Functional interpretations and applications

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Overview

Amuse-bouche

First course: BFI

Third course: functional interpretations for NSA

- Nonstandard analysis in proof theory
- Nonstandard Realizability
- Nonstandard Intuitionistic functional interpretation

Second course: a parametrised interpretation

- Parametrised interpretations of AL
- Parametrised interpretations of IL
- Instances

Dessert: realizability with stateful computations for NSA

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- In fact, there exist explicit examples ("Specker sequences") of sequences of computable reals with no computable limit and thus with no computable rate of convergence.

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Metastability

$$\forall \varepsilon > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \exists N \, \forall i, j \in [N, N + f(N)](\|x_i - x_j\| \leq \varepsilon)$$

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which is a Herbrandization of the Cauchy property of a sequence.

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Proof mining program $\rightarrow$ analyses of mathematical proofs with the help of proof theoretic techniques, including functional interpretations, in search of concrete new information: effective bounds, algorithms, weakening of premisses, ...

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- Allow to obtain explicit bounds

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- P. Pinto, D. : Fixed point theory (2019-...).

Functional interpretations

A functional interpretation is a mapping $f : S \to T$ such that a formula A (in classical logic) is mapped to a formula

$$A^f \equiv \forall x \exists y \, A_f(x, y)$$

such that theorems of S are mapped to theorems of T, i.e.

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Functional interpretations allow for the extraction of the (hidden) computational content (captured by t) in the proof of the theorem.

Interpretations with different flavours

- Kleene (numerical realizability) (1952)
- Gödel (Dialectica) (1958)
- Kreisel (modified realizability) (1959)
- Diller and Nahm (variant to avoid the contraction problem) (1974)
- Stein (family of interpretations) (1979)
- Kohlenbach (monotone functional interpretation) (1996)
- Ferreira and Oliva (bounded functional interpretation) (2005)
- Van den Berg, Briseid and Safarik (Herbrandized) (2012)
- ...

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- Completely new translation of formulas
- Independence on bounded parameters is made explicit (via the interpretation itself)

Majorizability

Let PA^{ω} be Peano Arithmetic in all finite types. Types are defined inductively as follows

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- $\leq^*_\sigma$ is not reflexive! We say that x^σ is monotone if and only if $x \leq^*_\sigma x$.

Majorizability

Proposition

1. $\mathrm{PA}^{\omega}_{\leq^*} \vdash x \leq^*_{\sigma} y \to y \leq^*_{\sigma} y$;
2. $\mathrm{PA}^{\omega}_{\leq^*} \vdash x \leq^*_{\sigma} y \wedge y \leq^*_{\sigma} z \to x \leq^*_{\sigma} z$.

Theorem (Howard's majorizability theorem)

For all closed terms t^{σ} *of* $\mathrm{PA}^{\omega}_{\leq^*}$*, there is a closed term* s^{σ} *of* $\mathrm{PA}^{\omega}_{\leq^*}$ *such that* $\mathrm{PA}^{\omega}_{\leq^*} \vdash t \leq^*_{\sigma} s$.

Quantifiers

The usual (unbounded quantifiers) $\forall x\, A(x)$ and $\exists x\, A(x)$.

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Formulas that don't contain unbounded quantifiers are called bounded formulas.

Bounded functional interpretation (Ferreira and Oliva)

Assign to each formula A of $\mathsf{PA}^{\omega}_{\leq^*}$ the formulas A^f and $A_f(a; b)$ of $\mathsf{PA}^{\omega}_{\leq^*}$ such that $A^f \equiv \tilde{\forall} a \, \tilde{\exists} b \, A_f(a; b)$ according to the following clauses.

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4. $(\forall x \, A(x))^f :\equiv \tilde{\forall} e \, \tilde{\forall} a \, \tilde{\exists} b \, \forall x \leq^* e \, A_f(x, a; b)$;

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5. $(\forall x \leq^* t \, A(x))^f :\equiv \tilde{\forall} a \, \tilde{\exists} b \, \forall x \leq^* t \, A_f(x, a; b)$.

Caracteristic Principles

Definition

1. $(\mathrm{mAC}^{\omega}_{\mathrm{bd}}) \equiv \tilde{\forall} x\, \tilde{\exists} y\; A_{\mathrm{bd}}(x,y) \to \tilde{\exists} f\, \tilde{\forall} x\, \tilde{\exists} y \leq^{*} fx\; A_{\mathrm{bd}}(x,y)$;

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3. $(\mathsf{MAJ}^{\omega}) \equiv \forall x \exists y (x \leq^* y)$.

Soundness

Theorem (soundness theorem of f)

For all formulas A of $\mathrm{PA}^{\omega}_{\leq^*}$, if

$$\mathrm{PA}^{\omega}_{\leq^*} + \mathrm{P} \vdash A,$$

then there are closed monotone terms t of appropriate types such that

$$\mathrm{PA}^{\omega}_{\leq^*} \vdash \tilde{\forall} a \, \tilde{\exists} b \, \leq^* ta \, A_f(a; b).$$

Abbreviation

$\mathrm{P} := \mathrm{mAC}^{\omega}_{\mathrm{bd}} + \mathrm{Coll}^{\omega}_{\mathrm{bd}} + \mathrm{MAJ}^{\omega}$.

Characterization

Theorem (characterization theorem of f)

For all formulas A *of* $\mathsf{PA}^{\omega}_{\leq^*}$, *we have*

$$\mathsf{PA}^{\omega}_{\leq^*} + \mathsf{P} \vdash A \leftrightarrow A^f.$$

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From arithmetic to Hilbert spaces

We add:

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- modulus (of convergence, of "Cauchyness", of asymptotic regularity, of metastability, etc.) witnessing problematic existential quantifiers.

As long as the new constants are majorizable and the new axioms are universal the proof of the Soundness theorem can be extended to this new theory.

An example: Browder's theorem

Theorem (Browder 1967)

Let H be an Hilbert space and $U : H \to H$ a non-expansive map. Suppose that C is a convex, closed and bounded subset of H, $0 \in C$ and that U maps C into C. For every $n \in \mathbb{N}$, let $U_n : H \to H$ the strict contraction $U_n(x) = (1 - \frac{1}{n+1})U(x)$ and let u_n the unique fixed point of U_n. Then the sequence (u_n) strongly converges for a fixed point $u \in C$ of U

A quantitative version of Browder's theorem

Theorem (Kohlenbach 2011; Ferreira, Leustean, Pinto 2019)

For all $k \in \mathbb{N}$ *and function* $f : \mathbb{N} \to \mathbb{N}$,

$$\exists n \leq \phi(k, f) \forall i, j \in [n, n + fn] \left(\|u_i - u_j\| \leq \frac{1}{2^k} \right).$$

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For f increasing one obtains the following rate of convergence

$$\boxed{\phi(k, f) := 2^{2g_k^{(r)}(0)+4+2d}}$$

where

- d is an upper bound of the diameter of C.
- $g_k(n) := 2k + d + 5 + \lceil \log_2(2^{2n+4+2d}) + f(2^{2n+4+2d}) + 1) \rceil$.
- $r := 2^{2k+4d+9}$.

Outline

Some (arithmetical) intuitions

- Conservative extension

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- Overspill and Underspill

The simplest example: ENA

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Internal formulas = "Without st".
External formulas = "With st".

The axioms of ENA

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- $\mathrm{st}(0)$
- $\forall n \in \mathbb{N}(\mathrm{st}(n) \Rightarrow \mathrm{st}(n+1))$
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The axioms of ENA

Axiom

- $\mathrm{st}(0)$
- $\forall n \in \mathbb{N}(\mathrm{st}(n) \Rightarrow \mathrm{st}(n+1))$
- $\exists \omega \in \mathbb{N}(\neg \mathrm{st}(\omega))$

  For each external formula Φ
- $(\Phi(0) \wedge \forall^{\mathrm{st}} n(\Phi(n) \Rightarrow \Phi(n+1))) \Rightarrow \forall^{\mathrm{st}} n\, \Phi(n)$

The axioms of ENA

Axiom

- $\mathrm{st}(0)$
- $\forall n \in \mathbb{N}(\mathrm{st}(n) \Rightarrow \mathrm{st}(n+1))$
- $\exists \omega \in \mathbb{N}(\neg \mathrm{st}(\omega))$

  For each external formula Φ
- $(\Phi(0) \wedge \forall^{\mathrm{st}} n(\Phi(n) \Rightarrow \Phi(n+1))) \Rightarrow \forall^{\mathrm{st}} n\, \Phi(n)$

$\rightsquigarrow$ $\forall^{\mathrm{st}} n\, \Phi(n)$ *abbreviates* $\forall n(\mathrm{st}(n) \Rightarrow \Phi(n))$.

How to be nonstandard?

- Model theory: Compactness theorem, ultrafilters, ultralimits, superstructures,... (Robinson, Luxemburg, Keisler, ...)
- Set theory: IST, HST,... Language $\{\in, \mathrm{st}\}$ (Nelson, Hrbacek, Kanovei, Reeken, ...)
- Algebraic: (Benci, Di Nasso and Forti, D. and Van den Berg)

Functional interpretations using NSA

- Pioneer works by Moerdijk, Palmgren and Avigad

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- "Intuitionistic nonstandard bounded interpretations" (D., Gaspar)
- "Realizability with stateful computations for NSA" (D., Miquey)

Most works are inspired by Nelson's IST

Internal set theory

- **Transfer:** $A(x)$ internal

$$\forall^{\mathrm{st}} x. A(x) \Longrightarrow \forall x. A(x)$$

- **Idealization:** $R(x, y)$ internal relation

$$\forall^{\mathrm{stfin}} z. \exists y. \forall x \in z. R(x, y) \Rightarrow \exists y. \forall^{\mathrm{st}} x. R(x, y)$$

- **Standardization:** For any $C(x)$

$$\forall^{\mathrm{st}} B. \exists^{\mathrm{st}} A. \forall^{\mathrm{st}} z. (z \in A \Leftrightarrow z \in B \wedge C(z))$$

$\text{E-HA}^{\omega}_{\text{st}}$

Enrich the language and the axioms of E-HA^{ω} as follows.

- $\mathrm{st}^{\sigma}(t^{\sigma})$ (for each finite type σ).

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  - $x =_\sigma y \wedge \mathrm{st}^\sigma(x) \to \mathrm{st}^\sigma(y)$;

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  - $\mathrm{st}^\sigma(y) \wedge x \leq^*_\sigma y \to \mathrm{st}^\sigma(x)$;
  - $\mathrm{st}^\sigma(t)$ for each closed term t;
  - $\mathrm{st}^{\sigma\to\tau}(x) \wedge \mathrm{st}^\sigma(y) \to \mathrm{st}^\tau(xy)$;

$\text{E-HA}^{\omega}_{\mathrm{st}}$

Enrich the language and the axioms of E-HA^{ω} as follows.

- $\mathrm{st}^{\sigma}(t^{\sigma})$ (for each finite type σ).
- Standardness axioms:
  - $x =_{\sigma} y \wedge \mathrm{st}^{\sigma}(x) \to \mathrm{st}^{\sigma}(y)$;
  - $\mathrm{st}^{\sigma}(y) \wedge x \leq^{*}_{\sigma} y \to \mathrm{st}^{\sigma}(x)$;
  - $\mathrm{st}^{\sigma}(t)$ for each closed term t;
  - $\mathrm{st}^{\sigma\to\tau}(x) \wedge \mathrm{st}^{\sigma}(y) \to \mathrm{st}^{\tau}(xy)$;
- External induction rule:

$$\frac{\Phi(0) \quad \forall x^0\,(\mathrm{st}^0(x) \to (\Phi(x) \to \Phi(x+1)))}{\forall x^0\,(\mathrm{st}^0(x) \to \Phi(x))}$$

Some abbreviations

- $\tilde{\forall} x\, \varphi(x)$ abbreviates $\forall x(x \leq^* x \rightarrow \varphi(x))$.
- $\tilde{\exists} x\, \varphi(x)$ abbreviates $\exists x(x \leq^* x \wedge \varphi(x))$.
- $\forall^{\mathrm{st}} x\, \varphi(x)$ abbreviates $\forall x(\mathrm{st}(x) \rightarrow \varphi(x))$.
- $\exists^{\mathrm{st}} x\, \varphi(x)$ abbreviates $\exists x(\mathrm{st}(x) \wedge \varphi(x))$.
- $\ldots$

Nonstandard bounded modified realizability (jww J. Gaspar)

Assign to each formula Φ of $\text{E-HA}^{\omega}_{\mathrm{st}}$ the formulas Φ^{b} and $\Phi_{\mathrm{b}}(a)$ of $\text{E-HA}^{\omega}_{\mathrm{st}}$ such that $\Phi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{b}}(a)$ according to the following clauses :

1. $\Phi^{\mathrm{b}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\mathrm{st}(t)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^{*} a]$;

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1. $\Phi^{\text{b}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\text{st}(t)^{\text{b}} :\equiv \tilde{\exists}^{\text{st}} a \, [t \leq^{*} a]$;

If $\Phi^{\text{b}} \equiv \tilde{\exists}^{\text{st}} a \, \Phi_{\text{b}}(a)$ and $\Psi^{\text{b}} \equiv \tilde{\exists}^{\text{st}} b \, \Psi_{\text{b}}(b)$, then:

Nonstandard bounded modified realizability (jww J. Gaspar)

Assign to each formula Φ of $\text{E-HA}^{\omega}_{\text{st}}$ the formulas Φ^{b} and $\Phi_{\mathrm{b}}(a)$ of $\text{E-HA}^{\omega}_{\text{st}}$ such that $\Phi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{b}}(a)$ according to the following clauses :

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If $\Phi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{b}}(a)$ and $\Psi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} b\, \Psi_{\mathrm{b}}(b)$, then:

3. $(\Phi \wedge \Psi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{b}}(a) \wedge \Psi_{\mathrm{b}}(b)]$;
4. $(\Phi \vee \Psi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{b}}(a) \vee \Psi_{\mathrm{b}}(b)]$;
5. $(\Phi \to \Psi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} B\, [\tilde{\forall}^{\mathrm{st}} a\, (\Phi_{\mathrm{b}}(a) \to \Psi_{\mathrm{b}}(Ba))]$;
6. $(\forall x\, \Phi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\forall x\, \Phi_{\mathrm{b}}(a)]$;
7. $(\exists x\, \Phi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\exists x\, \Phi_{\mathrm{b}}(a)]$.

Monotonicity

Lemma (monotonicity of b)

For all formulas Φ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, we have

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{b}}(a) \wedge a \leq^{*} c \to \Phi_{\mathrm{b}}(c).$$

$\tilde{\exists}^{\mathrm{st}}$-free formulas

Definition

We say that a formula of $\text{E-HA}^{\omega}_{\mathrm{st}}$ is $\tilde{\exists}^{\mathrm{st}}$-*free* if and only if it is built:

1. from atomic internal formulas $s =_0 t$;
2. by conjunctions $\wedge$;
3. by disjunctions $\vee$;
4. by implications $\rightarrow$;
5. by quantifications $\forall$ and $\exists$ (so also $\tilde{\forall}$ and $\tilde{\exists}$);
6. by monotone standard universal quantifications $\tilde{\forall}^{\mathrm{st}}$ (but, of course, not $\tilde{\exists}^{\mathrm{st}}$).

$\tilde{\exists}^{\mathrm{st}}$-free formulas

Lemma

- *For all* $\tilde{\exists}^{\mathrm{st}}$*-free formulas* $\Phi_{\tilde{\nexists}^{\mathrm{st}}}$ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$*, we have*
  - $(\Phi_{\tilde{\nexists}^{\mathrm{st}}})^{\mathrm{b}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}}(a)$;
  - $\text{E-HA}^{\omega}_{\mathrm{st}} \vdash (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}} \leftrightarrow \Phi_{\tilde{\nexists}^{\mathrm{st}}}$.

$\tilde{\exists}^{\mathrm{st}}$-free formulas

Lemma

- *For all $\tilde{\exists}^{\mathrm{st}}$-free formulas $\Phi_{\tilde{\nexists}^{\mathrm{st}}}$ of $\text{E-HA}^{\omega}_{\mathrm{st}}$, we have*
  - $(\Phi_{\tilde{\nexists}^{\mathrm{st}}})^{\mathrm{b}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}}(a)$;
  - $\text{E-HA}^{\omega}_{\mathrm{st}} \vdash (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}} \leftrightarrow \Phi_{\tilde{\nexists}^{\mathrm{st}}}$.
- *For all formulas Φ of $\text{E-HA}^{\omega}_{\mathrm{st}}$, the formula $\Phi_{\mathrm{b}}(a)$ is $\tilde{\exists}^{\mathrm{st}}$-free.*

Caracteristic Principles

Definition

- $\mathrm{mAC}^{\omega} \equiv \tilde{\forall}^{\mathrm{st}} x \, \tilde{\exists}^{\mathrm{st}} y \, \Phi \rightarrow \tilde{\exists}^{\mathrm{st}} Y \, \tilde{\forall}^{\mathrm{st}} x \, \tilde{\exists} y \leq^{*} Yx \, \Phi$;
- $\mathrm{R}^{\omega} \equiv \forall x \, \exists^{\mathrm{st}} y \, \Phi \rightarrow \tilde{\exists}^{\mathrm{st}} z \, \forall x \, \exists y \leq^{*} z \, \Phi$;
- $\mathrm{IP}^{\omega}_{\tilde{\nexists}^{\mathrm{st}}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \rightarrow \tilde{\exists}^{\mathrm{st}} x \, \Psi) \rightarrow \tilde{\exists}^{\mathrm{st}} y \, (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \rightarrow \tilde{\exists} x \leq^{*} y \, \Psi)$;
- $\mathrm{MAJ}^{\omega} \equiv \forall^{\mathrm{st}} x \, \exists^{\mathrm{st}} y \, (x \leq^{*} y)$.

Caracteristic Principles

Definition

- $\mathsf{mAC}^\omega \equiv \tilde{\forall}^{\mathrm{st}} x\, \tilde{\exists}^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} Y\, \tilde{\forall}^{\mathrm{st}} x\, \tilde{\exists} y \leq^* Yx\, \Phi$;
- $\mathsf{R}^\omega \equiv \forall x\, \exists^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} z\, \forall x\, \exists y \leq^* z\, \Phi$;
- $\mathsf{IP}^\omega_{\tilde{\nexists}^{\mathrm{st}}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \to \tilde{\exists}^{\mathrm{st}} x\, \Psi) \to \tilde{\exists}^{\mathrm{st}} y\, (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \to \tilde{\exists} x \leq^* y\, \Psi)$;
- $\mathsf{MAJ}^\omega \equiv \forall^{\mathrm{st}} x\, \exists^{\mathrm{st}} y\, (x \leq^* y)$.

Proposition

The principle R^ω *implies the principle* MAJ^ω, *that is* $\mathsf{E\text{-}HA}^\omega_{\mathrm{st}} + \mathsf{R}^\omega$ *proves all instances of* MAJ^ω

Soundness

Theorem (soundness theorem of b)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *if*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi,$$

then there are closed monotone terms t *of appropriate types such that*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{b}}(t).$$

Abbreviation

$\mathsf{P} := \mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\nexists\mathrm{st}} + \mathsf{MAJ}^{\omega}.$

Characterization

Theorem (Characterization theorem of b)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi \leftrightarrow \Phi^{\mathrm{b}}.$$

Abbreviation

$\mathsf{P} := \mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\nexists\mathrm{st}} + \mathsf{MAJ}^{\omega}$.

Intuitionistic nonstandard bounded functional interpretation

Assign to each formula Φ of $\text{E-HA}^{\omega}_{\text{st}}$ the formulas Φ^{B} and $\Phi_{\mathrm{B}}(a; b)$ of $\text{E-HA}^{\omega}_{\text{st}}$ such that $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{B}}(a; b)$ according to the following clauses.

1. $\Phi^{\mathrm{B}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\mathrm{st}(t)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a]$.

Intuitionistic nonstandard bounded functional interpretation

Assign to each formula Φ of $\text{E-HA}^{\omega}_{\text{st}}$ the formulas Φ^{B} and $\Phi_{\mathrm{B}}(a; b)$ of $\text{E-HA}^{\omega}_{\text{st}}$ such that $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{B}}(a; b)$ according to the following clauses.

1. $\Phi^{\mathrm{B}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\mathrm{st}(t)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, [t \leq^{*} a]$.

If $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{B}}(a; b)$ and $\Psi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} c \, \tilde{\forall}^{\mathrm{st}} d \, \Psi_{\mathrm{B}}(c; d)$ then:

Intuitionistic nonstandard bounded functional interpretation

Assign to each formula Φ of $\text{E-HA}^{\omega}_{\text{st}}$ the formulas Φ^{B} and $\Phi_{\mathrm{B}}(a; b)$ of $\text{E-HA}^{\omega}_{\text{st}}$ such that $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{B}}(a; b)$ according to the following clauses.

1. $\Phi^{\mathrm{B}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\mathrm{st}(t)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, [t \leq^{*} a]$.

If $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{B}}(a; b)$ and $\Psi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} c \, \tilde{\forall}^{\mathrm{st}} d \, \Psi_{\mathrm{B}}(c; d)$ then:

3. $(\Phi \wedge \Psi)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a, c \, \tilde{\forall}^{\mathrm{st}} b, d \, [\Phi_{\mathrm{B}}(a; b) \wedge \Psi_{\mathrm{B}}(c; d)]$;
4. $(\Phi \vee \Psi)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a, c \, \tilde{\forall}^{\mathrm{st}} e, f$
   $[\tilde{\forall} b \leq^{*} e \, \Phi_{\mathrm{B}}(a; b) \vee \tilde{\forall} d \leq^{*} f \, \Psi_{\mathrm{B}}(c; d)]$;
5. $(\Phi \to \Psi)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} C, B \, \tilde{\forall}^{\mathrm{st}} a, d$
   $[\tilde{\forall} b \leq^{*} Bad \, \Phi_{\mathrm{B}}(a; b) \to \Psi_{\mathrm{B}}(Ca; d)]$;
6. $(\forall x \, \Phi)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} b \, [\forall x \, \Phi_{\mathrm{B}}(a; b)]$;
7. $(\exists x \, \Phi)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} c \, [\exists x \, \tilde{\forall} b \leq^{*} c \, \Phi_{\mathrm{B}}(a; b)]$.

Monotonicity

Lemma (monotonicity of B)

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{B}}(a; b) \wedge a \leq^{*} c \rightarrow \Phi_{\mathrm{B}}(c; b).$$

Characteristic principles

Definition

- $\text{mAC}^\omega \equiv \tilde{\forall}^{\text{st}} x\, \tilde{\exists}^{\text{st}} y\, \Phi \to \tilde{\exists}^{\text{st}} Y\, \tilde{\forall}^{\text{st}} x\, \tilde{\exists} y \leq^* Yx\, \Phi$;
- $\text{R}^\omega \equiv \forall x\, \exists^{\text{st}} y\, \Phi \to \tilde{\exists}^{\text{st}} z\, \forall x\, \exists y \leq^* z\, \Phi$;
- $\text{I}^\omega \equiv \tilde{\forall}^{\text{st}} z\, \exists x\, \forall y \leq^* z\, \phi \to \exists x\, \forall^{\text{st}} y\, \phi$;
- $\text{IP}^\omega_{\tilde{\forall}^{\text{st}}} \equiv (\tilde{\forall}^{\text{st}} x\, \phi \to \tilde{\exists}^{\text{st}} y\, \Psi) \to \tilde{\exists}^{\text{st}} z\, (\tilde{\forall}^{\text{st}} x\, \phi \to \tilde{\exists} y \leq^* z\, \Psi)$;
- $\text{M}^\omega \equiv (\tilde{\forall}^{\text{st}} x\, \phi \to \psi) \to \tilde{\exists}^{\text{st}} y\, (\tilde{\forall} x \leq^* y\, \phi \to \psi)$;
- $\text{BUD}^\omega \equiv \tilde{\forall}^{\text{st}} u, v\, (\forall x \leq^* u\, \phi \vee \forall y \leq^* v\, \psi) \to \forall^{\text{st}} x\, \phi \vee \forall^{\text{st}} y\, \psi$;
- $\text{MAJ}^\omega \equiv \forall^{\text{st}} x\, \exists^{\text{st}} y\, (x \leq^* y)$.

Proposition

- $\text{E-HA}^{\omega}_{\text{st}} + \mathsf{I}^{\omega} \vdash \mathsf{BUD}^{\omega}$.
- $\text{E-HA}^{\omega}_{\text{st}} + \mathsf{R}^{\omega} \vdash \mathsf{MAJ}^{\omega}$.

Soundness

Theorem (soundness theorem of B)

For all formulas Φ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, if

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi,$$

then there are closed monotone terms t of appropriate types such that

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{B}}(t; b).$$

Abbreviation

$\mathsf{P} := \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{I}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\forall}^{\mathrm{st}}} + \mathsf{M}^{\omega} + \mathsf{BUD}^{\omega} + \mathsf{MAJ}^{\omega}$.

Characterization

Theorem (characterization theorem of B)

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi \leftrightarrow \Phi^{\mathrm{B}}.$$

Abbreviation

$\mathsf{P} := \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{I}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\forall}^{\mathrm{st}}} + \mathsf{M}^{\omega} + \mathsf{BUD}^{\omega} + \mathsf{MAJ}^{\omega}$.

Transfer Principles

Definition

1. $(T_\forall) \equiv \forall^{\mathrm{st}} f\, (\forall^{\mathrm{st}} x\, \phi \to \forall x\, \phi)$;
2. $(T_\exists) \equiv \forall^{\mathrm{st}} f\, (\exists x\, \phi \to \exists^{\mathrm{st}} x\, \phi)$;

where f are all the free variables in the internal formula ϕ.

Adding Transfer

Theorem

1. *Adding* $\mathsf{T}_\forall$ *or* $\mathsf{T}_\exists$ *to* $\mathsf{E\text{-}HA}^{\omega *}_{\mathrm{st}} + \mathsf{R} + \mathsf{HGMP}^{\mathrm{st}}$ *leads to nonconservativity over* **HA**.
2. *Adding* $\mathsf{T}_\forall$ *or* $\mathsf{T}_\exists$ *to* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ *leads to inconsistency.*

Krivine's negative translation

$A^{\mathrm{K}} :\equiv \neg A_{\mathrm{K}}$ (Φ_{at} is an atomic formula)

- $(\Phi_{\mathrm{at}})_{\mathrm{K}} :\equiv \neg \Phi_{\mathrm{at}}$,
- $(\neg \Phi)_{\mathrm{K}} :\equiv \neg \Phi_{\mathrm{K}}$,
- $(\Phi \vee \Psi)_{\mathrm{K}} :\equiv \Phi_{\mathrm{K}} \wedge \Psi_{\mathrm{K}}$,
- $(\forall x \, \Phi)_{\mathrm{K}} :\equiv \exists x \, \Phi_{\mathrm{K}}$.

Theorem (Soundness and characterization of K)

For all formulas Φ of the language of $\text{E-PA}^{\omega}_{\mathrm{st}}$, we have:

1. $\text{E-PA}^{\omega}_{\mathrm{st}} \vdash \Phi \;\Rightarrow\; \text{E-HA}^{\omega}_{\mathrm{st}} + \text{I-LEM} \vdash \Phi^{\mathrm{K}}$;
2. $\text{E-PA}^{\omega}_{\mathrm{st}} \vdash \Phi \leftrightarrow \Phi^{\mathrm{K}}$.

Factorization

Theorem (factorisation $\mathrm{U} = \mathrm{K}\,\mathrm{B}$)

For all formulas Φ *of the language of* $\text{E-PA}^{\omega}_{\mathrm{st}}$*, we have:*

1. $\text{E-HA}^{\omega}_{\mathrm{st}} + \text{I-LEM} \vdash \tilde{\forall} a, b\, (\Phi_{\mathrm{U}}(a; b) \leftrightarrow \neg \tilde{\forall} c \leq^{*} b\, (\Phi_{\mathrm{K}})_{\mathrm{B}}(a; c))$;
2. $\text{E-HA}^{\omega}_{\mathrm{st}} + \text{I-LEM} \vdash \tilde{\forall} a, B\, (\Phi_{\mathrm{U}}(a; Ba) \leftrightarrow (\Phi^{\mathrm{K}})_{\mathrm{B}}(a; B))$;
3. $\text{E-HA}^{\omega}_{\mathrm{st}} + \text{I-LEM} + \text{mAC}^{\omega}_{\mathrm{st}} \vdash \Phi^{\mathrm{U}} \leftrightarrow (\Phi^{\mathrm{K}})^{\mathrm{B}}$.

Application

- Using the factorization $\mathrm{U} = \mathrm{K}\,\mathrm{B}$ and the soundness theorem of B one gets new proofs of the soundness and characterization theorems of U.

Realizability with q-truth

Assigns to each formula Φ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ the formula $\Phi^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{bq}}(a)$ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ according to the following clauses, $\Phi^{\mathrm{bq}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{bq}}(a)$ and $\Psi^{\mathrm{bq}} \equiv \tilde{\exists}^{\mathrm{st}} b\, \Psi_{\mathrm{bq}}(b)$):

$$
\begin{aligned}
&\phi^{\mathrm{bq}} :\equiv [\phi],\\
&\mathrm{st}(t)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a],\\
&(\Phi \wedge \Psi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{bq}}(a) \wedge \Psi_{\mathrm{bq}}(b)],\\
&(\Phi \vee \Psi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [(\Phi_{\mathrm{bq}}(a) \wedge \Phi) \vee (\Psi_{\mathrm{bq}}(b) \wedge \Psi)],\\
&(\Phi \to \Psi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} B\, \tilde{\forall}^{\mathrm{st}} a\, [\Phi_{\mathrm{bq}}(a) \wedge \Phi \to \Psi_{\mathrm{bq}}(Ba)],\\
&(\forall x\, \Phi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\forall x\, \Phi_{\mathrm{bq}}(a)],\\
&(\exists x\, \Phi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\exists x\, (\Phi_{\mathrm{bq}}(a) \wedge \Phi)].
\end{aligned}
$$

Realizability with $\mathfrak{t}$-truth

$$\phi^{\mathrm{bt}} :\equiv [\phi],$$
$$\mathrm{st}(t)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a],$$
$$(\Phi \wedge \Psi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{bt}}(a) \wedge \Psi_{\mathrm{bt}}(b)],$$
$$(\Phi \vee \Psi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{bt}}(a) \vee \Psi_{\mathrm{bt}}(b)],$$
$$(\Phi \to \Psi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} B\, \tilde{\forall}^{\mathrm{st}} a\, [(\Phi_{bt}(a) \to \Psi_{bt}(Ba)) \wedge (\Phi \to \Psi)],$$
$$(\forall x\, \Phi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\forall x\, \Phi_{\mathrm{bt}}(a)],$$
$$(\exists x\, \Phi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\exists x\, \Phi_{\mathrm{bt}}(a)].$$

Theorem

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\text{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\text{st}} \vdash \forall^{\text{st}} a \, (\Phi_{\text{bt}}(a) \leftrightarrow \Phi_{\text{bq}}(a) \wedge \Phi).$$

Soundness of bq and bt

Theorem

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *if*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} \pm \mathsf{MAJ}^{\omega} \vdash \Phi,$$

then there are closed monotone terms t *such that*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} \pm \mathsf{MAJ}^{\omega} \vdash \Phi_{\mathrm{bq}}(t),$$
$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} \pm \mathsf{MAJ}^{\omega} \vdash \Phi_{\mathrm{bt}}(t).$$

Characterization of bq and bt

Theorem

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} + \mathsf{MAJ}^{\omega} \vdash \Phi^{\mathrm{bq}} \leftrightarrow \Phi,$$

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} + \mathsf{MAJ}^{\omega} \vdash \Phi^{\mathrm{bt}} \leftrightarrow \Phi.$$

Intuitionistic nonstandard bounded functional interpretation with q-truth

$$
\begin{aligned}
\Phi^{\mathrm{Bq}} &:\equiv [\Phi], \\
\mathrm{st}(t)^{\mathrm{Bq}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a], \\
(\Phi \wedge \Psi)^{\mathrm{Bq}} &:\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} b, d\, [\Phi_{\mathrm{Bq}}(a; b) \wedge \Psi_{\mathrm{Bq}}(c; d)], \\
(\Phi \vee \Psi)^{\mathrm{Bq}} &:\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} e, f \\
&\quad [(\tilde{\forall} b \leq^* e\, \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi) \vee (\tilde{\forall} d \leq^* f\, \Psi_{\mathrm{Bq}}(c; d) \wedge \Psi)], \\
(\Phi \to \Psi)^{\mathrm{Bq}} &:\equiv \tilde{\exists}^{\mathrm{st}} C, B\, \tilde{\forall}^{\mathrm{st}} a, d \\
&\quad [\tilde{\forall} b \leq^* Bad\, \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi \to \Psi_{\mathrm{Bq}}(Ca; d)], \\
(\forall x\, \Phi)^{\mathrm{Bq}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, [\forall x\, \Phi_{\mathrm{Bq}}(a; b)], \\
(\exists x\, \Phi)^{\mathrm{Bq}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} c\, [\exists x\, (\tilde{\forall} b \leq^* c\, \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi)].
\end{aligned}
$$

Intuitionistic nonstandard bounded functional interpretation with t-truth

$$
\begin{aligned}
\Phi^{\mathrm{Bt}} &:\equiv [\Phi], \\
\mathrm{st}(t)^{\mathrm{Bt}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a], \\
(\Phi \wedge \Psi)^{\mathrm{Bt}} &:\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} b, d\, [\Phi_{\mathrm{Bt}}(a; b) \wedge \Psi_{\mathrm{Bt}}(c; d)], \\
(\Phi \vee \Psi)^{\mathrm{Bt}} &:\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} e, f\, [\tilde{\forall} b \leq^* e\, \Phi_{\mathrm{Bt}}(a; b) \vee \tilde{\forall} d \leq^* f\, \Psi_{\mathrm{Bt}}(c; d)], \\
(\Phi \to \Psi)^{\mathrm{Bt}} &:\equiv \tilde{\exists}^{\mathrm{st}} C, B\, \tilde{\forall}^{\mathrm{st}} a, d \\
&\quad [\tilde{\forall} b \leq^* Bad\, \Phi_{\mathrm{Bt}}(a; b) \to \Psi_{\mathrm{Bt}}(Ca; d) \wedge (\Phi \to \Psi)], \\
(\forall x\, \Phi)^{\mathrm{Bt}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, [\forall x\, \Phi_{\mathrm{Bt}}(a; b)], \\
(\exists x\, \Phi)^{\mathrm{Bt}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} c\, [\exists x\, \tilde{\forall} b \leq^* c\, \Phi_{\mathrm{Bt}}(a; b)].
\end{aligned}
$$

Factorization

Theorem

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\text{st}}$*, we have*

$$\text{E-HA}^{\omega}_{\text{st}} \vdash \tilde{\forall}^{\text{st}} a, b \, (\Phi_{\text{Bt}}(a; b) \leftrightarrow \Phi_{\text{Bq}}(a; b) \wedge \Phi).$$

Soundnesses of Bq and Bt

Theorem

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *if*

$$\mathsf{P} \vdash \Phi,$$

then there are closed monotone terms t *such that*

$$\mathsf{P} \vdash \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{Bq}}(t; b),$$
$$\mathsf{P} \vdash \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{Bt}}(t; b).$$

Abbreviation

$\mathsf{P} := \mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{I}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\forall}^{\mathrm{st}}} \pm \mathsf{M}^{\omega} \pm \mathsf{BUD}^{\omega} \pm \mathsf{MAJ}^{\omega}$.

- No optimal characterisation theorem of Bq and Bt.

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  (optimal here means that it characterizes the *least* theory containing $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ and proving $\Phi^{\mathrm{Bq}} \leftrightarrow \Phi$ for all formulas Φ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$)

- No optimal characterisation theorem of Bq and Bt.

No surprise! It is well-known that there are difficulties in proving optimal characterisation theorems for functional interpretations with truth.

Outline

Functional interpretations: applications

- Relative consistency of HA (Gödel)
- Independence of Markov's principle (Kreisel)
- Proof mining (Kohlenbach)
- Interpretation of Weak König's Lemma (Ferreira, Oliva)
- Interpretation of principles of Nonstandard analysis (Van den Berg, Briseid, Safarik)

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Different interpretations for different purposes.

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Different interpretations for different purposes.

We try to capture their common structure.

A pot-pourri of interpretations

- Kleene (numerical realizability) (1952)
- Gödel (Dialectica) (1958)
- Kreisel (modified realizability) (1959)
- Diller and Nahm (variant to avoid the contraction problem) (1974)
- Stein (family of interpretations) (1979)
- Kohlenbach (monotone functional interpretation) (1996)
- Ferreira and Oliva (bounded functional interpretation) (2005)
- Van den Berg, Briseid and Safarik (Herbrandized) (2012)
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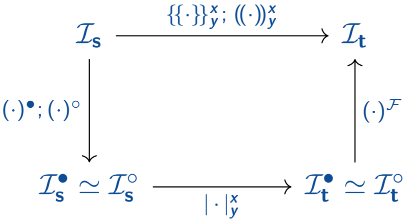
- Compare the various existing functional interpretations.
- Help explain subtle details of the more recent interpretations (BFI, Herbrandized,...)
- Obtain new interpretations

Parametrised interpretations of $\mathcal{I}_{\mathbf{s}}$ into $\mathcal{I}_{\mathbf{t}}$ (jww P. Oliva)

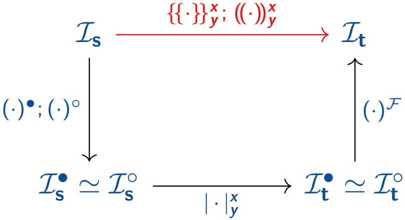


$\mathcal{I}_{\mathbf{s}}$: (intuitionistic) source theory

$\mathcal{I}_{\mathbf{t}}$: (intuitionistic) target theory

$(\cdot)^\bullet$; $(\cdot)^\circ$: Girard's translations

Parametrised interpretations of $\mathcal{I}_{\mathbf{s}}$ into $\mathcal{I}_{\mathbf{t}}$



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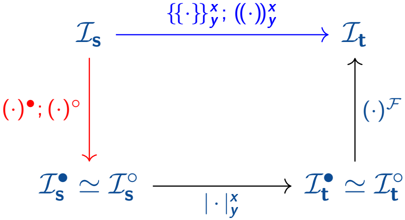

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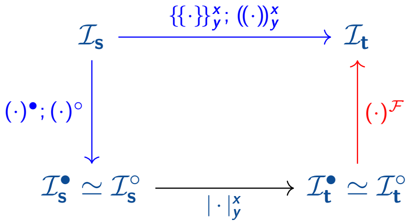

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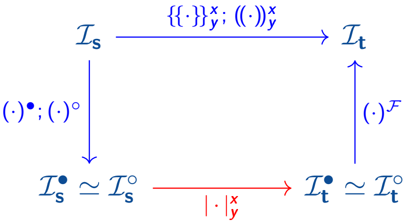

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AL Rules

$$\frac{}{A \vdash A}\,(\mathrm{id}) \qquad \frac{}{\Gamma, \bot \vdash A}\,(\mathrm{efq})$$

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,(\mathrm{cut}) \qquad \frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\,(\mathrm{per})$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes \mathrm{R}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes \mathrm{L})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap \mathrm{R}) \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,(\multimap \mathrm{L})$$

AL Rules

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \ (\forall \mathrm{R}, x \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B} \ (\forall \mathrm{L})$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \ (\exists \mathrm{R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B} \ (\exists \mathrm{L}, x \notin \mathrm{FV}(\Gamma, B))$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \ (\mathrm{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \ (\mathrm{wkn}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \ (!\mathrm{R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \ (!\mathrm{L})$$

AL Rules

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$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\mathrm{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ (\mathrm{wkn}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\mathrm{R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (!\mathrm{L})$$

AL Rules

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ (\forall R, x \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}\ (\forall L)$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ (\exists R) \qquad \frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B}\ (\exists L, x \notin \mathrm{FV}(\Gamma, B))$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\mathrm{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ (\mathrm{wkn}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!R) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (!L)$$

From $\mathbf{IL}^{\mathbb{B}}$ into $\mathbf{AL}^{\mathbb{B}}$

We use Girard's translations of $\mathbf{IL}^{\mathbb{B}}$ into $\mathbf{AL}^{\mathbb{B}}$:

$$
\begin{aligned}
(P(\boldsymbol{x}))^{\bullet} &:\equiv P(\boldsymbol{x}), \quad \text{if } P \not\equiv \bot \\
\bot^{\bullet} &:\equiv \bot \\
(A \wedge B)^{\bullet} &:\equiv A^{\bullet} \otimes B^{\bullet} \\
(A \to B)^{\bullet} &:\equiv !A^{\bullet} \multimap B^{\bullet} \\
(\forall x A)^{\bullet} &:\equiv \forall x A^{\bullet} \\
(\exists x A)^{\bullet} &:\equiv \exists x !A^{\bullet}
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$$

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\begin{array}{llll}
(P(\boldsymbol{x}))^{\bullet} & :\equiv P(\boldsymbol{x}) & (P(\boldsymbol{x}))^{\circ} & :\equiv\, !P(\boldsymbol{x}), \quad \text{if } P \not\equiv \bot \\
\bot^{\bullet} & :\equiv \bot & \bot^{\circ} & :\equiv \bot \\
(A \wedge B)^{\bullet} & :\equiv A^{\bullet} \otimes B^{\bullet} & (A \wedge B)^{\circ} & :\equiv A^{\circ} \otimes B^{\circ} \\
(A \rightarrow B)^{\bullet} & :\equiv\, !A^{\bullet} \multimap B^{\bullet} & (A \rightarrow B)^{\circ} & :\equiv\, !(A^{\circ} \multimap B^{\circ}) \\
(\forall x A)^{\bullet} & :\equiv \forall x A^{\bullet} & (\forall x A)^{\circ} & :\equiv\, !\forall x A^{\circ} \\
(\exists x A)^{\bullet} & :\equiv \exists x !A^{\bullet} & (\exists x A)^{\circ} & :\equiv \exists x A^{\circ}
\end{array}
$$

From $\mathbf{IL}^{\mathbb{B}}$ into $\mathbf{AL}^{\mathbb{B}}$

Proposition

If $\Gamma \vdash_{\mathcal{I}} A$ *then* $!\Gamma^{\bullet} \vdash_{\mathcal{I}^{\bullet}} A^{\bullet}$ *and* $\Gamma^{\circ} \vdash_{\mathcal{I}^{\circ}} A^{\circ}$.

From $\mathbf{IL}^{\mathbb{B}}$ into $\mathbf{AL}^{\mathbb{B}}$

Proposition

If $\Gamma \vdash_{\mathcal{I}} A$ *then* $!\Gamma^{\bullet} \vdash_{\mathcal{I}^{\bullet}} A^{\bullet}$ *and* $\Gamma^{\circ} \vdash_{\mathcal{I}^{\circ}} A^{\circ}$.

Proposition (Gaspar, Oliva (2010))

A° *is equivalent to* $!A^{\bullet}$ *in* $\mathbf{AL}^{\mathbb{B}}$. *More precisely,*

(*i*) $!A^{\bullet} \vdash_{\mathbf{AL}^{\mathbb{B}}} A^{\circ}$

(*ii*) $A^{\circ} \vdash_{\mathbf{AL}^{\mathbb{B}}} A^{\bullet}$

Back into $\mathbf{IL}^{\mathbb{B}}$: the forgetful function

Define a translation of formulas of $\mathbf{AL}^{\mathbb{B}}$ into formulas of $\mathbf{IL}^{\mathbb{B}}$ inductively as follows:

$$
\begin{aligned}
(P(\boldsymbol{x}))^{\mathcal{F}} &:\equiv P(\boldsymbol{x}), \quad \text{for the predicate symbols } P \\
(A \otimes B)^{\mathcal{F}} &:\equiv A^{\mathcal{F}} \wedge B^{\mathcal{F}} \\
(A \multimap B)^{\mathcal{F}} &:\equiv A^{\mathcal{F}} \to B^{\mathcal{F}} \\
(!A)^{\mathcal{F}} &:\equiv A^{\mathcal{F}} \\
(\forall x A)^{\mathcal{F}} &:\equiv \forall x A^{\mathcal{F}} \\
(\exists x A)^{\mathcal{F}} &:\equiv \exists x A^{\mathcal{F}}
\end{aligned}
$$

Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_{\mathbf{s}}$ into $\mathcal{A}_{\mathbf{t}}$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(\boldsymbol{x})$, associate, $\boldsymbol{x} \prec^P a$.

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2. **Domain of witnesses and counter-witnesses**. For each finite type τ we associate in $\mathcal{A}_{\mathbf{t}}$ a formula $\mathsf{W}_\tau(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses.
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   $(\mathbf{Q}_1)$ If $A \vdash_{\mathcal{A}_{\mathbf{t}}} B$ then $!\forall \boldsymbol{x} \sqsubset_{\tau} \boldsymbol{a}\, A \vdash_{\mathcal{A}_{\mathbf{t}}} \forall \boldsymbol{x} \sqsubset_{\tau} \boldsymbol{a}\, B$

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   (Q$_1$**)** If $A \vdash_{\mathcal{A}_\mathbf{t}} B$ then $!\forall \boldsymbol{x} \sqsubset_\tau \boldsymbol{a}\, A \vdash_{\mathcal{A}_\mathbf{t}} \forall \boldsymbol{x} \sqsubset_\tau \boldsymbol{a}\, B$

   (Q$_2$**)** $\vdash_{\mathcal{A}_\mathbf{t}} \forall \boldsymbol{x} \sqsubset_\tau \boldsymbol{a}\, \mathsf{W}(\boldsymbol{x})$

Towards the parametrised interpretation

Finally, for each formula, terms $\eta(\cdot), (\cdot) \sqcup (\cdot)$ and $(\cdot) \circ (\cdot)$ satisfying conditions

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$(\mathbf{C}_\eta)$ $\rightsquigarrow$ to deal with substitutions.

$(\mathbf{C}_\sqcup)$ $\rightsquigarrow$ to have a sort of union/maximum of two terms.

$(\mathbf{C}_\circ)$ $\rightsquigarrow$ to deal with application of terms.

Parametrised **AL**-interpretation

For each formula A of $\mathcal{A}_{\mathbf{s}}$, let us associate a formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ of $\mathcal{A}_{\mathbf{t}}$, with two fresh lists of free-variables $\boldsymbol{x}$ and $\boldsymbol{y}$, inductively as follows:

$$
\begin{array}{lcl}
|P(\boldsymbol{x})|^{a} & :\equiv & \boldsymbol{x} \prec^{P} a, \qquad (P \text{ computational}) \\
|P(\boldsymbol{x})| & :\equiv & P(\boldsymbol{x}), \qquad (P \text{ non-computational}) \\
|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{g}\boldsymbol{x}\boldsymbol{w}} \multimap |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv & \exists z |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z A|^{\boldsymbol{x}}_{\boldsymbol{y}} & :\equiv & \forall z |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} & :\equiv & !\forall \boldsymbol{y} \sqsubset_{\tau_A^-} \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

Witnessable **AL** sequents

A sequent $\Gamma \vdash A$ of $\mathcal{A}_{\mathbf{s}}$ is said to be witnessable in $\mathcal{A}_{\mathbf{t}}$ if there are closed terms $\gamma, \boldsymbol{a}$ of $\mathcal{A}_{\mathbf{t}}$ such that

(i) $\vdash_{\mathcal{A}_{\mathbf{t}}} \mathrm{W}(\gamma)$ and $\vdash_{\mathcal{A}_{\mathbf{t}}} \mathrm{W}(\boldsymbol{a})$

(ii) $!\mathrm{W}(\boldsymbol{x}, \boldsymbol{w}), |\Gamma|^{\boldsymbol{x}}_{\gamma \boldsymbol{x} \boldsymbol{w}} \vdash_{\mathcal{A}_{\mathbf{t}}} |A|^{\boldsymbol{a}\boldsymbol{x}}_{\boldsymbol{w}}$

Soundness

Theorem (Soundness)

If $\mathcal{A}_{\mathsf{t}}$ is adequate and the axioms of $\mathcal{A}_{\mathsf{s}}$ are witnessable in $\mathcal{A}_{\mathsf{t}}$, then the parametrised **AL***-interpretation is sound.*

IL-interpretations

Given an **AL**-interpretation $A \mapsto |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ based on the translated parameters we can derive two **IL**-interpretations, namely

$$A \quad \mapsto \quad (|A^{\bullet}|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\mathcal{F}} \qquad \text{and} \qquad A \quad \mapsto \quad (|A^{\circ}|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\mathcal{F}}$$

We will abbreviate these compound interpretations as

$$\{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}} \equiv (|A^{\bullet}|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\mathcal{F}} \qquad \text{and} \qquad (\!(A)\!)^{\boldsymbol{x}}_{\boldsymbol{y}} \equiv (|A^{\circ}|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\mathcal{F}}$$

Parametrised interpretations of IL

Proposition

$$
\begin{array}{lcl}
\{\![P(\boldsymbol{x})]\!\}^{a} & \equiv & \boldsymbol{x} \prec^{P} a \quad \textit{if } P \in \mathbf{Pred}^{c}_{\mathcal{A}_s} \\
\{\![P(\boldsymbol{x})]\!\} & \equiv & P(\boldsymbol{x}) \quad \textit{if } P \in \mathbf{Pred}^{nc}_{\mathcal{A}_s} \\
\{\![A \to B]\!\}^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & \equiv & \forall \boldsymbol{y} \sqsubset \boldsymbol{f}\,\boldsymbol{x}\,\boldsymbol{w}\, \{\![A]\!\}^{\boldsymbol{x}}_{\boldsymbol{y}} \to \{\![B]\!\}^{\boldsymbol{g}\,\boldsymbol{x}}_{\boldsymbol{w}} \\
\{\![A \wedge B]\!\}^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & \equiv & \{\![A]\!\}^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge \{\![B]\!\}^{\boldsymbol{v}}_{\boldsymbol{w}} \\
\{\![\exists z A]\!\}^{\boldsymbol{x}}_{\boldsymbol{y}} & \equiv & \exists z \forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, \{\![A]\!\}^{\boldsymbol{x}}_{\boldsymbol{y}'} \\
\{\![\forall z A]\!\}^{\boldsymbol{x}}_{\boldsymbol{y}} & \equiv & \forall z \{\![A]\!\}^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{array}
$$

Parametrised interpretations of IL

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$$
\begin{array}{lcl}
\{\!\{P(\mathbf{x})\}\!\}^{a} & \equiv & \mathbf{x} \prec^{P} a \quad \textit{if } P \in \mathbf{Pred}^{c}_{\mathcal{A}_s} \\
\{\!\{P(\mathbf{x})\}\!\} & \equiv & P(\mathbf{x}) \quad \textit{if } P \in \mathbf{Pred}^{nc}_{\mathcal{A}_s} \\
\{\!\{A \to B\}\!\}^{\mathbf{f},\mathbf{g}}_{\mathbf{x},\mathbf{w}} & \equiv & \forall \mathbf{y} \sqsubset \mathbf{f}\,\mathbf{x}\,\mathbf{w}\, \{\!\{A\}\!\}^{\mathbf{x}}_{\mathbf{y}} \to \{\!\{B\}\!\}^{\mathbf{g}\mathbf{x}}_{\mathbf{w}} \\
\{\!\{A \wedge B\}\!\}^{\mathbf{x},\mathbf{v}}_{\mathbf{y},\mathbf{w}} & \equiv & \{\!\{A\}\!\}^{\mathbf{x}}_{\mathbf{y}} \wedge \{\!\{B\}\!\}^{\mathbf{v}}_{\mathbf{w}} \\
\{\!\{\exists z A\}\!\}^{\mathbf{x}}_{\mathbf{y}} & \equiv & \exists z \forall \mathbf{y}' \sqsubset \mathbf{y}\, \{\!\{A\}\!\}^{\mathbf{x}}_{\mathbf{y}'} \\
\{\!\{\forall z A\}\!\}^{\mathbf{x}}_{\mathbf{y}} & \equiv & \forall z \{\!\{A\}\!\}^{\mathbf{x}}_{\mathbf{y}}
\end{array}
$$

In particular, we have that for computational predicate symbols P*:*

$$
\begin{array}{lcl}
\{\!\{\exists z^{P} A\}\!\}^{c,\mathbf{x}}_{\mathbf{y}} & \equiv & \exists z \prec^{P} c \forall \mathbf{y}' \sqsubset \mathbf{y}\, \{\!\{A\}\!\}^{\mathbf{x}}_{\mathbf{y}'} \\
\{\!\{\forall z^{P} A\}\!\}^{\mathbf{f}}_{b,\mathbf{y}} & \equiv & \forall z \prec^{P} b\, \{\!\{A\}\!\}^{\mathbf{f}\,b}_{\mathbf{y}}
\end{array}
$$

Parametrised interpretations of IL

Proposition

$$
\begin{array}{lcl}
(\!(P(\boldsymbol{x}))\!)^{a} & \Leftrightarrow & \boldsymbol{x} \prec^{P} a \quad \textit{if } P \in \mathbf{Pred}^{c}_{\mathcal{A}_s} \\
(\!(P(\boldsymbol{x}))\!) & \Leftrightarrow & P(\boldsymbol{x}) \quad \textit{if } P \in \mathbf{Pred}^{nc}_{\mathcal{A}_s} \\
(\!(A \to B)\!)^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & \Leftrightarrow & \forall \boldsymbol{x}', \boldsymbol{w}' \sqsubset \boldsymbol{x}, \boldsymbol{w}\, ((\!(A)\!)^{\boldsymbol{x}'}_{\boldsymbol{f}\,\boldsymbol{x}'\,\boldsymbol{w}'} \to (\!(B)\!)^{\boldsymbol{g}\,\boldsymbol{x}'}_{\boldsymbol{w}'}) \\
(\!(A \wedge B)\!)^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & \Leftrightarrow & (\!(A)\!)^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge (\!(B)\!)^{\boldsymbol{v}}_{\boldsymbol{w}} \\
(\!(\exists z A)\!)^{\boldsymbol{x}}_{\boldsymbol{y}} & \Leftrightarrow & \exists z (\!(A)\!)^{\boldsymbol{x}}_{\boldsymbol{y}} \\
(\!(\forall z A)\!)^{\boldsymbol{x}}_{\boldsymbol{y}} & \Leftrightarrow & \forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, \forall z (\!(A)\!)^{\boldsymbol{x}}_{\boldsymbol{y}'}
\end{array}
$$

Parametrised interpretations of IL

Proposition

$$((P(\boldsymbol{x})))^a \Leftrightarrow \boldsymbol{x} \prec^P a \quad \textit{if } P \in \mathbf{Pred}^c_{\mathcal{A}_s}$$

$$((P(\boldsymbol{x}))) \Leftrightarrow P(\boldsymbol{x}) \quad \textit{if } P \in \mathbf{Pred}^{nc}_{\mathcal{A}_s}$$

$$((A \to B))^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \Leftrightarrow \forall \boldsymbol{x}', \boldsymbol{w}' \sqsubset \boldsymbol{x}, \boldsymbol{w}\, (((A))^{\boldsymbol{x}'}_{\boldsymbol{f}\boldsymbol{x}'\boldsymbol{w}'} \to ((B))^{\boldsymbol{g}\boldsymbol{x}'}_{\boldsymbol{w}'})$$

$$((A \wedge B))^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \Leftrightarrow ((A))^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge ((B))^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$((\exists z A))^{\boldsymbol{x}}_{\boldsymbol{y}} \Leftrightarrow \exists z ((A))^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$((\forall z A))^{\boldsymbol{x}}_{\boldsymbol{y}} \Leftrightarrow \forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, \forall z ((A))^{\boldsymbol{x}}_{\boldsymbol{y}'}$$

In particular, we have that for computational predicate symbols P

$$((\exists z^P A))^{\boldsymbol{x},c}_{\boldsymbol{y}} \Leftrightarrow \exists z \prec^P c\, ((A))^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$((\forall z^P A))^{\boldsymbol{f}}_{c,\boldsymbol{y}} \Leftrightarrow \forall c', \boldsymbol{y}' \sqsubset c, \boldsymbol{y}\, \forall c'', \boldsymbol{y}'' \sqsubset c', \boldsymbol{y}'\, \forall z \prec^P c''\, ((A))^{\boldsymbol{f}c''}_{\boldsymbol{y}''}$$

Comparing the interpretations

Theorem

For each formula A there are tuples of closed terms $\boldsymbol{s}_1, \boldsymbol{t}_1$ and $\boldsymbol{s}_2, \boldsymbol{t}_2$ such that

$$(i)\quad \mathrm{W}(\boldsymbol{x}, \boldsymbol{y}), \forall \boldsymbol{y}' \sqsubset \boldsymbol{s}_1 \boldsymbol{x} \boldsymbol{y}\ \{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash_{\mathbf{IL}^\omega} ((A))^{\boldsymbol{t}_1 \boldsymbol{x}}_{\boldsymbol{y}}$$

$$(ii)\quad \mathrm{W}(\boldsymbol{x}, \boldsymbol{y}), ((A))^{\boldsymbol{x}}_{\boldsymbol{s}_2 \boldsymbol{x} \boldsymbol{y}} \vdash_{\mathbf{IL}^\omega} \forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\ \{\{A\}\}^{\boldsymbol{t}_2 \boldsymbol{x}}_{\boldsymbol{y}'}$$

$$(iii)\quad \vdash_{\mathbf{IL}^\omega} \mathrm{W}(\boldsymbol{s}_1) \wedge \mathrm{W}(\boldsymbol{s}_2) \wedge \mathrm{W}(\boldsymbol{t}_1) \wedge \mathrm{W}(\boldsymbol{t}_2)$$

Instances

$\forall x \sqsubset_\tau a\, A$	$x \prec^\tau a$	$W_\tau(a)$	**Interpretation**
$A[a/x]$	$x = a$	true	Dialectica interpretation
$\forall x A$	$x = a$	true	Modified realizability
$\forall x \leq^* a\, A$	$x = a$	true	(*combination not sound*)
$\forall x \in a\, A$	$x = a$	true	Diller-Nahm interpretation
$A[a/x]$	$x \leq^*_\tau a$	$a \leq^*_\tau a$	(*combination not sound*)
$\forall x\, A$	$x \leq^*_\tau a$	$a \leq^*_\tau a$	Bounded modified realizability
$\forall x \leq^* a\, A$	$x \leq^* a$	$a \leq^* a$	Bounded functional interpretation
$\forall x \in a\, A$	$x \leq^*_\tau a$	$a \leq^*_\tau a$	**Bounded Diller-Nahm interpretation**
$A[a/x]$	$x \in a$	true	**Herbrand Dialectica** ($\simeq$ Dialectica)
$\forall x\, A$	$x \in a$	$\tau^*(a)$	Herbrand realizability (for IL)
$\forall x \leq^* a\, A$	$x \in a$	$a \leq^*_\tau a$	**Herbrandized bfi**
$\forall x \in a\, A$	$x \in a$	$\tau^*(a)$	Herbrand Diller-Nahm interpretation

Questions and future work

- Other ways to instantiate the parameters?

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$$|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} :\equiv !\forall \boldsymbol{y} \sqsubset_{T} \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes A.$$

Questions and future work

- Other ways to instantiate the parameters?
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$$|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} :\equiv !\forall \boldsymbol{y} \sqsubset_{\tau} \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes A.$$

- Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.

Questions and future work

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- Characterization theorem?
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$$|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} :\equiv !\forall \boldsymbol{y} \sqsubset_{\tau} \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes A.$$

- Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.
- Composing with Krivine's negative translation does one obtain classical interpretations? Factorization?

Outline

Realizability with stateful computations for NSA (jww É. Miquey)

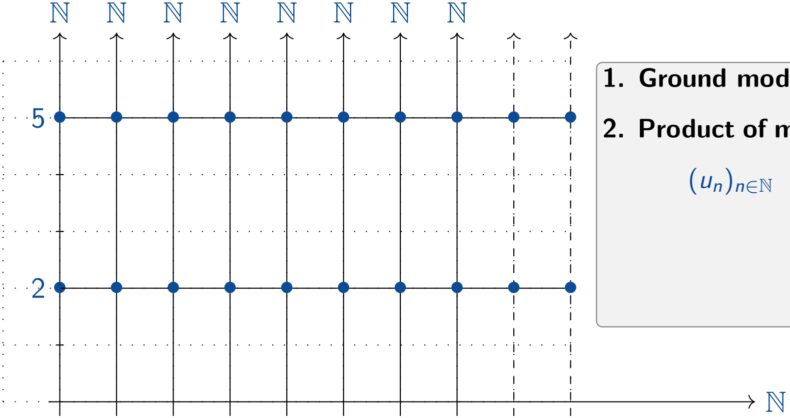
- Goal: to deal with nonstandard analysis in the context of intuitionistic realizability, focusing on the Lightstone-Robinson construction of a model for nonstandard analysis through an ultrapower.

In particular, we consider an extension of the λ-calculus with a memory cell, that contains an integer (the state), in order to indicate in which slice of the ultrapower $\mathcal{M}^{\mathbb{N}}$ the computation is being done.

Nonstandard models

1. Ground model

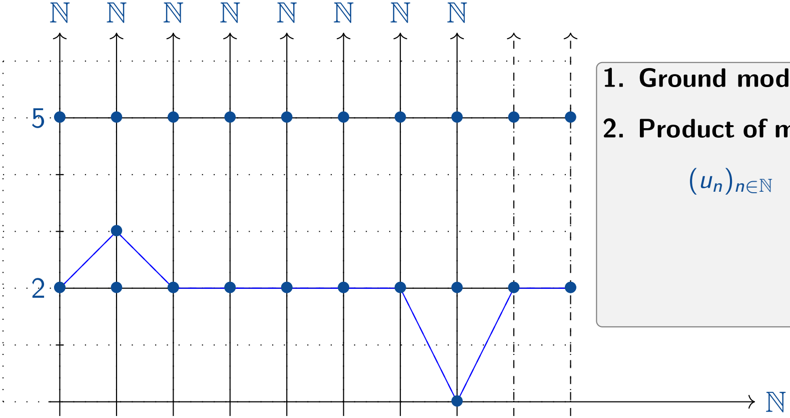
Nonstandard models



1. **Ground model**
2. **Product of models**

$$(u_n)_{n\in\mathbb{N}}$$

Nonstandard models



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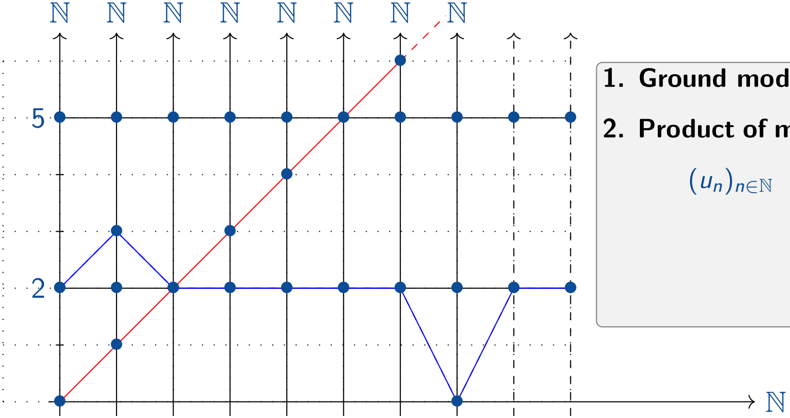

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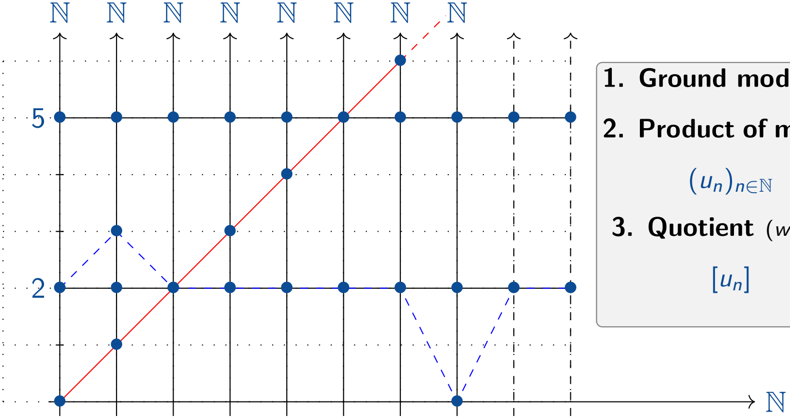
Nonstandard models



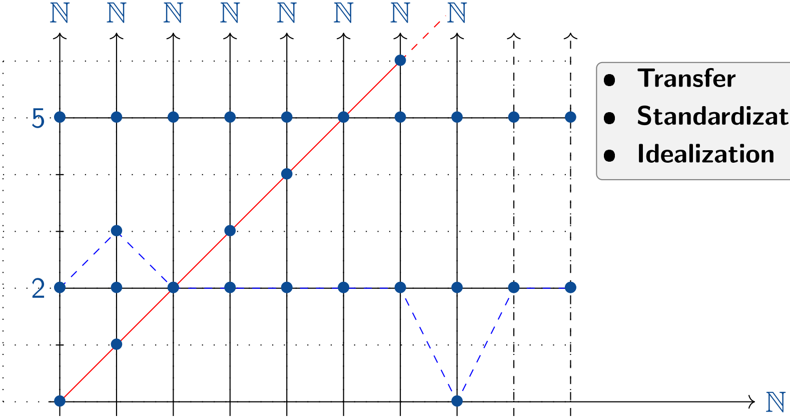
1. **Ground model**
2. **Product of models**

$$(u_n)_{n\in\mathbb{N}}$$

3. **Quotient** (*w.r.t.* $\mathcal{U}$)

$$[u_n]$$

Nonstandard models



- **Transfer**
- **Standardization**
- **Idealization**

The first step in the Lightstone-Robinson construction aims at getting a product $\mathcal{M}^{\mathbb{N}}$ of the (initial) model $\mathcal{M}$.

- Add a memory cell to our calculus that contains an integer, which we call the *state*.
- The state keeps track of which "slice" of the product is the interpretation being done.

This product allows us to interpret first-order individuals as functions in $\mathbb{N}^{\mathbb{N}}$, so that the interpretation accounts for new elements – the so-called nonstandard elements – for instance the diagonal function.

$$
\begin{array}{llll}
\textbf{Formulas} & A, B & ::= & \mathrm{st}(e) \mid X(e_1, \ldots, e_n) \mid \mathrm{Nat}(e) \mapsto A \\
 & & & \mid A \to B \mid A \wedge B \mid A \vee B \\
 & & & \mid \forall x.A \mid \exists x.A \mid \forall X.A \mid \exists X.A \\
\textbf{Terms} & t, u & ::= & \ldots \mid \mathsf{get} \mid \mathsf{set} \\
\textbf{States} & \mathfrak{S} & := & \mathbb{N}
\end{array}
$$

- get allows to read the current state
- set allows to increase the value of the current state
- With the exception of the get/set instructions, the syntax of terms does not account for states.

The interpretation of a formula A together with a valuation ρ is the set $|A|^{\mathfrak{S}}_{\rho}$ defined inductively according to the following clauses:

$$
\begin{array}{rcl}
|\mathrm{st}(e)|^{\mathfrak{S}}_{\rho} & \triangleq & \begin{cases} \Lambda \times \mathfrak{S} & \textit{if } [\![e]\!]_{\rho} \textit{ is standard} \\ \emptyset & \textit{otherwise} \end{cases} \\
|X(e_1, \ldots, e_n)|^{\mathfrak{S}}_{\rho} & \triangleq & \rho(X)@([\![e_1]\!]_{\rho}, \ldots, [\![e_n]\!]_{\rho}) \\
|\{\mathrm{Nat}(e)\} \mapsto A|^{\mathfrak{S}}_{\rho} & \triangleq & \{(t; \mathfrak{s}) \in \Lambda \times \mathfrak{S} : (t\,\overline{n}; \mathfrak{s}) \in |A|^{\mathfrak{S}}_{\rho},\ \textit{where } n = [\![e]\!]_{\rho}(\mathfrak{s})\} \\
|A \to B|^{\mathfrak{S}}_{\rho} & \triangleq & \{(t; \mathfrak{s}) \in \Lambda \times \mathfrak{S} : \forall u.\big((u; \mathfrak{s}) \in |A|^{\mathfrak{S}}_{\rho} \Rightarrow (t\,u; \mathfrak{s}) \in |B|^{\mathfrak{S}}_{\rho}\big)\} \\
|A_1 \wedge A_2|^{\mathfrak{S}}_{\rho} & \triangleq & \{(t; \mathfrak{s}) \in \Lambda \times \mathfrak{S} : (\pi_1(t); \mathfrak{s}) \in |A_1|^{\mathfrak{S}}_{\rho} \wedge (\pi_2(t); \mathfrak{s}) \in |A_2|^{\mathfrak{S}}_{\rho})\}
\end{array}
$$

$$
|A_1 \vee A_2|^{\mathfrak{S}}_{\rho} \triangleq \{(t; \mathfrak{s}) \in \Lambda \times \mathfrak{S} : \exists i \in \{1, 2\}.(\mathsf{case}\ t\,\{\iota_1(x_1) \mapsto x_1 | \iota_2(x_2) \mapsto x_2\}; \mathfrak{s}) \in |A_i|^{\mathfrak{S}}_{\rho}\}
$$

$$
\begin{array}{rclrcl}
|\forall x.A|^{\mathfrak{S}}_{\rho} & \triangleq & \bigcap_{f \in \mathbb{N}^{\mathfrak{S}}} |A|^{\mathfrak{S}}_{\rho, x \mapsto f} & |\forall X.A|^{\mathfrak{S}}_{\rho} & \triangleq & \bigcap_{F : \mathbb{N}^k \to \mathbf{SAT}} |A|^{\mathfrak{S}}_{\rho, X \mapsto F} \\
|\exists x.A|^{\mathfrak{S}}_{\rho} & \triangleq & \bigcup_{f \in \mathbb{N}^{\mathfrak{S}}} |A|^{\mathfrak{S}}_{\rho, x \mapsto f} & |\exists X.A|^{\mathfrak{S}}_{\rho} & \triangleq & \bigcup_{F : \mathbb{N}^k \to \mathbf{SAT}} |A|^{\mathfrak{S}}_{\rho, X \mapsto F}
\end{array}
$$

This interpretation realizes (in a non-trivial way):

- Usual properties of nonstandard natural numbers (including external induction)
- The diagonal as a nonstandard element
- Idealization
- Transfer
- Overspill and Underspill

It does not validate Standardization: for that a quotient is necessary (work in progress).

Questions and future work

- What applications are there for the interpretations with truth? Can they give additional information about Transfer?

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- What applications are there for the interpretations with truth? Can they give additional information about Transfer?
- Is it possible to use any of these interpretations in Proof Mining?
- Is it possible/interesting to extend nonstandard interpretations to the feasible context?
- Adapt the interpretation with slices to Krivine's classical realizability (in progress)

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Thank you!