# Anamorphosis Reformed: from Optical Illusions to Immersive Perspectives

António B. Araújo

**Abstract** We discuss a definition of conical anamorphosis that sets it at the foundation of both classical and curvilinear perspectives. In this view, anamorphosis is an equivalence relation between three-dimensional objects, which includes two-dimensional representatives, not necessarily flat. Vanishing points are defined in a canonical way that is maximally symmetric, with exactly two vanishing points for every line. The definition of the vanishing set works at the level of anamorphosis, before perspective is defined, with no need for a projection surface. Finally, perspective is defined as a flat representation of the visual data in the anamorphosis. This schema applies to both linear and curvilinear perspectives, and is naturally adapted to immersive perspectives, such as the spherical perspectives. Mathematically, the view here presented is that the sphere and not the projective plane is the natural manifold of visual data up to anamorphic equivalence. We consider how this notion of anamorphosis may help to dispel some long standing philosophical misconceptions regarding the nature of perspective.

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## Introduction

This chapter presents an unusual treatment of a familiar concept: anamorphosis. It is an elaboration on a set of recent works, especially (Araújo 2018c), that reconsider the notion of anamorphosis with a view towards a rigorous and useful definition of the conical spherical perspectives. The title of this chapter is an allusion to that foundational role.

This view turns out to also have a clarifying role with regards to the philosophical debate on the mimetic nature of classical perspective. It can be seen as reconciling concepts often thought to be in opposition: Euclidean optics, linear perspective, and curvilinear (spherical) perspectives.

The proposed definition of anamorphosis can be thought of as Euclid's optics with vanishing points: it emphasizes topological aspects, providing a canonical definition for vanishing points that does not depend on a choice of projection surface. This gives anamorphosis a role that precedes that of perspective both logically and in importance.

Perspective proper we define as a two-step process, the first and most important of which is anamorphosis, the second step being a representation step, a *flattening* of the visual data onto the plane, analogous to cartography.

This is rather the opposite of the usual state of affairs. Expressing the common view on anamorphosis, (Collins 1992) states that "any explanation of its mechanisms necessarily begins with a discussion of classical perspective". We affirm here the antipodal view, that perspective is the derived concept, and that its most important features - namely the definition of vanishing points - are more adequately defined at the level of anamorphosis.

Historically, anamorphosis has been seen as the trickster, the charming but ultimately fatuous sibling indulging in party tricks while perspective handles the respectable work; the latter fit for the architectural firm, the former for the cabinet of curiosities. In historical analysis, as in perspective textbooks, it portrayed as a derivative and degenerate special case of perspective, or even a perversion of it, a wonderful monstrosity, though a charming one, like the scares of a B-movie can be charming. We feel the fear is justified and telling; it is the trepidation of the human mind contemplating the palpable ambiguity of its senses. This view of anamorphosis is expertly treated in other places (Baltrušaitis 1983; Collins 1992) (see also a excellent chapter of the present book, Anamorphosis:Between Perspective and Catopritics). Here we take a very different view: we say that anamorphosis is a charmingly simple consequence of a single axiom, the *principle of radial occlusion*, and a perfect example the rational study of perception, with its origins in Euclid's optics, and that linear perspective is the slightly awkward, utilitarian derived concept.

Traditionally, also, anamorphosis is mostly seen as an *inverse problem*, a game of hide and seek, where the spectator is bent on finding the single observational point from which an image makes sense, from which it can be "formed again", as the etimology of the word "anamorphosis" suggests. It is the territory of Niceron and of Holbein. But here we emphasize a parallel tradition that treats anamorphosis as a *direct* problem, that starts with Brunelleschi and at which Pozzo excelled. The distinction is one of emphasis and intention rather than principle. In this second approach the observational point is proffered at the start, and the emphasis is on integration with the physical environment. With Brunelleschi, standing at the doors of Florence with his tavoletta, anamorphosis became a seminal experiment in visual psychophysics, the study of the connection between stimuli and sensation. With Andrea Pozzo, with anamorphic ceilings such as the one of the church of Sant'Ignazio in Rome (Fig. 1), it becomes immersive, a discipline of *architecture by other means* . No peeking through peepholes here, no mirrors or contraptions. Indeed Pozzo made use of anamorphosis as part of a lived in space, and projected it both as true architecture and as a replacement for such (Kemp 1990; Fasolo and Mancini 2019). Although the view is not fully immersive, it is wide enough to make clear that there are no limitations of principle to the field of view, only practical ones. This view of anamorphosis as a living part of architecture is of course not exclusive to Pozzo (see for instance Rossi (2016)), although he is a prime example of that tradition. Besides architecture proper, the use of such devices in scenography is well known (Čučaković and Paunović 2016).

We note that some works here called anamorphoses (e.g. Pozzo's ceilings, Brunelleschi's panels) are often, and even historically called *perspectives* . The term anamorphosis was a belated 17th-century invention and in the mean time the term perspective became rather polymorphous, destined to denote both plane and curvilinear projections, 2D pictures and 3D objects. This semantic overloading can become a source of confusion, and Araújo (2016) claims it as important for the debate between the realist and conventionalist schools of the philosophy of perspective but also for the late development of spherical perspective. We will here follow Araújo (2018c) in using the freedom of mathematicians to plunder old words for new purposes. Definitions are after all just syntactic sugar - they are good if they help us to think clearly. We will organize concepts by restricting the term "perspective" to plane pictures and using "anamorphosis" for the more general mimetic device. It is a choice justified by how natural it makes the terminology of spherical perspectives while decently in accordance with common usage and historic precedence; Taylor Andersen (1992) for instance defined perspective as (emphasis mine) "the art of drawing *on a plane* the appearances of figures (...)".

The separation of perspective in two steps, first a conical projection onto the sphere, the next a cartographic flattening of this projection, is practically inevitable when dealing with spherical perspectives, and has been done implicitly in Barre and Flocon's seminal work (Barre and Flocon 1968; Barre et al 1964) "La Perspective Curviligne", although with a reluctance in considering the whole sphere in the perspective construction (hence this first spherical perspective was in fact only *hemispherical*). In artistic applications, Dick Termes made a career out of spherical anamorphoses, but also limited himself to the hemisphere in his plane projections (Termes 1998). It is hard to know how much of the limitation was due to technical difficulties, how much to philosophical reluctance, how much to aesthetical preference. The matter of constructing total spherical perspectives is taken up in another chapter of this book (see the chapter Spherical Perspective in the present volume). Here we emphasizes the role of anamorphosis as an equivalence between 3D objects and not just drawings; as a radial equivalence, without a preferred axis, hence with no field of view limitations. Mathematically, this can be seen as replacing the projective plane by the sphere as the visual data manifold (on the projective formalization of perspective, see (Morehead Jr 1955) and also the chapter Looking Through the Glass in the present volume).

In computer graphics the spherical view - free of the practical difficulties of handmade drawing on the sphere - is very natural. Barnard (1983) saw it as a natural setting for the algorithmic interpretation of perspective images. In (Correia et al 2013; Correia and Romão 2007) a two-step process of projection onto a curvilinear surface followed by a plane representation is used for free-form wide-view visualization of architecture. As the surfaces in question can be deformed homeomorphically onto the sphere, this is analogous to the method considered here. The approach of (Araújo 2018c) is of particular use to us here for explicitly relating the first step of this entailment to a general notion of anamorphosis and for setting it within a tradition of *rational drawing* as practiced by the draughtsman rather than in the field of computer rendering. This work used spherical anamorphosis as the first step of a construction for handmade spherical perspective drawings of the azimuthal equidistant (or fisheye) spherical perspective, and defined a general strategy for solving spherical perspectives, later applied to the equirectangular Araújo (2017d, 2018b) and cubical Araújo et al (2019b) spherical perspectives.

The entailment in the definition proposed, separating the anamorphic aspect from the plane projection has the side effect of dispelling the state of tension between the concepts of anamorphosis and perspective, automatically solving equivocations like the so-called paradox of Leonardo (Araújo 2016) that so bothered the philosophy of art.

This view is especially important in face of a growing awareness of the connection between the digital techniques of visual immersion with the sometimes misunderstood methods of the past (Gay and Cazzaro 2019; Araújo 2017b; Grau 1999; Tomilin 2001), a connection often made obscure by the fog of confusing and inadequate terminology incidental to their historical development. The details of the machinery apart, the geometry used in VR, AR, and MR, in immersive photography, video mapping and full dome projections is still the same as what was used to draw the anamorphoses of Niceron and Pozzo, or Robert Barker's immersive panoramas that so dazzled 19th century crowds (Grau 1999; Huhtamo 2013). Hence these concepts, properly reformulated, have didactic, artistic, and technological possibilities still to be explored, and indeed several researchers and artists have lately investigated, through mathematical, technological and artistic works, the possibilities of such connections between the digital and physical notions of anamorphosis, and the hybrids in between Araújo et al (2019a); Michel (2013); Rossi et al (2018). This chapter arose from such explorations, as well as from an extensive teaching experience of these concepts to a varied audience ranging from school children just starting out in their studies of perspective and descriptive geometry (Araújo 2017c), to working illustrators, to Ph.D. students of digital media art looking to better understand the underlying concepts hidden in the often opaque digital black boxes that serve as their tools Araújo (2017b). It is hoped that both the artist and the geometer will find something of interest here.

**Fig. 1** Architecture by other means. Right: Fisheye view of the ceiling of St. Ignazio's Church in Rome. Illusionistic columns prolong the real ones seamlessly. A painting on a flat ceiling simulates a dome. Left: on the ground a golden disc marks the spot where the observer should stand. In a coincidental but quite charmingly appropriate manner, the disc and the surrounding reddish strips look like an eye when photographed with a fisheye camera. Photograph by the author.

## Anamorphosis formed again

We present a formulation of anamorphosis following (Araújo 2018c) and Araújo (2017a). We intend this chapter to be readable both by mathematicians and artists. We therefore will alternate between the terse style of the mathematician and longer explanations for the artist and philosopher, flirting with that doubtful balance wherein all are fed and none is satisfied. To make the reading fluid we present it uncritically and naively in a first section, then render it mathematically, and finally discuss the assumptions that were made in the process.

## *The empirical principle: radial occlusion*

To start with, let's recall the usual definition of (conical) anamorphosis. It is usually some variation of "a distorted image that will look normal when viewed from a particular point". We are unsatisfied with this. It will do as a dictionary definition but it won't pass muster as a mathematical one. What is meant by distortion? What is an image? What does it mean to look normal? It is claimed in (Araújo 2016) that this carelessness in definition is a piece of historical baggage that has caused all sorts of conceptual mischief. We will discard it entirely and try to recapture its spirit in a more rigorous fashion that encapsulates both the actual operational aspects of anamorphosis and exposes its mathematical structure.

The study of conical anamorphosis is a subset of the study of *mimesis*, the imitative representation of the world. We find that there are classes of objects that although very different as three dimensional forms, *look the same* to human observers under adequate experimental conditions. The expression *look the same* has here a very specific and operational meaning: a human observer, under stipulated experimental conditions, cannot tell the difference between the two objects. If one object was to be suddenly replaced by the other, the subject would not notice the switch.

This mimicry, to really deceive the eye, depends on several factors, regarding shape, colour, lighting, among others. Conical anamorphosis, as defined here, studies only the most basic of these factors: the appearance of objects with regard to their apparent contour, or the form of the region that they occupy in the visual field. This is by itself enough for certain effective illusions requiring only the matching of form, and is a necessary condition for more complex illusions involving matters of color or more complex optical conditions.

That visual mimesis of form occurs under certain conditions is not demonstrable from first principles of geometry but is rather an empirical fact about human vision that relates human visual sensations to measurable, geometrical properties of visual stimuli. It can be made to be demonstrable only when certain empirical principles are abstracted onto geometrical assumptions.

In the case of conical anamorphosis, the whole subject derives from a single empirical observation, regarding how object occlude (hide) each other. We may imagine the following experiment: an observer, his eye at a fixed point $O$, is presented with two very small ("point-like"), black, featureless balls, set in various positions relative to each other. The observer is asked how many balls he sees. For most such positions the observer reports seeing two balls. But when the balls are geometrically aligned with his eye, he will report seeing only one. From this we abstract a principle we call *radial occlusion*:

**Principle of radial occlusion:** Two points $P$ and $Q$ look the same to an observer at a point $O$ if they are on the same ray with origin in $O$.

By *look the same* we mean literally that they look like the same point. You could say that they occupy the same location in the visual field. Or you could state the principle as "optical alignment coincides with geometric alignment". The choice of words here is not where precision lies. The meaning is operational, so precision comes from the specification of the conditions of the experiment: "Look the same" means that the subject of some specific experiment reports something like "I see only one ball" instead of "I see two balls". We are dealing here with psychophysics, the study of human perceptions as a function of (controlled, measured) stimuli. The study of perception ultimately relies on the response of human subjects to experiments. These are very concrete experiments made under well-specified conditions, typically meant to elicit simple yes-or-no responses. In the study of color, for instance, a subject may be presented with two adjacent patches of color from different power spectra and asked if they are the same color or not (or, equivalently, if they look like one or two different patches). With such experiments, a principle of linear color addition may be tested. In the same way, the principle of radial occlusion is not demonstrable by mathematics. It is a mathematical abstraction of the results of experiments on human subjects. Brunelleschi's demonstration of perspective may be seen as one such experiment, showing that the lines of a drawing made in linear perspective match the lines of the real object when one is superimposed on the other.

Of course, we do have precise models of both optics and physiology that allows us to model and understand when and why the principle of radial occlusion is valid (such models were in turn tested against the empirical facts of perception). We know the principle depends on the way light rays move and how they are processed by the human visual system, and we therefore also know when to expect it to fail. Both refraction and reflection cause it to fail. In Fig. 2 we see that points $A$ and $B$ are aligned with both a light rays and a geometrical ray, but points $C$ and $D$, which are on different optical media (say water and air) are therefore on the same light ray but not on the same geometrical ray. Then $C$ will occlude (hide) $D$ to an observer at $O$, although the points are not aligned. This is what optics leads us to expect, and perception confirms it (when refractions or reflections are important we are led to a different sort of anamorphosis, which we will not treat here).

Now, since a lens works through refraction and that our very eye has a lens, we realize that radial occlusion cannot be but approximately true. This is common fare with statements regarding perception. A principle borne out of experiment is only strictly trusted to be valid under the conditions of the experiment - how far from that it will extend is an open question. So, more than explicit statements of absolute truth, these principles are implicit definitions of the gamut of experimental conditions under which they are verified. And they are interesting concepts if that gamut is interesting. That is the case with radial occlusion. Although far from universally valid, it is the default supposition of our interaction with light and the world. We will no longer be concerned in this section on the intricacies of when the principle works, but will reason on its consequences. All that lies ahead follows geometrically from this single assumption of radial occlusion, that we will later abstract into our definition of anamorphosis (definition 4).

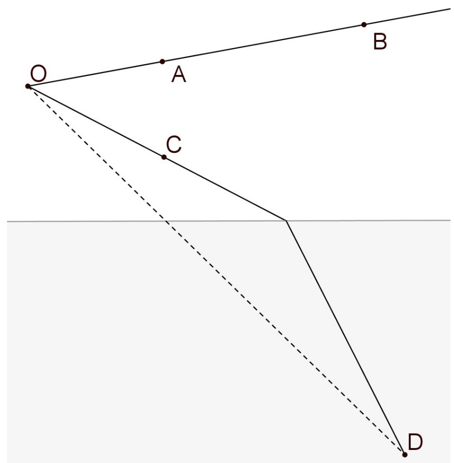


**Fig. 2** Points $A$ and $B$ are geometrically aligned and visually occluded relative to the observer at $O$. Points $C$ and $D$ are visually occluded relative to $O$ but not geometrically aligned due to refraction on the transition between optical media (e.g. water and air).

## *Anamorphosis formed fast*

We make a quick and informal summary here, before we proceed to the formal definitions and results.

If we accept the principle of radial occlusion, then it follows that many different $3D$ objects will look the same from $O$. The points of an object define a cone of rays from the observer $O$, the visual cone. If two objects define the same cone of rays, they should look the same. Such objects we will call *anamorphs* of each other relative to $O$ (in fact we will require a technical condition for this *anamorphic equivalence* or *anamorphosis* that is slightly looser than equality of visual cones, but that is the subject of the next section; here we speak loosely).

The important things to note is that, unlike in perspective, the equivalence here is between 3D objects, and not just between objects and drawings. There is no need here to mention projection surfaces.

For instance, take the wireframe cube of Fig. 3, and an observer at $O$. If we slide each vertex along its ray from $O$ we get the object to the right of the cube. These two shapes are very different, but they define exactly the same cone of rays from $O$, so, if we accept the principle of radial occlusion, we must accept that the two objects will look the same from $O$. Of course, the segments joining the vertices don't have to be straight. You could bend each segment freely within the cone of rays it generates, and it would preserve the equivalence. For instance, arc $A''B''$ in Fig. 3 is equivalent to rays $AB$ and $A'B'$ as it lies within the cone of $AB$. This cone is just the angle $\angle AOB$.

This equivalence is radial and isotropic in nature. Unlike classical perspective, where there is a preferential axis defined by the perpendicular to the plane of the

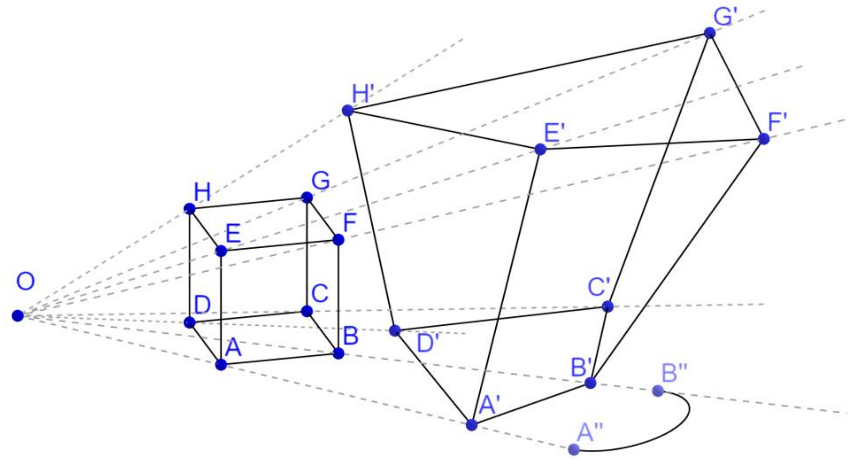


**Fig. 3** Two very different 3D objects with the same visual cone. They will look the same when seen from *O* if the principle of radial occlusion is valid.

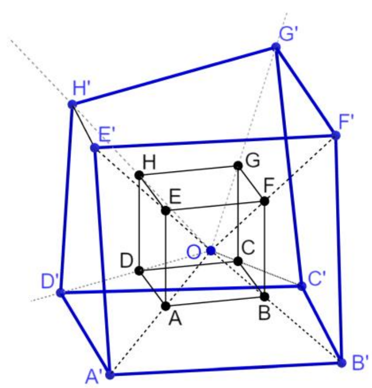


**Fig. 4** Two very different 3D objects with the same visual cone, one surrounding the other. Anamorphosis is radial and isotropic, with no preferential axis.

picture, here there is no such axis, plane, or even picture, hence no limitation of field of view: this is just an equivalence between objects. So the example of Fig. 4 works just as well as the previous one. This allows give equal footing to immersive anamorphoses, that is, anamorphs that completely surround the observer (in turn, this proves foundational for spherical perspectives (Araújo 2018c)).

*Remark 1.* The limitation of a preferred axis is often kept in the literature from attachment either to the habits of perspective or to the convenient availability of the mathematical framework of projective geometry (see for instance (Sánchez-Reyes and Chacón 2016)). But in projective geometry all points on a line are equivalent, and this is a limited framework for anamorphosis. In anamorphosis the classes of equivalence are rays, not lines, so the sphere and not the projective plane will represent the visual data. The projective plane ignores exactly half of this visual data.

The principle of radial occlusion implies the existence of 2D anamorphs to any 3D object. Suppose $S$ is a surface, not containing $O$. Then the intersection of the visual cone of a 3D object $X$ with $S$ is a 2D object that subtends the same cone as $X$ itself. Hence it is a "drawing" that looks exactly like $S$ when seen from $O$.

You could loosely say that the principle of radial occlusion is what makes *drawing* possible.

The most obvious case is the intersection of an object's cone with a plane. We see two examples in Fig. 5, the same object being projected onto a "vertical" and a "horizontal" plane.

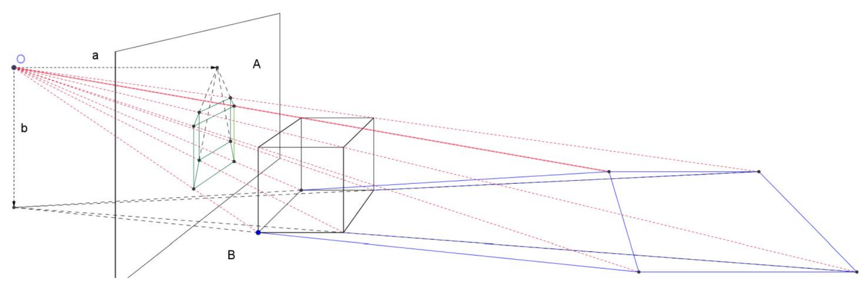


**Fig. 5** Projection onto a plane. There is no absolute difference between the linear perspective and the oblique anamorphosis. Both are anamorphosis onto a plane, the latter merely projects the object to a region further away from the foot of the perpendicular.

There is no fundamental difference between horizontal and vertical, of course, since these are just named relative to the gravity field, which is quite irrelevant for optics. And yet the former anamorph is usually called a classical perspective while the latter is called an "oblique anamorphosis". What is the difference? Any projection plane defines a preferred axis, the perpendicular to the plane through $O$. The difference is the angular distance of the object's cone with regard to that axis. When the object's cone is closely aligned with the main axis, the drawing is called a perspective, when the angular separation is "large", it is called an oblique anamorphosis. We see that the difference is one of degree, and both are special cases of anamorphosis, the projection being just another anamorph that happens to be contained in a plane, whatever the plane's orientation may be. The projection is operationally the same in both cases, what makes it a "perspective" or "oblique anamorphosis" is the relative position of the body being projected, not the functional form of the projection itself. It is also quite indefensible to say, as is usually naively put, that the anamorphosis is "deformed" while the perspective isn't. What deformation is meant? By the principle of radial occlusion, none of them is *optically* deformed, being indistinguishable from $O$, while metrically both of them are deformed; the elongation of the edges in the oblique anamorphosis has a counterpart in their compression in the perspective. Of course some type of metrical deformation is inevitable if you lose one dimension.

Let us move to more complex setups. Nothing stops us from using several planes at once, as in Fig. 6. The cone projects onto this union of planes as several images that should all join seamlessly when viewed from $O$. We will see that this changes the global properties (such as the number of vanishing points) in a fundamental manner while preserving the local properties of classical perspective. This multi-plane projection is historically used in the so-called *perspective boxes* . Verweij (2010); Spencer (2018); Gay and Cazzaro (2019); Čučaković and Paunović (2016)

**Fig. 6** Anamorphosis of a cube onto three planes. Locally these are just plane perspectives but globally, some lines in such multi-plane configurations may present up to two vanishing points. Temporary installation in Óbidos, Portugal, joint work by the author and Maria Bianchi Aguiar.

But of course we are not limited to planes. If we intersect the visual cone of an object with a curved surface we still obtain a 2D anamorph. Now the line projections will themselves be curved in space, yet they will seem straight when seen from $O$. Ahead we will see how to investigate the shapes of these lines analytically and geometrically, and how to draw them in practice. In Fig. 7 we see an example of an anamorph onto a composite surface made up of a cylinder and a plane. We notice that the lines of the anamorph don't even have to be connected.

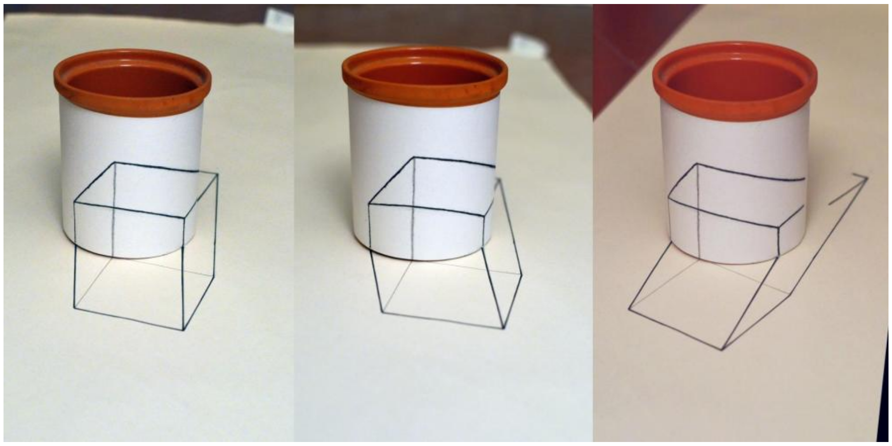

**Fig. 7** Anamorphosis of a cube onto a union of a plane and a cylinder. Drawing by the author.

Further, if we wish to obtain an immersive anamorphosis, that is, an anamorphosis that surrounds the viewer, we may place $O$ inside a surface, for example, in the axis of a cylinder. Then the observer may look in all directions around the vertical axis and have the illusion of being surrounded by an immersive landscape. This is the principle behind the famous panoramas of the 19th century, illusions so striking at the time that the word em *panorama* , originally created to signify these illusory pictures, became in time associated to the physical landscape itself. In fact the word seems to appear in print for the first time in 1791 in an advertisement promoting Robert Barker's large displays of immersive, cylindrical anamorphoses at his "panorama building", erected on Leicester square, in London (Huhtamo 2013). From the platform at the center of this building, large paintings were displayed, to immerse the spectators in foreign landscapes or representations of famous battles, such as that of Waterloo, that was pictured in the Leicester Square panorama in 1816, having been fought less than a year before. Such panoramas became widespread worldwide in the 19th century, and even today a panorama of the battle of Waterloo is on display at the museum erected on the field of battle, and a few depictions of civil war battles survive in the United States of America. It is important to note here that Barker's panoramas, striking as they were, were not strict cylindrical anamorphoses. According to (Kemp 1990, p. 214) they were made by joining a series of adjacent linear drawings, with some care to soften the transitions. These are perfectly adequate as long as there are no exceedingly long architectural lines involved, which is not a problem for wide vistas where each building will occupy a small angle of view and the general landscape is one of natural forms that the eye cannot judge for perspectival exactness.

Of course, the most isotropic anamorphic view would be a spherical one, and this too has been exploited in various ways. The visual illusions of the planetarium or of a modern, digital full dome allow for hemispherical anamorphoses, and no more than architectural convenience and economy stops us from applying the same principle to build a full sphere, to provide a truly immersive anamorphosis, completely surrounding the observer. In fact, fully spherical immersive illusions were tried architecturally. Charles Delangard proposed the concept of the *Georama* to the Geographical Society of Paris in 1822, and the first exemplar was erected in that city in 1826 - a sphere of approximately 12 meters in diameter, whose interior was painted with a representation of the Earths geography, seen from within (Belisle 2015; Oettermann and Schneider 1997). Delangard's Georama, like the panorama before it, became widespread and was implemented elsewhere , the most famous example being James Wyld's Great Globe in Leicester Square. The concept was also used to represent the celestial globe. There was one such globe in the Paris exhibition of 1900, that visitors could enter to see the firmament rotate around their station point. Some Georama's paired these geographic and astronomical views, as in the case of Wyld's great globe, whose interior displayed the Earth's surface, while having the celestial firmament painted on the exterior dome. This was to be expected, since the map making tradition has a long standing pairing between the charts of the stars and those of the Earth (see for instance the chapter on Spherical Perspective in the present volume). Again, like the panoramas before them, the Georamas were not strict anamorphoses, having often deformations to facilitate particular readings, or putting the viewers in observational platforms not necessarily located at the center of the sphere (Belisle 2015). Still, they demonstrate the architectural possibilities of a completely immersive anamorphosis. Of course in the present day, a type of immersive anamorphoses are daily achievable through digital means. Fist-person games or simulations work by creating a flat anamorphosis dynamically onto the screens of the 3D glasses, centered on a moving observational point $O$, the screen serving as a moving window into the virtual world that will be in anamorphosis if the distance to the screen is adequate. Of course, since the distances are small, the principle of radial occlusion will not really be verified, so the illusion is not really immersive. Better results are obtained with the help of 3D helmets, where lenses create a sense of size and distance and illusions pairs are created for each eye at a moving pair of observational points $O_L$ and $O_R$. This is but another step in the evolution of moving panoramas.

## *Some considerations on anamorphosis*

### The point of observation

It is important to stress that anamorphic equivalence is always related to an observer $O$. It makes no sense to ask if something is an anamorphosis without context. Everything is an anamorphosis of some other thing relative to some point. It only makes

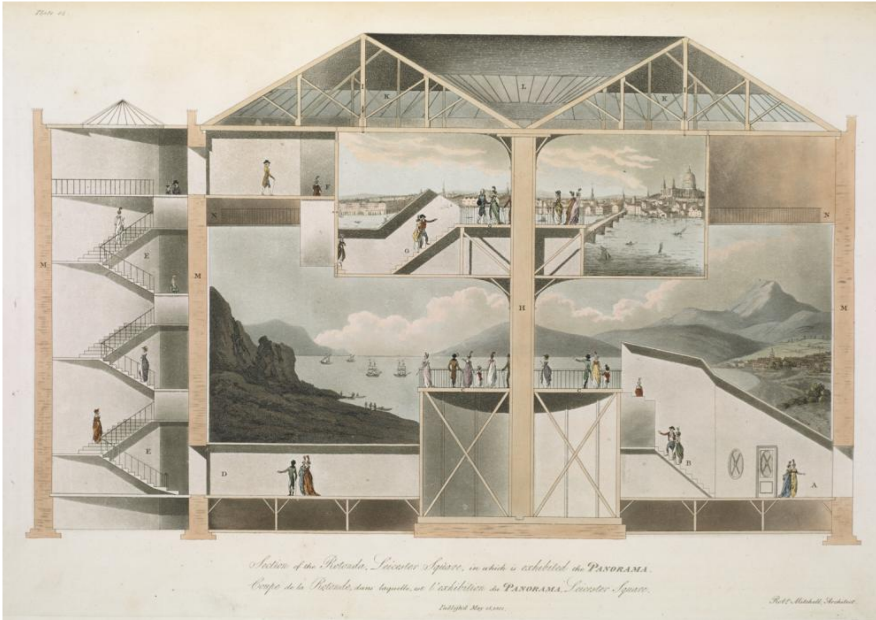


**Fig. 8** Cross-section of the Rotunda in Leicester Square, showing the viewing platforms for the main and secondary panoramas. A dark passage between the two cleared the viewing palate. The building was designed by Robert Mitchell to exhibit the panoramas of Robert Barker. Etching/Colored Aquatint by Robert Mitchell, 1801.Mitchell (1801a,b)

sense to ask if object $X$ is an anamorphosis of object $Y$ with relation to $O$. This is what is implied when in the usual dictionary definition of anamorphosis we say that the object "looks right" from a certain point. But this vagueness causes all sorts of misconceptions. The issue is not that it looks "correct" (meaning recognizable as "something") but that it looks like something that has be prescribed in advance. The question is how much can be prescribed, or to put it more rigorously, how much does a prescribed anamorph determine the set of all possible anamorphs of it with relation to given viewpoints.

## Multiple points of observation

We can prescribe the different appearances for the same object from two different points within some rather forgiving bounds. When one prescribes that $X$ be anamorphic to $Y_1$ from $O_1$, this determines a cone of rays from $O_1$ to $X$, but does not determine $X$ itself. Each point of $X$ on that cone may still be freely moved along its ray. That degree of freedom allows one to specify another object (under constraints) $Y_2$ such that $X$ will be anamorphic to $Y_2$ from another point $O_2$.

As beautifully demonstrated by the works of Kokichi Sugihara, the degrees of freedom of the conic projection allow quite some leeway. Sugihara wonderfully ex-

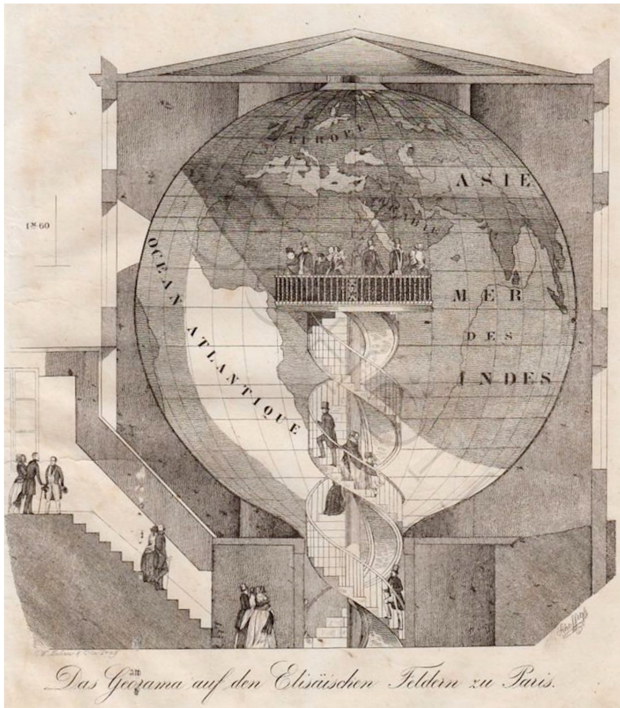


**Fig. 9** Illustration of Delangard's Georama.

plored such double anamorphic constructions with ambiguous 3D objects that from one point may look cylindrical and from another box-like or star-shaped (Sugihara 2015a,b, 2016), and not only the geometry of such ambiguous cylinders but even their topology may be changed, when for instance a pair of cylinders may appear to be intersecting or not according to the choice of observation point (Sugihara 2018).

## "Impossible" objects

It may seem paradoxical, but a real object may be anamorphic to an *impossible* one, that is, it may look like an object that makes sense locally, but not globally, like Penrose's triangle (Penrose and Penrose 1958; Draper 1978), M. C. Escher's Belvedere cube (Escher 1958, 1972), or Huffman's various specimens of wireframe constructions (Huffman 1968, 1971). This is achieved by by visually mimicking incompatible self-connections and other impossible features through hidden discontinuities or deformations of a real object (see Sánchez-Reyes and Chacón (2020) for a modern software-based treatment of this problem). Of course the apparent impossibility comes not so much from what we see as from what we assume, that is, not from the geometric constraints determined by the appearance of the object but from the psychological assumptions of how our brain makes sense of the limited visual information supplied by the view from a single point (Kulpa 1983; Sugihara 1982, 2000). Human visual processing relies on shortcuts and rules-of-thumb meant to work on the generic rather than the general case. For a simple example of an apparently impossible object, note that the rightmost object of Fig. 3 might be classified as one of Huffman's impossible polyhedra, but only if we assume that the implied faces are planar. The reader will verify that there is in fact no polyhedron whose faces are delimited by the edges shown. However, if we triangulate across the vertices, as in Fig. 10, then get a perfectly consistent polyhedron; only the number of planar faces is greater than the six originally assumed. Due to the ambiguity of radial occlusion, it is no wonder that even the simplest of wireframe objects, like Necker's cube (Necker 1832), should make our brains oscillate between two readings undecisively; the wonder is that we may make any reading at all with confidence from among the infinite anamorphs that are mathematically consistent.

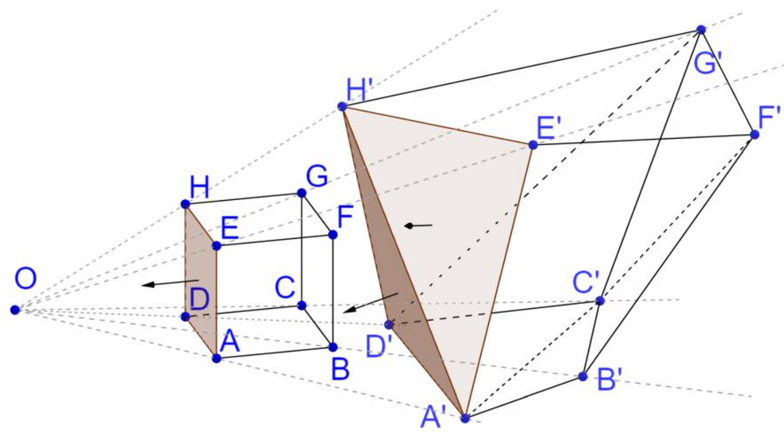


**Fig. 10** The rightmost object of figure Fig. 3 can be made into a consistent polyhedron by a choice of triangulation. Corresponding triangulations would result in the anamorphically equivalent edges in the cube and the rightside object but the faces themselves would be shaded differently under a light source. In a Lambert model of illumination, the value (gray level) of the two faces $A'H'D'$ and $A'E'H'$ will vary with the inner product of the faces normals to the incidence direction of the light. This could be compensated by painting the faces with different shades of gray so as to make them seem uniform when seen from $O$, achieving "color anamorphosis".

## On color

It is important to note that anamorphosis in our present sense refers only to mere equivalence of contours or outlines. This limited principle is necessary but not sufficient for proper anamorphoses in the usual sense, as most optical illusions going by that name - Pozzo's ceiling or Holbein's painting - require also mimesis of color to be effective. This aspect is usually unmentioned in the literature, as we assume that color will take care of itself once we use conical anamorphoses to define the contours of the objects we wish to represent in painting. This is true with careful diffuse lighting, but in a general setup color will certainly *not* take care of itself. Color anamorphoses (so to speak) is a whole new layer of complexity for mimesis, that we will not treat it here with any detail, but would like to mention briefly. Color anamorphosis would require consideration of models of illumination, material properties, and, of course, consideration of color itself as a perceptual theory with its own principles. Color has its own mimetic principle, its own "anamorphosis" of sorts. The space of visual color stimuli may be identified with the set of all power distributions in the visible spectrum, which, as a vector space of functions is of very high dimension (we could say infinite dimensional if we identify it with the set of all functions on the interval $[0,1]$, but there are restrictions so the space is probably better seen as very high - but finite - dimensional). This space is reduced by our perception to only a convex set in a space of three dimensions, identifiable (for instance) as something like value, saturation, and hue. This reduction in dimension is analogous to the way anamorphosis reduces the three dimensions of the real space to a two-dimensional perceptual space (though the reduction is much more drastic in color).

But all of this is beyond our scope. To consider the simplest of cases, take again the two anamorphic wireframe objects of Fig. 10. The situation is simple enough if we just consider the wireframe objects with ideal segments joining the points. If we consider a real object, then already we have some complications, as those lines will need some thickness to be visibly represented. We can naively conceive of them optically as painted matte black cylinders (with no specular reflections), thin, and featureless. If we now consider the polyhedron defined by a triangulation then both objects would just look like a black mass defined by their identical outlines. Let us now go a bit further and suppose the objects are made of triangles painted in some neutral gray. Suppose also they reflect light equally in every direction and with an intensity proportional to the inner product of the surface normal to the incident light ray direction. This is a simple Lambert model of illumination, adequate for rendering matte surfaces (Foley et al 1996).

Within such a model, the gray level of each face will be determined by its angle to the light source. Hence, mimicry of colour will require us to paint each face in such a way as to compensate the difference in their normal vectors. For instance, in Fig. 10, face $ADHE$ of the cube is anamorphic to the union of the two triangular faces $A'D'H'$ and $A'H'E'$. Notice the difference in the normal vectors on the faces. If the faces were to be painted equally, and assuming light comes from front and high, the face that points downwards would look darker than the cube's face and the face that points upward would look brighter. So to compensate for this and achieve apparent equality of value ("color anamorphosis") we would have to paint the upward facing triangle darker and the downward one lighter than the square face.

In Fig. 11 we see a simple hand-drawn example of color anamorphosis: a plane anamorph of a cube, side by side with a real cube of the same apparent dimensions. The physical cube is painted uniformly with white paint. The plane anamorph's normal vector coincides with the normal vector of the top face, so these corresponding faces can be painted with the same value. But the other "faces" of the plane anamorph had to be painted darker in order to match the appearance of the faces of the 3D cube that are turned at a larger angle from the incident light.

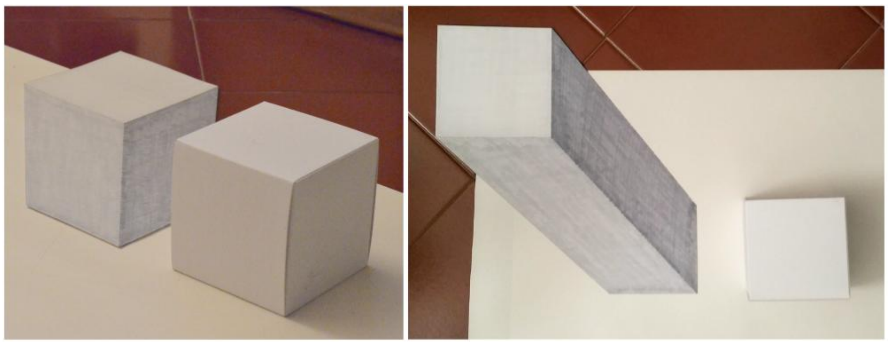

**Fig. 11** A 3D cube and a plane anamorph.

The rules of this color anamorphosis game depend on many factors, and will vary with the type of light, its form, its location, the type of reflection we get from the materials, and so on. In most cases it will be impossible to match apparent color by simple changing the local color, as the ranges will be too large. Lambert reflection and a uniform light is just about the easiest situation for matching. Diffuse lighting helps, as cast shadows will create additional complications (or artistic opportunities).

If we were to attempt a formal treatment of color anamorphosis we would start with something akin to the following definition: Two objects $X_1$ and $X_2$ are color-anamorphic relative to point $O$ and light-sources $L_1, L_2$ if $X_1$ seen from $O$ under light source $L_1$ is visually indistinguishable from $X_2$ seen from point $O$ under light source $L2$. Hence we have a dependence on the light sources quite analogous to the dependence on $O$ in conical anamorphosis.

Such a concept allows us to speak, for instance, of a ball under an incandescent bulb being color-anamorphic to a disc under a halogen lamp, the disc being painted so as to simulate the color gradations of the ball. We will not, however, attempt here any formal development of this, but it the concept is intuitively what any painter does when he paints a daylight scene to be seen under gallery lights, and, more analytically, what a restoration expert has in mind when making sure that he uses not just the right color but the right pigment to restore a patch of color, so as to ensure color matching under change of light source spectrum.

We can sum up the observations of this section by saying (unpardonably boldy and omitting many qualifiers): anamorphosis is what makes drawing possible; color anamorphosis is what makes painting possible.

**Binocular anamorphoses**

The abstract eye of anamorphosis is cyclopic, yet we can still easily work with binocular vision. Anamorphosis acts before the light hits the eye - it is a manipulation not so much of the eye, but of the light that reaches it. For this reason, arguments too concerned with what happens at the retina or the brain are missing the point. Whatever the complex workings of the visual system, feeding it equivalent inputs will result in equal perceptions. Hence, we don't need to understand how the brain combines binocular images; we just need to fool each eye separately, by figuring out what light packets it would be receiving from its position if the imaginary object was present. Then the brain will take care of the mixing. The problem is reduced to the technical one of showing each eye its own anamorph, built in the standard way, tailored to its specific position. The biggest problem is a suppressive one: to make sure that each eye *only* gets *only* its corresponding picture. This can be achieved simply (but with limitations) by using a physical wall (septum) to constrain each eye to its own compartment. This simple method, used by Dutour (1760) in the 18th century, is still an element used in contemporary solutions. Wheatstone (1838, 1852) achieved the separation through his mirror stereoscope in 1832 (Wade and Ono 2012). From that idea comes a long line of evolutions and simplifications, through the Holmes stereoscope in the 1860s and then the view-master in the late 1930s. These are not very different from modern VR viewers like Google Cardboard, except that today the illusions can move. All of these use in some measure either diffraction or reflection, so strictly speaking they are not working according to conical anamorphosis (although the distinction is superficial). A more strictly conical device is the anaglyph. Perspective anaglyphs were first published by Vuibert (1912) in a book with drawings by Henri Richard (Cabezos Bernal 2015). We are all familiar with the red-blue or red-cyan 3D anaglyphic glasses. The filter over each eye eliminates the opposite image, therefore separating the two anamorphoses, resulting in pictures that pop out of their frames. Doing the same procedure with an "oblique anamorphosis" instead of a "perspective" creates a far more startling result (Araújo 2017a; Cabezos Bernal 2015). For instance, Fig. 12, seen from the correct point, at a sharp angle to the page, will look like a cube popping out of the page, and young students get surprisingly amazed - in spite, or perhaps because of their familiarity of computer graphics - to see what resembles a wireframe cube floating midair and wobbling as they move their head slightly. Moving the head in a vertical over the obervation point gives the illusion of a bar chart animation, with a parallelepiped growing before one's eyes. We will not remark further upon binocular vision, as it can be handled by this adaptation of monocular considerations. The reader can refer to Cabezos Bernal (2015) for more on this matter.

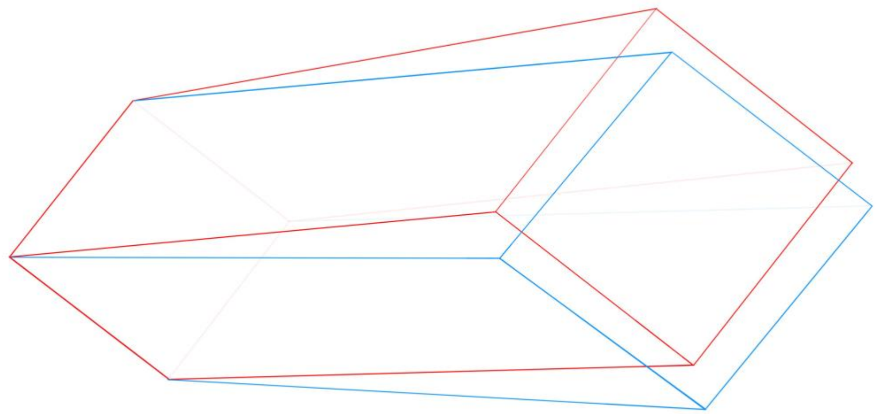

**Fig. 12** Anamorphic anaglyph of a cube, to be viewed with red-blue 3D glasses.

## Anamorphosis formally reformed

### *Mathematical preliminaries*

We now proceed to the mathematical development of what we loosely described above.

We need some geometric background. We will model our three-dimensional world abstractly as an euclidean three-dimensional space, denoted $\mathscr{E}$. You can think of $\mathscr{E}$ as $\mathbb{R}^3$. A point $O$ of $\mathscr{E}$ will represent the location of the observer's eye. Let $\mathscr{E}_O = \mathscr{E}_O \setminus \{O\}$ denote the three-dimensional space except for point $O$. We will want to model objects. We define an *object* to mean a closed set of $\mathscr{E}_O$, and a *scene* to mean a finite union of objects. The term "closed" is a topological term. We recall some terminology briefly for the mathematician and then explain it at length. We beg the artist for bravery at this point. Hold the line for these will all be made more concrete later.

**Definition 1.** Let $S$ be a subset of a topological space. We say that $P$ is a *limit point* of $S$ if every neighbourhood of $P$ contains a point $Q$ of $S$ other than $P$. The *closure* of $S$ is the union of $S$ with its limit points, and is denoted by $cl(S)$. $S$ is *closed* if $S = cl(S)$. The *residue* of $S$ is the closure of $S$ minus the set itself, denoted $Res(S) = cl(S) \setminus S$.

Intuitively: a limit point of a set is a point that can be approached indefinitely close without leaving the set. Closed sets are those that contain all their limit points. A point, a segment, a line, a plane, a circle, a sphere, are all closed sets. For a counterexample, take the set $S = \overline{AB} \setminus \{A, B\}$, a segment minus its endpoints. Both $A$ and $B$ can still be approached indefinitely close without leaving $S$, so they are limit points of $S$. But they are not in $S$, hence $S$ is not closed.

We will find ahead sets that naturally will be missing limit points. For various reasons we do not like that, so we will add them in. Mathematicians have a fancy name for adding in those missing points - they call it *taking the closure* . The *closure* of $S$ is the smallest closed set that contains it. So, going back to our example of the segment without endpoints $S$, the closure of $S$, denoted by $cl(S)$ is the full segment $\overline{AB}$, which contains all its limit points and is therefore a closed set. Finally, we call residue of $S$ to the set of limit points that we have to add to a set $S$ in order to close it. So, $Res(S) = cl(S) \setminus S$. In our example, the residue was the set of the missing endpoints of the segment, $\{A, B\}$.

We cannot really speak about closed sets, compacts, and so on, without having a topology. A topology on a set is a specification of which subsets are open (open sets are the complement of closed sets). In what follows we will be interested in studying the set of visual rays from the eye $O$. We will see how to put a topology on the set of these rays. First let us define what a ray is.

We say that a set $S$ is *convex* if for any two points $A$ and $B$ of $S$, the segment $\overline{AB}$ is in $S$.

A point $O$ on a line $l$ divides $l \setminus \{O\}$ into two disjoint convex sets $l_1, l_2$. We say that $l_1$ and $l_2$ are each a ray over the line $l$, with origin (or vertex) in $O$. We interpret rays as directed half-lines going from $O$ to infinity (in the case of $l_1$ and $l_2$, going in opposite directions from $O$). Note that rays are missing the point of origin $O$. Vectorially, if $\overrightarrow{v}$ is a vector, a ray is the set $O + a\overrightarrow{v}, a \in \mathbb{R}^+$. For $P \neq O$, $\overrightarrow{OP}$ denotes the ray with origin in $O$ that passes through $P$. It is the same notation as for a vector, but we will allow context to disambiguate. Note that the usual definition of ray includes the origin point; ours doesn't because we don't want different rays to intersect each other. So a ray with origin in $O$ is contained in $\mathscr{E}_O$.

Let $\mathscr{R}_O$ denote the set of all rays from $O$. Let $S^2_O$ denote the unit sphere with center $O$. There is a canonical isomorphism between the set of rays and the sphere: we identify each ray with the point where it intersects the sphere. This bijection endows the set of rays with the topology of the sphere, giving it a notion of closed sets, limit points and so on. We will therefore freely refer to rays or points on the sphere interchangeably.

## *Anamorphosis as a mathematical object*

The principle of radial occlusion defined above expresses a notion of equivalence between points with regard to a reference point $O$, the eye of the observer. When the principle is valid, two points that are geometrically aligned along a ray from the eye are equivalent as far as visual perception goes. It follows that two objects should be equivalent if all their points are aligned, or, equivalently, if they generate the same set of visual rays from $O$. We express this set as follows:

**Definition 2.** We say that the visual cone of an object $\Sigma$ relative to $O$ is the set of rays subtended by $\Sigma$ with origin in $O$, which we denote $C_O(\Sigma)$. Hence $C_O(\Sigma) = \{\overrightarrow{OP}, P \in \Sigma\}$.

Now, from radial occlusion, it is natural to define that two sets "look the same" if they have the same visual cone. Later we will show that a somewhat looser equivalence is more interesting, which we will call *anamorphosis* . For now, for the purpose of motivation, we will concentrate on the stricter notion of conical equivalence.

**Definition 3.** We say that two objects $\Sigma_1$ and $\Sigma_2$ are *conically equivalent* relative to $O$ if they have the same visual cone relative to $O$. We write this as $\Sigma_1 \overset{O}{=} \Sigma_2$.

It is clear that conical equivalence is indeed an equivalence relation between 3D objects, meaning a relation that is reflexive, symmetric and transitive. We are interested in studying the classes of equivalence by this relation, and how to construct the members of these classes.

**Proposition 1.** *Let $P,Q$ be points of $\mathscr{E}_O$. $P \overset{O}{=} Q$ if and only if $P$ and $Q$ are on the same ray from $O$.*

Proposition 1 shows that the principle of radial equivalence for pairs of points, is a particular case of definition 3. This trivially follows from the fact that the cone of a point $P$ is the ray $\overrightarrow{P}$.

**Proposition 2.** *if $P,Q,P',Q'$ are points of $\mathscr{E}_O$ such that $P' \overset{O}{=} P$ and $Q' \overset{O}{=} Q$ then $\overline{PQ} \overset{O}{=} \overline{P'Q'}$.*

*Proof.* By proposition 1, triangles $POQ$ and $P'OQ'$ define the same angle with vertex at $O$. Each ray inside the angle intersects sides $\overline{PQ}$ and $\overline{P'Q'}$ at a single point, which establishes a bijection between both the sides and the rays of the cone.

We can state this as: the cone of rays of a segment $AB$ is the angle $\angle AOB$.

*Remark 2.* The proof does not work for lines as the triangles do not capture the full cone of the line. In fact the proposition is false for lines.

**Corollary 1.** *If $\gamma$ is a continuous curve with parametrization $g : [0,1] \mapsto \angle AOB$ with $g(0) \overset{O}{=} A$ and $g(0) \overset{O}{=} B$ then $\gamma \overset{O}{=} \overline{AB}$.*

*Remark 3.* This is the case of curve $A''B''$ in Fig. 3.

**Proposition 3.** *if $\Delta PQR$ and $\Delta P'Q'R'$ are triangles with vertices in $\mathscr{E}_O$, such that $P' \overset{O}{=} P$, $Q' \overset{O}{=} Q$, and $R' \overset{O}{=} R$ then $\Delta PQR \overset{O}{=} \Delta P'Q'R'$.*

*Proof.* For the sides of the triangles the result follows from proposition 2. Let $I$ be an interior point of one of the triangles, say triangle $\Delta PQR$. Then the ray from $A$ through $I$ finds a point $J$ on $\overline{QR}$. Since $\overline{QR} \overset{O}{=} \overline{Q'R'}$, there is a point $J'$ on $\overline{Q'R'}$ such that $J' \overset{O}{=} J$.

As in the previous proposition, this does not generalize to the full plane defined by a triangle.

Note that segments are convex combinations of two points and triangles are convex combinations of three. The result is valid for general convex combinations.

As a corollary of the propositions above, if $\Sigma$ is a graph composed of vertices and segments (say, the cube of Fig. 3) we can obtain slide every vertex freely along its ray from $O$ and the graph obtained will have the same cone as the original graph. The same is true of the polyhedra obtained from the graph by triangulation (there are several choices of flat faces compatible with a graph). Moving the vertices will create a new polyhedron, equivalent to the first.

**Corollary 2.** *If $S$ is a continuous surface with boundary on a continuous closed curve anamorphic to triangle $\triangle ABC$ and $S$ is contained in the solid angle defined by $\triangle ABC$ then $S \stackrel{\Omega}{=} \triangle ABC$.*

*Remark 4.* When dealing in computer graphics, the propositions above can be used to obtain anamorphs very easily from a reference object. For instance it is very easy, using the Geogebra software (Hohenwarter et al 2013; Hohenwarter 2002), to slide points on their rays from $O$ to obtain anamorphic polyhedra as in Fig. 3, and even to deform these sides into simple curves. Sánchez-Reyes and Chacón (2016) do something analogous to obtain anamorphic deformations of a reference object using Bézier triangles and NURBS.

As we have seen, proposition 2 does not generalize to full lines. A line $l$ on a plane $H$ divides the plane into a disjoint union of convex sets $\pi = \pi_1 \cup l \cup \pi_2$, where $\pi_1$ and $\pi_2$ the half-planes on either side of $l$. The cone of $l$ is characterized by the following proposition.

**Proposition 4.** *Let $l$ be a line in $\mathscr{E}_O$. $O$ and $l$ define a plane $\pi_O(l)$. Let $l_O$ be the line parallel to $l$ passing through $O$. $l_O$ divides $\pi_O(l)$ in two half-planes. Then $C_O(l)$ is the set of rays through the half-plane of $\pi_O(l)$ that contains $l$.*

We remark that although $l$ is closed in $\mathscr{E}$, its cone is not closed in $\mathscr{R}_O$. This will be important ahead. There are two rays missing to make the cone closed, these being the two diametrically opposite rays whose union is $l_O$. These mark the limits of the cone of $l$ but do not belong to it. Intuitively, the line of sight approaches parallelism to $l$ more and more as it follows it to infinity, but never reaches it. The reader probably recognizes line $l_O$ as the line that in classical perspective defines the vanishing point of $l$ by intersection with the plane of projection. Here we have no plane of projection, but this line is still important:

**Proposition 5.** *Let $l$ and $l'$ be lines in $\mathscr{E}_O$. Then $l \stackrel{\Omega}{=} l'$ if and only if $l$ and $l'$ define the same plane with $O$ and $l_O = l'_O$.*

In linear perspective line $l_O$ is used to construct the projection of $l$. The construction is made by joining the intersection of $l$ and $l_O$ with the projection plane. This is a fundamental theorem of linear perspective (Andersen 1992, page 12), but it cannot even be stated for anamorphoses since we have no projection plane. Further we will see ahead that a general projection surface will not intersect $l$ anyway. But we have the following analogous construction that uses $l_O$ and $l$ to obtain an arc of circle that is related to $l$ in a canonical way.

**Proposition 6.** *There is a single circle $\mathscr{C}$ of center O that is tangent to l. Let $\mathscr{C}_O(l)$ be the intersection of $\mathscr{C}$ with the half-plane of $\pi_O(l)$ on the same side of $l_O$ as l. Then $\mathscr{C}_O(l)$ and l define the same cone of rays from O.*

*Proof.* The perpendicular to $l$ from $O$ intersects $l$ at the point $I$ which is minimizes the distance from $l$. Hence the circle through $I$ of center $O$ will be tangent to $l$. This circle intersect $l_O$ at two points $V$ and $V^\star$ diametrically opposite on $l_O$. Consider the half-cicle obtained from $VIV^\star$ by excluding its endpoints $V$ and $V^\star$. There is a one-to-one correspondence between points on this arc and points on the cone $C_O(l)$, obtained by intersection of the arc with $\overrightarrow{OP}$, for each $P$ in $l$.

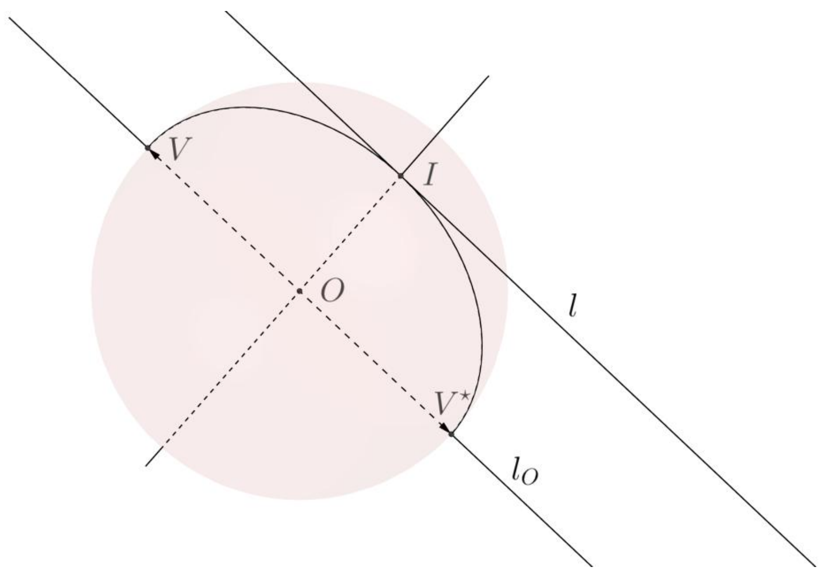


**Fig. 13** The canonical meridian representative of a line's visual cone.

This is nice result at first sight. There is a canonical arc of circle that has the same cone as the line. But there is a problem. The reader may have noticed that we didn't say that these were equivalent objects. That is false both for the arc with the endpoints and for the arc without them. The arc without the endpoints has the same cone of rays as the line, but it is *not* a closed set, hence it is not what we defined as an object. The arc with the endpoints is closed, but it is not conically equivalent to the line, as the rays corresponding to the endpoints are not a part of the line's cone.

We could solve this in several ways, for instance by changing the definition of object, but the most satisfying seems to be to change our notion of equivalence. Hence we define anamorphic equivalence by slightly relaxing conical equivalence:

**Definition 4.** We say that two objects $\Sigma_1$ and $\Sigma_2$ are *equivalent by conical anamorphosis* relative to $O$ their visual cones are equal up to topological closure, that is, if $cl(C_O(\Sigma_1)) = cl(C_O(\Sigma_2))$. We write this as $\Sigma_1 \mathrel{\mathcal{Q}} \Sigma_2$.

Some nomenclature: ”equivalent by conical anamorphosis relative to $O$” is a mouthful, so for short we’ll just say that two objects are *anamorphic*, or that they are *anamorphs* (of each other), or that one is an anamorphosis of the other. This definition fulfills well both the common usage of these expressions and our technical needs. These concessions to brevity and custom should not make us forget that any anamorphic equivalence is always relative to a specific observation point, so each of these phrases should be mentally ended with “relative to $O$”.

**Proposition 7.** *If $\Sigma_1, \Sigma_2$ are objects such that $\Sigma_1 \overset{O}{=} \Sigma_2$, then $\Sigma_1 \overset{O}{\sim} \Sigma_2$.*

Obviously, conic equivalence implies anamorphic equivalence. Hence the results we obtained for conical equivalence above are still valid for anamorphosis.

Now let us reconsider the canonical semicircle we defined above. If we add to it its endpoints, by taking its closure, it becomes an anamorph to the line $l$. These two points are very important. They are the intersection of $l_O$ with the sphere, hence they are the same for every line parallel to $l$. Hence they will be meeting points for all lines parallel to $l$. The two rays of $l_O$ generalize the notion of vanishing points from classical perspective.

We can generalize the construction we did for the line, so as to get a canonical anamorph of any object, as well as a vanishing set:

**Definition 5.** We say that the *anamorphosis* of an object $\Sigma$ relative to $O$ is $\Lambda_O(\Sigma) = cl(C_O(\Sigma))$. We identify it with its projection onto the unit sphere, $cl(C_O(\Sigma)) \cap S^2_O$, when context is clear. The latter we also call the *canonical anamorph* of $\Sigma$.

It follows trivially from 4 that an object is anamorphic to its anamorphosis thus defined. Its spherical anamorph is called canonical because it the most natural of its 2D anamorphs, arising from the identification of the space of rays with the sphere.

**Definition 6.** We say that $\mathscr{V}_O(\Sigma) = Res(C_O(\Sigma)) = cl(C_O(\Sigma)) \setminus C_O(\Sigma)$ is the *vanishing set* of object $\Sigma$ relative to $O$. We call *vanishing* points to the points of the vanishing set. We identify the rays of the vanishing set with the corresponding points on the unit sphere $S^2_O$.

The vanishing set is the residue of the visual cone of the object, that is, the points you must add to make it closed. This definition has the advantage that it works for any object, and, when restricted to lines and planes, reduces to what you would expect from the classical definition, as we will see.

*Remark 5.* When teaching the concept of vanishing points to artists it is hard to deal with topological concepts explicitly, especially on a short timeframe, such as a workshop. Then it is more practical to generalize the tradition of perspective started by Taylor and simply define vanishing points of lines as the points associated with the two opposite rays that make up $l_O$, and then notice that these are at the end of the canonical semicircle; then it follows from the definition that parallel lines have the same vanishing points and that these are meeting points for those lines.

**Proposition 8.** *The canonical anamorph of a line $l \subset \mathscr{E}_O$ is a meridian of the unit sphere. The vanishing set of $l$ is a pair of antipodal points located at the ends of the meridian, where $l_O$, the parallel to $l$ through $O$ intersects the sphere.*

This proposition turns perspective into spherical geometry, so we can apply the tools of that discipline. See Catalano (1986) and Araújo (2018c). In particular we will be talking a lot about antipodal points, so let us recall that two points are called *antipodal* if they are diametrically opposite on a sphere. Given a point $P$ on the sphere we denote its antipodal point by $P^\star$. The notion extends trivially to the corresponding rays through the center of the sphere.

This proposition has the obvious corollary that not only do parallel lines have the same vanishing points, but these are actual meeting points of their anamorphosis. This is a very concrete alternative to the abstract notion of parallel lines meeting at infinity - their spherical anamorphs actually meet on the sphere.

**Corollary 3.** *If $l$ and $l'$ are parallel lines, their canonical anamorphs intersect at their vanishing points.*

This is a beautifully unifying view of vanishing points, unlike the classical perspective view where a line has sometimes one and sometimes no vanishing points, or even the (hemi)spherical perspective of Barre and Flocon (Barre and Flocon 1968) where lines have either one or two vanishing points. This search for constancy in the number of vanishing points can be compared to the insistence of mathematicians in redefining numbers to ensure that an $n$-degree equation always has exactly $n$ roots. The projective plane view of perspective answers the same desire for symmetry, but there we have always a single vanishing point. Our present notion of anamorphosis (and later of perspective) splits that point in two as we want no preferential axis and wish to capture the whole visual environment.

Now we consider the matter of construction of these anamorphs. Since the canonical anamorph of a line is an arc of circle, we might expect it to require three points for its construction. But since it is a meridian, and the two vanishing points are antipodal, two points are enough.

Recall that two points on a sphere define a plane through its center. The intersection of that plane with the sphere is a circle with the same radius as the sphere, and to this is called a *great circle* of the sphere, or a *geodesic*. A meridian is a connected half of a great circle, so it can be specified by choosing two two antipodal points and then picking one of the two possible halves of the geodesic. This picking of a meridian can be done by choosing a point in its interior. But since the two endpoints are mutually antipodal (hence knowing one is enough) all we really need to know to determine a meridian is two points of it, as long as one of them is specified to be an endpoint. For the same reason, only two points are needed to determine the image of a line, as long as one of them is specified to be a vanishing point.

**Proposition 9.** *The anamorphosis of a line $l$ of $\mathscr{E}_O$ is determined by one vanishing point and by the projection on the sphere of one point of $l$.*

*Proof.* The vanishing point $V$ and the measured point $P'$ projected from $l$ onto the sphere determine a plane through $O$, hence a geodesic. On this geodesic, the line image will be the meridian $VP'V^{\star}$.

Note that if the line $l$ crosses $O$ then its image is just a set of two points, these being located at the place of the vanishing points, which in this case are actual points of the projection. It is the line itself that vanishes. Taylor remarked on much the same situation in classical perspective, and that is one of the justifications for the terms "vanishing point", as the line "vanishes" into it in this degenerate case. There is a technical difference in our definitions here, since by definition 6, these two points are not in our vanishing set at all, while in classical perspective the vanishing point exists and coincides with the single projected point.

*Example 1.* In Figures 14 and 15 we see two examples of pairs of spatial lines $l$ and $j$ relating to their vanishing points and to their spherical anamorphs.

In Fig. 14 we see two parallel lines. They have therefore the same translation to the origin, so $l_O \equiv j_O$, hence they have the same pair of antipodal vanishing points, $V$ and $V^{\star}$. Their canonical anamorphs on the sphere are meridians ending at these vanishing points. Hence the vanishing points are actual meeting points for the canonical anamorphs.

As for construction of these spherical anamorphs, two projected points $P$ and $Q$ from $l$ and $j$ respectively define each meridian, as in proposition 9. These points may be measured anywhere, but one way would be at the intersection of the meridian with the geodesic plane orthogonal to $l_O$.

In Fig. 15 we see two perpendicular lines. Now lines $l_O$ and $j_O$ are themselves orthogonal, crossing at $O$, and we get two pairs of vanishing points. The meridians are obtained from these by measuring a single additional point, the common point where the two lines cross. Hence we see that only vanishing point $V_1$ and point $P$ need be measured to obtain the whole construction.

In Fig. 16 we see the spherical anamorph of a cube. The cube defines a set of six distinct vanishing points, which are meeting points for the lines that prolong the edges of the cube..

**Proposition 10.** *Let $\sigma$ be a plane not containing O. Let $\sigma_O$ be the plane parallel to $\sigma$ through O. Then $\sigma_O$ defines a great circle of the sphere, which is the vanishing set of $\sigma$ (Fig 17). This great circle divides the sphere in two hemispheres. The anamorphosis of $\sigma$ is the hemisphere on the side of $\sigma_O$ that contains $\sigma$.*

*Proof.* The construction is analogous to that of the canonical circle. Take a perpendicular from $O$ to $\sigma$, and this defines a sphere tangent to $\sigma$, half of which will be the image of the visual cone of $\sigma$. To get the anamorphosis scale it down to unit radius.

The vanishing points of lines and planes are quite analogous to the linear perspective case, in that they are obtained by translation to the observation point followed by intersection with the projection surface. But our definition 6 applies also to general objects that cannot be treated in this fashion, such as the following example.

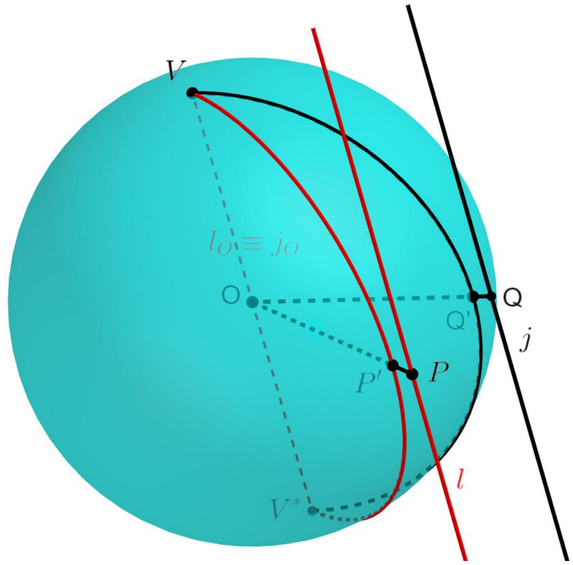


**Fig. 14** Anamorphosis onto the sphere of two parallel lines. The lines project as meridians that meet at two common vanishing points $V$ and $V^{\star}$, antipodal to each other.

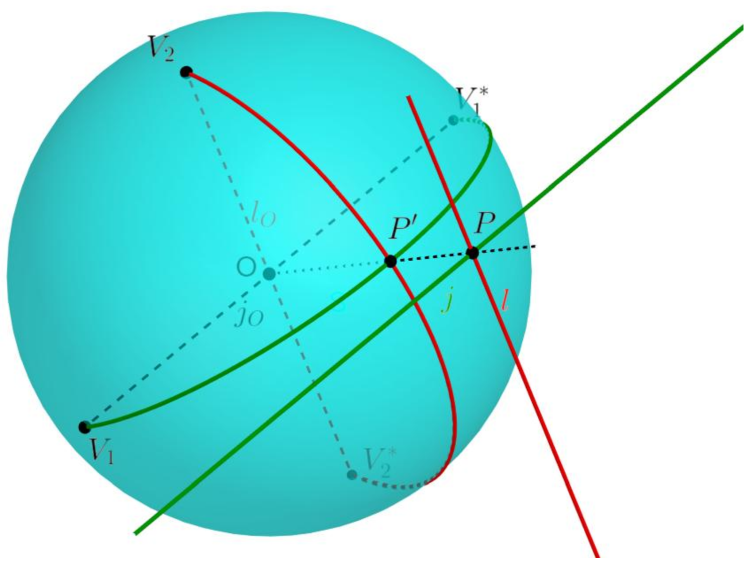


**Fig. 15** Two lines $l$ and $j$ intersecting at a right angle. Their canonical anamorphs on the sphere go to pairs of vanishing points distributed at a regular separation of 90 degrees along the geodesic defined by the orthogonal lines $l_O$ and $j_O$.

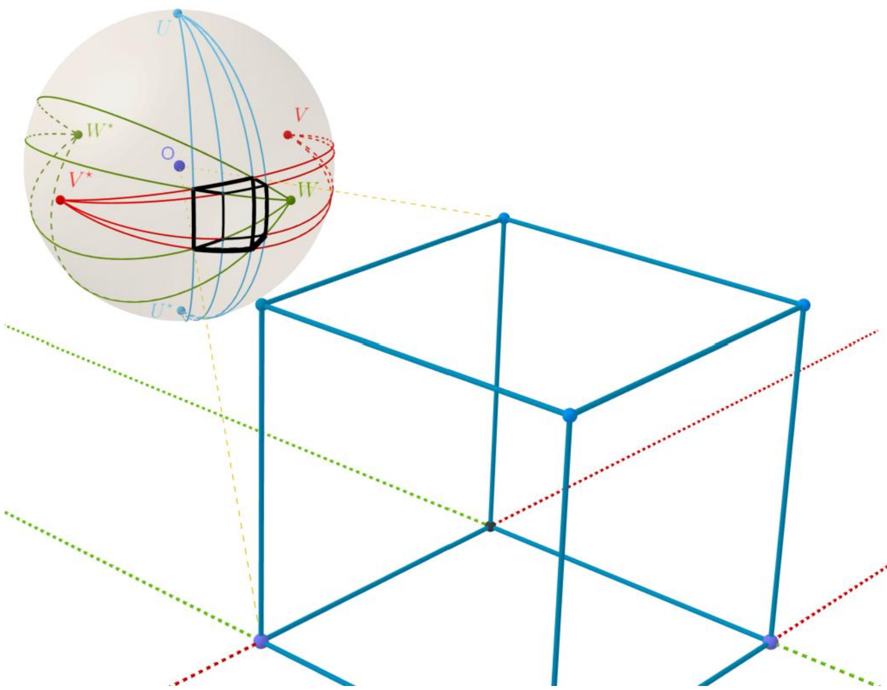


**Fig. 16** Anamorph of a cube on a sphere. The cube defines a set of six distinct vanishing points, two for each set of parallel edges.

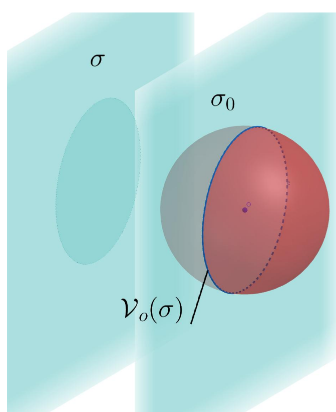


**Fig. 17** A plane $\sigma$ and its vanishing set $\mathscr{V}_O(\sigma)$ on the unit sphere.

*Example 2.* Let $l$ be the curve $t \mapsto (t, t^2, 0)$ and let $O = (0, 1, 0)$. Then the cone of $l$ will the great circle defined by $z = 0$, minus point $(0, 0, 1)$. But the closure of this set on the sphere is the whole of the great circle. So the vanishing set is $(0, 0, 1)$ and the anamorphosis is the great circle through $z = 0$.

*Example 3.* Let $S$ be the surface $z = x^2 + y^2$ and let $O = (0, 0, 1)$. Then the cone of $S$ will the whole sphere, minus the point $(0, 0, 1)$. But the closure of this set on the sphere is the whole of the sphere, so the vanishing set is $(0, 0, 1)$ and the anamorphosis is the sphere itself.

This is a different perspective from the more traditional notion of "following the line to infinity". Although the vanishing set is the same, we can look for the vanishing points in the anamorphic projection itself, by looking for "missing points", or, if you will, at "vanished" points. This is also a better definition when we consider sets whose limit rays cannot be obtained by simple curve parametrizations like in our second example above.

We would now like to define an anamorphosis onto an arbitrary surface, and its vanishing points.

In what follows we will define surface to mean a manifold of dimension two, that is, something that locally maps to a region of a plane. These will usually be manifolds with boundary, such as a sheet of paper, or a half-sphere. Most of the time they will be smooth, but we only require them to be topological manifolds. Most of the time they will be connected, but not always.

We use the canonical anamorphosis onto the sphere as the blueprint from which all others are derived as projected images, usually partial ones. The idea is simple: the anamorphosis of an object onto a surface is just the closure of the conical projection of the object onto the surface. The vanishing points on the surface will be the subset of the canonical vanishing points that happen to fall on the surface. Technically, it takes some care not to count points multiple times if the surface has a complex shape that folds over itself or has several connected components.

To avoid these problems, and because that work is centered on spherical perspectives, Araújo (2018c) defines the anamorphosis only for compact starred surfaces, which are just radial deformations of the sphere. These are quite enough for most purposes, so we start with them and then discuss possible generalizations briefly.

**Definition 7.** We say that a compact surface $S$ is starred relative to $O$, or that it is an $O$-starred surface, if every ray from $O$ touches $S$ at most once. A compact surface is said to be locally $O$-starred at $P \in S$ if there is a neighbourhood $B$ of $P$ such that every ray through $B$ intersects $S \cap B$ at most once.

You can think of an $O$-starred surface as being locally defined by $f(u) = h(u)u, u \in S^2_O$ where $h : U \subset S^2_O \to \mathbb{R}^+$ on a region $U$ of the sphere $S^2_O$. Intuitively, $h(u)$ represent a ratio that pushes each point of the sphere closer or further away from the center $O$.

We can define anamorphosis onto an $O$-starred surface $S$ as follows.

**Definition 8.** The anamorphosis relative to $O$ of an object $\Sigma$ onto an $O$-starred surface $S$ is $\Lambda_{O,S}(\Sigma) = cl(C_O(\Sigma) \cap S)$. The vanishing set of $\Sigma$ on $S$ is $\mathscr{V}_{O,S}(\Sigma) = \Lambda_{O,S}(\Sigma) \setminus (C_O(\Sigma) \cap S)$.

Since the conical projection is a continuous map onto an $O$-starred surface, we have the following result.

**Proposition 11.** *Let $S$ be a locally starred surface at $V$. Then $V$ is a meeting point for the anamorphic images of lines with vanishing point $\overrightarrow{OV} \in \mathscr{R}_O$.*

When $S$ is a starred surface, we can express these objects very simply as intersections.

**Proposition 12.** *Let $S$ be an $O$-starred surface. Then $\Lambda_{O,S}(\Sigma) = \Lambda_O(\Sigma) \cap S$, and $\mathscr{V}_{O,S}(\Sigma) = \mathscr{V}_{O,S}(\Sigma) \cap S$.*

That is, the anamorphosis (resp. vanishing set) of $\Sigma$ on $S$ are just the intersection of the rays of the anamorphosis (resp. vanishing set) with $S$. When considering the actual construction of an anamorphosis on a surface, we can construct one anamorph - the simplest one we can find - and then project it onto the required surface. In particular, the canonical anamorphosis onto the unit sphere is a good candidate as a prototype for others, since it is so symmetrical. So it is natural to solve the anamorphosis there, and then project it conically onto the required surface, using the fact that $\Lambda_{S,O}(\Sigma) \overset{\varrho}{\sim} C_{0,S}(\Lambda_{S^2,O}(\Sigma))$.

For this reason, we are compelled to study the projections of lines and planes on the sphere with special interest, as well as its projections onto various kinds of surfaces. Lines, as we have discussed, project as meridians onto the sphere, each meridian ending at two antipodal vanishing points. We will now consider how these meridians in turn project onto various surfaces.

*Example 4.* Consider the anamorphosis of a line $l$ onto a compact region $S$ of a plane $\pi \in \mathscr{E}_O$: this can be identified with (a compact subset of) linear perspective or with so-called oblique anamorphosis. Let $\pi_O$ be the plane through $O$ such that $\pi_O \parallel \pi$. $l$ projects on the sphere as a meridian with vanishing set $\{V, V^\star\}$. Suppose the vanishing set is on $\pi_O$. Then if $l$ is on the side of $\pi$, it projects onto a line; if on the other side, it projects as the empty set. In either case the rays corresponding to the vanishing points do not intersect $\pi$, so the line has no vanishing points on $\pi$. If the vanishing points are not on $\pi_O$, then the line intersect the plane $\pi$ at a point $I$ and one and only one of the vanishing points projects onto $\pi$ as a point $V_\pi$. Then the line projects into the ray with origin at $V_\pi$ that passes through $I$, and its vanishing point on $\pi$ is $V_\pi$. Since we assume $S$ to be a compact (hence bounded) subset of $\pi$, both the vanishing point and point $I$ (as well as an unbounded section of the line projection) may be actually outside the anamorphosis proper, but they can still be used for the construction.

*Remark 6.* We can contrast classical perspective (or plane anamorphosis) with spherical anamorphosis by saying that in the former a line projects either as a line or as a ray, while in the latter a line projects as a "segment" (i.e., a meridian ending at vanishing points).

*Example 5.* Suppose $S$ is a compact region of three intersecting planes disposed as the floor and walls in the corner of a room (figure 6). This is locally the same as a plane anamorphosis, but globally very different, as some lines may have two vanishing points. In rendering lines, we may consider the following strategy: in each plane plot two points of the line, thus finding two lines joining at the intersection of the projection planes. More efficiently, find only three points per line, one point on each plane and one on the intersection of the planes. But since two non-antipodal points define a geodesic, in fact two points anywhere on the planes must be enough to define the line across all projection planes. These constructions are maximally symmetric in the case of the anamorphosis onto a cube, which has been studied in (Araújo et al 2019b) as a special case of spherical perspective. Then the geodesic associated with a line $l$ projects as a set of six or of four segments on the cube, obtainable from any two points through simple descriptive geometry constructions.

*Example 6.* Let $S$ be a compact region of a cylinder, with $O$ outside the cylinder (figures 18 and 7). This is a good example of how we can make do with only starred surfaces. Although the cylinder is not starred from $O$ we can consider as projection surface only the proximal region of the cylinder, which is visible from $O$ and exclude the distal, occluded part, thus obtaining a starred subset. A generic non-vertical line will define a plane that cuts the cylinder in an ellipse. A line will project as an arc of an ellipse, with at most one vanishing point. A large class of lines will have no vanishing points at all on the cylinder, as $l_O$ will not intersect it. If the cylinder were to be cut at a vertical line and opened isometrically (unrolled) the ellipses would transform into segments of sinusoidal curves. See Apostol and Mnatsakanian (2007).

*Example 7.* Let $S$ again be a cylinder as in the previous example, but now $O$ is on its axis (Fig. 19). Now generic geodesics will project onto ellipses that are half above and half below the horizontal plane through $O$. The line will project as an arc of its ellipse, ended by two diametrically opposed vanishing points. Lines on the horizontal plane are special cases, projecting as horizontal half-circles. Lines with vanishing points along the axis of the cylinder will project as verticals. All geodesics, projected as sinusoids in the opened cylinder, will share the same horizontal axis of symmetry, as the planes of their geodesics all intersect the axis of the cylinder at $O$. This was not true in the previous case, where the plane of a generic line could intersect the cylinder's axis at any point.

## More general surfaces

The following section is technically nitpicking that should be ignored by the artist (and probably by everyone on a first reading).

Although starred surfaces are quite enough for most purposes, and provide the most elegant theoretical results, they can hardly model the large variety of real surfaces on which artists have constructed anamorphoses. Often these can be rather

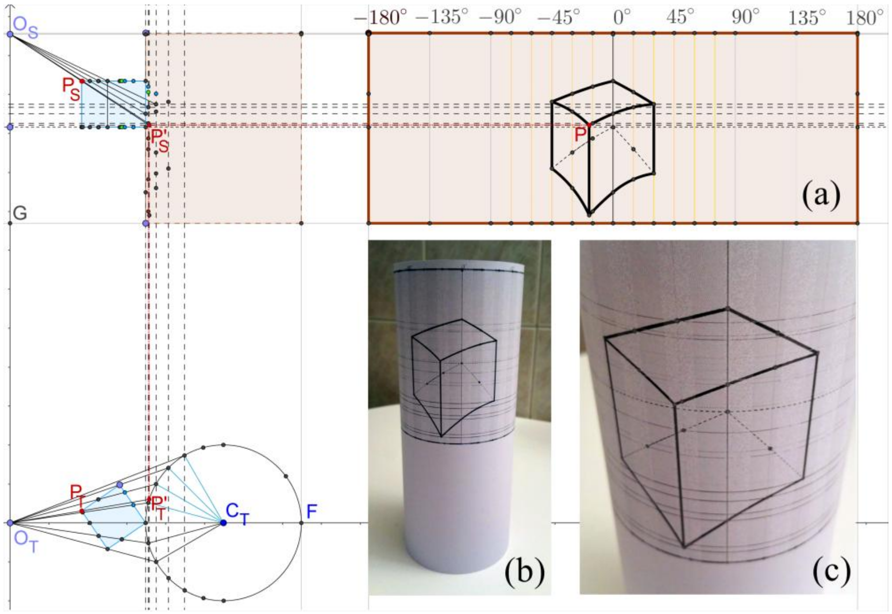


**Fig. 18** Descriptive geometry construction of the anamorphosis of a cube onto a cyilinder. Each point is found on the top view (Bottom left) then lifted to the side view (Upper Left) and found on the developed (cut and unrolled) cylinder by transforming the angular coordinates from the top view and transporting the heights from the side view (a). The rectangle can be cut and rolled up to obtain the cylindrical anamorphosis (b) which will be correctly observed from the point $O$ determined by the two projections $O_S$ and $O_T$. From this point the edges of the cube appear straight (c).

complex surfaces that fold over themselves in such a way as cut a ray from $O$ multiple times, hence they are non-starred. Often they will have several connected components, sometimes so many that they might be better modeled as a point cloud.

It is in fact quite up for debate what the most desirable definition would be for a general anamorphic projection surface. We will not aim at closing that discussion here but just at exploring it a little.

First of all, we note that we could certainly extend our definitions to include all sorts of surfaces, even non-compact surfaces such as the whole euclidean plane (in that case the formalism under consideration just becomes that of linear perspective). We would lose some of the elegance but things still work. It is interesting to consider what happens when we remain with compacts but relax conditions just a little.

One way to deal with non-starred surfaces is by cropping or changing the surfaces into their minimal starred equivalents by considering only the subset that is visible from $O$. We did this informally with the cylinder in example 6. We can formalize this procedure by generalizing conical projection to take occlusion into account, in the following way: given a point $P$ and a surface $S$, the *proximal conical projection* sends $P$ to the intersection with $S$ which is closest to $O$ (the reasoning is that the distal ones are occluded).

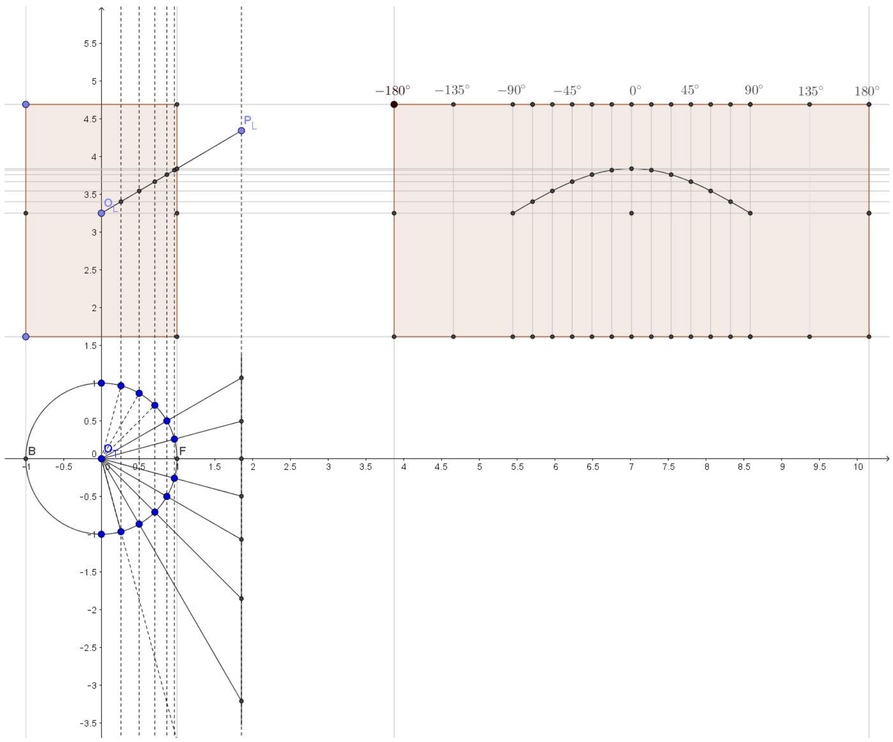


**Fig. 19** Descriptive geometry construction of the anamorphosis of a line onto a cylinder with $O$ on the axis. The image of the line on the unrolled cylinder is one half of a sinusoid whose axis of symmetry is the horizon line, at the height of $O$.

**Definition 9.** The proximal conical projection from $O$ to a surface $S \subset \mathscr{E}_O$ is the map from $\mathscr{R}_O$ to $S$ defined by $\varphi_{O,S}(\vec{r}) = \{P \in \vec{r} \cap S : \forall Q \in \vec{r} \cap S, |OQ| \geq |OP|\}$

*Remark 7.* When the surface is $O$-starred, the proximal conical map reduces to the ordinary conical projection $\vec{r} \mapsto \vec{r} \cap S$. We identify the map from $\mathscr{R}_O$ with the corresponding map from $\mathscr{E}_O$ defined by $P \mapsto \overrightarrow{OP} \cap S$, that maps each spatial point to its conical projection on the surface.

Identifying point of $S$ with their rays, we can use the proximal conical projection map to define the minimal starred equivalent of $S$ itself, i.e., the smallest locally starred surface that has the same anamorphosis as $S$. In fact $\varphi_{O,S}(S)$ has no double points, preserving only the *proximal* (closest to $O$) intersection of each ray going through a point of $S$. But since we insist on compact sets, we define the minimal $O$-starred equivalent of $S$ to be $S^\star_O = cl(\varphi_{O,S}(S))$. Because of the possible existence of multiple connected components, this surface will generally have a non-empty set of repeated intersections, although one of measure zero. It will be locally starred *almost everywhere* , i.e. , there will be a set $\delta$ of measure zero such that $S^\star_O$ is locally starred in $S^\star_O \setminus \delta$. We say that a surface with this property is *quasi-starred* .

*Example 8.* The object of Fig. 20, defined by the union of rectangles $A$, $B$, $C$, and $D$ is not $O$-starred. Taking the proximal projection $\varphi_{O,S}(S)$ eliminates sections $B$ and $C$, which are occluded by $A$, leaving two connected components $A$ and $B$, minus the segment $h_2$ of $B$, which is occluded by segment $h_1$ of $A$. This segment is recovered by taking the closure, so that $S^{\star}_O = A \cap B$ is the disjoint union of two compact rectangles. It is only quasi-starred, since $h_1 \wr h_2$ is a set of double points relative to $O$.

These double points have to be handled carefully. Continuing example 8, consider figure 21. The vanishing points $V$ and $V'$, which are aligned with $O$, both belong to the surface and correspond to the same vanishing point on $\mathscr{R}_O$, but are meeting points for different sets of lines. We must take some care in defining vanishing points for general surfaces, to avoid duplication or the appearance of false vanishing points, as the equivalent of proposition 12 is no longer valid.

Let us consider our requirements: the vanishing points of a set $\Sigma$ on $S$ are determined by projection of $\mathscr{V}_O(\Sigma)$, so clearly $\mathscr{V}_{O,S}(\Sigma)$ is a subset of the intersection of $\mathscr{V}_O(\Sigma) \subset \mathscr{R}_O$ with $S$. But this cannot be an equality otherwise we'd have double vanishing points. Also, we cannot define them as $\varphi_{O,S}(\mathscr{V}_O(\Sigma))$ as sometimes the proximal intersection with $S$ of the vanishing rays are not the meeting point of the projected lines, which is located in a distal connected component. The following solves all these difficulties:

**Definition 10.** The anamorphosis of an object $\Sigma$ onto a projection surface $S$ relative to $O$ is $\Lambda_{O,S}(\Sigma) = cl(\varphi_{O,S}(C_O(\Sigma) \cap S^{\star}_O))$, where $S^{\star}_O = \varphi_{O,S}(S)$ is the minimal compact starred subset of $S$. The vanishing set of $\Sigma$ on $S$ is $\mathscr{V}_{O,S}(\Sigma) = \Lambda_{O,S} \setminus (C_O(\Sigma) \cap S^{\star}_O)$.

Let us see how this works with the example of figure 21. $S = S^{\star}_O$ is the union of two disjoint compact rectangles. Points $V$ and $V'$ are on the same ray from $O$, on the edge of the proximal and distal rectangle respectively. Let $\Sigma$ be the set of lines $s_1, s_2$. By definition 10, the vanishing set is $V'$, the meeting point of the lines projected on the distal rectangle, and $V$ doesn't figure in either the vanishing set or the anamorphic image. That is because neither $V$ nor $V'$ are present in $\varphi_{O,S}(C_O(\Sigma) \cap S^{\star}_O)$, the latter only appearing when we take the closure. In the same way, $V'$ will be ignored when calculating the vanishing set of lines $s_1, s_2$ by definition 10, and only $V$ will remain. This is why we define the vanishing set through topological closure and not mere projection from the canonical vanishing set. In this way we both avoid duplication of points and ensure that the vanishing set preserves its meaning as a location for the meeting of parallel line images.

**Proposition 13.** *If $S$ is a surface, and its minimal quasi-starred equivalent $S^{\star}_O$ is locally starred at a point $V$, then $V$ is a meeting point for the anamorphic images of lines that have the ray $\overrightarrow{OV}$ as a vanishing point.*

Note that if $V$ is in $\delta \subset S^{\star}_O$ then it will be a meeting point for only a subset of those lines (see figure 21 ).

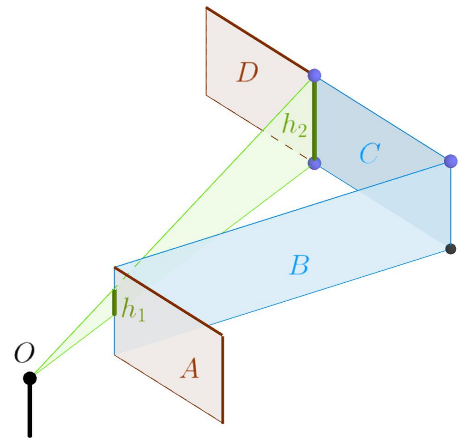


**Fig. 20** The object composed by the rectangles $A$, $B$, $C$, $D$ is not starred. Taking the proximal projection $\varphi_{O,S}(S)$ eliminates sections $B$ and $C$, leaving two connected components $A$ and $D$, minus segment $h_2$, which is recovered by taking the closure.

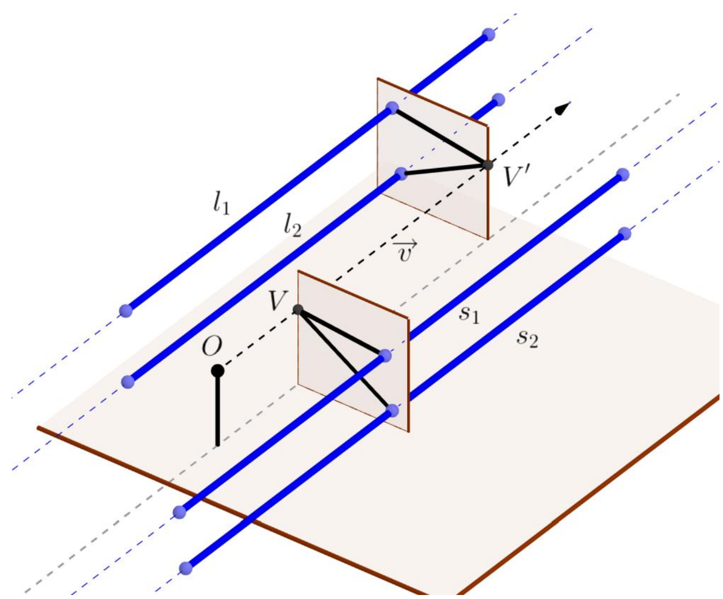


**Fig. 21** Anamorphosis onto a quasi-starred set with two connected components. Points $V$ and $V'$ are in the ray of the same canonical vanishing point, but definition 10 ensures that $V$ (resp. $V'$) is only a vanishing point for the family parallel lines $s_1, s_2$ (resp. $l_1, l_2$).

## *Simplifications: Talking to artists*

We would like to keep a balance between the concerns of artists and those of mathematicians. The technical details of definition 10 are somewhat overwhelming for the former, so it behooves us to find some simplification. The trick is how to simplify without outright lying. The matter can be presented thus to an artist:

We are interested in drawing an anamorphosis of an object $\Sigma$ onto a surface $S$. The surface can be anything. The artist is instructed that for the purposes of drawing, only the part that is visible from $O$ must be considered. Points of $S$ in the conical shadow of other points of $S$ which are closer to $O$ are to be discarded. The exception are points at the edge (boundary) of the shadow. These are allowed to remain. From this cropping we get a working surface $S'$, the minimal surface, in which we will actually draw.

Each line $l$ determines exactly two vanishing "points", $V$ and $V^\star$ which are the rays from $O$ that are parallel to $l_O$ (you can also call them vanishing rays for clarity). These are always exactly two, and diametrically opposite to each other. These vanishing rays can be represented as actual points in a canonical way by intersecting them with the surface of an the imaginary "visual sphere" around $O$. When drawing an anamorphosis onto the minimal surface $S'$, each of the two vanishing rays may intersect it several times. The multiple intersections of each ray with $S'$ should be seen as multiple manifestations of the same vanishing point. So for instance $V$ will manifest as points $V_1,\dots,V_k$, which we can order by distance to $O$. Then all lines parallel to $l$ which meet at $V$ on the sphere will meet at one of the $V_i$ on $S'$. Which one can be found a posteriori from drawing the lines near to their meeting points and seeing in which subsurface they are found (it will be the one that contains the projection of the germ of $l$ around $V$ - but here we'd be going all abstract again, so we won't).

This statement avoids the most difficult terminology and requires minimal abstraction. It can be easily explained graphically to even very young students, and yet it carries the essentials, as far as drawing is concerned, of the discussion above. In a later section we will see how to implement these concepts graphically through descriptive geometry constructions.

## *On Compactness*

Why have we insisted on compact projection surfaces? If we think about it, this is quite strange. This was introduced by Araújo (2018c) for the needs of immersive perspectives as it is a very natural setting for the most usual such perspectives, namely the cylindrical and spherical cases. Rather shockingly, however, it excludes classical perspective in its usual formulation, as the euclidean plane, infinite and unbounded, is not a compact set. So it reformulates linear perspective onto a bounded region of the plane, though this region may as large as you want it. You can imagine this region as a rectangle, for instance, representing a sheet of paper. Since we want it to be closed, it will technically be a manifold with boundary. Classical perspective will then become a degenerate limiting case as the region grows to infinity. This is analogous to the way in which orthogonal projection is usually considered a limiting case of conical projection as $O$ becomes more and more distant from the projected object.

As we have seen in example 5, classical perspective still remains, in our view, a fundamental object, as our constructions of the compact plane anamorphosis are better done considering the full plane, but only as a construction device, with the final result being cropped to a region that is bounded, though as large as we want it (reminding us of the Aristotelian philosophical distinction between *potential* and *actual* infinity). Even so, this is a frightful demotion of the infinite plane of classical perspective, so we must consider what we get in return.

Compactness is a desirable property. Ask a mathematician what linear perspective is as a mathematical object, and he will insist it is the projective plane. Like the sphere, the projection plane is a compact. Like the sphere, it gives you a definition of vanishing points that attributes a constant number of vanishing points to a line (this number being one, in the projective case, two in our case). This uniformity is pleasing to the mathematician's eye, like that of the number of roots in a polynomial equation of given degree. In the projective plane case this is done by identifying the actual perspective drawing as a chart of the projective plane. Hence the drawing lives naturally in the infinite plane. On the other hand this implies that the vanishing point of lines parallel to the chart's plane actually lies nowhere in the chart. It lives only on the "point at infinity" of the chart, that is, on the 1-dimensional projective line that lies outside of the chart's domain. We find that this is a rather immaterial satisfaction. The projective plane is not a surface that the draughtman can "see", the gained vanishing point is more than a little ethereal. By contrast, the sphere is not only an abstraction, it lives in the actual scene to be drawn, it is as concrete a surface as the plane itself, - in fact more so, because it is bounded. It is the natural object, that requires the least abstraction, just a bounded surface of ordinary euclidean space, and yet affords a more pleasing result than either the infinite plane or the comparably abstract projective plane, affording a distinction between directions that encompasses a fully immersive view.

What is formalized by the anamorphosis onto a compact surface is the notion of a *drawing* itself: the result is a real, drawing that can be materially executed and yet captures two vanishing points for every line. Mathematicians, of course, have no problem with the abstract notion of the infinite plane, but sometimes we want to deliberately keep to the finite. Just as we can abstract a notion of an infinite proof and yet for the most part decide that finite, actually executable proofs are our main object of interest, so here too, although plane perspective is certainly not a problematic concept - we could certainly accept the full euclidean plane as a projection surface and still do our work - we decide that compact surfaces are the object of most interest to abstract the notion of drawing. This in fact corresponds neatly to the way in which plane perspective, or plane anamorphosis, is actually used by draughtsmen. Every drawing is made onto a finite canvas, although the canvas may be extended at will with no absolute limitation. And the draughstman knows the difference well between the bounded plane of the drawing and the unbounded plane that contains it and may be useful as a construction device though not part of the drawing itself. Often the draughtsman will have to use methods that find a vanishing point outside the drawing sheet in order to construct the lines inside it. And it is nice to have an abstraction of perspective that relates directly to the practical distinction.

Let us be more general: in a way we can say that compact - bounded and closed - objects are all that we can draw. Can we really draw a line? No. It is unbounded. We cannot draw infinite things. We draw a segment and end it with little dots to signify continuation. This is representing infinity, not drawing it. But when we project the line conically onto the sphere around $O$, it becomes a meridian. Can we now draw it? Not yet. As we have seen, it is not a closed meridian - it is missing its endpoints, as the vanishing points do not belong to the visual cone. We cannot draw open ended arcs. We can again only represent them, usually by drawing a small circles to signify that the ends are missing, but we cannot really perform the feat of drawing an arc without its endpoints. How to solve this? We could just add on those endpoints. In mathematics we do that and call it that fancy name: taking the closure. By taking the closure of the conical projection of the line, the cone becomes a closed half-plane, the meridian gets its endpoints. We now have a closed meridian on the closed and bounded - compact - sphere. A compact arc of circle. And compact sets are the right mathematical abstraction for that which can actually be drawn.

## Descriptive Geometry construction of anamorphoses

In this section we will be concerned with constructing anamorphoses onto surfaces from the point of view of the draughtsman, using simple descriptive geometry. We focus on the mechanical device of a modified Dürer machine, both in physical form and abstracted through descriptive geometry constructions.

This is a matter intrinsic interest for our purposes, but we start with some lateral considerations of a didactic nature.

### *Handmade vs Digital anamorphoses*

It is undeniable that at present, digital means greatly simplify practical construction of anamorphosis, both in 2D, with *projection mapping* Monroe and Redmann (1994); Inglis (2018) and in sculpture through 3D modelling and printing (we have already mentioned Sugihara (2015a), but see also Ballegooijen and Kuiper's work in Dunham (2019)). These are the quickest and more efficient options for production purposes and there is nothing wrong with that; the working artist knows that there is no such thing as cheating, and anyway a vast gamut of anamorphosis would be unreachable or at least untried without such means. However, premature or excessive reliance on technological shortcuts not only lacks charm and intellectual appeal, but can lead to limiting results, as the tools that happen to be on offer define a scope of action and thought, the terms of the language of creativity, so to speak (Papert and Turkle 1991). This is always so, even if the tool is the brush, but digital tools, due to their complexity and opaqueness, tend to so in a more insidious manner. The expressive scope thus delimited may be inadequate for some artists , who then have little recourse to developers if they lack the understanding of the basics that is required for communication. The tendency will be for artists to adapt to tools rather than the other way around, and end up chasing ever changing flawed interfaces (Norman 1990; Coates et al 2010), so it is hard to say how much effort is really saved if time economies obtained by neglecting fundamentals are spent chasing transient knowledge of tools that never really *just work* as advertised (Kim and Chin 2019).The tendency is for these interfaces to help artists work with implicit principles they do not understand deeply, but there are limits to what can be achieved before the confused artist hits the walls of his ignorance. For instance, Tran Luciani and Lundberg (2016) reports both on the achievements of digital interfaces in helping artists create spherical panoramas, and on the difficulties that always eventually arise from not knowing the essentials of the spherical perspective underlying the interface. Araújo (2019a) has argued that there is a need for more interfaces that force (and help) the artist to learn the fundamentals rather than helping them to avoid such learning; and that production software should ideally come in after the user has already mastered the discipline in its rudimentary form, so as to avoid the black box frame of mind. Further, the knowledge of the mathematical and geometrical fundamentals underlying their tools, is better acquired, for most artists, not through formalism, but through manual practice in an embodied form. For this purpose, Araújo (2017b) proposed anamorphosis, in the form here discussed, as the central concept behind technologies such as virtual and augmented reality, 360-degree photography, and projection mapping; further, it proposed the modified Dürer machine, both physical and “virtualized” through descriptive geometry constructions, as an adequate physical embodyment for artistic exploration of the principle. This resulted in a teaching process that has been tested in students of a Ph.D. program in digital media arts and yet also, reworked and simplified, as a new way of introducing young students to the subjects of linear perspective and descriptive geometry (Araújo 2017c). In the latter case, making anamorphosis the fundamental concept and linear perspective the derived concept, in a reversal of the usual practice, and expressing this concept with constructions that are handmade, yet display their anamorphic properties through digital means (such as photography and VR), resulted in an enticing process to students that have grown with a view centered on digital media. This has shown promise in preliminary tests with Portuguese 9th grade students (approximately 15 years old). In what follows we present the descriptive geometry constructions of anamorphosis in a way that reflects the learning paths here mentioned.

## *Dürer machines running back and forth*

Descriptive geometry constructions are adequate for constructing both planar and curvilinear anamorphoses. We will describe a path of growing generality leading swiftly from plane anamorphosis to curvilinear perspectives. This is both a descriptive path for our subject and a learning path a student might follow with exercises of growing difficulty and generality. Using conical anamorphosis as the central concept integrates the process and simplifies it to the point where the whole apparatus of both anamorphosis and spherical perspectives can be learned by an artist in a few weeks, even if starting from no formal knowledge of perspective. (Araújo 2017b).

A practical way to teach anamorphic constructions to artists is to start by implementing a simple Dürer machine with a thread fixed on a tripod or other fixed point that will serve as the observer's point $O$ (Fig. 22). We make the Dürer machine work forward to the ground plane rather than, as usual, backward to the perspective plane. For each reference point $P$ on the object, we extend the thread from $O$ through $P$ and mark the image point $R$ where the thread hits the ground plane. In this way Dürer's perspective machine becomes a machine for making anamorphoses, which can be used to obtain an oblique anamorphosis such as those of Figs. 11 and 23. The same setup can also be used to obtain curvilinear anamorphoses such as that of Fig. 7, onto a cylinder, only requiring more points for drawing each line segment, that will now project as a curve.

This mechanical implementation of the Dürer machine is important to establish in the mind of the artist a concrete, physical, and operational notion of anamorphic equivalence. This makes it easier to then abstract it diagramatically, doing away with the thread. A simple diagram with a side view and top view - the most basic of descriptive geometry concepts - allows for construction of the oblique anamorphosis with diagrammatic "thread" (see figure 24). The process integrates the anamorph and its construction on the same piece of paper.

Take the example of Fig. 24, a student's exercise. We start by establishing orthographic top and side views (a "plan and elevation") of the object to be projected, using the ground plane as a horizontal folding line. We establish the position of $O$ by plotting its projection $O_T$ on the horizontal plane and fixing its height at $O_S$ in the side view. For the example presented, it is easier to start the object's drawing with the top view of the vertices of the base and then lifting them to the side view, through the folding line, for instance starting from the top view projection of point $P_T$ in the figure, lifting it through a vertical to get its side view $P_S$, by setting its height above the folding line. Then to plot the anamorph of the object onto the ground plane, start with the side view, as rays in the side view have *true* intersections with the ground plane. An intersection is said to be *true* if the intersection of the projection is the projection of the intersection. For instance, taking the vertex $P$, $\overrightarrow{OP}$ projects as $\overrightarrow{OP_S}$, and since the ground plane is perpendicular to the folding line on the side view, ray $\overrightarrow{OP_S}$ has a *true* intersection with the ground line, that is, the side view of the intersection of $\overrightarrow{OP}$ with the ground plane is the intersection of the projected ray $\overrightarrow{OP_S}$ with the ground line. By contrast, the intersection of that ray with a vertical edge of the cube, for instance, would not necessarily be true. Obtaining the intersection with the ground line, drop it on a vertical to intersect ray $\overrightarrow{OP_T}$. Since $\overrightarrow{OP_T}$ is the projection of the vertical plane through $\overrightarrow{OP}$, the intersection is true, and equal to the top view of the anamorphic image of $P$ onto the ground plane.

In this way we obtain the image of all the vertices on the top view of the ground plane, which we can connect with line segments to get the full anamorph of the cube. Then the anamorphic illusion will be obtained if we observe it with our eye over point $O_T$ at the height determined by $O_S$.

Of course this effect only happens in our perception if the principle of radial occlusion is verified. This will not be verified by the naked eye if the anamorphosis is too small, and the typical classroom anamorphosis will generally be far too small for the principle to be valid. However, the camera - especially the small sensor camera of mobile phones, with their large depth of field and close focusing distance - is just the right tool to simulate the eye of the perfect monocular observer. Most anamorphosis we have displayed here, being drawn in small A4 to A3 sheets of paper, are in this situation: small constructions that cannot really fool the naked eye, but which display their magic perfectly on camera. Students, especially younger students, seem to find this extra step not a mere concession to convenience, but actually an added charm, a mathemagical trick that is inherently *instagrammable* .

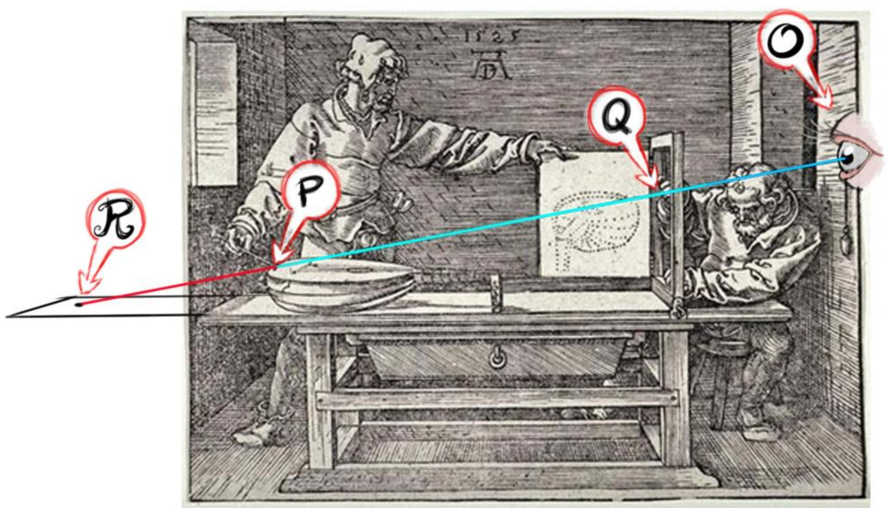


**Fig. 22** A Dürer perspective machine can be run back or forth, to make either a “perspective” onto the vertical plane or an “oblique anamorphosis” onto the horizontal one. Points P, R and Q are all anamorphically equivalent from $O$, so should be indistinguishable to a viewer at $O$ (assuming radial occlusion). Reproduction of a print by Albrech Dürer, defaced by the author.

We can extend this descriptive geometry construction to many other situations; in fact we could use it as an pretext to teach all the usual operations of descriptive geometry, projecting objects against planes slanted at arbitrary angles, or against cones, cylinders, spheres, or various other classes of surfaces. One can also consider

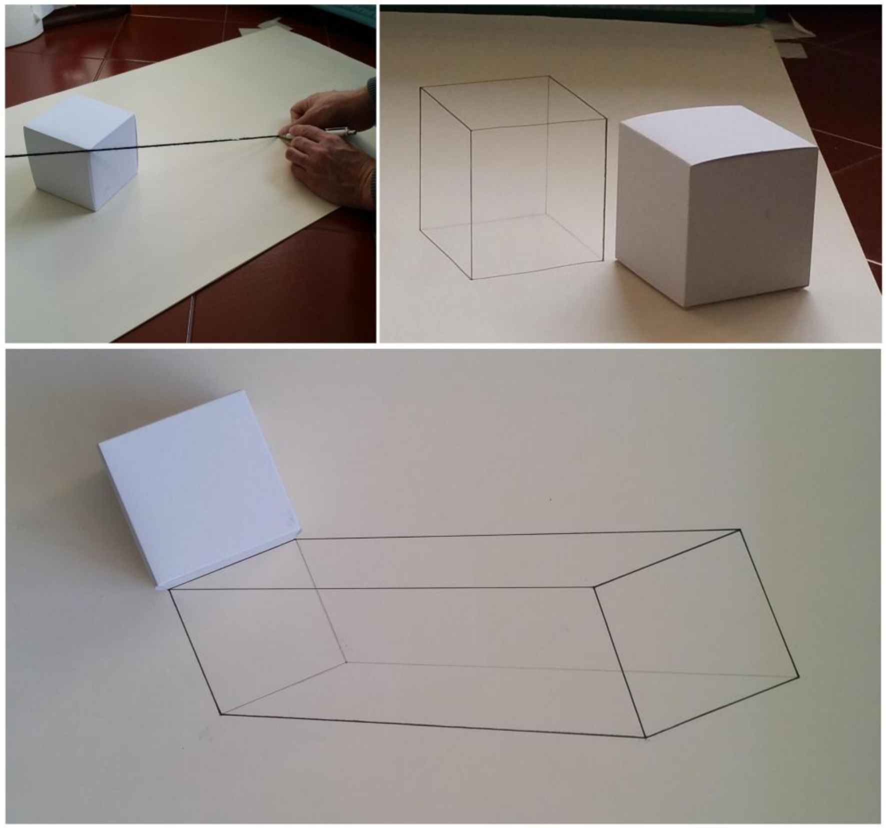

**Fig. 23** Making a cube anamorph with a thread fixed to a point.

the problem of shadows, both volumetric shadows as we have considered above, and also projected shadows (See Fig. 25), that can be constructed in much the same way as we construct oblique anamorphosis, as the mechanism is exactly the same, requiring only that we treat a point light source as an additional conic apex in the diagram of Fig. 24. Anaglyphic anamorphoses (12) are also easily constructed in this way.

So many are the achievable constructions and the descriptive geometry techniques put to use that Araújo (2017c) proposes anamorphoses as a path into descriptive geometry for young students, given its motivational possibilities, and the fact that, unlike many other constructions in this discipline, anamorphoses provides exercises without a need for an *oracle* , meaning that a student can, for the most part, independently *see* if the exercise was well executed, without a teacher to tell him so. This solves a good deal of the difficulties of descriptive geometry, the main ones being the student's inability to visualize the spatial construction that is being achieved, the other the lack of compelling incentives to achieving the result.

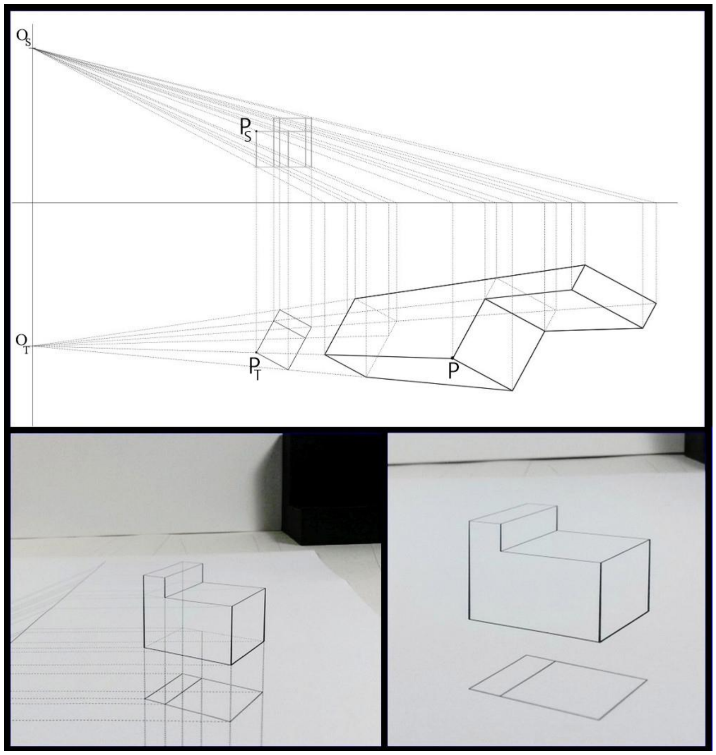


**Fig. 24** Top: Abstracting a Dürer machine with plan and elevation views of the threads going through a cube's vertices. The anamorphosis is realized on the top view and should be oberved with the eye above $O_T$ at the height of $O_S$. Bottom: The anamorphic view from $O$. Student work by Manuel Flores.

Let us consider, among the many constructions accessible through this scheme, a couple that will be of use in our transition to the study of perspective. We revisit, with the aid of descriptive geometry, the case of the cylinder.

First, revisit example 6 (Fig. 18), where a cube is projected onto a cylinder with $O$ located outside of the cylinder. In this case, and in contrast to the case of the oblique anamorphosis onto the ground plane, it is the top view that provides for the easier starting point, as rays from $O_T$ have true intersections with the cylinder wall, which projects as a circle. So, as we see in Fig. 18 (a), a vertex $P$ of the cube, with top view $P_T$ may be projected from $O_T$ in the top view until it hits a point $P'_T$ in the cylinder's circular boundary; raise this point on a vertical to meet the side view of the same ray, $\overrightarrow{O_S P'_S}$, finding $P'_S$. Transport the height of $P'_S$ by sending a horizontal across to the flattened cylinder view, and find there the image of $P$ where this horizontal intersect the vertical that marks the position of the radial plane through $O$ defining angle $\alpha = \angle(P_T C_T F)$ in the top view, where point $F$ is located at 180 degrees to the point where the cylinder is cut.

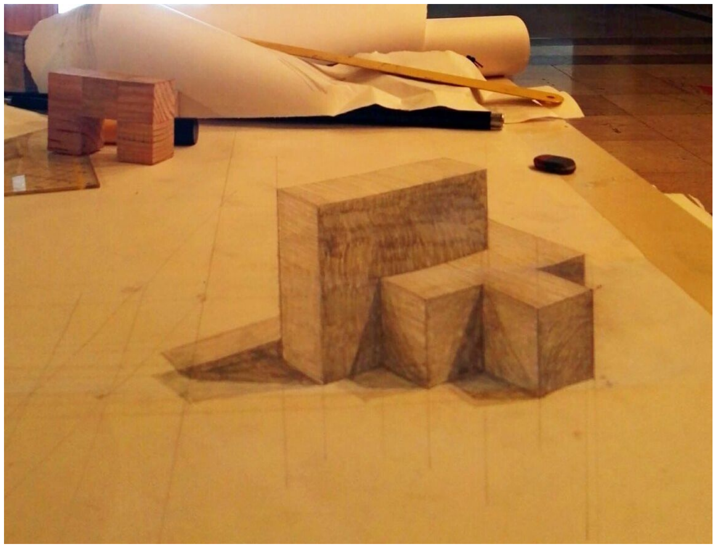

**Fig. 25** Anamorphic structure with projected shadows, using orthographic projection. Student work by Maria Bianchi Aguiar,

Consider next the case of example 7 (Figs. 19 and 26) in which $O$ is in the cylinders axis. The projection is made in the same way, illustrated in the figure with the plot of a vertex $P$ of a cube. Consider for instance the horizontal edge of the cube going through point $P$ and ending above point $F$, in the figure. The segment is contained in a line that projects as the curve $VPV^\star$.

As mentioned in example 7 this curve is a sinusoid. It is easy to see why using descriptive geometry, in the particular case of horizontals (and the general case is easily obtained through a little trigonometry). One can always rotate the horizontal to assume it projects as a point in sideview, as in Fig. 19. Then any point $Q$ along that line, whose orthogonal projection $Q_T$ subtends an angle $\alpha = Q_T O_T F$ in the top view, will project as $\cos(\alpha)$ on the horizontal line $O_T F$ in top view, and this measure will be transferred up to the line $O_L P_L$ on the side view, hence being equal to $m\cos(\alpha)$ with $m$ determined by the slope of $O_L P_L$ and hence constant for all $Q$ on the line. Once the cylinder is flattened, the vertical measurement is preserved, and the horizontal measurements are linear with the angle $\alpha$, so the curve obtained as $Q$ slides along the line is proportional to $\cos(\alpha)$, hence a sinusoidal curve.

Returning to Fig. 19 and to the line defined by the horizontal edge through $P$, its vanishing points $V$ and $V^\star$ are obtained by translation to $O$ and intersection with the cylinder. There are several ways to plot the sinusoid itself. In a variation of what was done in Fig. 19, one can simply plot points corresponding to regular angular intervals on the top view circle and project these onto the flat view, then interpolate. More interesting may be to note that the sinusoid is defined by any two of its (non-antipodal) points, for instance $V$ and $P$. A good choice is $V$ and the apex of the curve, where it reaches it maximum. This apex projects in top view at the point where the top view of the line meets its orthogonal through $O_T$. Given $V$ and the apex image it is easy to plot the sinusoid in the flat view.

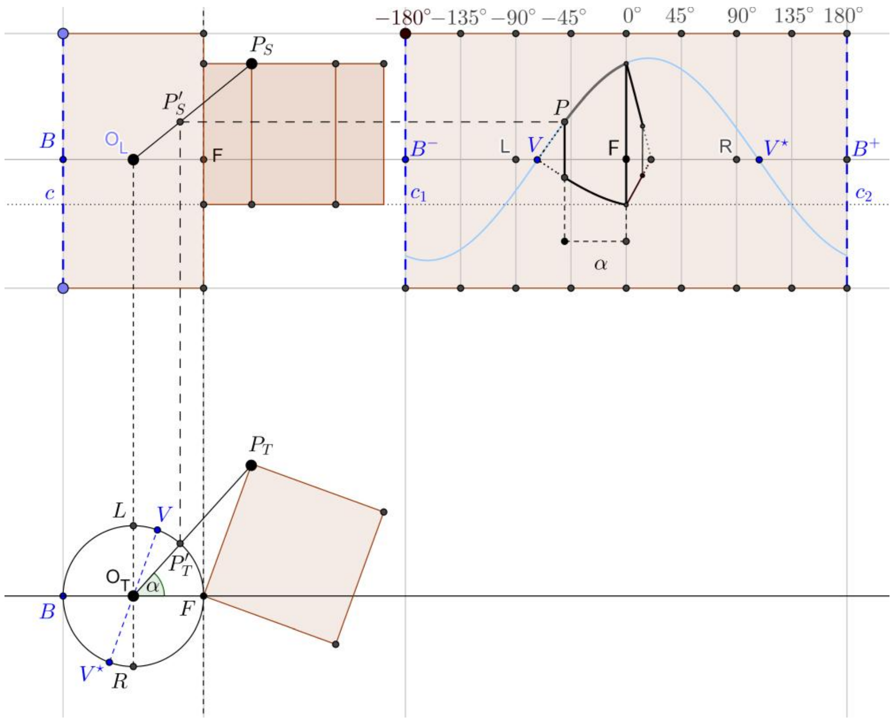


**Fig. 26** Cylindrical anamorphosis of a cube onto a cylinder. The image on the flattened cylinder (upper right) is a cylindrical perspective. It transforms non-vertical lines into sinusoidal curves. Notice the blue sinusoidal curve defining the cube's top edge through point *P*.

## Perspectives

We have talked from the start of considering perspectives, both linear and curvilinear, as a derived concept from anamorphosis. We are now in a position to develop that idea.

Let us begin once more intuitively and loosely: in our view, a perspective is just a flat representation of a surface anamorphosis, like a map is a flat representation of the Earth's globe. That is all. We want these maps to have certain properties, so we will need some technical details, but this is the main idea to keep in mind.

The notion of *curvilinear perspective* has a complicated past with a history that is often confusingly reported upon. The difficulties come not so much from the misconceptions of the past, but from those remaining in the present, under whose burden the historian must work. It is hard to argue whether Leonardo or Jean Fouquet are precursors of curvilinear perspective (Andersen 2007) without a proper definition of what those perspectives are. Often both the original sources and the historian fail to distinguish the perspective from the corresponding anamorphosis. This distinction, that so clarifies the subject, is our focus here. The matter of a perspective being "curvilinear" is comparatively minor. We will define as curvilinear any perspective that projects spatial straight lines into plane curves. This of course includes the cylindrical perspective and the usual spherical ones. Note that "curves" cannot mean only smooth curves, otherwise this has the curious result that at least one perspective (the cubical case) manages to be spherical without being curvilinear (in this perspective spatial lines project as sets of line segments). But we will not concern ourselves much with this. Curves will arise from definition 11 as side effects of the need to project compact anamorphoses onto compact plane sets while preserving their main topological characteristics, such as vanishing points. The perspectives we will define are not so much characterized by being *curvilinear* as they are by being *central* (a side-effect of being derived from conical anamorphosis) and *compact* .

We will motivate our definition of perspective by considering the simplest of curvilinear perspectives: cylindrical perspective. It has a long history. Whether you consider it defined by Herdman's 1853 treatise (Herdman 1853) or implicitly defined by Baldassare Lanci's 1557 (Kemp 1990, p. 175) perspective machine, curvilinear perspective was certainly textbook matter in the 1900 Ware (1900), and in the very early 20th century was used for military balloon airmen spotting for artillery (Dept. of Military Aeronautics 1918). Its aspect is familiar to the modern reader, as it corresponds to that of the common panoramic photography.

We have already silently met cylindrical perspective, in Figs. 19 and 26. We have seen that in order to construct the anamorphosis of an object (in this case a cube) onto the cylinder, with $O$ at the axis, it is convenient to first draw the scene in plan and elevation, flatten the cylinder into a rectangle by cutting and unrolling it on the plane, and project the object onto this rectangle (upper right corner of Fig. 26). Then, once the projection is drawn, the rectangle can be rolled up again and glued at the vertical edges to form the compact cylinder in 3D space, and the resulting picture on its surface will thus become the cylindrical anamorphosis. That auxiliary plane drawing on the rectangle, from which the anamorphosis is rolled up, is what we call the *cylindrical perspective*. Note that this is purely an affair of convenience. By definition, the anamorphosis is obtained on the spatial cylinder, through conical projection. But in practice, we find it convenient to draw on planes. This is analogous to cartography, where the exact shape and metric of features on a globe is sacrificed to the convenience of a plane chart. So, to us, perspective is merely this: a plane chart - a flat representation - of the spatial anamorphosis. The flat representation is not only easier to draw, it allows for they eye to capture the whole picture at a glance, at the price of introducing optical deformations. This is often the reason to use it, as in the case of Jean Fouquet's *Arrival of the Emperor Charles IV at the Basilica in Saint Denis* , where a deformation quite similar to cylindrical perspective allows us to see at a glance further down the street than would be possible with linear perspective.

Analogously to the case of cartography, we must pay for this convenience with deformations, that result in perception inaccuracies. This is true even in the case of the cylinder, that unrolls isometrically onto the plane. See for instance Fabritius' *View of Delft* (1652) (Fig. 27).The flat cylindrical perspective drawing, although it contains all the information of the cylindrical anamorphosis, and is isometric to it, is no longer an optical illusion; there is no point $O$ from which the points can recreate the effect of the original object - the cones no longer align, hence it is no longer an anamorphosis. So the straight lines in the picture all get deformed. The line image of Fig. 19 looks nothing like a line (as we have seen, it is a sinusoidal curve), and the cube on Fig. 26 looks nothing like a cube. The non-vertical lines are bent; that in itself is not the problem, as they are also bent in the anamorphosis, and still they will look straight when seen from $O$. The problem is that in the perspective they are not just metrically but fundamentally *optically* deformed; there is no point $O'$ from which they can all look straight. The perspective drawing is a *representation*, a storage of visual information - but it is not mimetic.

Of course it can still be *read* up to a point, even without training. But I challenge anyone to distinguish with perfect accuracy - a task easy in classical perspective - which lines in a cylindrical perspective are really bent and which are only bent through projection. The task can hardly be done by the untrained eye unless with an extreme reliance on context. The cylindrical perspective still evokes the scene, but no longer mimics it.

**Fig. 27** A View of Delft, with a Musical Instrument Seller's Stall. Carel Fabritius, 1652. Currently at the National Gallery in London. Apparently a cylindrical perspective, although nothing is known regarding the method of its construction.

Let us be more precise. We view the perspective as a map from 3D space to a compact region of the plane, defined by a two-step process: the first step being the anamorphosis, from the 3D space to the surface $S$, the second step being the flattening of $S$. The anamorphosis is completely defined by the choice of $S$ and $O$; the flattening is a much more arbitrary map, just like in cartography. Because we will want to preserve the visual *information* , if not the visual aspect, we have some requirements. First, we would like it to be a bijection, so that no information is lost. Then, we'd like it to be a homeomorphism (continuous with continuous inverse) so as to preserve the topology, and hence the vanishing sets and their meeting properties. Finally, we would like it to be as smooth as possible. We have a second set of more vague, yet crucial requirements: we would like it to preserve as much of the mimesis as possible; though it will not be an anamorphosis, it should be in some way recognizable as a picture; it should summon the visual presence - or at least the intellectual recognition - of what it represents. Finally, it should be solvable by elementary means. By this we mean that you should be able to draw it by hand, by ruler and compass, or some other simple means with adequate precision and in reasonable time, i.e., we would like it to be a perspective for humans and not just for computers.

We find that even the first three technical requirements are asking too much. We need to interpret them with some care. Look at the case of the cylinder in Fig. 26 and it will become apparent why. The cylinder is a developable surface; it can be unrolled onto the plane, preserving distances on the surface. Intuitively we would like to say that the flattening is the unrolling map that sends the cylinder to the rectangle. But this is not a bijection on the compact rectangle. Let $c$ be the segment through $B$ where the cylinder is cut, and let $c_1$ and $c_2$ be the corresponding vertical edges of the rectangle. Each point of $c$ is sent to a point on $c_1$ and another equivalent one on $c_2$, so this "projection" is not even a map at all.

We solve this by noting first that the flattening $\pi$ is well defined *almost everywhere*, meaning, on an open dense subset of the cylinder $S$. Namely, it is well defined on $S$ minus the union of its boundary with the vertical segment $c$. This projects to an open rectangle, that excludes $c_1, c_2$, and is a homeomorphism onto that open rectangle. Further, the inverse $\pi^{-1}$ can be extended uniquely to a continuous map $\tilde{\pi}$ from the full compact rectangle to the complete cylinder. We just map both $c_1$ and $c_2$ to $c$, since in this direction the map has no problem of uniqueness. Apart from technicalities, we have the map we wanted. We define the cylindrical perspective to be the composition of $\pi$ with the conical projection, and the perspective image/vanishing set to be the image of the anamorphosis/vanishing set, where we identify as the same point any points with equal images through $\tilde{\pi}$.

We will use this example as a template for the general case and say that $\pi$ is a flattening if it can be extended to a continuous map in the way we just exemplified. We formalize this by the following definition, adapted from Araújo (2015):

**Definition 11.** Let $S$ be an $O$-starred surface. Let $U$ be an open dense subset of $S$. We say that $\pi : U \to \mathbb{R}^2$ is a *flattening* of $S$ if $\pi$ is an homeomorphism onto $\pi(U)$, and there is a continuous map $\tilde{\pi} : cl(\pi(U)) \to S$ such that $\tilde{\pi}|_{\pi(U)} = \pi^{-1}$. We say that $p = \pi \circ \varphi_{O,S}$ is the *perspective* associated to the flattening $\pi$, where $\varphi_{O,S}$ is the conical projection. Let $\tilde{p} = \varphi_{O,S}^{-1} \circ \tilde{\pi}$. Given an object $\Sigma$, we say that $\tilde{p}^{-1}(\Sigma)$ is the strict perspective image of $\Sigma$, that $\tilde{p}^{-1}(\mathscr{V}_O(\Sigma))$ is the vanishing set of $\Sigma$, and that the perspective image of $\Sigma$ is the union of its strict perspective image with its vanishing set.

The following set is often useful in discussing perspectives. It is the minimal closed set we must take out of $S$ for $\pi$ to be well defined.

**Definition 12.** Given a flattening $\pi$ of $S$, we call blowup of $S$ to the subset of points of $cl(\pi(S))$ where $\tilde{\pi}$ is not injective.

The geometer may notice that we are here abusing the term "blowup", common in other areas of geometry, where it means, roughly, the replacement of a point by the projective line. Here it is in fact just the set where gluing (identification) of edges takes place. We use the term blowup by analogy to the important case of fisheye perspective where a point will identify to a circle, or more precisely to the set of rays or directions defined by that circle. This is analogous to the usual sense of blowup of a point, which roughly is the replacement of the point with a projective line.

*Example 9.* Take again the case of the cylinder in Fig. 26. Let $O = (0,0,0)$, let the radius of the cylinder be $r = 1$ and its top boundary lie at $z = \pm h$. Suppose we cut the cylinder at the vertical segment $c$ that passes through $B = (-1,0,0)$. Then, in cylindrical coordinates, we define $\tilde{\pi}(\theta,z) = (\cos(\theta),\sin(\theta),z)$, so the blowup set is $\mathscr{B} = c$, and $\tilde{\pi}$ implicitly defines $\pi$ in the $S \setminus c$. Having defined $\pi$, the cylindrical perspective is defined by the entailment $p = \pi \circ \varphi_{O,S}$.

As seen above, spatial verticals are anamorphic to verticals on the cylinder and non-verticals to ellipses. Their perspective images by $\pi$ are the curves we obtained above through the cylinder's unfolding: arcs of sinusoidal curves. For instance, in Fig. 26, the line that goes through $P$ and has vanishing points $V$ and $V^\star$ defines a plane through $O$ that intersects the cylinder on an ellipse, and that ellipse maps in perspective to the full sinusoid in blue. The line itself is the arc $VPV^\star$, one half of the sinusoid. These can be seen as the images through the cylindrical flattening of the great circle (resp. meridian) of the canonical anamorph of the plane (resp. line) in question.

Notice that although $\pi$, the flattening, is the more intuitive concept, $\tilde{\pi}$, the closure of its inverse, is the more natural map, defined everywhere and usually easier to work with analytically, as in the example 9. This is very common in geometry.

Definition 11 is just a technical way of making sure that the perspective image of an object and its vanishing points are just the sets in the perspective image that correspond to the ones previously defined in the anamorphosis. As we stated before, everything important is already defined at the level of anamorphosis, perspective being relegated to a final step of convenient representation. This of course does not in deny the importance, both technical and aesthetic, of perspective, nor the intricacies of its construction, but it clarifies it meaning. In particular it does away with long-standing conceptual mistakes expressed in questions such as "do we see in classical/spherical perspective" or "does linear perspective cause deformations". It is not so much that we answer these questions; instead we show they are badly posed. There is no meaning to the question because we simply do not

”see in perspective”, whatever that perspective may be. Perspective is a map of the information in an anamorphosis and the anamorphosis is, when radial occlusion is verified, a mimetic object (a *trompe l’oeil* ).

It is true that we can speak, more or less vaguely, of some perspectives preserving the aspect of the scene better than others, that is, of being ”closer” to the mimesis than others. How close depends on the measure we use. Araújo (2016) speaks of the *reading mode* of a perspective. For instance, a cylindrical perspective has a natural reading mode in the sense that when the cylinder of radius $r$ is unrolled we can think of the eye point $O$ being transformed into a horizontal line floating parallel to the central axis of the picture, at a constant distance $r$. The natural reading mode is for the eye to travell horizontally, scanning the picture as it moves, one vertical thin strip at a time. At each point of the motion the vertical line in front of the viewer - and only that one - will be in anamorphosis. By contrast, in a fisheye perspective, only the central point in in anamorphosis, and a small neighbourhood around it will show minimal differences if overlapped with the observation of the actual object. Notice that we take care to say ”with the observation of the actual object” not with ”its image”, as otherwise we would be in the difficult situation of trying to define what the ”image” means. Mimesis is checked not by comparing ”images” but by looking at pairs of objects (that may be volumes or drawings on surfaces) and being unable to tell the difference between them, at least in some well specified experimental sense (for example, do the vertices of a wireframe anamorph of a cube seem to occupy the same spot as the vertices of the real cube it seeks to mimic).

In this sense we see that indeed classical perspective occupies a special place in the bestiary of perspectives, as it is the only one that is still an anamorphosis. This happens because the anamorphosis is already on a plane, so the flattening map is just the identity map. The reading mode of a perspective is the same as that of the anamorphosis: it consists of putting your eye in the point $O$ and rotating it freely towards any spot of the plane of the picture. The so-called ”perspective deformations” that happen for large angles of vision happen simply because the pictures in question are made in such a scale that they make the viewer unable to occupy point $O$ without breaking the requirements of the principle of radial occlusion. Usually this happens because the perspective is draw at such a scale that the eye must be too close to the picture plane. And of course, the illusion will not work of the viewer leaves point $O$ to look closer at some detail, or if he looks at the picture from some arbitrary point in a crowded room. But anamorphosis only works from point $O$. To blame it for not working otherwise is to blame a fork for not being a spoon.

### Spherical perspectives

We have seen that among anamorphoses, the spherical one holds a special place, due to its natural identification with the space of rays $\mathscr{R}_O$, or with the concept of *visual sphere*. For the same reasons, the spherical case also holds special importance among perspectives.

Of course, while spherical anamorphosis is unique, spherical perspectives are innumerable. Every chart of the sphere defines a flattening and hence a spherical perspective according to definition 11. In fact every central perspective (even classical perspective, if restricted to a compact subset of the plane) can be seen formally as a *spherical* perspective, hence the term is only as interesting as the qualifications we add to it: for instance, we can distinguish *total* spherical perspectives, i.e., one those in which all points of the sphere have a perspective image. Each spherical perspective, although it holds the same visual information and would generate the same anamorphosis, holds its own visual characteristics and artistic possibilities as a drawing type. These visual characteristics hold all sort of representational and expressive possibilities that have been explored in traditional media (Barre and Flocon 1968; Casas 1983; Moose 1986; Michel 2013; Araújo 2019b) as well as in purely digital visualizations (Correia et al 2013), and are more and more being investigated as *hybrid immersive media* (Araújo et al 2019a; Olivero and Sucurado 2019), a mode of artistic expression that allows handmade drawings to be visualized through immersive, digital means.

It is interesting to ask how we can solve a spherical perspective, where by *solving* we mean giving a a systematic method to find and draw all vanishing sets of lines and planes through simple geometrical constructions (such as ruler and compass constructions).

Notice that since by proposition 10 the perspective images of lines are subsets of the vanishing sets of planes, and since plane images are delimited by their vanishing sets, obtaining all vanishing sets also obtains all line and plane images, hence we can say that solving vanishing sets is the true subject of perspective.

We have seen that the vanishing sets of spherical perspective are all determined by the geodesics of the sphere. This suggests, if not an algorithm, at least a strategy for solving a spherical perspective: solving a given spherical perspective should be always attempted by focusing not on lines themselves, but on full geodesics; on classifying the geodesics according to the properties of their projection by the flattening, and on finding an efficient method to plot the geodesics of each class.

In this, we will find there are features common to all spherical perspectives, and features peculiar to each. What common is the first step of the perspective projection, since it is just anamorphosis onto the sphere; what is particular is that each spherical perspective will have its own metric characteristics, which may vary widely. It will also have its own topological properties, but these are by design more limited in variation than the metric ones. Definition 11 ensures that $\pi$ is continuous outside of the blowup set, so the topological variations will depend entirely on this set, which means that the blowup is a fundamental feature of perspective.

This view of perspective, first stated and applied to the case of the azimuthal equidistant perspective (Araújo 2018c) was later used to solve the equirectangular perspective case (Araújo 2018b) and then the case of cubical perspective (Araújo et al 2019b). The splitting of the anamorphosis from the flattening, and the focus on full geodesics instead of individual lines, proved an elegant solution for these perspectives. Previously, the only spherical perspective that had been solved systematically was the azimuthal equidistant in the hemispherical case, meaning that in fact only half of it had been solved. This is the perspective of Barre and Flocon (Barre and Flocon 1968), solved in the 60s, which renders into images similar to those of a "fisheye" camera. The full azimuthal equidistant case had only been solved in either a qualitative way (Casas 1983) or through informal grid methods Michel (2013) which lack the generality of Barre and Flocon's solution. So there stood this curious situation in which the first spherical perspective was not (fully) spherical, and the subsequent ones were not (formal) perspectives.

The approach here described, and implemented in (Araújo 2018c) was able not only to extend Barre and Flocon's perspective to the full sphere, but also simplified their solution in the frontal hemisphere, reducing it to two distinct classes, the focus on geodesics instead of lines providing a clearer, more unified view of the problem, for even when plotting only the frontal part of a line, it is helpful to be able to call on the antipode of every point (and the pair of any vanishing point) for auxiliary constructions.

As for the the equirectangular case, it had previously been treated only computationally or through fixed grid constructions, the equirectangular grid being calculates at fixed intervals and used as a guide for drawing. The more general approach in Araújo (2018b) leads naturally to an exploitation of the translational symmetries of the perspective , which can be used with a system of dynamical moving grids that allow for the plotting of all lines in good approximation (Araújo 2018a, 2019a).

The cubical perspective case is an interesting one, since seeing it as a spherical perspective (Araújo et al 2019b) greatly simplifies previous approaches (Rossi et al 2018; Olivero et al 2019) that saw it as a conjunction of six related classical perspectives. It might not at all seem obvious that seeing a set of classical perspectives as a spherical one would be a simplification.

In all these three cases, the approach passes through understanding the flattening and its blowup set, classifying the geodesics of the sphere according to how they are projected in the plane, and finding a method to render each class through simple constructions. For this, the understanding of antipodes and how to plot them is essential , as the duality of vanishing point pairs usually results in symmetries that can be exploited. As these are curvilinear perspectives, approximations will be required to draw line projections, and in these the number of operations should be kept to a minimum. This will require a careful exploitation of the symmetries arising from the flattening. A general philosophical principle is that not only the perspective drawings themselves but also the auxiliary drawings should happen in a compact subset of the plane; in short, everything should be bounded.

We will not explore in any detail here the actual constructions of spherical perspectives, leaving that for another chapter in the present volume (see Spherical Perspective). We will merely relate the fundamental properties of two of these cases, to see how they fit on our general scheme described above.

In the case of the azimuthal equidistant perspective (Fig. 28 (top)) the flattening works by choosing a reference point $B$ (for "Back") and loosening the meridians there, straightening them without stretching to obtain a disc centered on the point $F$ (for "'Forward") at the antipode of $B$. If we reference $F$ and $B$ as the "poles" of the sphere, then $FB$ meridians flatten as rays of the disc, and "parallels" flatten as circles, with the image of the equator separating an inner circle (the frontal view, rendered yellow on Fig. 28 (right)) from an outer ring, where the back view is rendered (see the examples in Fig. 29). The flattening is one-to-one at all points except at $B$. Taking the closure we get a compact disc, with the outer boundary circle (the blowup set) mapping entirely onto $B$. The flattening preserves the metric along each individual $FB$ meridian, though of course not globally. This preservation of the metric is what allows measurements to be made, so drawing this perspective requires careful attention to these special meridians in the construction process. The natural measurements for this perspective are angle pairs, each point on the sphere being measured by one angle choosing the $FB$ meridian where it lies and the other the length traversed from $F$ along that chosen meridian. As for geodesics, these can be classified in two classes: geodesics through $F$ render as diameters, and all others as closed curves (See Fig. 28 (right)). For the latter, half the geodesics will be rendered in the anterior disc (yellow in Fig. 28 (right)) as an arc of circle $c$ in good approximation, as shown by Barre and Flocon (1968); the other half can be rendered, as the locus of a point $P$ that moves so as to keep at a distance of half a diameter from the point of $c$ that is across from it along the line that joins it to $F$. This locus can be drawn using a mechanical process involving a ruler sliding on a fixed point, as explained (Araújo 2018c) and in the chapter on Spherical Perspective in the present volume. This can be simplified by a moving grid system using the rotational symmetries of the perspective around $F$ (Araújo 2019b).

In the case of equirectangular perspective, we blowup the sphere at two points $U$ and $D$ (representing for instance the directions "up" and "down"), and straighten the meridians, keeping them fixed at the circle halfway between $U$ and $D$ (the "equator"). In this way we obtain a cylinder, which we can now cut and unroll onto a rectangle. Unlike the case of cylindrical perspective, this rectangle will contain a record of the whole view around $O$. There are two classes of geodesics: Geodesics through $U$, that render onto straight vertical lines, and all the others. The first (resp. second) type includes the image of vertical lines (resp. horizontals) but is not restricted to these. If we consider the set of all geodesics through two antipodal points on the equator (say, points $L$ and $R$, corresponding to the observer's Left and Right), we can generate all other geodesics from translations of elements of this set. This allows for a construction of complex scenes such as that of Fig. 30 without explicit use of ruler and compass and without the limitations of a fixed grid (see (Araújo 2018a, 2019a)). As pointed shown in Araújo (2018b) These geodesics are very similar to sinusoidal curves when their apex (point of highest angular elevation) is low, hence this perspective looks like cylindrical perspective for low elevations (see Fig. 30). As the elevation rises, the geodesic images converge to a square wave, but even for high elevation they can still be rendered in good approximation with descriptive geometry constructions. In Fig. 30 we can see a drawing done on location of an equirectangular spherical perspective using the sliding grid method of (Araújo 2018a). We can also see a ball covered with a print of this drawing, duly transformed by a computer into a sinusoidal projection, then printed and glued onto a ball. This again shows that often the point of a perspective is to be a blueprint for the anamorphosis, instead of the other way around. We defined the perspective as the flattening of the anamorphosis because the anamorphosis is the uniquely defined object; but just as often the anamorphosis will be in practice treated as the folding of the flat perspective.

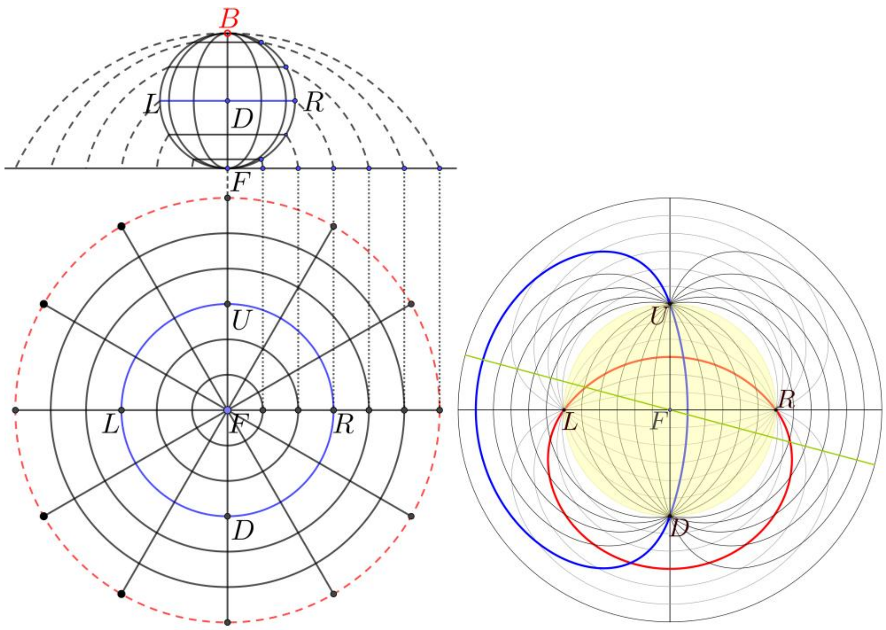


**Fig. 28** Azimuthal equidistant flattening of the sphere. Left: Blowing up at point $B$ maps the sphere to a disc centered on the image of $F = B^{\star}$. Every $F$-geodesic maps isometrically to a diameter of the disc. Right: Geodesics. Green is a $F$-geodesic. Blue and Red are $U$ and $R$-geodesics respectively. Both are approximately arcs of circles in the yellow inner disc, which represents all that lies to the front of the observer.

## *The problem with perspective*

The view of anamorphosis we studied here was motivated mainly by the wish to solve conceptual problems with perspective. We will start by discussing what those problems are.

One might be surprised to hear are any conceptual problems with perspective at all. Linear perspective, is universally used today to represent a 3D scene on a plane. Most working artists and architects take it for granted as an objective picture of an environment, identifying it with photographic representation. Mathematicians take it for a well-understood concept, made rigorous as an application of projective geometry (see Looking Through the Glass).

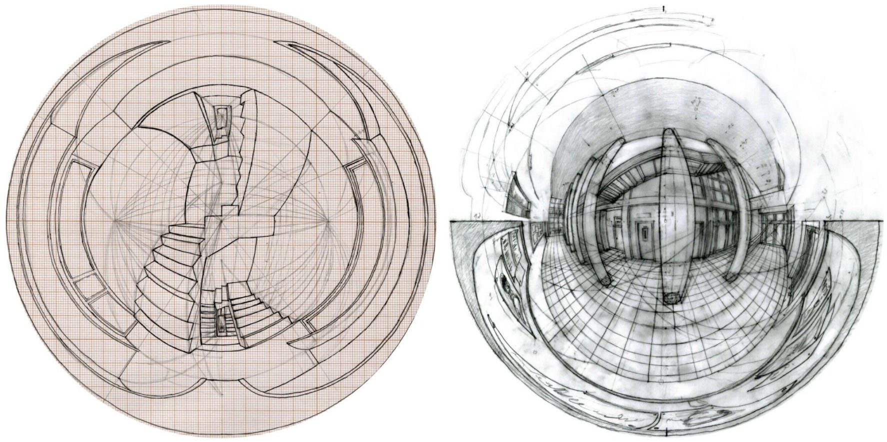

**Fig. 29** Two handmade drawings of urban scenes, obtained through the process described in (Araújo 2018c)(left) and (Araújo 2019b) (right). All lines are contained in geodesic images of the azimuthal equidistant flattening. Drawings by the author.

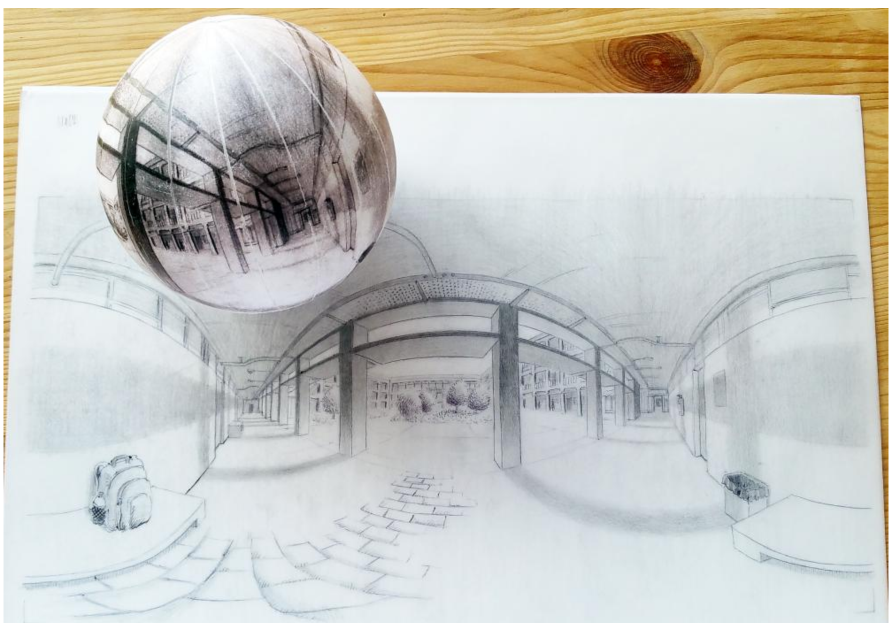

**Fig. 30** A spherical anamorphosis and its equirectangular perspective. The spherical perspective drawing was done on location by the author. It was scanned, transformed on a computer, then printed and glued on a ball to obtain the corresponding spherical anamorphosis. The author would like to thank Ph.D. student Lucas F. Olivero for making the 3D model on the top left of the picture.

Yet the nature and meaning of perspective has been a point of contention since its inception. We can see this as a contention between the views of Euclid and those of Leonardo.

## Euclid and Psychophysics

We could argue that our view of anamorphosis in the present work is a recovery of concepts first formulated in Euclid's Optics with the addition of vanishing points. Or, put in another way, that the fourth axiom of Euclid's optics is the earliest known precedent for the concept of anamorphosis.

**Axiom 1** *(4th axiom of Euclid's Optics) and that those things seen within a larger angle appear larger, and those seen within a smaller angle appear smaller, and those seen within equal angles appear to be of the same size (Burton 1945)*

Euclid here is postulating that for visual perception, the visual angle subtended by an object and its perceived size are the same thing. This is clearly true for a simple object in isolation of other stimuli. That is, if I take a black ball on a white background and I make it subtend progressively larger angles an observer will certainly report a larger apparent size. But it is too daring a proposition when related to complex visual environments. For instance, Ebbinghaus illusion (Ebbinghaus 1902) shows that the perceived size of a colored circle can depend strongly on the disposition of other colored circle's in it's vicinity. What exactly is happening is still a matter for contention, with several theories extant (Roberts et al 2005). This is not uncommon for visual phenomena whenever global variables - rather than very well controlled local experiments - are taken into account. Analogous problems occur in color theory, where the local phenomena are well understood and mathematically elegant while the global theory is confusing and qualitative. So we must not take the term "apparent size" at face value. Astronomers use the term in the Euclidean sense to mean simply the size of the solid angle subtended, while in studies of perception, apparent size means the subjective impression of size as reported by human subjects.

In the latter sense, Euclid asserts too much, ignoring contextual matters. Still, a careful localized interpretation of the fourth axiom does hold water as empirical fact. Like we said before, it defines an implicit empirical scope wherein it is true, and therefore it is as interesting a concept as that scope happens to be relevant.

Still, in our own work here, we took a different, although related path. We are not concerned with size but with form; not with the size of the solid angle subtended by the cone of an object but with the cone itself. Our principle of radial occlusion neither implies nor is implied by Euclid's 4th axiom of optics. But the emphasis on the visual angle, without reference to any projection surface, the isotropy of view, and the notion of visual cone as a function of the object rather than on some central axis is already present in Euclid, and a reading of his theorems shows that he has in mind something similar to us. The difference is his avoidance of infinity. Therefore we might say that the work here developed clarifies Euclid and adds to it the notion of points at infinity - of compactness, if you will. It is impressive to read Euclid in retrospect and notice the generality of his view at such an early time, and how the late arrival of linear perspective seems by comparison a step back in generality, being a theory of representation of too particular a kind.

## Leonardo's axiom and paradox

Leonardo considered there were many conceptual problems with perspective as formulated by Alberti. He distinguished between "natural" and "accidental" perspective. This is related (but not identical) with the more usual distinction between *natural* perspective (meaning optics, as a study of both light and visual perception) and *artificial* perspective (meaning linear perspective). Leonardo objected to the requirement in linear perspective of eye placement at single point of observation, and thought that a spherical canvas, would make for a more natural perspective; and he seems to have thought that in the absence of this a very distant canvas might do. In this he seemed to be concerned with the inclination of the surface normal with regard to the angle of view (in Kemp's (Kemp 1990) view, the foreshortening of the canvas just as much as the foreshortening of the picture). "Foreshortening of the canvas" is of course no more than a reference to the effects of oblique anamorphosis.

Leonardo's greatest difficulty, or at least the one that survived to our time, was based on the following axiom, which he considered a requirement of a natural perspective:

**Axiom 2** *Leonardo's axiom: Among objects of equal size that which is most remote from the eye will look the smallest.*

This in itself is not contentious. But he then confusingly identified this notion with the completely different and in fact incompatible one that they should therefore be represented by metrically smaller projections. This in contrast with Euclid's fourth axiom of optics.

He gave the example still called *Leonardo's paradox* : the conical projection of a sphere that moves along the plane projection actually grows as it moves away from the observer (see Andersen (2007); Araújo (2016)). This "paradox" really isn't one. True, the perspective picture grows, but so does the distance of this projection to the observer, in such a way that the apparent size of the projection diminishes. By Euclid's fourth axiom, both the distal sphere and its projection will look smaller than the proximal one. Leonardo's real objection may be the disparity between the properties of the metric representation and the angular/visual properties. He would like a perspective that carried the proper visual properties of mimesis without the requirement of standing at a single point $O$. And he would like that the metric properties, along with the angular ones, should satisfy his axiom. Hence his preference for a spherical canvas. In this, there is a conflating of the two concepts that we separated in the present work: perspective and anamorphosis. This conflation would endure and persist as a philosophical difficulty until the 20th century. It is in fact still a difficulty today in the way these affairs are discussed. This leads to asking the

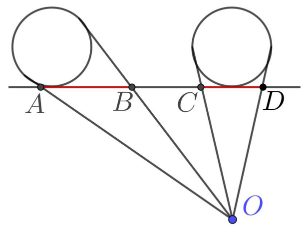


**Fig. 31** Leonardo's paradox: $|AB| > |CD|$ even though the corresponding sphere is further away from $O$.

wrong questions such as whether a ”perspective” has ”deformations” or whether it is ”natural”, or whether it corresponds to the way we see. These questions vanish if we only separate the two concepts of mimesis and representation, anamorphosis and its flattening into perspective. So that the point may be made that the paradox is no such thing - it does not point out to any failure of linear perspective in achieving what it sets out to do, but is merely a complaint that linear perspective does not do as much as Leonardo wishes from a perspective. But it is hard to see this as a flaw of perspective as much as a failure of Leonardo to settle on a consistent set of requirements. One cannot ask at the same time to preserve both angular and linear measurements. That linear perspective (as plane anamorphosis) does not preserve the latter is not a bug but a feature mathematically required to preserve the former.

This enduring failure to distinguish the mimetic role of anamorphosis from the representational role of perspective may be due to the fact that linear perspective and plane anamorphosis are the same physical object (the ”flattening” being trivial); this identification between the anamorphosis and the corresponding perspective only happens in the case of linear perspective, and this coincidence probably made it all the harder to understand the distinction between the two concepts. It is rather unfortunate that the most ”obvious” of perspectives is in fact a quite peculiar one. The matter was still contentious in dealing with cylindrical perspective, much later, and even when the practical difficulties of the drawing itself were well handled, mechanically and theoretically (Kemp 1990, p. 175)).

### Effects on the development of spherical perspective

In the 20th century, the discussion started by Leonardo took a particularly confusing turn and produced reams of papers in a polemic between the *realist* and *conventionalist* schools of thought. A simplified view of the matter is that the realist position, represented by Ernst Gombrich, attributes an objective quality to the ability of perspective pictures to mimic reality. This ability comes from the fact that the picture realized by the rules of perspective furnishes a properly located observer with the same bundle of light rays that he would receive from the original object. The conventionalist position, as championed by Goodman, argues that instead we see classical perspective pictures as realistic simply because we have been acculturated to them.

Like in most such debates, things are not so clear as in this rapid caricature. First of all Gombrich's "realist" position is hardly clear-cut. In *Art and Illusion*(Gombrich 1960) he considered that there is no reality without interpretation and that seeing is inseparable from conceptualization: there is no *innocent eye*. Gilman (1992) argues that there is no *naturalistic* position on either side of the debate and sees it rather as a contention between conventionalist factions, Gombrich being merely a conventionalist with particular scruples where perspective is concerned, which Goodman saw as puzzling contradiction within the context of Gombrich's general position. Gombrich's objection was not so much against the contention that perception required interpretation but that this interpretation was just a cultural construct. According to Mitrović (Mitrović 2013b) it was this strong thesis of cultural relativism, that the totality of perceived reality is culturally constructed, as well as associated collectivist notions, that repulsed Gombrich. The positions are complex and Vergsten (Verstegen 2011) argues that both Gombrich and Goodman are "guilty of letting extraneous issues color their discussion of the (non)conventionality of perspective". Be all this as it may, the opposing positions have been argued confusingly and in so many skirmishes that scholar's careers can be erected on the mere accountancy of blows. As in most such debates, the boxers have reached belatedly and by exhaustion the obvious conclusion that, while fresh, they refused to reach by good sense: that neither extreme position is tenable (Frigg and Hunter 2010, page xx) and that the interesting question is how exactly purely optical mimesis and convention interact.

Fortunately, however, we are not here concerned with fleshing out such intricate business. Goodman argued famously that representation cannot be mere resemblance, since resemblance is symmetric (A resembles B implies B resembles A) while representation is not (a picture may represent a horse but a horse doesn't represent its picture).

We argue that in our present formulation we have cut this gordian knot by separating the matter of *resemblance* (anamorphic equivalence, which is indeed symmetric) from that of *representation* (perspective, which is not; A being a perspective of B doesn't make B a perspective of A).

So we are not interested in dealing with this debate in the terms set by Goodman. We have answered it: linear perspective *is not* conventional - it is the only perspective that is also an anamorphosis, and anamorphosis is an optical principle that stands on top of an objective empirical fact: the principle of linear occlusion.

We still care about this debate, however, because it influenced contemporary understanding of the purely mimetic possibilities of perspective and anamorphosis. This is especially unfortunate as Goodman was at its weakest when intruding into the field of mimesis proper, and has since been extensively refuted in detail - see for instance Mitrović (2013a). Consider the infamous phrase from *Languages of Art* (Goodman 1968). Goodman concedes for the moment the mimetic properties of perspective, but decried what he calls the "remarkable" requirements for such an illusion to occur:

"The picture must be viewed through a peephole, face on, from a distance, with one eye closed and the other motionless. The object also must be observed through a peephole, from a given (but not usually the same) angle and distance, and with a single unmoving eye. Otherwise, the light rays will not match." (Goodman 1968, page 12)

The conditions of observation, as described, seem indeed remarkable, even reminiscent of an infamous scene from Kubrik's *A Clockwork Orange* when Alex is bound to a chair with eyes clamped wide open, his gaze forcedly fixed forward. But more remarkable is that the argument easily crumbles not only under the scrutiny of the empiricist but merely under that of the tourist. For all it takes is a casual stroll into the church of Saint Ignazio de Loyola in Rome and a look at its illusionary ceiling, where Andrea Pozzo indeed used "perspective" (anamorphosis in our view) as an alternative to architecture proper (Fig. 1). The Jesuits wanted a grander building than it could afford, and Pozzo constructed it in virtuality, going through all the steps of an architectural project, with plan and elevation constructions, perspective renderings and finally anamorphic constructions . Here, the tourist looks to painted columns that extend the real ones so expertly that the seams are hard to identify. So expertly are these illusions constructed that it is hard to discern the true form of the curved ceiling. There is no peephole here, no need to close an eye and to keep the other unmoving. Goodman is stuck on the experience of small scale anamorphoses, and confuses the practical limitations of size with theoretical limitations. With a high ceiling at his disposal Pozzo constructs an *immersive* anamorphosis. At that large scale, the observer's two eyes work monocularly in good approximation. Also, the head can and is expected to rotate freely to contemplate the surrounding illusion. It is rather deceiving to say that the anamorphosis has to be seen "from a certain angle and distance". It has to be seen *from a certain point*, but on that point the eye is allowed to rotate freely. In such a large scale illusion as in Pozzo's ceiling at St. Ignazio's (or as in the large scale 19th century panoramas) the observer can even break a little with the theoretical impositions and get away with it. A large disk marks the spot from where the illusion of Pozzo should be observed, but a couple of steps away from it (or a few heads too tall or short) won't break the illusion substantially. It is quite remarkable that the arguments of Goodman could survive scrutiny in the 20th century when Pozzo not only understood in practice the full possibilities of anamorphosis but wrote on them so eloquently and clearly in his treatise. If the eye is not innocent, far less is the mind so. It was simply the right cultural moment for such an argument to be entertained, in spite of the evidence of the eye. In the present work we would hardly engage with these arguments, which in our view have aged badly, except that they were also used in much the same way by none other than Barre and Flocon as a justification for their development of spherical perspective (Barre and Flocon 1968) , or at least in the paper they wrote with Bouligand aiming at its mathematical and conceptual justification (Barre et al 1964). In the latter work the author's seek to justify their choice of spherical projection (the azimuthal equidistant projection, limited to one hemisphere) in the context of the search for a natural perspective, which in their view would satisfy the rather vague requirement of "ordering visible elements on a surface to form an image that causes on the spectator a sensation of volume and space" ("Ordonner sur une surface des éléments visibles, formant une image qui procure au spectateur des sensations de volumes e d'espace"). This desideratum (called "demande $\Delta$") being somewhat vague, the authors seek to approach it by the constraint of an ”axiome fondamental” $A$, a more geometrical axiom that should be, in their view, a necessary condition for such a perspective. This ”axiome $A$” turns out to be just Leonardo’s axiom (axiom 2). Watching what the authors proceed to do one finds that the ”surface” mentioned in the requirement $\Delta$ is in fact assumed to be a plane, and that ”seems smaller” means metrically smaller. This is the exact same conflation of concepts that was made by Leonardo. The author’s give much the same objections to linear perspective as Leonardo, as Goodman, and in particular, two objections usually attributed to Panofsky (Panofsky (1927, 1991)), but actually much older, which pretend to justify that whatever the ”natural” perspective is, it can’t be classical perspective:

The first objection is that the projection of lines in the retina is curved, not straight, hence we must in fact not see lines as straight. This is a rather blunt misconception that implicitly assumes some sort of theater of the mind where some homunculus in the visual brain sits to see the retinal image (and then we might ask, in infinite regression, how does *its* visual system work?). This is in fact an old objection already aimed at Kepler’s model of vision, which Descartes answered by pointing out that the shape of the stimulus in the retina and the sensation caused by the stimuli are two different things. The retinal image is also inverted and yet we don’t see the world upside down (Kemp 1990, p. 234).

A second objection is that the example of two parallel lines going to infinity goes against Leonardo’s axiom, as we know from experience that the lines seem to converge yet in linear perspective the distance between their projections remains constant. This confuses the metric properties of the projection with the perception of these when seen from $O$. Araújo (2016) refutes this by pointing out that if we put the plane of projection on top of the lines then the lines will coincide with their perspective. Hence to claim that the perspective of the lines does not seem to converge would be to claim that the lines themselves don’t seem to converge. To put it more strikingly, if the perspective drawing of a long wall is not visually realistic then the real wall itself is not visually realistic. Of course the answer is that both the lines and their perspectives do appear to converge because the subtended angle from $O$ decreases with distance in exactly the right way precisely because the drawn lines remain at a constant distance from each other. But much the same question had been raised by Schickhardt in 1623 and refuted by Kepler in a similar fashion (Kemp 1990, p.477-8). We can add that Leonardo’s paradox of the spheres has exactly the same answer: it doesn’t matter that the distal sphere has the largest projection, since the projection is seen from further away, in exactly the right measure to make it mimic the real sphere.

Being charitable and removing the layers of confusion and contradiction, one finds in the arguments of Barre, Flocon and Bouligand one concrete and well-defined objective: to find a projection onto the plane that should cover a 180 degree angle of vision around an axis, project onto a bounded region of the plane, and be anamorphic in the whole representation. This would imply that the metric properties should be such as to save appearances (the angular properties). This is a concrete interpretation of desideratum $\Delta$ and Leonardo’s axiom. The only problem is that it can’t be done, which is also, in the end, the conclusion of Bouligand. So the azimuthal equidistant projection is chosen as simply the better compromise among the available candidates. It is all about minimizing metric deformations as far as possible. Hence the vagueness of desideratum $\Delta$ - the more strict requirement of actual mimesis could not be satisfied.

But of course, no matter what word juggling we use, nobody in the end mistakes a spherical perspective for the real thing. There is certainly a visual evocation of the object, but not mimesis in the strict sense. An this is not, as the authors seem to assume, a problem that comes merely from the sphere being non-developable. The cylinder is developable (isometric to the plane) and yet unrolling it destroys the anamorphic effect. In the case of Barre and Flocon's spherical perspective, there is simply no point $O$ from which the drawing has a mimetic effect in the sense of the cone of the represention being the same as that of the represented object. The authors have it exactly the wrong way around: spherical perspective, unlike classical perspective (and unlike spherical *anamorphosis* ), is entirely conventional. It requires an agreed upon, rather sophisticated intellectual translation in order to be read. It's achievements are of a different order: it is a flat record of one half of a spherical anamorphosis, which has good metrical properties and - crucially - is easy to draw by hand, since its line projections are arcs of circle. The authors do injustice to their won work by chasing a quixotic goal while underselling their true achievements - the rational description of a construction method for the first rigorous spherical perspective.

These philosophical misconceptions are not merely academic. Araújo (2016) argues that looking exclusively at the minimization of distortion is probably the main reason why Barre and Flocon did not extend their perspective beyond the hemispheric view.Extending their perspective to a full 360-degree view would require acceptance of rather egregious metric deformations, which would defeat their standard objections against linear perspective.

In the view we present in this work, these deformations are not a problem, as the spherical perspective does not intend to contend for the place of "natural perspective", and is by definition flawed in its mimetic properties. Mimesis is left to the anamorphic step, and we bind all the complicated penomenological, optical, and physiological concepts purely within our axiom of radial occlusion. Rules of interpretation and empirical motivations are not allowed inside once the mathematical development begins.

In this view there is no contradiction between anamorphic mimesis and metric deformations on the perspective, just as there is no paradox between surveying and map making. The view that perspective is an entailment of an anamorphosis and a flattening also clarifies the fundamental role of anamorphosis and the arbitrary nature of perspective, giving a dual role to spherical anamorphosis and linear perspective,

Among anamorphoses, it gives the spherical one a canonical place, the points of the sphere being identified with rays, hence with the classes of anamorphic equivalence of spatial points; and to linear perspective it gives a special place, justifying the realist's view in its most naturalistic bent: linear perspective is indeed special, because it is the only perspective for which the flattening is trivial, that is, the only perspective that is still an anamorphosis. It therefore keeps a dual role of representation of visual information and of mimetic object, objectively creating a visual illusion, independently of any visual convention.

## Conclusion

We have presented a theoretical framework that separates and delimits the notions of perspective and anamorphosis in such a way as to render mute long points of contention regarding whether or not curvilinear perspectives are “natural” or ”correspond to the way we see”. The question is found to be not well posed, and the difficulties vanishes upon reformulation. Our view of anamorphosis may be seen as an update of Euclid’s optics with a reworking of the assumptions involved and the addition of vanishing points, hence taming the notion of infinity, which Euclid is careful to avoid. We make a single empirical assumption - the principle of linear occlusion - abstracted into a simple geometrical notion - anamorphic equivalence - and proceed from there to obtain both practical geometrical constructions and an elegant, symmetric notion of vanishing set. We derive from this notion of anamorphosis a definition of compact central perspectives that includes all the usual curvilinear perspectives. This family, strictly speaking, excludes classical perspective, since the infinite plane is not a compact, yet it includes the restrictions of linear perspective to any compact, with classical perspective as a degenerate limiting case. Yet this notion - by viewing perspectives as an entailment of two separate steps, one being mimetic (anamorphosis) and the other representational (flattening) - clarifies the special role of linear perspective, giving it a distinguished position in the bestiary of perspectives, as the perspective that is simultaneously an anamorphosis. In this way we provide a language that naturally dissolves a philosophical misconception dating back at least to the Renaissance, simply by making the terms of discourse correspond to the operational practice of perspective.

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