Gamma(-\alpha(t))} \int_0^t (t-\tau)^{-\alpha(t)-1} \mathrm{d}\tau \\ &= \frac{t^{-\alpha(t)}}{\Gamma(-\alpha(t)+1)} \varepsilon(t), \end{aligned} \tag{78}$$

which is similar to the constant order case [63].

For functions with LT, we can define a regularised Liouville VOFD by [105], [23]

$$D_f^{\alpha} f(t) = \int_0^{\infty} \frac{\tau^{-\alpha(t)-1}}{\Gamma(-\alpha(t))} \left[f(t-\tau) - \varepsilon(\alpha(t)) \sum_{0}^{N(t)} \frac{(-1)^m f^{(m)}(t)}{m!} \tau^m \right] d\tau, \tag{79}$$

where $N(t) = \lfloor \alpha(t) \rfloor$. With this derivative, we can define VO LS. For example, the VO FARMA reads

$$\sum_{k=0}^{N} a_k D^{\alpha_k(t)} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k(t)} x(t) \tag{80}$$

with $t \in \mathbb{R}$. We can introduce the VO IR and VO TF [105], [23].

2) Fractional stochastic processes: Consider a continuous-time linear system with TF given by $G(s)$, having no poles on the imaginary axis (regular system). Assume that the input $x(t)$ to the system is a stationary stochastic process with autocorrelation function

$$R_{xx}(t) = E\left[x(\tau + t)x(\tau)\right]. \tag{81}$$

The output is given by $y(t) = g(t) * x(t)$, and the corresponding autocorrelation is

$$R_{yy}(t) = g(t) * g(-t) * R_{xx}(t). \tag{82}$$

Let $S_{xx}(s) = \mathcal{L}\left[R_{xx}(t)\right]$ represent the LT of the autocorrelation, and let us define the *power spectral density* (or simply the *spectrum*) of $x(t)$ as

$$S_{xx}(j\omega) = \mathcal{F}\left[R_{xx}(t)\right], \tag{83}$$

obtained restricting s to the imaginary axis, i.e. $s = j\omega$. The relation (83) states the Wiener-Khintchin-Einstein theorem [106]. We get

$$S_{yy}(s) = G(s)G(-s)S_{xx}(s), \tag{84}$$

In the integer order case, $S_{yy}(s)$ has a non empty ROC that includes the imaginary axis, but, in general, the ROC is empty, since $H(s)$ exists only for $Re(s) > 0$. This leads us to define

$$S_{yy}(j\omega) = \lim_{s\to j\omega} G(s)G(-s) \cdot S_{xx}(j\omega) = \left|G(j\omega)\right|^2 S_{xx}(j\omega), \tag{85}$$

that relates the input with the corresponding output power spectral densities, $S_{xx}(j\omega)$ and $S_{yy}(j\omega)$.

In applications, mainly in modelling real data, we assume that the input is white noise, $w(t)$. In this case, the autocorrelation is an impulse, usually written as

$$R_{ww} = \sigma^2 \delta(t). \tag{86}$$

Therefore, the output spectrum of a linear system is

$$S_{yy}(j\omega) = \sigma^2 \left|G(j\omega)\right|^2, \tag{87}$$

stating an important relation suitable for stochastic modeling and identification. Let us go back to equation (84) and substitute there the expression of the TF (10). We have

$$S_{yy}(s) = \frac{\sum_{k=0}^{M} b_k s^{k\alpha}}{\sum_{k=0}^{N} a_k s^{k\alpha}} \cdot \frac{\sum_{k=0}^{M} b_k (-s)^{k\alpha}}{\sum_{k=0}^{N} a_k (-s)^{k\alpha}} S_{xx}(s). \tag{88}$$

However, we must take into account that, if $\alpha \neq 1$, these relations are only valid in the limit when $s \to j\omega$. Consequently, we have

$$\sum_{k=0}^{N}\sum_{m=0}^{N} a_k a_m (j\omega)^{k\alpha}(-j\omega)^{m\alpha} S_{yy}(j\omega) = \sum_{k=0}^{M}\sum_{m=0}^{M} b_k b_m (j\omega)^{m\alpha}(-j\omega)^{k\alpha} S_{xx}(j\omega). \quad (89)$$

With the inverse FT and introducing a two-sided derivative D_θ^γ

$$D_{\alpha-\beta}^{\alpha+\beta} f(t) = \mathcal{F}^{-1}\left[(j\omega)^\alpha(-j\omega)^\beta F(j\omega)\right], \quad (90)$$

we obtain a differential equation

$$\sum_{k=0}^{N}\sum_{m=0}^{N} a_k a_m D_{k\alpha-m\alpha}^{k\alpha+m\alpha} R_{yy}(t) = \sum_{k=0}^{M}\sum_{m=0}^{M} b_k b_m D_{k\alpha-m\alpha}^{k\alpha+m\alpha} R_{xx}(t), \quad (91)$$

that defines a new LS relating the autocorrelation functions of input and output signals. Noting that

$$\Psi_{\alpha-\beta}^{\alpha+\beta}(j\omega) = (j\omega)^\alpha(-j\omega)^\beta = |\omega|^{\alpha+\beta} e^{j(\alpha-\beta)\frac{\pi}{2}\operatorname{sgn}(\omega)}, \quad (92)$$

the frequency response of such system is

$$|G(j\omega)|^2 = \frac{\sum_{k=0}^{M}\sum_{m=0}^{M} b_k b_m |\omega|^{(k+m)\alpha} e^{j(k-m)\frac{\alpha\pi}{2}\operatorname{sgn}(\omega)}}{\sum_{k=0}^{N}\sum_{m=0}^{N} a_k a_m |\omega|^{(k+m)\alpha} e^{j(k-m)\frac{\alpha\pi}{2}\operatorname{sgn}(\omega)}}. \quad (93)$$

In the following we study briefly the two-sided derivatives. These were formally introduced in [107], [108], [63] and were unified in a formulation that included the one-sided (forward/backward) Grünwald-Letnikov derivatives [109], [110].

Definition III.1. *Let $f(t)$, $t \in \mathbb{R}$, be a real function and $\gamma, \theta \in \mathbb{R}$ two real parameters. We define a two-sided GL type FD of $f(t)$ by*

$$D_\theta^\gamma f(t) = \lim_{h\to 0^+} h^{-\gamma} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \Gamma(\gamma+1) f(t-nh)}{\Gamma(\frac{\gamma+\theta}{2}-n+1)\Gamma(\frac{\gamma-\theta}{2}+n+1)}, \quad (94)$$

where γ is the derivative order, and θ is an asymmetry parameter. The situation corresponding to $\gamma = -N$, $N \in \mathbb{N}$ deserves particular attention [110], [23].

For absolutely or square integrable functions, the FT of (94) is given by

$$\mathcal{F}\left[D_\theta^\gamma f(t)\right] = \Psi_\theta^\gamma(\omega)\mathcal{F}\left[f(t)\right], \quad (95)$$

where

$$\Psi_\theta^\gamma(\omega) = |\omega|^\gamma e^{i\theta\frac{\pi}{2}\operatorname{sgn}(\omega)}. \quad (96)$$

Definition III.2. *The general two-sided fractional derivative (TSFD), D_θ^γ, can be expressed by its FT [109], [110]*

$$\mathcal{F}\left[D_\theta^\gamma f(t)\right] = |\omega|^\gamma e^{i\frac{\pi}{2}\theta\cdot\operatorname{sgn}(\omega)} F(\omega), \quad (97)$$

where γ and θ are the derivative order and asymmetry parameter, respectively.

3) The fractional Brownian motion: As an application of the presented formalism, we consider the fractional Brownian motion (fBm). This process was studied first by Mandelbrot and Van Ness [13] that suggested it as a model for non-stationary signals, but with stationary increments. These are suitable to understand phenomena exhibiting long range or $1/f^\alpha$ dependences. Let $H \in (0,1)$ be the so-called Hurst parameter [111] and let $b_0 \in \mathbb{R}$. The fBm, $B_H(t)$, with parameter H is defined by

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)} \left\{\int_{-\infty}^{0}\left[(t-\tau)^{H-1/2} - (-\tau)^{H-1/2}\right] \mathrm{d}B(\tau) + \int_0^t (t-\tau)^{H-1/2}\,\mathrm{d}B(\tau)\right\}, \quad (98)$$

where $B_H(0) = b_0$, and $B(t)$ is the standard Brownian motion. Note that $B(t)$ is not differentiable, but we can assign it a generalised derivative, the *white noise*, $w(t),\ t \in \mathbb{R}$, so that $\mathrm{d}B(t) = w(t)\,\mathrm{d}t$. We can show that the fBm can be defined in a way similar to the classic Brownian motion:

$$v_H(t) = B_H(t) - B_H(0) = \int_0^t D_f^\alpha w(\tau)\,\mathrm{d}\tau, \quad (99)$$

where

$$r_\alpha(t) = D_f^\alpha w(t) = \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t} w(\tau)(t-\tau)^{-\alpha-1}\,\mathrm{d}\tau, \quad (100)$$

is the Liouville forward derivative of order $-H+1/2$ (55). As $H \in (0,1)$, then $\alpha \in (-1/2, 1/2)$. Expression (99) is similar to the current definition of Brownian motion, provided that $\alpha = 0$ (i.e. that $H = 1/2$). This formula suggests the use of other FD definitions alternative to the Liouville definition, as the GL formulation. If the white noise is gaussian, then $r_\alpha(t)$ is a *fractional Gaussian noise* (FGN). The signal $r_\alpha(t)$ has an infinite power, but its mean value is constant (and null). Moreover, the autocorrelation function $R_r(t,\tau) = E\left[r_\alpha(t+\tau)r_\alpha(\tau)\right]$ depends only on t, not on τ. Therefore, the fractional noise $r_\alpha(t)$ is a wide sense stationary stochastic process and its autocorrelation function is

$$R_\alpha(t) = \sigma^2\frac{|t|^{-2\alpha-1}}{2\Gamma(-2\alpha)\cos(\alpha\pi)}. \quad (101)$$

Relation (101) shows that we only have a (wide sense) stationary (hyperbolic) noise if

$$2\alpha+1 > 0 \qquad \text{and} \qquad \Gamma(-2\alpha)\cos(\alpha\pi) > 0. \quad (102)$$

The other cases do not lead to a valid autocorrelation function of a stationary stochastic process, since it does not have a maximum at the origin. Then, for $-1/2 < \alpha < 0$ and $\alpha \in (2n, 2n+1), n \in \mathbb{Z}^+$, we obtain valid autocorrelation functions. We conclude that, if $|\alpha| < 1/2$, we obtain a stationary process in the anti-derivative case, $\alpha < 0$, and a nonstationary process in the derivative case, $\alpha > 0$ [112], [113].

The above-defined process is a somehow strange process with infinite power. However, the power inside any finite frequency band is always finite. Its spectrum is

$$S_\alpha(\omega) = \frac{\sigma^2}{|\omega|^{2\alpha}}, \tag{103}$$

and thus a "$1/f$ noise". From the fractional noise, $r_\alpha(t)$, we can generate a fractional Brownian motion, using expression (99).

The process introduced in (99) enjoys the properties usually attributed to fBm [112], [113], namely

1) $v_\alpha(0) = 0$ and $E[v_\alpha(t)] = 0$, for every $t \geq 0$.
2) The covariance is

$$E[v_\alpha(t)v_\alpha(s)] = \frac{V_H}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2\alpha+1}\right], \tag{104}$$

where

$$V_H = \frac{\sigma^2}{\Gamma(2H+1)\sin H\pi}. \tag{105}$$

3) The process has stationary increments.
4) The incremental process has a $1/f^\beta$ spectrum.
Consider the process corresponding $v_{H+1/2}(t)$. The incremental process defined, for t and $t-T$, with $t \in \mathbb{R}$ and $T \in \mathbb{R}^+$, by

$$d_h(t) = v_{H+1/2}(t) - v_{H+1/2}(t-T), \tag{106}$$

has the following autocorrelation function:

$$R_d(t) = \frac{V_H}{2}\left[|t+T|^{2H} + |t-T|^{2H} - 2|t|^{2H}\right]. \tag{107}$$

If $\beta > 0$, then

$$\mathcal{F}\left[\frac{1}{2\Gamma(\beta)\cos(\beta\pi/2)}|t|^{\beta-1}\right] = \frac{1}{|\omega|^\beta}, \tag{108}$$

so that the FT of $R_d(t)$ leads to the spectrum of the incremental process:

$$S_d(\omega) = \sigma^2\,\frac{\sin^2(\omega \mathrm{T}/2)}{|\omega|^{2H+1}}. \tag{109}$$

For $|\omega| \ll \pi/T$, the spectrum can be approximated by

$$S_d(\omega) \approx \frac{\sigma^2 T^2}{4}\frac{1}{|\omega|^{2H-1}}. \tag{110}$$

This result shows that:

- If $0 < H < 1/2$, then the spectrum is parabolic and corresponds to an antipersistent fBm, because the increments tend to have opposite signs; this case corresponds to the integration of a stationary fractional noise.
- If $1/2 < H < 1$, then the spectrum has a hyperbolic nature and corresponds to a persistent fBm, because the increments tend to have the same sign; this case corresponds to the integration of a nonstationary fractional noise.

IV. Tempered Fractional Linear Systems

A. Tempered fractional derivatives

As it is well known, a stable CT-ARMA (integer order) system has an impulse response that decreases to zero exponentially when the argument goes to infinite: it has a short memory. A CT-FARMA has an impulse response that is a sum of one integer and one fractional components (31). The fractional part decreases like a power function of the argument and that is the reason why we claim that fractional systems are of long range. However, we find phenomena that are neither of short, nor long range. They are medium range systems and can be obtained by embeding the two types of responses. Therefore, a multiplication of the derivative kernels by an exponential gives the required behaviour. This reasoning leads to the so-called tempered derivatives. Due to the similarity to the results introduced above, we will consider here only the forward derivatives. For the backward see [24]. For $\alpha \in \mathbb{R}$ we can write

$$D^\alpha_{\lambda,f} f(t) = \lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}e^{-n\lambda h}f(t-nh), \tag{111}$$

that has LT

$$\mathcal{L}\left[D^\alpha_{\lambda,f} f(t)\right] = (s+\lambda)^\alpha F(s), \quad Re(s) > -\lambda. \tag{112}$$

The inverse LT of this expression can be obtained from the properties of the LT and of the Gamma function. It is given by:

$$\mathcal{L}^{-1}\left[(s+\lambda)^\alpha\right] = \pm e^{-\lambda t}\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(\pm t). \tag{113}$$

Equation (113) and the convolution property of the LT allow us to introduce the integral version of the TFD as

$$\begin{aligned} D^\alpha_{\lambda,f} f(t) &= \int_0^\infty f(t-\tau)e^{-\lambda\tau}\frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}d\tau \\ &= e^{-\lambda t}\int_{-\infty}^{t} f(\tau)e^{\lambda\tau}\frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)}d\tau, \end{aligned} \tag{114}$$

The regularised tempered derivative is defined by

$$D^\alpha_{\lambda,f} f(t) = \int_0^\infty \left[f(t-\tau) - \varepsilon(\alpha)\sum_0^N \frac{(-1)^m f^{(m)}(t)}{m!}\tau^m\right]\frac{e^{-\lambda\tau}\tau^{-\alpha-1}}{\Gamma(-\alpha)}d\tau, \tag{115}$$

that generalises the causal expression (114) to real orders and where $N = \lfloor\alpha\rfloor$. For stability reasons, we consider always $\lambda \in \mathbb{R}^+_0$.

B. On the tempered LS

We introduced in (14) the notion of a tempered LS through its TF. After having the derivatives defined in the previous subsection we can write the corresonding differential equation. Let $x(t)$ and $y(t)$ be two functions assumed almost everywhere continuous, with bounded variation, and of exponential order.

Therefore, they have LT with non empty regions of convergence. We define a tempered fractional LS with input $x(t)$ and output $y(t)$ as the one following the differential equation

$$\sum_{k=0}^{N} a_k D^{\alpha_k}_{\lambda_k,f} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k}_{\gamma_k,f} x(t), \tag{116}$$

where $t \in \mathbb{R}$, $a_k,\ k = 0, 1, \cdots, N$, and $b_k,\ k = 0, 1, \cdots, M$, are real valued constant coefficients. The parameters α_k and β_k are the derivative orders that, without loss of generality, we assume to form strictly increasing sequences of positive real numbers. The exponential coefficients $\lambda_k,\ k = 0, 1, \cdots, N$, and $\gamma_k,\ k = 0, 1, \cdots, M$, are real numbers. The differential equation (116) is very general in the sense that we can use forward, backward or both types of derivatives. However, for most pratical applications, where we deal with causal systems and, therefore, the use of the forward tempered GL or L derivatives is more appropriate. For stability reasons, the parameters $\lambda_k,\ k = 0, 1, \cdots, N$, and $\gamma_k,\ k = 0, 1, \cdots, M$, must be positive.

The TF (14) corresponding to (116) poses difficulties for an analytic manipulation. Therefore, just consider the commensurate case defined with $\alpha_k = \beta_k = k\alpha,\ k \in \mathbb{N}_0$. Furthermore, this case is only manageable if $\lambda_k = \gamma_k = \lambda_0, k = 1, 2, 3, \cdots$. The TF becomes:

$$G(s) = \frac{\sum\limits_{k=0}^{M} b_k (s+\lambda_0)^{k\alpha}}{\sum\limits_{k=0}^{N} a_k (s+\lambda_0)^{k\alpha}}, \tag{117}$$

or, equivalently

$$G(s) = K_0 \frac{\prod\limits_{k=1}^{M} [(s+\lambda_0)^{\alpha} - z_k]}{\prod\limits_{k=1}^{N} [(s+\lambda_0)^{\alpha} - p_k]}, \tag{118}$$

where p_k and z_k, $k = 1, 2, \cdots$ are the pseudo-poles and -zeroes and K_0 is a constant. The corresponding differential equation can be written as

$$\sum_{k=0}^{N} a_k D^{k\alpha}_{\lambda_0} y(t) = \sum_{k=0}^{M} b_k D^{k\alpha}_{\lambda_0} x(t). \tag{119}$$

If we denote by $g_0(t)$ the IR of this system, corresponding to $\lambda_0 = 0$, and use the shift property of the LT, we conclude that the IR of (119) is given by

$$g(t) = e^{-\lambda_0 t} g_0(t). \tag{120}$$

This shows that the tempering procedure leads to an increase in the stability domain of a system.

Example IV.1. *The fractional lead* ($+\alpha$) *compensator used in Control to increase the phase of a system around a chosen frequency and the lag* ($-\alpha$) *compensator, used to increase the static gain of a plant, are defined by the TF [114], [23], [115]*

$$C(s) = \left(\frac{\tau s + a}{s + a}\right)^{\pm\alpha},\ \alpha, a \in \mathbb{R}^+,\ \tau > 1. \tag{121}$$

Let us compute the impulse response merely for the lead controller. We can write

$$C(s) = \tau^{\alpha} \left(\frac{s + \frac{a}{\tau}}{s + a}\right)^{\alpha} = \tau^{\alpha} \left(s + \frac{a}{\tau}\right)^{\alpha} (s+a)^{-\alpha}.$$

The inverse LT gives

$$c(t) = \tau^{\alpha} \left[e^{-\frac{at}{\tau}} \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \varepsilon(t) \right] * \left[e^{-at} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \varepsilon(t) \right] \tag{122}$$

that can be written as

$$c(t) = \frac{\tau^{\alpha} e^{-at}}{\Gamma(\alpha)\Gamma(-\alpha)} \int_0^t u^{-\alpha-1} (t-u)^{\alpha-1} e^{-au\left(\frac{1}{\tau}-1\right)} \mathrm{d}u\, \varepsilon(t). \tag{123}$$

This expression shows clearly the presence of two factors with different charcteristics described by exponential and fractional power functions [116].

V. Conclusions

The fractional continuous-time linear systems were presented. Two classes were introduced, namely, the fractional ARMA and the Tempered systems. For both classes we showed how to handle the standard tools, like impulse response, transfer function, and frequency response. We considered also the stability and the initial-condition problem. Additionally, for backward compatibility with classic systems, suitable fractional derivatives were introduced having in mind an adequate definition of system theory for Engineering applications.

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References

[1] O. Heaviside, *Electromagnetic Theory 2*, 1925, vol. 2.

[2] S. Westerlund, *Dead Matter has Memory!* Kalmar, Sweden: Causal Consulting, 2002.

[3] R. L. Magin, *Fractional Calculus in Bioengineering*. Begell House, 2006.

[4] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. London: Imperial College Press, 2010.

[5] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, ser. Nonlinear Physical Science. Beijing, Heidelberg: Springer, 2011.

[6] J. T. Machado, "And I say to myself: "What a fractional world!"," *Fractional Calculus and Applied Analysis*, vol. 14, no. 4, pp. 635–654, 2011.

[7] C. M. Ionescu, *The Human Respiratory System: An Analysis of the Interplay between Anatomy, Structure, Breathing and Fractal Dynamics*, ser. BioEngineering. London: Springer, 2013.

[8] R. Herrmann, *Fractional Calculus*, 3rd ed. World Scientific, 2018.

[9] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers. Vol. I: Background and Theory. Vol. II: Applications*, ser. Nonlinear Physical Science. Heidelberg: Springer and Higher Education Press, 2012.

[10] I. S. Jesus and J. Tenreiro Machado, "Application of integer and fractional models in electrochemical systems," *Mathematical Problems in Engineering*, vol. 2012, 2012.

[11] V. Uchaikin and R. Sibatov, *Fractional Kinetics in Solids: Anomalous Charge Transport in Semiconductors, Dielectrics and Nanosystems*. Singapore: World Scientific Publishing Company, 2013.

[12] A. C. J. Luo and V. S. Afraimovich, Eds., *Long-range Interaction, Stochasticity and Fractional Dynamics - Dedication to George M. Zaslavsky (1935-2008)*. Beijing and Dordrecht: Higher Education Press and Springer, 2010.

[13] B. Mandelbrot and J. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, no. 4, pp. 422–437, 1968. [Online]. Available: https://doi.org/10.1137/1010093

[14] N. Nossenson and B.-Z. Bobrovsky, "Analysis of direct spectrum measurement of a sinusoidal signal impaired by either fractional gaussian phase noise or fractional Brownian phase motion," *IEEE transactions on ultrasonics, ferroelectrics, and frequency control*, vol. 56, no. 11, pp. 2351–2362, 2009.

[15] J. M. Lilly, A. M. Sykulski, J. J. Early, and S. C. Olhede, "Fractional Brownian motion, the Matérn process, and stochastic modeling of turbulent dispersion," *Nonlinear Processes in Geophysics*, vol. 24, no. 3, pp. 481–514, 2017.

[16] S. Westerlund and L. Ekstam, "Capacitor theory," *IEEE*.

[17] N. Laskin, "Fractional quantum mechanics," *Physical Review E*, vol. 62, no. 3, p. 3135, 2000.

[18] ——, "Fractional Schrödinger equation," *Physical Review E*, vol. 66, no. 5, p. 056108, 2002.

[19] V. E. Tarasov, "Fractional vector calculus and fractional Maxwell's equations," *Annals of Physics*, vol. 323, no. 11, pp. 2756–2778, 2008.

[20] M. D. Ortigueira, M. Rivero, and J. J. Trujillo, "From a generalised Helmholtz decomposition theorem to fractional Maxwell equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 22, no. 1-3, pp. 1036–1049, 2015.

[21] J. Gulgowski, D. Kwiatkowski, and T. P. Stefański, "Signal propagation in electromagnetic media modelled by the two-sided fractional derivative," *Fractal and Fractional*, vol. 5, no. 1, 2021. [Online]. Available: https://www.mdpi.com/2504-3110/5/1/10

[22] M. D. Ortigueira, "An introduction to the fractional continuous-time linear systems: the 21st century systems," *IEEE Circuits and Systems Magazine*, vol. 8, no. 3, pp. 19–26, 2008.

[23] M. D. Ortigueira and D. Valério, *Fractional Signals and Systems*. Berlin, Boston: De Gruyter, 2020.

[24] M. D. Ortigueira, G. Bengochea, and J. A. T. Machado, "Substantial, tempered, and shifted fractional derivatives: Three faces of a tetrahedron," *Mathematical Methods in the Applied Sciences*, vol. n/a, no. n/a. [Online]. Available: https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.7343

[25] S. Dugowson, "Les différentielles métaphysiques (histoire et philosophie de la généralisation de l'ordre de dérivation)," PhD, Université Paris Nord, 1994.

[26] J. Liouville, "Memóire sur le calcul des différentielles à indices quelconques," *Journal de l'École Polytechnique, Paris*, vol. 13, no. 21, pp. 71–162, 1832.

[27] ——, "Memóire sur l'usage que l'on peut faire de la formule de Fourier, dans le calcul des différentielles à indices quelconques," *Journal für die reine und angewandte Mathematik (Journal de Crelle)*, vol. 13, no. 21, pp. 219–232, 1835.

[28] ——, "Note sur une formule pour les différentielles à indices quelconques à l'occasion d'un mémoire de M. Tortolini," *Journal de Mathématiques Pures et Appliquées*, vol. 20, pp. 115–120, 1855.

[29] B. Riemann, "Versuch einer allgemeinen auffassung der integration und differentiation," *Gesammelte Werke*, vol. 62, no. 1876, 1876.

[30] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*. Yverdon: Gordon and Breach, 1993.

[31] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier, 2006.

[32] P. J. Nahin, "Oliver Heaviside, Fractional Operators, and the Age of the Earth," *IEEE Transactions on Education*, vol. 28, no. 2, pp. 94–104, 1985.

[33] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulations*, vol. 16, no. 3, pp. 1140–1153, 2011.

[34] J. T. Machado, A. M. Galhano, and J. J. Trujillo, "On development of fractional calculus during the last fifty years," *Scientometrics*, vol. 98, no. 1, pp. 577–582, 2014.

[35] D. Valério, J. T. Machado, and V. Kiryakova, "Some pioneers of the applications of fractional calculus," *Fractional Calculus and Applied Analysis*, vol. 17, no. 2, pp. 552–578, 2014.

[36] J. A. Machado and V. Kiryakova, "The chronicles of fractional calculus," *Fractional Calculus and Applied Analysis*, vol. 20, no. 2, pp. 307–336, 2017. [Online]. Available: https://doi.org/10.1515/fca-2017-0017

[37] A. Oustaloup, "Fractional order sinusoidal oscillators: optimization and their use in highly linear FM modulation," *IEEE Transactions on Circuits and Systems*, vol. 28, no. 10, pp. 1007–1009, 1981.

[38] ——, *Systèmes asservis linéaires d'ordre fractionnaire: Théorie et Pratique*. Masson, 1983.

[39] A. Oustaloup, P. Melchior, and A. El Yagoubi, "A new control strategy based on the concept of non integer derivation: application in robot control," in *Information Control Problems in Manufacturing Technology 1989*. Elsevier, 1990, pp. 641–648.

[40] A. Oustaloup and P. Melchior, "The great principles of the CRONE control," in *Proceedings of IEEE Systems Man and Cybernetics Conference-SMC*, vol. 2. IEEE, 1993, pp. 118–129.

[41] A. Oustaloup, P. Lanusse, and F. Levron, "Frequency-domain synthesis of a filter using Viète root functions," *IEEE*.

[42] J. Sabatier, M. Aoun, A. Oustaloup, G. Gregoire, F. Ragot, and P. Roy, "Fractional system identification for lead acid battery state of charge estimation," *Signal processing*, vol. 86, no. 10, pp. 2645–2657, 2006.

[43] R. N. Hughes, "Design and analysis of electrical circuits that produce fractional-order differentiation," Air Force Inst of Tech Wright–Patterson AFB OH School of Engineering, Tech. Rep., 1992.

[44] G. Bohannan, "Analog fractional order controller in a temperature control application," in *Fractional Differentiation and its Applications*, Porto, 2006.

[45] A. Verma and J. James, "Novel realization of voltage transfer function with fractional characteristics using CFAs," in *APCCAS 2008-2008 IEEE Asia Pacific Conference on Circuits and Systems*. IEEE, 2008, pp. 1502–1505.

[46] A. G. Radwan, A. S. Elwakil, and A. M. Soliman, "On the generalization of second-order filters to the fractional-order domain," *Journal of Circuits, Systems, and Computers*, vol. 18, no. 02, pp. 361–386, 2009.

[47] A. Lahiri and T. K. Rawat, "Noise analysis of single stage fractional-order low-pass filter using stochastic and fractional calculus," *ECTI Transactions on Electrical Engineering, Electronics, and Communications*, vol. 7, no. 2, pp. 47–54, 2009.

[48] A. G. Radwan and M. E. Fouda, "Optimization of fractional-order RLC filters," *Circuits, Systems, and Signal Processing*, vol. 32, no. 5, pp. 2097–2118, 2013. [Online]. Available: https://doi.org/10.1007/s00034-013-9580-9

[49] K. Biswas, G. Bohannan, R. Caponetto, A. Lopes, and T. Machado, *Fractional-Order Devices*. Cham: Springer, 2017.

[50] M. D. Ortigueira, "Introduction to fractional linear systems. part 1: Continuous-time case," *IEE Proceedings-Vision Image and Signal Processing*, vol. 147, no. 1, pp. 62–70, January 2000.

[51] R. Magin, M. D. Ortigueira, I. Podlubny, and J. Trujillo, "On the fractional signals and systems," *Signal Processing*, vol. 91, no. 3, pp. 350–371, 2011.

[52] P. Nazarian, M. Haeri, and M. S. Tavazoei, "Identifiability of fractional order systems using input output frequency contents," *ISA transactions*, vol. 49, no. 2, pp. 207–214, 2010.

[53] M. S. Tavazoei, "From traditional to fractional PI control: a key for generalization," *IEEE Industrial electronics magazine*, vol. 6, no. 3, pp. 41–51, 2012.

[54] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, ser. Springer Monographs in Mathematics. Berlin Heidelberg: Springer-Verlag, 2014.

[55] T. Kailath, *Linear Systems*, ser. Information and System Sciences Series. Prentice-Hall, 1980. [Online]. Available: https://books.google.pt/books?id=ggYqAQAAMAAJ

[56] A. V. Oppenheim, A. S. Willsky, and S. Hamid, *Signals and Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1997.

[57] M. Roberts, *Signals and Systems: Analysis using transform methods and Matlab*, 2nd ed. McGraw-Hill, 2003.

[58] M. D. Ortigueira, "The fractional quantum derivative and its integral representations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, pp. 956–962, April 2010.

[59] ——, "On the fractional linear scale invariant systems," *IEEE Transactions on Signal Processing*, vol. 58, no. 12, pp. 6406–6410, December 2010.

[60] ——, "A simple approach to the particular solution of constant coefficient ordinary differential equations," *Applied Mathematics and Computation*, vol. 232, pp. 254 – 260, 2014. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0096300314000733

[61] R. Nigmatullin and Y. E. Ryabov, "Cole-Davidson dielectric relaxation as a self-similar relaxation process," *Physics of the Solid State*, vol. 39, no. 1, pp. 87–90, 1997.

[62] J. A. Tenreiro Machado, A. M. Lopes, and R. de Camposinhos, "Fractional-order modelling of epoxy resin," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 378, no. 2172, p. 20190292, 2020. [Online]. Available: https://royalsocietypublishing.org/doi/abs/10.1098/rsta.2019.0292

[63] M. D. Ortigueira, *Fractional Calculus for Scientists and Engineers*, ser. Lecture Notes in Electrical Engineering. Dordrecht, Heidelberg: Springer, 2011.

[64] M. D. Ortigueira and G. Bengochea, "Non-commensurate fractional linear systems: New results," *Journal of Advanced Research*, vol. 25, pp. 11–17, 2020, recent Advances in the Fractional-Order Circuits and Systems: Theory, Design and Applications. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S2090123220300151

[65] M. D. Ortigueira and J. T. Machado, "Revisiting the 1D and 2D Laplace transforms," *Mathematics*, vol. 8, no. 8, p. 1330, 2020.

[66] I. M. Gelfand and G. P. Shilov, *Generalized Functions*. New York: Academic Press, 1964, 3 volumes, English translation.

[67] A. M. AbdelAty, D. A. Yousri, L. A. Said, and A. G. Radwan, "Identifying the parameters of Cole impedance model using magnitude only and complex impedance measurements: A metaheuristic optimization approach," *Arabian Journal for Science and Engineering*, vol. 45, no. 8, pp. 6541–6558, 2020. [Online]. Available: https://doi.org/10.1007/s13369-020-04532-4

[68] I. Schäfer and K. Krüger, "Modelling of coils using fractional derivatives," *Journal of Magnetism and Magnetic Materials*, vol. 307, no. 1, pp. 91–98, 2006. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0304885306006834

[69] J. Machado and A. Galhano, "Fractional order inductive phenomena based on the skin effect," *Nonlinear Dynamics*, vol. 68, no. 1, pp. 107–115, 2012.

[70] J. T. Machado, "Fractional generalization of memristor and higher order elements," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 2, pp. 264–275, 2013.

[71] J. Machado and A. Lopes, "Multidimensional scaling locus of memristor and fractional order elements," *Journal of Advanced Research*, 2020.

[72] R. Sikora, "Fractional derivatives in electrical circuit theory – critical remarks," *Archives of Electrical Engineering*, vol. 66, no. 1, pp. 155–163, 2017.

[73] R. Sikora and S. Pawlowski, "Problematic applications of fractional derivatives in electrotechnics and electrodynamics," in *2018 14th Selected Issues of Electrical Engineering and Electronics (WZEE)*, 2018, pp. 1–5.

[74] A. Oustaloup, *La commande CRONE: Commande Robuste d'Ordre Non Entier*. Paris: Hermès, 1991.

[75] A. Oustaloup, F. Levron, B. Matthieu, and F. M. Nanot, "Frequency-band complex noninteger differentiator: characterization and synthesis," *IEEE*.

[76] J. Walczak and A. Jakubowska, "Resonance in series fractional order $RL_\beta C_\alpha$ circuit," *Przeglad Elektrotechniczny*, vol. 90, no. 4, pp. 210–213, 2014.

[77] R. Caponetto, S. Graziani, and E. Murgano, "Realization of a fractional-order circuit via constant phase element," *International Journal of Dynamics and Control*, 2021. [Online]. Available: https://doi.org/10.1007/s40435-021-00778-4

[78] V. Martynyuk and M. D. Ortigueira, "Fractional model of an electrochemical capacitor," *Signal Processing*, vol. 107, pp. 355 – 360, 2015, special Issue on ad hoc microphone arrays and wireless acoustic sensor networks Special Issue on Fractional Signal Processing and Applications. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0165168414000917

[79] V. Martynyuk, M. D. Ortigueira, M. Fedula, and O. Savenko, "Methodology of electrochemical capacitor quality control with fractional order model," *AEU - International Journal of Electronics and Communications*, vol. 91, pp. 118 – 124, 2018. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S143484111732928X

[80] P. Henrici, *Applied and Computational Complex Analysis*. Wiley-Interscience, 1991, vol. 2.

[81] M. D. Ortigueira, J. T. Machado, M. Rivero, and J. J. Trujillo, "Integer/fractional decomposition of the impulse response of fractional linear systems," *Signal Processing*, vol. 114, pp. 85–88, 2015.

[82] M. D. Ortigueira, A. M. Lopes, and J. A. T. Machado, "On the numerical computation of the Mittag-Leffler function," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 20, 2019.

[83] R. C. Dorf and R. H. Bishop, *Modern Control Systems*, ser. Electrical & Computing Engineering: Control Theory. Heidelberg: Pearson, 2017.

[84] S. Liang, S.-G. Wang, and Y. Wang, "Routh-type table test for zero distribution of polynomials with commensurate fractional and integer degrees," *Journal of the Franklin Institute*, vol. 354, no. 1, pp. 83 – 104, 2017. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0016003216302940

[85] J. Mendiola-Fuentes and D. Melchor-Aguilar, "Modification of Mikhailov stability criterion for fractional commensurate order systems," *Journal of the Franklin Institute*, vol. 355, no. 5, pp. 2779–2790, 2018.

[86] A. H. Zemanian, *Distribution Theory and Transform Analysis: An Introduction to Generalized Functions, with Applications*, ser. Lecture Notes in Electrical Engineering, 84. New York: Dover Publications, 1987.

[87] M. D. Ortigueira, J. J. Trujillo, V. I. Martynyuk, and F. J. Coito, "A generalized power series and its application in the inversion of transfer functions," *Signal Processing*, vol. 107, pp. 238 – 245, 2015, special Issue on ad hoc microphone arrays and wireless acoustic sensor networks Special Issue on Fractional Signal Processing and Applications. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0165168414001753

[88] M. D. Ortigueira and J. T. Machado, "What is a fractional derivative?" *Journal of Computational Physics*, vol. 293, no. 15, pp. 4–13, 2015.

[89] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. San Diego: Academic Press, 1999.

[90] G. S. Teodoro, J. T. Machado, and E. C. De Oliveira, "A review of definitions of fractional derivatives and other operators," *Journal of Computational Physics*, vol. 388, pp. 195–208, 2019.

[91] M. D. Ortigueira, R. L. Magin, J. J. Trujillo, and M. P. Velasco, "A real regularised fractional derivative," *Signal, Image and Video Processing*, vol. 6, no. 3, pp. 351–358, Sep 2012.

[92] M. D. Ortigueira and J. A. T. Machado, "Which derivative?" *Fractal and Fractional*, vol. 1, no. 1, 2017. [Online]. Available: http://www.mdpi.com/2504-3110/1/1/3

[93] K. H. Lundberg, H. R. Miller, and D. L. Trumper, "Initial conditions, generalized functions, and the Laplace transform troubles at the origin," *IEEE Control Systems Magazine*, vol. 27, no. 1, pp. 22–35, 2007.

[94] N. Heymans and I. Podlubny, "Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives," *Rheologica Acta*, vol. 45, no. 5, pp. 765–771, 2006. [Online]. Available: https://doi.org/10.1007/s00397-005-0043-5

[95] C. F. Lorenzo and T. T. Hartley, "Initialization in fractional order systems," in *2001 European Control Conference (ECC)*. IEEE, 2001, pp. 1471–1476.

[96] M. D. Ortigueira, "On the initial conditions in continuous-time fractional linear systems," *Signal Processing*, vol. 83, no. 11, pp. 2301–2309, 2003.

[97] J.-C. Trigeassou, N. Maamri, J. Sabatier, and A. Oustaloup, "State variables and transients of fractional order differential systems," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3117–3140, 2012.

[98] G. Montseny, J. Audounet, and D. Matignon, "Diffusive representation for pseudo-differentially damped nonlinear systems," in *Nonlinear control in the year 2000 volume 2*. Springer, 2001, pp. 163–182.

[99] M. D. Ortigueira and F. J. V. Coito, "System initial conditions vs derivative initial conditions," *Computers & Mathematics with Applications*, vol. 59, no. 5, pp. 1782–1789, 2010.

[100] C. F. Lorenzo and T. T. Hartley, "Variable order and distributed order fractional operators," *Nonlinear dynamics*, vol. 29, no. 1, pp. 57–98, 2002.

[101] C. Coimbra, "Mechanics with variable-order differential operators," *Annalen der Physik*, vol. 12, no. 11-12, pp. 692–703, 2003.

[102] R. Almeida and D. F. Torres, "An expansion formula with higher-order derivatives for fractional operators of variable order," *The Scientific World Journal*, vol. 2013, 2013.

[103] B. P. Moghaddam and J. A. T. Machado, "Extended algorithms for approximating variable order fractional derivatives with applications," *Journal of Scientific Computing*, vol. 71, no. 3, pp. 1351–1374, Jun 2017. [Online]. Available: https://doi.org/10.1007/s10915-016-0343-1

[104] W. Malesza, M. Macias, and D. Sierociuk, "Analytical solution of fractional variable order differential equations," *Journal of Computational and Applied Mathematics*, vol. 348, pp. 214 – 236, 2019. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0377042718305223

[105] M. D. Ortigueira, D. Valério, and J. A. T. Machado, "Variable order fractional systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 71, pp. 231 – 243, 2019.

[106] S. S. Haykin and B. V. Veen, *Signals and Systems*, 1st ed. New York, NY, USA: John Wiley & Sons, Inc., 1998.

[107] M. D. Ortigueira, "Riesz potential operators and inverses via fractional centred derivatives," *Int. J. Math. Mathematical Sciences*, vol. 2006, pp. 48 391:1–48 391:12, 2006.

[108] ——, "Fractional central differences and derivatives," *Journal of Vibration and Control*, vol. 14, no. 9-10, pp. 1255–1266, September 2008.

[109] M. D. Ortigueira and J. A. T. Machado, "Fractional derivatives: The perspective of system theory," *Mathematics*, vol. 7, no. 2, 2019. [Online]. Available: http://www.mdpi.com/2504-3110/1/1/3

[110] M. D. Ortigueira, "Two-sided and regularised Riesz-Feller derivatives," *Mathematical Methods in the Applied Sciences*. [Online]. Available: https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.5720

[111] L. Ding, Y. Luo, Y. Lin, and Y. Huang, "Revisiting the relations between Hurst exponent and fractional differencing parameter for long memory," *Physica A: Statistical Mechanics and its Applications*, vol. 566, p. 125603, 2021. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0378437120309018

[112] M. D. Ortigueira and A. G. a. Batista, "A fractional linear system view of the fractional Brownian motion," *Nonlinear Dynamics*, vol. 38, no. 1-4, pp. 295–303, December 2004.

[113] ——, "On the relation between the fractional Brownian motion and the fractional derivatives," *Physics Letters A*, vol. 372, no. 7, pp. 958–968, February 2008.

[114] D. Valério and J. S. da Costa, *An Introduction to Fractional Control*, ser. Control Engineering. Stevenage: IET, 2012.

[115] G. Maione, "Design of cascaded and shifted fractional-order lead compensators for plants with monotonically increasing lags," *Fractal and Fractional*, vol. 4, no. 3, 2020. [Online]. Available: https://www.mdpi.com/2504-3110/4/3/37

[116] F. Tricomi, "Sulle funzioni ipergeometriche confluenti," *Annali di Matematica Pura ed Applicata*, vol. 26, no. 1, pp. 141–175, 1947.

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