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The concept of function up to the middle of the 19th century

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1. Preliminary Remarks

Up to now the history of functionality has remained insufficiently studied. This important subject is actually avoided even by C. Boyer, whose book [1] on the history of the main concepts of the calculus ran into three editions. It goes without saying that this work, as well as others on the history of mathematics, does contain a number of statements on isolated features of the evolution of the concept of functional dependence and on several scholars' interpretation of this dependence. While undoubtedly valuable, such statements, even taken together, do not provide the whole picture. In addition, the opinions of various authors often differ from each other; in particular, they do not agree about the time when the concept of function actually originated. Perhaps the commonest point of view was voiced in the well known book of D. E. Smith ([2], p. 376) who stated, some fifty years ago:

> *... after all, the real idea of functionality, as shown by the use of coordinates was first clearly and publicly expressed by Descartes*

However, BOYER's opinion ([1], p. 156), formulated in connection with the works of FERMAT, a scholar contemporary with DESCARTES, is that

> *... the function concept and the idea of symbols as representing variables does not seem to enter into the work of any mathematician of the time.*

On the other hand, W. HARTNER & M. SCHRAMM ([3], p. 215) suppose that

> *The question of* [the] *origin and development* [of the concept of function] *is usually treated with striking one-sidedness: it is considered almost exclusively in relation to Cartesian analysis, which in turn is claimed (erroneously, we believe) to be a late offspring of the scholastic latitudines formarum.*

And, further,

> *...operating with functions had already reached a high degree of perfection by the time the first attempts were made to form a general conception of functions.*

Operations with functions, these authors contend, may be found in astronomical calculations of ancient scholars (*e.g.*, in those of PTOLEMY), then in Arabic science and indeed in AL-BĪRŪNĪ's works (to whom the authors' article is devoted).

In a book [4] published later than the one quoted above [1] and devoted to the history of analytic geometry C. BOYER points out other prototypes of functions in ancient Greek mathematics. Thus, considering the use of proportions, he says (p. 5):

> *This was somewhat equivalent to the modern use of equations as expressions of functional relationships, although far more restricted.*

The same author ([4], p. 46) as well as J.E. HOFMANN ([5], pp. 80–81), A.C. CROMBIE ([6], vol. ii, pp. 88–89) and others relate geometrical expressions of functions and computation of their values with the *theory of calculations* and with the *theory of latitudes of forms* of the 14th century. However, H. WIELEITNER ([7], p. 145) supposed that the idea of a function in the last theory contained

> *... nicht die geringste Vorstellung der zahlenmäßigen Abhängigkeit einer Grösse von einer anderen*

while E.T. BELL ([8], p. 32) credited even Babylonian mathematicians with an *instinct for functionality*. Lastly, an opinion on the existence of an idea of a function in antique mathematics has been put forward recently by O. PEDERSEN [9].

I shall not extend this list of opinions, some concordant and some discordant one with another, sometimes correct and sometimes incorrect or at least incomplete. I shall only add that, as regards the 19th century, the classical definition of a function included in almost every current treatise on mathematical analysis is usually attributed either to DIRICHLET or to LOBATCHEVSKY (1837 and 1834, respectively). However, historically speaking, this general opinion is inaccurate because the general concept of a function as an arbitrary relation between pairs of elements, each taken from its own set, was formulated much earlier, in the middle of the 18th century.

The importance of a historical analysis of the concept of functionality, especially bearing in mind contemporary discussions of this very concept, is obvious. Not attempting such a goal, I shall offer brief remarks describing only the main stages of development of the idea of function up to the middle of the 19th century. As I see it, these stages are:

(1) Antiquity, the stage in which the study of particular cases of dependences between two quantities had not yet isolated general notions of variable quantities and functions.

(2) The Middle Ages, the stage in which, in the European science of the 14th century, these general notions were first definitely expressed both in geometrical and mechanical forms, but in which, as also in antiquity, each concrete case of dependence between two quantities was defined by a verbal description, or by a graph rather than by formula.

(3) The Modern Period, the stage in which, beginning at the end of the 16th century, and, especially, during the 17th century, analytical expressions of functions began to prevail, the class of analytic functions generally expressed by sums of infinite power series soon becoming the main class used.

It was the analytical method of introducing functions that revolutionized mathematics and, because of its extraordinary efficiency, secured a central place for the notion of function in all the exact sciences.

Still, with all its fruitfulness, by the middle of the 18th century this interpretation of functions as analytic expressions proved itself inadequate so that a new, general definition of a function, which later became universally accepted in mathematical analysis, was introduced during that very period.

In the second half of the 19th century this general definition opened up widest possibilities for the development of the theory of functions but at once betrayed logical difficulties which in the 20th century caused the essence of the concept of function to be reconsidered (as, indeed, were the other main concepts of mathematical analysis). The struggle between different points of view continues; however, as I stated above, I will not discuss this period (or, rather, these two periods, connected respectively with the theory of functions and with mathematical logic), which have been described by A. F. MONNA [10].

Here I shall as a rule discuss single-valued functions of one real variable. Such functions are introduced in modern treatises on mathematical analysis in somewhat various wordings which have a common meaning. In the most general sense a function y of the variable x, $y=f(x)$, is a relation between pairs of elements of two number sets, X and Y, such that to each element x from the first set X one and only one element y from the second set Y is assigned according to some definite rule. Leaving aside logical difficulties inherent in the definition just given[1], I remark only that the functional rule, or "law", might be introduced in various forms: verbally; by a table of values of x and y; by an analytic expression; by a graph, *etc.*, subject only to the condition that this rule be definite and, once the value of x be given, sufficient for finding y.

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[1] A study of some aspects of the idea of function as represented in this definition (but not of the difficulties mentioned!) and, also, of the traditional terminology, aimed at a broader circle of readers is contained in KENNETH O. MAY's book ([11], pp. 253–262).

The idea of function understood in one or another sense is implicitly contained in rules for measuring areas of the simplest figures such as rectangles, circles, *etc.*, known even at the outset of civilization and, also, in the very first tables (some of them being tables of functions of two variables) of addition, multiplication, division, *etc.* used so as to facilitate calculations.

Obviously relations between numbers or, more generally, quantities, are encountered at every step in the realm of what is called elementary mathematics. However, this trivial fact is in itself fruitless in our search for the formation of the idea of function, its generalization and gradual comprehension, the concrete meaning which it acquires with the progress of scientific and philosophical thought and, lastly, for the role it plays during various stages of this progress.

2. Tabulated Functions and the "Symptoms" of Conic Sections in Antiquity

As stated above, the first stage of the concept of function is that of antiquity. Even in 2000 B.C. Babylonian mathematicians used widely for their calculations sexagesimal tables of reciprocals, squares and square roots, cubes and cube roots as well as some other tables. Tables of functions of two different types, the *step-function* and the *linear-zigzag-function*, as O. NEUGEBAUER ([12], Chap. 5) called them, were used in Babylonian astronomy during the reign of the SELEUCIDS for compilation of ephemerides of the sun, moon, and the planets. Empirically tabulated functions thereafter became the mathematical foundation for the whole subsequent development of astronomy.

New shoots of the concept of function made their appearance in Greek mathematics and natural science. Attempts attributed to the early PYTHAGOREANS to determine the simplest laws of acoustics are typical of the search for quantitative interdependence of various physical quantities, as, for example, the lengths and the pitches of notes emitted by plucked strings of the same kind, under equal tensions. Later on, during the ALEXANDRIAN epoch, astronomers developed a whole trigonometry of chords corresponding to a circumference of a fixed radius and, using theorems of geometry and rules for interpolation, calculated tables of chords actually tantamount to tables of sines such as those that came into use by the Hindus a few centuries afterward. The earliest of extant table of chords is found in PTOLEMY's *Almagest*, in which numerous astronomical tables of other quantities, equivalent to rational functions and, also, the simplest irrational functions of the sine are inserted [9].

However, the Greeks did not restrict themselves to use of tabulated functions. The main role in the theory of conics was played by their *symptoms* (*συμπτώματα*), *i.e.* by those basic planimetric properties of corresponding curves that follow immediately from their original (though actually unused) stereometric definition as being plane sections of the cone. A symptom of some conic section represents, a modern mathematician would say, for each point of the given curve one and the same functional dependence between its semichord y and the segment x of the diameter conjugate with the chord, the ends of this segment being the point of intersection of the diameter with the chord and the corresponding vertex. Antique geometers described symptoms verbally and, also, by means of *geometrical algebra* (the term is due to H. G. ZEUTHEN ([13], p. 7)), in which identities and equations of the first two degrees were represented by equalities of areas of certain rectangles. The meaning of these symptoms, the verbose antique description of which seems unusual to the modern ear, could be conveyed absolutely accurately in the language of analytic geometry by equations of curves of the second order with respect to their vertices,

$$y^2 = 2px \mp \frac{p}{a} x^2, \qquad y^2 = 2px.$$

However, opportunities provided by geometrical algebra were insufficient for conveying similarly the properties of curves of the third and fourth orders (cissoid and conchoid) and of some other algebraic curves known to Greek mathematicians, who had to define all these curves and also certain transcendent curves such as the quadratrix and the equiangular spiral, by means of special geometric or mechanical (kinematic) constructions.

Antique mathematicians introduced a peculiar classification of curves and of problems solved by means of these curves. Even before EUCLID they singled out three classes of geometric loci: plane (ἐπίπεδοι) loci – straight lines and circles; solid (στερεοί) loci – conic sections; and linear loci (τόποι γραμμικοί) – all other curves. It is really impossible to study here the origin and meaning of this classification, so remote it is from ours, which originated in the 17th century ([14], § 25).

In ancient Greece and in Hellenistic countries later to become Roman provinces functions introduced in connection with mathematical and astronomical problems were subjected to studies similar to those carried out in the mathematical analysis of modern times. According to the goal pursued, functions were tabulated by use of linear interpolation, and, in the simplest cases, limits of ratios of two infinitely small quantities were found as, *e.g.*, the limit of $\sin x/x$ as $x \to 0$.

Problems on extremal values and on tangents were solved by methods equivalent to the differential method; areas, volumes, lengths, and centres of gravity were calculated by integral methods equivalent to the calculation of integrals, *e.g.*, $\int_0^a x\,dx$ and $\int_0^a x^2\,dx$.

Lastly, problems in which roots of cubic polynomials had to be calculated were solved by using conic sections (curves of the second order). For this purpose roots of the corresponding equations were considered as coordinates of points of intersection, or contact, of two such appropriate curves. In this description I use the common modern terminology and notation, foreign to antique mathematics. I emphasize this fact as distinctly as possible.

Greek symbolism until about the third century A.D., apart from the use of digits, confined itself to denoting various quantities by different letters of the alphabet. No algebraic formula, no kind of literal algorithm, no analytical expression was ever introduced. Only in the works of the late Alexandrian mathematician DIOPHANTUS and, possibly, in those of his immediate predecessors, whose names have been forgotten, do some algebraic signs appear, as, for example, signs for the first six powers of the unknown quantity, a sign of equality, *etc.* However, with the downfall of antique society, this notation was not developed.

3. A General Notion of Function in Antiquity

Apart from the lack of symbolism, which impeded the whole progress of mathematics, the achievements of the Greeks both in increasing the number of functional dependences used and in discovering new methods to study them were indeed substantial and played a prominent role in the later development of mathematics right up to the creation of the new algebra, analytic geometry and the infinitesimal calculus in the 16th and 17th centuries. Nevertheless, I must repeat that there was no general idea of functionality in ancient times.

The problem of whether antique mathematicians possessed a general concept of function has been considered in detail also by O. PEDERSEN in his paper devoted to PTOLEMY's *Almagest* [9]. Quite correctly, PEDERSEN notices that, according to the PTOLEMAIC system of the world, positions of the sun, moon and planets are considered to change continuously and periodically in time; that the determination of these positions is accomplished by PTOLEMY by means of standard procedures, sometimes explained by a numerical examples or, alternatively, formulated verbally in a quite general manner; that, lastly, these standard procedures are used to compile various astronomical tables, *i.e.*, to tabulate corresponding functions (not only of one, but even of two, and, in several instances, of three variables). Noticing that the word *function* itself first appeared not in the works of antique mathematicians but much later, PEDERSEN ([9], p. 35) asks the next question:

> *But are we for that reason justified in concluding that they had no idea of functional relationships?*

His own answer is that everything depends on what actually is meant by a function. If, together with many mathematicians of bygone days, one is to interpret a function as an analytical expression, then the conclusion is that the ancients did not know functions.

> *But if,* continues PEDERSEN (p. 36), *we conceive a function, not as formula, but as a more general relation associating the elements of one set of numbers (viz, points of time $t_1, t_2, t_3, \ldots$) with the elements of another set (for example some angular variable in a planetary system), it is obvious that functions in this sense abound throughout the Almagest. Only the word is missing: the thing itself is there and clearly represented by the many tables of corresponding elements of such sets.*

I almost agree with all this. Of course, PTOLEMY, like other astronomers of that age and of earlier ones, knew that celestial coordinates of moving heavenly bodies periodically change with time, or that, in a given circle, chords of unequal lengths are related to arcs of unequal lengths. Above (see § 2) I have considered other, earlier instances of functions studied by Greek mathematicians who did not compile tables for the purpose. Also, two thousand years before PTOLEMY, tabular relationships were well known to Babylonians. All this notwithstanding, antique mathematical literature lacks not only words tantamount to the term *function* but even an allusion to that more abstract and more general idea which unifies separate concrete dependences between quantities or numbers in whichever form (verbal description, graph, table) these dependences happen to be considered. There is a good distance between the *instinct for functionality* (BELL) and the perception of it, and the same is true in regard to particular functions and the emergence of the concept of a function in one or another degree of generality. The use of the singular (*the thing itself*, *i.e.* the functional relation *represented* by various tables) by PEDERSEN in connection with the *Almagest* (see quotation above) seems to be incorrect in that it allows the whole passage to be interpreted as implying that functions corresponding to these tables were considered as particular instances of functional relationship in general.

A similar situation may be found in Greek mathematics as a whole. Its procedures of calculating or of determining individual concrete limits never led to an explicit formulation of general concepts of a sequence, variable, limit, infinitely small quantity, integral, or of general theorems concerning these objects[2]. Appropriate examples are quadratures and cubatures accomplished by ARICHIMEDES. Indeed, solving several problems (determining the area of a turn of a spiral, the volume of a spheroid, the area of a segment of a hyperboloid of revolution), he actually calculated one and the same integral $\int_0^a x^2\,dx$ or, to put it otherwise, the limit of one and the same "RIEMANN-DARBOUX" sum, completely carrying out the procedures required by the method of exhaustion each time anew. Noticing that also some other problems solved by ARCHIMEDES (quadrature of a parabola, determination of the centre of gravity of a triangle) could have been reduced to the calculation of the same integral, N. BOURBAKI ([15], p. 208) continues:

> ... *nous ignorons jusqu'à quel point il a pris consience des liens de parenté qui unissent les divers problèmes dont il traite (liens que nous exprimerions en disant que la même intégrale revient en maints endroits, sous des aspects géométriques variés), et quelle importance il a pu leur attribuer.*

It is impossible to answer this question, but of course ARCHIMEDES could not have failed to notice that the procedures of calculation in the first three problems were identical. Still, even for the case of the one function he used, $y = x^2$, he did not introduce a general notion of a definite integral (*cf.* [16]).

Generally speaking, studying mathematics of bygone ages, one often not only estimates its importance for the further development of this science (which is necessary) but also, not infrequently, one impermissibly broadens the interpretation of its ideas, linking them with modern, much more general, notions and conceptions. And it really happens that, as GOETHE'S FAUST remarked to his pupil WAGNER, the historian equates the spirit of the times with its reflection in his own mind:

Was ihr den Geist der Zeiten heisst,
Das ist im Grund der Herren eigner Geist,
In dem die Zeiten sich bespiegeln.

---

[2] One of the few exceptions is Proposition I from Book X of EUCLID'S *Elements* according to which (in our terminology), beginning from a certain term, each subsequent term of any sequence

$$a, aq_1, aq_1q_2, aq_1q_2q_3, \ldots \quad (q_k \leqq \tfrac{1}{2},\ \kappa = 1, 2, 3, \ldots)$$

becomes smaller than any given quantity b.

In particular, it would have been an impermissible modernization to see the idea of a variable quantity in the proper sense in the works of DIOPHANTUS, who did use substitutions for calculation of rational roots of indeterminate equations and whose method does make it possible in many instances to calculate an infinite number of values of the unknown of the indeterminate problem. At best, it is possible to speak, as D.T. WHITESIDE ([17], p. 197) does, about the notion or, rather, about the actual use of a *substitution variable*, but not about the *fully free variable* characteristic of the algebra of VIÈTE.

Ideas of change and of variable quantity were not foreign to Greek thought. Problems of motion, continuity, infinity, have been considered since the times of HERACLITUS and ZENO of ELEA, and to the study of these notions was devoted most of the ARISTOTELIAN *Physics* or natural philosophy (φύσις means nature). Using the term *motion* of matter in the broad sense of *change*, ARISTOTLE[3] distinguished three main forms of the world processes: alteration or change of quality; change of magnitude or quantity, *e.g.* growth or decrease; and local motion *(motus localis)*, this being the lowest form of motion, which necessarily accompanies the two other, higher forms of changes of matter. The local motion was subdivided into uniform motion, in which equal distances (segments or, say, arcs of a circumference) are travelled in equal times and difform motion; however, neither the (mean) velocity, such as the quotient s/t, nor, much less, the instantaneous velocity, was introduced in antiquity. Hence, neither the quantitative change nor the local motion, both of which have eventually found their representation in a more abstract notion of a variable quantity, became an object of mathematical study for the Greeks. This fact could be partially accounted for by the influence of controversies brought about by ZENO's paradoxes.

The connection of this fact with the general direction of the development of Greek mechanics and astronomy is striking. Neither of these sciences overstepped the limits of uniform motion, for the irregular motions of heavenly bodies were reduced in antique systems of the world to combinations of uniform circular motions. Irregular motion was not studied as such. Wherever possible, kinematic ideas were banished from the realm of pure mathematics. Isolated propositions found in EUCLID in which motion and superposition are used, as well as isolated cases of kinematic definitions of curves (say, of the quadratrix or of the equiangular spiral) do not change the general picture.

I have remarked above that even the so-called PYTHAGOREANS had glimpsed quantitative laws of nature. Apart from kinematic models of the system of the world, this quantitative aspect of laws of nature was little developed in Greek science.

Whatever the ideological or social causes and circumstances which brought about the features of ancient science just described, the mathematical thought of antiquity created no general notion of either a variable quantity or of a function. In the field of applications, mainly in astronomy, in which quantitative methods of research underwent the greatest development, the chief goal was the tabular representation of functions conceived as relations between discrete sets of given constant quantities isolated for practical purposes from continua of numerical values of quantities functionally related one to another.

[3] ARISTOTLE used the term μεταβολή (change) on a par with κίνησις (motion).

In this context, a similarity with the statical conception of CANTOR'S set theory, in which the intuitive idea of a variable quantity is reduced to an idea of a set of constant quantities given beforehand, suggests itself. In any case the thoughts of Greek mathematicians taken in general were far, far from the kinematic conception of a flowing quantity, characteristic of the infinitesimal calculus of the 17th, 18th and 19th centuries.

4. Kinematic and Geometrical Representation of Functional Relations. Theories of Calculations and of Latitudes of Forms

Occurring some time after the downfall of antique society, the new flowering of science in countries of Arabic culture did not, as far as is known, bring about essentially new developments in functionality. Still, the number of functions used increased, and methods of studying them improved. Thus every one of the main trigonometrical functions was introduced, methods of tabulating them were perfected (in particular, quadratic interpolation came to be used along with linear interpolation), and the study of positive roots of cubic polynomials by means of conic sections advanced essentially. Further progress was made in optics and astronomy. An exception, it seems, and an especially remarkable one from my point of view, was the analysis of accelerated motion in the *Mas'ūdic Canon* (ca. 1030) of AL-BĪRŪNĪ, which was partly preceded in 9th century by THĀBIT IBN QURRA ([3], pp. 212–214; [17a], p. 37–38).

Still, AL-BĪRŪNĪ's analysis and ideas did not exert much influence on his successors. The notion of function first occured in a more general form three centuries later, in the schools of natural philosophy at Oxford and Paris. Following such thinkers as ROBERT GROSSETESTE and ROGER BACON, these two schools, which flourished in the 14th century, declared mathematics to be the main instrument for studying natural phenomena. Departing from the ARISTOTELIAN doctrine of intension and remission of qualities and forms *(intensio et remissio qualitatum et formarum)*, they proceeded to the mathematical study of non-uniform quantitative and local motion.

Qualities or forms are phenomena such as heat, light, color, density, distance, velocity, *etc.*, which can possess various degrees *(gradus)* of intensity *(intensio)* and which, generally speaking, change continuously within some given limits. Intensities of forms are considered in relation to their extensions *(extensio)* such as, for example, quantity of matter, time *etc.* During such considerations a whole series of most important concepts came to be introduced, *e.g.*, instantaneous, or punctual, velocity *(velocitas instantanea, punctualis)*, acceleration (*intensio motus localis*, also *velocitatio*), and variable quantity, conceived as being a degree or a flux of quality *(gradus qualitatis, fluxus qualitatis)*. In all this, a dominant role was played by a synthesis of kinematic and mathematical thought.

> *Toute cinématique*, notices N. BOURBAKI ([15], p. 292), *repose sur une idée intuitive, et en quelque sorte expérimentale, de quantités variables avec le temps, c'est-à-dire de fonctions du temps.*

Simultaneously, an idea that quantitative laws of nature were laws of functional type gradually ripened in natural philosophy.

The doctrine of intensity of forms, or, otherwise, the theory of "calculations" *(calculationes)* and its most important part, kinematics, had been developed in England by WILLIAM HEYTESBURY, RICHARD SWINESHEAD, and others, mostly in the kinematic-arithmetical direction, while in France, where its main representative was NICOLE ORESME, it developed also in the geometrical direction. Of special interest is the theory of configurations of qualities *(de configurationibus qualitatum)*, or, in other words, of uniformity and difformity of intensities, or, in still other words, of latitudes of forms *(de latitudinibus formarum)*, developed by ORESME in the middle of the 14th century.

> *Every measurable thing*, wrote ORESME ([18], pp. 164–165), *except numbers* [which he, like the ancient Greeks, understood to be a set of units] *is imagined in the manner of continuous quantity (Omnis res mensurabilis exceptis numeris ymaginatur ad modum quantitatis continue).*

Therefore points, lines, and surfaces, in which, according to ARISTOTLE, the measure or ratio *(mensura seu proportio)* is initially found, are needed so as to measure these *things;* in all other things measure or ratio is learned by their mental relation with points, lines, and surfaces.

ORESME represents degrees of intensity by segments of corresponding lengths, "latitudes" *(latitudo)* perpendicularly erected upon the line of "longitudes" *(longitudo)*, the segments of which represent extensions; the ratio of two intensities of some quality is the same as that of the corresponding latitudes, so that, as ORESME himself says, latitudes and longitudes of some quality could be considered instead of its intensity and extension. The upper ends of the latitudes of some quality generate the "line of intensity" *(linea intensionis)* or, in other words, the "line of summit" *(linea summitatis)* which, as does also the figure bounded by this line, by the segment of the line of longitudes under consideration, and by the two extreme latitudes, represents the given quality and its "degrees". The angle between the latitudes and the line of longitudes could be chosen arbitrarily, although latitudes are most conveniently constructed perpendicular to the line of longitudes.

One of ORESME's remarks should be specially noticed, *viz*, that intensities could be called longitudes, so extensions should then be named latitudes. In this context "linear" *(linearis)* qualities are considered, the intensities of which are distributed among points of a line, but there exist also "surface" *(superficialis)* and "corporeal" *(corporalis)* qualities, distributed among points of a two-dimensional or three-dimensional continuum. Surface qualities are represented by solids with flat bases; as to corporeal qualities, the problem of their geometrical representation naturally presented ORESME with extraordinary difficulties, so that his remarks about them are far from clear ([18]; see especially Pt. 1, Chapters i-iv and x).

Thus these theories, developed in the 14th century, seem to be founded on a conscious use of general ideas about independent and dependent variable quantities; though direct definitions of these quantities are lacking, each of them is designated by a special term. The latitude of a "quality" is interpreted in a most general manner as being a variable quantity dependent on its longitude and, similarly, the "line of summit" is understood to be a graphical representation of some continuous functional relation ([6], Vol. II, p. 88; [19], p. 341). Thus, in these theories, a function is defined either by a verbal description of its specific property or directly by a graph.

In the mathematical language of modern times the latitude and longitude and also the corresponding semichords and segments of diameters of the antique theory of conic sections (see § 2) could well be called the ordinate and abscissa, respectively, with only one, albeit substantial, reservation: coordinates used in the 14^{th} century were always related to points of some curve rather than to arbitrary points of the plane. However, the same reservation applies even to DESCARTES. It really seems that coordinates of arbitrary points having no connection with some curve first appear in FR. VAN SCHOOTEN's commentary on the Latin edition of DESCARTES' *Geometry* (published in 1649), in the context of deducing the first known formulae for transformation of coordinates ([20], p. 191 and ff.).

The theory of latitude of forms is distinctive for its absolutely abstract preliminary interpretation of the problems solved, no significance being attached to the concrete form or quality. But then ORESME introduces also a kind of classification of the main kinds of linear qualities, to the study of which he essentially restricts himself. This classification is as follows ([18], Pt. 1, Chap. xi–xvi):

(1) Uniform quality *(qualitas uniformis)* with a constant latitude and the line of intensity being parallel to the line of longitudes. The corresponding figure is a rectangle.

(2) Uniformly difform *(uniformiter difformis)* quality ([18], pp. 192–193)

> *is one in which if any three points* [of the line considered] *are taken, the ratio of the distance between the first and the second to the distance between the second and the third is as the ratio of the excess in intensity of the first point over that of the second point to the excess of that of the second point over that of the third point; I call the first of those three points the one of greatest intensity. (Est cuius omnium trium punctorum proportio distantie inter primum et 2^m ad distantiam inter 2^m et 3^m est sicut proportio excessus primi supra 2^m ad excessum 2^i supra 3^m intensione, ita quod punctum intensiorem illorum trium voco primum.)*

Corresponding to this verbal description is our equation of a straight line passing through two given points $(x_1; y_1)$ and $(x_2; y_2)$:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

The line of intensity is here represented by the hypotenuse of a rectangular triangle or, alternatively, by the inclined upper side of a quadrangle having two right angles at its base, the difference between these cases being occasioned by whether this line meets the given segment of the line of longitudes at one of its ends (in ORESME's terminology, in this case the line is terminated at no-degree, *terminatur ad non gradum, i.e.*, at the zero point of latitude) or does not meet the given segment (is terminated in both extremes at some degree, *terminatur utrobique ad gradum*).

(3) Difformly difform *(difformiter difformis)* qualities, to which all other cases belong. This, the most extensive class of qualities could be "described negatively" *(potest describi negative)* as belonging neither to uniform nor to uniformly difform qualities ([18], pp. 194–195).

First, ORESME here distinguishes four simple *(simplex)* kinds of qualities, these being convex and concave (relative to the line of longitudes) arcs of a circle, not larger than the semicircle, and, also, similar arcs of an ellipse. (The word itself is not used; what is actually discussed is a curve proportional in altitude to a circular figure). Then, in the second place, ORESME discusses 63 "composite" *(compositae)* difform difformities the lines of intensity of which are composed of two or more arcs of previously described curves or of segments of a straight line. These combined lines somewhat resemble EULER's "mixed" curves *(lineae mixtae)*, see § 9; ORESME even uses the same term, *mixtio, mixture*.

An important component of the theory of calculations or latitudes of forms was the study of functions of time. Correctly pointing out the rudimentary nature of these studies, N. BOURBAKI ([15], p. 217) notices that obviously they have been carried out *sans considérations infinitésimales*. This, however, is not exactly so. Infinitesimal considerations were not only latently present in the concepts of instantaneous velocity and acceleration themselves but also explicitly used in solving a whole series of problems, such as, *e.g.*, problems of determining the areas of some figures unbounded in their extent, or the mean velocity of bodies the (instantaneous) velocities of which change by leaps according to some definite law an infinite number of times during a given interval of time divided into such parts as form a geometric progression. In these problems the main method of calculation was exactly the summation of infinite geometric progressions; later on, in the framework of the same theory, mathematicians encountered more complicated series, the sums of which were represented by (still unknown) transcendental quantities which they had to estimate approximately both from above and below (A. THOMAS, in 1509).

An achievement most important for mechanics if not for mathematics was the determination of the mean velocity of uniformly difform (uniformly accelerated) motion, notwithstanding failure to connect this problem with the problem of the free fall of heavy bodies. This achievement, first accomplished at Oxford, was described in the works of W. HEYTESBURY (in 1335?), R. SWINESHEAD, and J. DUMBLETON written almost simultaneously; they concluded that the uniformly difform motion is equivalent to a uniform motion with a velocity equal to the velocity of the accelerated movement at the middle moment of time[4]. Since all three scholars worked in the same place, at Merton College in Oxford, modern literature usually refers to their conclusion as the "Merton theorem" ([19], Chapter 5).

ORESME, also, proved this theorem. He represented the past distance or the proportional quantity, the total (mean) velocity *(velocitas totalis)*, by the area of a triangle or a trapezium ([18], Pt. iii, Chap. vii). Actually ORESME (21], pp. 37–39 and 122–124) went still further and determined that, for a zero initial velocity, the

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[4] A special feature of SWINESHEAD's research was his attempt to study a rectilinear motion, the velocity of which is proportional to the distance from a fixed point ([17], p. 217).

distance increases proportionally to the square of time and, also, that the distances travelled during equal intervals of time increase in proportion to odd numbers (1:3:5:7:...). As a matter of fact ORESME arrived at these results much as GALILEO was to do in his study of the free fall of heavy bodies *in vacuo*, published in the *Dialogo* (in 1632) and, again, in the *Discorsi e dimostrazioni matematiche* (in 1638). However, GALILEO's proof of the "Merton theorem" explicitly rests upon the method of indivisibles, whereas in ORESME's derivation infinitesimal considerations are only implied.

In the 15th century and also in the first half of the 16th the theory of latitudes of forms and calculations enjoyed wide fame, especially in England, France, Italy and Spain. It had been expounded in university courses and to it not only manuscript works but also a number of printed books had been devoted. Nevertheless, it was not much enriched at that time and, in particular, applications of its methods in physics and mechanics did not go beyond isolated, artificially posed, problems. As A.C. CROMBIE ([6] vol. ii, p. 89) puts it:

> *In the 14th century the idea of functional relationships was developed without actual measurements and only in principle.*

A survey of the general achievements of the theory under discussion might well conclude that in the development of some of the basic concepts of mathematics and mechanics, that of function included, in generalization and in abstraction the natural philosophers of the 14th century advanced far beyond all their predecessors taken together. Also, particular results of fundamental importance were arrived at; thus for example, the existence of figures of unbounded extent but of finite areas and the divergence of the harmonic series were discovered (ORESME). But then, potential possibilities provided by the new concepts were not widely exploited either in mathematics or in its applications. The schools of Oxford and Paris disposed only of scant means for concrete mathematical research; neither the representatives of these schools nor their immediate successors introduced any substantial novelties in computational techniques, algebra (except in the theory of proportions and the work of BRADWARDINE and ORESME), trigonometry, or methods of quadrature and cubature. An obvious disproportion developed between the high level of abstract theoretical speculations and the weakness of mathematical apparatus.

To determine the influence exerted by the theories of calculations and latitude of forms upon the mathematics of modern times is a rather complicated problem, the materials at our disposal being insufficient for an accurate and comprehensive solution. In many instances the similarity between the common concepts and particular results of the two is so great as hardly to be attributed to ordinary coincidence. More naturally, we may perceive here persistence of traditions sometimes transmitted by complicated means, *e.g.* by migration through a number of countries. Information could have been transmitted not only in written or printed form but also by means of lectures or even private conversations (some indubitable evidence of which exists).

An example is provided by GALILEO's study of the free fall of heavy bodies: Even the general resemblance of GALILEO's mathematical interpretation of the corresponding law to ORESME's interpretation of the Merton theorem implies a continuity of ideas; this implication becomes certitude in view of the fact that M. CLAGETT has found the Merton theorem in no less than seventeen books printed in the 16^{th} century.

Just as striking is the resemblance of some basic principles of DESCARTES' universal mathematics with ORESME's theory of latitudes of forms. What I mean here is the representation of all quantities and relations among them by means of geometric forms and, ultimately, by means of segments of straight lines, as DESCARTES himself stated in his *Regulae ad directionem ingenii*, written as early as 1629. We do not know whether DESCARTES actually read ORESME's works, but we do know how important for DESCARTES were his conversations with his friend, I. BEECKMAN, whose familiarity with ORESME's ideas and in particular with the Merton theorem is testified by his diary for the year 1618 ([19], pp. 417–418). Thus, some influence of ORESME upon DESCARTES is very probable; of course it is not contradicted by the direct connection between DESCARTES' coordinate method with the symptoms of conic sections as described by APOLLONIUS of Perga [5].

Also, it could hardly be doubted that the kinematic ideas of English calculators persisted in England and influenced the works of NEPER, BARROW and NEWTON. In particular, we know that SWINESHEAD was not forgotten even in the 17^{th} century; among those who read SWINESHEAD and who admired him highly was LEIBNIZ, ([1], p. 88).

5. Descartes' Variable Quantity; Algebraic Functions

Sure as I am that the ideas of both the Oxford and the Paris schools of natural philosophy played a noticeable role in the making of mathematics of modern times and, in particular, in the development of the general notion of function, still I do not maintain that this role was dominant, the more so as a new interpretation of functionality came to the fore in the 17^{th} century.

Decisive significance for the further development of the doctrine of functions was played, on the one hand, by the impetuous growth of computational mathematics and, on the other, by the creation of literal, symbolic algebra along with the corresponding extension of the concept of number, so as, by the end of the 16^{th} century, to embrace not only the whole field of real numbers but also imaginary and complex numbers. These were, so to say, preliminaries in mathematics itself to the introduction of the concept of function as a relation between sets of numbers rather than "quantities" and for analytical representation of functions by formulae. It is sufficient in this context to mention the progress in trigonometry and discovery of logarithms; what should be especially emphasized, though, is the introduction of numerous signs for mathematical operations and relations (in the first place, those of addition, substraction, of powers and of equality) and, above all, of signs for unknown quantities and parameters, which VIÈTE in 1591 denoted by vowels $A, E, I, \ldots$ and consonants $B, G, D, \ldots$ of the Latin alphabet, respectively. The importance of this notation, which, for the first time ever, made it possible to put on paper in symbolic form algebraic equations and expressions containing un-

[5] This connection has been recently pointed out again by M. SCHRAMM in a polemic with A.C. CROMBIE, who supposes that ORESME had made a step toward founding analytic geometry and that DESCARTES probably knew ORESME's works ([22], pp. 90–91).

known quantities and arbitrary coefficients (a word also originating with VIÈTE), could be hardly overestimated. However, the creator of the new algebra did not use his remarkable discovery to further the concept of function; "functional thought" was not characteristic of his mind.

VIÈTE's symbolism suffered from serious shortcomings and soon was amended by number of scholars, then extended beyond the realms of algebra and used in the infinitesimal calculus. DESCARTES, NEWTON, LEIBNIZ (who attached utmost importance to the appropriate selection of signs), EULER and other scholars of the highest calibre participated in this process of perfecting mathematical symbolism; this process continues in our time in all branches of mathematics.

On the other hand, in the exact sciences of former times, especially from the beginning of the 17$^{\text{th}}$ century, the new conception of quantitative laws of nature (see § 4) as establishing functional relations between numerical values of physical quantities had been gathering strength in ever-increasing measure and becoming more and more distinctive. In this process the creation of a broader and broader field of physical metrology with the introduction of quantitative measures of heat, pressure *etc.* played an important role; so did the swift gain in the precision of experiments and observations, brought about by the invention of various scientific instruments. Among the sciences mechanics, overtaking astronomy, came to the fore and, with it, its new branch, dynamics, soon to be joined by celestial mechanics. To study the relation between curvilinear motion and the forces affecting motion had become the chief problem of science. This problem gave rise to a series of problems in infinitesimal analysis, the solution of which had to be carried through to numerical answers.

As a consequence of all this, a new method of introducing functions was brought into being, to become for a long time the principal method in mathematics and, especially, in its applications. As before, functions not infrequently were introduced verbally; by a graph; kinematically; and, as before, tables of functions continued to be used most extensively. However, in theoretical research, the analytical method of introducing functions by means of formulae and equations came to the foreground.

We are able to tell almost exactly when this reversal of ideas took place. Even by the turn of the 16$^{\text{th}}$ century functions were being introduced only by means of old methods. In just this way the logarithmic function (the most important, along with the trigonometric functions), was introduced. J. BÜRGI calculated his logarithmic tables (published in 1620), starting from the relation, emphasized earlier by M. STIEFEL (in 1544) but known even to ARCHIMEDES, between the geometric progression of the powers of some quantity (*e.g.*, $q, q^2, q^3, \ldots$ and the arithmetic progression of its powers $(1; 2; 3; \ldots)$. This relation, as is evidenced by the interpolation process used by him, BÜRGI intuitively understood to be continuous. However, J. NEPER, whose work was published in 1614–1619, proceeded from a comparison of two continuous rectilinear motions, one being that of a point (L) moving uniformly and the other being that of a second point (N) the velocity of which is presumed proportional to its distance from some fixed point.[6] In this case, the distance travelled by point L is the (NAPIERIAN) logarithm of the distance travelled by point N.

[6] See Footnote 4 on the corresponding work of SWINESHEAD.

But then, only fifteen to twenty years after this, independently of each other, both FERMAT and DESCARTES in applying the new algebra to geometry presented the analytical method of introducing functions, thus opening a new era in mathematics.

In his *Introduction to plane and solid loci (Ad locos planos et solidos isagoge)* written somewhat before 1637 but published only in 1679, FERMAT ([23], p. 91) says:

> *As soon as two unknown quantities appear in a final equation, there is a locus, and the end point of one of the two quantities describes a straight or a curved line.*
>
> *(Quoties in ultima aequalitate duae quantitates ignotae reperiuntur, sit locus loco et terminus alterius ex illis describet lineam rectam aut curvam.)*

Here both the argument and the function are just called unknown quantities, this term actually meaning line-segments of continuously varying length.

Using VIÈTE's notation and also a rectilinear coordinate system, FERMAT then writes down equations of a straight line and, drawing upon APOLLONIUS' *Conics*, of some curves of the second order.

In more detail the idea of introducing a function analytically was developed by DESCARTES in his celebrated *Geometry* (*La géométrie*, 1637). His main purpose was to reduce the solution of all algebraic problems and equations to some standard procedures for constructing their real roots – *i.e.*, the coordinate segments of points of intersection of appropriate plane curves of the lowest possible order.

Relating a plane algebraic curve with an equation between the coordinates of its points, the coordinates being again understood as line-segments, DESCARTES ([24], p. 386) wrote:

> *Prenant successivement infinies diverses grandeurs pour la ligne y, on en trouvera aussi infinies pour la ligne x, et ainsi on aura une infinité de divers points tels que celui qui est marqué C, par le moyen desquels on décrit la ligne courbe démandée.*

Here, for the first time and completely clearly, is maintained that an equation in x and y is a means for introducing a dependence between variable quantities in such a way as to enable calculation of the values of one of them corresponding to a given values of the other one.

A little further on DESCARTES singles out the class of algebraic curves (which he calls geometric curves). All the points of these curves, as DESCARTES noticed, bear some relation to all the points of a straight line, it being possible to represent this relation by some equation, the same for each point of a given curve. By an equation DESCARTES, not being able to write down in symbols any equations of other kinds, actually meant an algebraic equation. Calling non-geometrical curves mechanical, DESCARTES then and there introduced his not yet perfect classification of geometrical curves in kinds *(genres)*, those of the first kind being lines described by equations of second degree; those of second kind, described by equations of the third and fourth degrees; those of the third kind, by equations of fifth and sixth degrees *etc.*[7]

[7] The universally adopted classification of algebraic curves introduced by NEWTON about the year 1670 was published only in his *Enumeration of lines of the third order (Enumeratio linearum tertii ordinis)* in 1704.

Introduction of functions in the form of equations effected a real revolution in the development of mathematics. The use of analytical expressions, the operations with which are carried out according to strictly specified rules, imparted a feature of a regular calculus to the study of functions, thus opening up entirely new horizons. Originating in the course of applying algebra to geometry, this method of representing functions was immediately extended to other branches of mathematics and in the first place to the realm of infinitesimal calculations.

In notes written approximately a hundred years ago but first published only in 1925 F. ENGELS ([25], p. 275), the great thinker, maintained that

> *Der Wendepunkt in der Mathematik war Descartes' **variable Größe.** Damit die **Bewegung** und **damit die Dialektik** in der Mathematik, und **damit auch sofort mit Notwendigkeit die Differential- und Integralrechnung,** die auch sofort anfängt ...*

The opinion of the noted mathematician H. HANKEL expressed approximately at the same time ([26], pp. 44–45) is much the same as the assertion just quoted:

> *... während die Alten den Begriff der Bewegung, des räumlichen Ausdruckes der Veränderlichkeit ... in ihrem strengen Systeme niemals und auch in der Behandlung phoronomisch erzeugten Kurven nur vorübergehend verwenden, so datiert die neuere Mathematik von dem Augenblicke, als Descartes von der rein algebraischen Behandlung der Gleichungen dazu fortschritt, die Grössenveränderungen zu untersuchen, welche ein algebraischer Ausdruck erleidet, indem eine in ihm allgemein bezeichnete Grösse eine stetige Folge von Werten durchläuft.*

Exactly at the time of DESCARTES and FERMAT functional thought became predominant in mathematical creative work. In connection with this I notice in passing also that the analytic geometry of DESCARTES and FERMAT, poor as it was at first in discovery if compared with the achievements of the theory of conic sections of the ancients, is potentially superior to the analytic geometry of APOLLONIUS and differs from it as much as the new symbolic algebra differs from the antique "geometrical algebra" (*cf.* [17], p. 294).

At the beginning the range of analytically expressed functions was restricted to algebraic ones, and DESCARTES even excluded from his geometry all mechanical curves as not being amenable to his method of analysis. However, a discovery made somewhat later, in the middle of the 17^{th} century by P. MENGOLI, N. MERCATOR, J. GREGORY and I. NEWTON independently made it possible to represent analytically any functional relation studied in those times.

What I mean here is the discovery of how to develop functions into infinite power series. Other infinite expressions of functions were afterwards added – infinite products, continued fractions *etc.* In an embryonic form the idea that an infinite expression was a "function" was not new, the infinite decreasing geometric progression having long been known (see § 4), but only in the second half of the 17^{th} century did the power series become the most fruitful and, as was supposed even a great while afterwards, the universal means for analytic expression and study of any function. P. BOUTROUX ([27], p. 117) even considered the theory of development of functions into power series to be the most original, remarkable, and fruitful component of the new mathematics as discovered by NEWTON and LEIBNIZ. In any case, exactly because of power series the conception of function as an analytic expression occupied the central place in mathematical analysis. Not without reason one of NEWTON's principal works was called *The method of Fluxions and infinite Series (Methodus fluxionum et serierum infinitarum).*

6. The Concept of Function as Understood by Newton (c. 1670) and Leibniz (1673–1694)

There was no great distance between first descriptions of the new concepts of function and the formulation of corresponding definitions which at first bore mechanical or geometrical features, both by force of tradition and because the methods of infinitesimal calculus were created mainly in the course of solution of problems in mechanics and of related geometrical problems.

The logarithmic function was a hyperbolic area; the elliptic function, an arc of a conic section; integrals were represented by distances, areas, arcs, volumes; differentials, by infinitely small coordinate segments; derivatives, by velocities or ratios of sides of infinitely small (characteristic) rectangle triangles, *etc.*

An especially clear kinematic-geometric interpretation of the basic conceptions of mathematical analysis was presented by NEWTON, who developed the ideas of his teacher, I. BARROW, as explicated in lectures delivered at Cambridge in 1664–1665 but published only later [28], which describe conceptions of time and motion and of their geometrical presentation originating with GALILEO and ORESME ([15], p. 220; [29], p. 240).

Like BARROW, NEWTON chooses time as a universal argument and interprets dependent variables as continously flowing quantities possessing some velocity of change.

In two letters to J. WALLIS, dated 27 August and 17 September 1692 (old style), NEWTON concisely explained his conception of the infinitesimal calculus, the development of which he had begun as early as 1664–1666. Somewhat shorter versions of these were published in 1693, in the Latin, enlarged edition of WALLIS' algebraic tract (English edition 1685). Here one reads that NEWTON ([30], p. 391) reduced his method to the solution of two problems:

> *Data aequatione fluentes quotcunque quantitates involvente, fluxiones invenire: et vice versa. Per fluentes quantitates intelligit indeterminatas, id est quae in generatione Curvarum per motum localem perpetuo augentur vel diminuuntur, et per earum fluxionem intelligit celeritatem incrementi vel decrementi.*[8]

In more detail NEWTON expounded these same ideas in a number of other works, as for example in the above-mentioned *Method of fluxions and infinite*

[8] A somewhat loose English translation made at the end of the 17th or at the beginning of the 18th century, published in 1961 ([31], p. 222 and ff), is included in the following passage:

The illustrious Mr Newton has reduced the Doctrine of Fluxions to two Prop: 1 Any Equation given wherein are Flowing Quantities to find the Fluxions, and ye Contrary. By flowing quantities he understands Indeterminate Quantities, that is which in ye Generation of a Curve by local motion perpetually Encrease or Decrease, & by ye Flux: he means the Celerity of their Increm't or Decrem't.

series, written ca. 1670 but published in an English translation from a Latin manuscript only in 1736 [32]. As is evident, even the two principal problems of the infinitesimal calculus were expressed in mechanical terms, *viz*, given the law for the distance, to determine velocity of motion (differentiation), and, given the velocity of motion, to determine the distance travelled (integration of differential equations and, in particular, of functions). However, NEWTON's conceptions plainly incline towards a more abstract understanding of philosophical and mechanical terms. Thus, concerning the universal argument, time, NEWTON says in his *Method of fluxions* ([32a], pp. 72–73) (I am here quoting his original Latin version, dating back to 1670–1671):

> *We can have, however, no estimate of time except in so far as it is expounded and measured by an equable local motion, and furthermore quantities of the same kind alone, and so also their speeds of increase and decrease, may be compared one with another. For these reasons, in what follows I shall have no regard to time, formally so considered, but from quantities propounded which are of the same kind shall suppose some one to increase with an equable flow: to this all the others may be referred as though it were time, and so by analogy the name of 'time' may not improperly be conferred upon it.*
>
> *(Cùm autem temporis nullam habeamus aestimationem nisi quatenus id per aequabilem motum localem exponitur et mensuratur, et praeterea cùm quantitates ejusdem tantùm generis inter se conferri possint et earum incrementi et decrementi celeritates inter se, eapropter ad tempus formaliter spectatum in sequentibus haud respiciam, sed e propositis quantitatibus quae sunt ejusdem generis aliquam aequabili fluxione augeri fingam cui caeterae tanquam tempori referantur, adeoque cui nomen temporis analogicè tribui mereatur.)* [9]

Somewhat further ([32a], pp. 88–91) NEWTON calls the fluent, which plays the role of independent variable, a correlated quantity *(quantitas correlata)*; the dependent quantity he calls related *(relata)*. Thus only the basic notions are introduced kinematically, so actually the method of fluxions is developed for the fluents, expressed analytically either in a finite form or by sums of infinite power series, those decimal fractions of mathematical analysis.

At the outset, LEIBNIZ also arrived at the basic notions of differential and integral calculus, developing them from the geometry of curves. It is sufficient to recall that as early as in his basic memoir on the differential calculus, *A new method for maxima and minima as well as tangents, ... and a remarkable type of Calculus for them (Nova methodus pro maximis et minimis, itemque tangentibus ..., et singularis pro illis calculi genus)*, in 1684, he described the differential (dy) of an ordinate

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[9] *Cf.* ORESME ([18], pp. 274–275):

... therefore time so stated is in no way "difform" or even properly "uniform", as time also is not said to be "quick" or "slow". However, time can be said improperly to be uniform, since that duration which is time in the aforesaid way is not properly measured except by uniform motion, i.e. regular motion. (... idcirco tempus sic dictum nullo modo est difforme nec etiam proprie uniforme, sicut etiam tempus non dicitur velox vel tardum. Verumptamen improprie tempus potest dici uniforme quoniam illa duratio que tempus est modo predicto non mensuratur proprie nisi per motum uniformem, id est, regularem).

of some curve ([33], v, p. 220) as being a segment whose ratio to dx, an arbitrary increment of the abscissa, is equal to the ratio of its ordinate to the subtangent.

The word "function" first appears in LEIBNIZ' manuscripts of August, 1673, and in particular in his manuscript entitled *The inverse method of tangents, or about functions (Methodus tangentium inversa, seu de functionibus).* At first the determination of subtangents, subnormals and other segments related to variable points of a curve is here treated both for "geometrical" and "non-geometrical" curves for which ([34], p. 44)

> *the relation between its applicate [ordinate] ED and abscissa AE is represented by some equation known to us (in qua Relatio applicatae ED ad abscissam AE aequatione quadam nobis cognita explicatur).*

Then LEIBNIZ ([34], p. 47) goes on to consider the inverse problem of determining applicates (ordinates) from a given property of the curve's tangent or of

> *other kinds of lines which, in a given figure, perform some function (ex aliis linearum in figura data functiones facientium generibus assumtis).*

It should be remembered that the Latin verb *fungor, functus sum, fungi* means, to perform, to fulfil (execute) an obligation, *etc.* As D. MAHNKE remarks ([34], p. 47):

> LEIBNIZ *gebraucht allerdings in der vorliegenden Handschrift für diese gesetzliche Beziehung, in der die Ordinate einer Kurve zu ihrer Abszisse ... steht, noch nicht das Wort Funktion; aber wie der Anfang der Handschrift beweist, hat er den Funktionsbegriff schon im weitesten Sinne gebildet und benennt ihn mit dem Wort relatio. Auch an der vorliegenden Stelle, bei der allgemeinen Formulierung der dem umgekehrten Tangentenproblem ähnlichen Probleme, hat das Wort Funktion noch nicht ganz den heutigen mathematischen Sinn, sondern eher den, den wir in der Sprache des täglichen Lebens mit ihm verbinden; es bedeutet also etwa die "Verrichtung", die ein Glied eines Organismus oder ein Teil einer Maschine zu leisten hat, seine Aufgabe, Stellung oder Wirkungsweise. "In figura functionem facere" bedeutet also z.B.: die Kurve berühren, auf ihr senkrecht stehen, ihre Subtangente oder Subnormale bilden usw., wobei natürlich immer ein begrenztes Stück der so oder so "funktionierenden" Linie, z.B. das Tangentenstück zwischen Berührungspunkt und X-Achse, in Betracht zu ziehen ist.*

But further on in the same manuscript the term function takes on a new meaning as a general term for different segments connected with a given curve.

> *So spricht er* [LEIBNIZ], said D. MAHNKE (p. 48), *an späteren Stellen der Handschrift von dem regressus a Tangentibus aut aliis functionibus ad ordinatas, und in diesem Sinne ist auch der Ausdruck de functionibus in der Überschrift zu verstehen.*

In the same relatively broad sense of differential geometry a definition of a function first appeared in print in a few articles of LEIBNIZ published in 1692 and

1694. There he calls functions (functiones, fonctions) any parts of straight lines, *i.e.*, segments obtained by constructing infinite straight lines corresponding to a fixed point and to points of a given curve[10]. He explains that he actually means abscissae, ordinates, chords, segments of tangents and of normals cut off by coordinate axes, segments of subtangents and subnormals *etc.*, and in the same sense the word function was used by JAKOB BERNOULLI in his work in the *Acta Eruditorum* for October 1694.

However, such a definition of a function did not correspond to any broader analytical context. The correspondence of LEIBNIZ with JOHANN BERNOULLI during 1694–1698 actually traces how the want of a general term to represent arbitrary quantities dependent on some variable soon brought about the use of the term function in the sense of an analytical expression.

7. A Function as an Arbitrary Analytic Expression: Johann Bernoulli (1694–1718) and Euler (1748)

In his letter of 2 September 1694 BERNOULLI ([33], iii, p. 150), telling LEIBNIZ about his discovery of the development of $\int n\,dz$ into an infinite series

$$nz - \frac{1}{1\cdot 2} z\cdot z\cdot \frac{dn}{dz} + \frac{1}{1\cdot 2\cdot 3} z^3 \frac{ddn}{dz^2} - \cdots$$

(which, however, LEIBNIZ already knew) wrote:

> *by n I understand a quantity somehow formed from indeterminate and constant* [quantities] *(per n intelligo quantitatem quomodocunque formatam ex inderminatis et constantibus).*

In the same year this discovery, expressed in the same words, appeared in BERNOULLI's article ([35], i, p. 126) in the *Acta Eruditorum*. The term function is not yet used. It is lacking also in BERNOULLI's letter of 25 Aug. 1696 ([33], iii, p. 324) where he proposes to denote by

$$\overset{1}{X},\ \overset{2}{X}$$

> *diverse quantities given somehow by an indeterminate* [quantity] *x and by constants* ... [either] *algebraically or transcendentally (quantitates diversas utcunque datas per indeterminatam x et constantes ... vel algebraica, vel transcendenter).*

JOHANN BERNOULLI first uses the word function only two years later, in an article appended to his letter of 5 July 1698 and devoted to the solution of the isoperimetric problem posed by his brother JAKOB: among all the curves *BFN*

[10] See LEIBNIZ, *De linea ex lineis numero infinitis ordinatim ductis inter se concurrentibus formata* ..., Acta Eruditorum, Apr. 1692 ([33], V, p. 268); *Nova Calculi differentialis applicatio et usus* ..., Acta Eruditorum, July 1694 ([33], V, p. 306); *Considérations sur la différence qu'il y a à observer entre l'Analyse ordinaire et le nouveau Calcul des Transcendentes*, Journal des Sçavans, Aug. 1694 ([33], V, p. 307–308). *E.g.* ([33], v, p. 306)

Functionem voco portionem rectae, quae, ductis ope sola puncti fixi et puncti curvae cum curvedine sua dati rectis, abscinduntur.

of given length and base BN to find a curve any powers of the ordinates FP of which generate ordinates PZ of (another) curve BZN of a maximum, or minimum, area.

Actually, JOHANN BERNOULLI ([33], iii, pp. 506–507) even generalizes this problem supposing it to be

> *to find* [a curve] *BFN, the ordinates FP of which, raised to a given power or, in general, some functions of these ordinates, etc.* (*illa* [curva] *BFN, cujus applicatae FP ad datam potestatem elevatae seu generaliter earum quaecunque functiones etc.*).

In a French translation published in 1706 in the *Mém. Acad. sci. Paris* ([35], t. 1, p. 424) this passage from the original reads thus:

> *trouver la courbe BFN telle, que ses appliquées FP élevées à une puissance donnée, ou généralement telle, que les fonctions quelconques de ces appliquées PZ, exprimées par d'autres appliquées PZ etc.*

BERNOULLI does not explain in what sense he takes "some" *(quaecunque)* functions; nevertheless, he could hardly have meant anything other than analytic expressions already known by that time.[11]

[11] It seems that the first approach to a general definition of a function as being an "analytic" expression and, moreover, allowing an infinite process to be involved, is found in J. GREGORY's *Veritable quadrature of the circle and hyperbola (Vera circuli et hyperbolae quadrature)*, published in 1667. This book being unavailable, I shall describe the corresponding definition introduced by GREGORY as expounded in the article of M. DEHN & E. HELLINGER ([36], p. 477):

> *we call a quantity x composed (compositum) of other quantities a, b, ... if x results from a, b, ... by the four elementary species, extracting of roots or by any other imaginable operation (quacunque alia imaginabili operatione).*

By these last words GREGORY meant composition of convergent sequences, he himself having introduced the term *convergens*, possibly transplanting it into mathematics from optics, with which he occupied himself a good deal. Note that GREGORY used the term *terminatio* for the limit of a convergent sequence *(series convergens)*.

Addendum. Having forwarded this article to the Editor, I am now able to add the relevant passage from GREGORY's work *Vera circuli et hyperbolae quadratura* (1667), for which I am greatly indebted to Dr. D.T. WHITESIDE [37a, p. 9]:

Definitiones

> 5. *Quantitatem dicimus à quantitatibus esse compositum; cum à quantitatum additione, subductione, multiplicatione, divisione, radicum extractione, vel quacunque alia imaginabili operatione, fit alia quantitas.*
>
> 6. *Quando quantitas componitur ex quantitatum additione, subductione, divisione, radicum extractione; dicimus illam componi analyticè.*
>
> 7. *Quandò quantitates à quantitatibus inter se commensurabilibus analyticè componi possunt, dicimus illas esse inter se analyticas.*

Definition 5 corresponds to the definition published by J. BERNOULLI in 1718 (see § 7); only the *any other imaginable operation* means for GREGORY some rather general infinite process called by him *our sixth operation (nostra sexta operatio).*

Definition 6, which defines the *quantity composed analytically (analyticè)*, corresponds to a certain degree to our algebraic function. It is difficult to agree with M. BARON who says [29, p. 8] that: *The expression analytic was first used by James Gregory who defined an analytic quantity as one obtainable by algebraic operations together with passage to the limit.* The word *analyticè* is employed here by GREGORY in VIÈTE's sense. As C. J. SCRIBA says [37b, p. 13–14]: "*Analytisch*" *nennt er dabei eine Grösse, die durch endlich viele der fünf Grundoperationen aus zueinander kommensurablen Grössen zusammengesetzt ist.*

On 29 July 1698 LEIBNIZ expressed his satisfaction with JOH. BERNOULLI's use of his (LEIBNIZ') term "function" ([33], iii, p. 526) after which both correspondents exchanged their opinions a few times more about the most appropriate notation for a function of one or many variables. Both favored distinguishing functions by means of indices, not in the way we do it now but thus: $\overline{x}^{\llcorner 1 \lrcorner}$, $\overline{x}^{\llcorner 2 \lrcorner}$, $\overline{x; y}^{\llcorner 1 \lrcorner}$, $\overline{x; y}^{\llcorner 2 \lrcorner}$ and so on ([33], iii, p. 537).

In the same place LEIBNIZ proposed to write $\dot{d}z$ for the ratio $dz{:}dx$. This notation did not endure.

Simultaneously or somewhat earlier LEIBNIZ introduced into general use the words "constant" and "variable",[12] "coordinates" (in 1692 ([33], v, p. 268)), and "parameter" in the sense of an arbitrary constant segment or quantity [in a manuscript written ca. 1679 ([33], iii, p. 103) and, in 1692, in a printed work (*ibidem*, p. 268)], *etc.* Lastly, he found inconvenient the terminology which DESCARTES had introduced, and so he changed it. DESCARTES had classified curves as "geometrical" and "mechanical", erroneously excluding the latter from geometry as being insusceptible to study by means of his (algebraic) method; see also § 5.

LEIBNIZ instead divided functions and curves into two classes: algebraic, namely, those which could be represented by an equation of a certain order *(certi gradus)*, and transcendental. Transcendental functions and curves could also be subjected to an exact study and calculus, although of a different nature, by their representation by equations of an indefinite *(gradus indefiniti)* or infinite order which ([33], V, pp. 123–124 and 228, 1684 and 1686 respectively)

> *transcend any algebraic equation (omnem aequationem algebraicam transcendant)*[13].

LEIBNIZ' definition of transcendental functions as non-algebraic ones has been repeated in textbooks right up to our day. As to the intrinsic property of transcendental complex analytic functions (possession of at least one singular point besides poles and branch points of finite order), this was to be established only in the middle of the 19th century. However, twenty years had to pass until the new definition of a function appeared in print. All this time the term function itself remained little known. It is lacking in CHR. WOLFF's *Mathematisches Lexicon*, published in 1716, in which, nevertheless, two related articles were included, *Quantitas constans, eine unveränderliche Grösse*, and *Quantitates variabiles, veränderliche Grössen.* The second article mentions that the distinction between the two kinds of quantities is essential in LEIBNIZ' new analysis ([38], columns 1144 and 1149–1150).

Expression of one variable quantity by means of another one is also treated in the same source, though in another article, *Abscissa, die Abscisse*, it is as follows ([38], columns 3–4):

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[12] These two words happened to enjoy a wider fame because of the first printed treatise on differential calculus, written by L'HOSPITAL and published in 1696, in which [37] the *quantités constantes* and *quantités variables* are defined right from the beginning.

[13] In the manuscript, dated 1679, LEIBNIZ ([33], iii, p. 103) called algebraic curves "analytic" *(curva analytica)*; in the same place also the term "transcendent curve" is found.

Durch die Relation der Abszisse AP zu der halben Ordinate [we should have preferred: to the (whole) ordinate] *PM pfleget man die krummen Linien von einander zu unterscheiden.*

A few examples of functions are presented in such articles as *Aequatio exponentialis, eine Exponential-Gleichung; Aequatio indeterminata, eine undeterminirte Gleichung* and *Aequatio transcendens, eine Transcendentische Gleichung.*

The idea of functional relationship is not even mentioned in such articles as *Calculus differentialis, die Differential-Rechnung* and *Calculus integralis, seu summatorius, die Integral-Rechnung.* The idea that mathematical analysis is a general science of variables and their functions seems to be due to EULER, who said just this in the preface to his famous *Introductio in analysin infinitorum,* completed ca. 1744 and published in 1748 [39].

The first explicit definition of a function as an analytic expression to appear in print is in J. BERNOULLI's article *Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes sur les isopérimètres,* published in the *Mém. Acad. roy. sci. Paris* for 1718. Here it is that one finds ([35], ii, p. 241)

Définition. On appelle fonction *d'une grandeur variable une quantité composée de quelque manière que ce soit de cette grandeur variable et de constantes.*

In the same place BERNOULLI also proposed the Greek letter φ as a notation for a *caractéristique* of a function (the term is due to LEIBNIZ), still writing the argument without brackets: φx. Brackets, as well as the sign f for function are due to EULER who used them in his article *E.* 45, communicated in 1734 and published in 1740.

In his definition BERNOULLI gave no indication of how to constitute functions from the independent variable. But then, it is obvious that he actually meant analytic expressions of functions, this being in accord with the basic tendency in the development of the infinitesimal analysis which, retaining and even strengthening its connections with geometry, mechanics and physics, during the 18th century became a scientific discipline more and more self-contained in its principles. All the initial concepts of the calculus gradually lose their geometrical and mechanical shell, are formulated arithmetically or algebraically, and begin to be apprehended as logically preceding similar concepts of other exact sciences.

The process of making mathematical analysis into an autonomous scientific discipline, which in the 19th century turned into a process of arithmetizing it, was protracted. At first it subdued mechanics, making it a part of mathematical analysis: indeed, for NEWTON a fluxion of a quantity was the velocity of its change; for LAGRANGE velocity was a derivative of the function which represented distance in terms of time. Moreover, in his *Mécanique analytique* (in 1788) LAGRANGE declared mechanics to be a part of mathematical analysis the exposition of which demanded neither figures nor geometrical or mechanical considerations in general. There was a similar trend concerning the relation of mathematical analysis to geometry, the methods of which ceased to be applied not only for defining, but even for illustrating basic concepts of the calculus.

This is testified by even a most cursory comparison of L'HOSPITAL's *Analyse des infiniments petits, etc.* (published in 1696) with EULER's and LAGRANGE's courses, in which geometrical illustrations are not used at all. Of course, geometrical intuition did continue to play its constructive role; of course, there always were scholars who substantiated analytical "existence theorems" by referring to geometrical obviousness; and, of course, the educational value of geometrical and mechanical analogies came to be understood once again.

The general tendency does not change, however, so that in due time (though only in the second half of the 19th century) it became necessary to define analytically such geometrical notions as the area of a surface, the length of a curve, *etc.*, which before that seemed to be intuitively obvious.

Further essential development of the concept of function was effected by LEONHARD EULER, the pupil of JOH. BERNOULLI. In Chapter I of Volume I of his *Introductio in analysin infinitorum*, in 1748 (*E.*101) EULER subjected to more detailed study the concept of function as actually used in mathematical analysis. He began by defining initial notions. According to EULER, a constant is a definite quantity always assuming one and the same value while a variable is introduced as the set (sometimes as one or another subset) of complex numbers.

> *A variable quantity*, wrote EULER ([39], p. 17), *is an indeterminate, or universal, quantity, which comprises in itself absolutely all determinate values.*
>
> *(Quantitas variabilis est quantitas indeterminata seu universalis, quae omnes omnino valores determinatos in se complectitur.)*
>
> *Thus*, he continues (p. 18), *a variable quantity comprises in itself absolutely all numbers, both positive and negative, both integer and fractional, both rational and irrational and transcendent. Even zero and imaginary numbers are not excluded from the meaning of a variable quantity.*
>
> *(Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales et transcendentes. Quin etiam cyphra et numeri imaginarii a significatu quantitatis varabilis non excluduntur.)*

In his definition of a function EULER once more followed his teacher, JOH. BERNOULLI, changing however the word "quantity" into "analytic expression" *(ibidem)*:

> *A function of a variable quantity is an analytic expression composed in any way from this variable quantity and numbers or constant quantities.*
>
> *(Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitatibus constantibus.)*

I shall have to leave aside both EULER's introduction of functions of a complex variable on a par with those of a real variable (a step of utmost importance) and also some formal inconvenience occasioned by the fact that EULER did not consider constants to be functions in their own right. To me, it is important that EULER was the first to attempt to answer the question, what is the extent of the term *analytic expression?* Or, which methods of its composition are actually meant? [14]

[14] This problem had been encountered even in the 17th century (see Footnote 11) when, in his own way, J. GREGORY attempted to solve it.

Enumerating operations by means of which analytical expressions are composed, EULER starts with algebraic operations (to which he refers also the solution of algebraic equations) and then names various transcendent ones, arriving in particular at exponential and logarithmic functions and at an infinite number of other functions furnished by the integral calculus, integration of differential equations included.

Then, EULER singles out explicit and implicit functions, the latter being those originated by solution of equations, and formulates theorems on the existence of a function inverse to a given one and of a function represented parametrically (given y and x as functions of z, y is a function of x and, inversely, x is a function of y). Practically speaking ([39], p. 25), because of the imperfection of algebra such functions are not always capable of being represented explicitly;

> *meanwhile, nevertheless, this reciprocity of functions is understood as if all equations could be solved.*

> *(interim tamen nihilominus, quasi omnes aequationes resolvi possent, haec functionum reciprocatio perspicitur).*[15]

I shall show in § 8 how EULER classifies these last methods of introducing functions under his first general definition of a function. For the time being, I remark that EULER's classification of functions (described above, to be sure, not in every detail) was put to use in its entirety.

8. Analytic Functions

Obviously, it seemed impossible to enumerate various methods of expressing functions analytically so in Chapter 4 of his *Introductio* EULER reduces them all to a single one, declaring the universal and, simultaneously, the most convenient form of an analytic expression of a function to be an infinite power series of the type

$$A+Bz+Cz^2+Dz^3+\cdots.$$

Being of course unable to prove that any function could be developed into such a series, he offered the challenge ([39], p. 74):

> ... *should anyone doubt,* [his] *doubt will be eliminated by the very development of one or another function.*

> *(si quis dubitet, hoc dubium per ipsam evolutionem cuiusque functionis tolletur).*

> However, added EULER, *to render this explication broader, not only positive integral powers of z should be admitted, but any powers. Thus there will be no doubt that any function of z could be transmuted into an infinite expression of the type*
> $$Az^{\alpha}+Bz^{\beta}+Cz^{\gamma}+Dz^{\delta}+\cdots$$
> *the exponents α, β, γ, δ etc. denoting any numbers.*

[15] What is here said about algebraic equations holds, *mutatis mutandis*, also for any other equations.

> *(Quo autem haec explicatio latius pateat, praeter potestates ipsius z exponentes integros affirmativos habentes admitti debent potestates quaecunque. Sic dubium erit nullum, quin omnis functio ipsius z in huiusmodi expressionem infinitam transmutari possit* (see above) *denotantibus exponentibus* α, β, γ, δ, *etc. numeros quoscunque.)*

Indeed, the overwhelming majority of functions used in mathematical analysis in EULER's time were analytic (in our sense of the term) in the whole domain of their definition, except perhaps at isolated values of the argument and, in special cases, could have been developed in series of terms containing fractional or negative powers of the argument.[16] No wonder that power series and, to a lesser extent, infinite products and developments into sums of partial fractions or continued fractions are used in Volume 1 of the *Introductio* as the main instrument for studying various classes of elementary functions.

As noticed above, theorems on the existence of implicit or parametric functions from EULER's viewpoint could have been considered within the limits of a general definition of a function. The point is that, according to EULER, an arbitrary algebraic equation of any power is solvable in radicals. In a more general case, because each function, y, could be represented by some series of terms containing powers of the argument, z, this argument could be expressed in terms of y by inverting the series; procedures of inverting series had been introduced by NEWTON.

JOH. BERNOULLI and EULER's definition of a function as being an analytic expression the most general form of which is a power series was accepted by many other mathematicians right up to LAGRANGE who, referring in his *Théorie des fonctions analytiques* (in 1797) to LEIBNIZ and BERNOULLI called a function any *expression de calcul*[17].

In passing I shall notice that LAGRANGE, like EULER and other mathematicians of the 18th century, considered it beyond doubt that any function of mathematical analysis could be represented by a series of terms proportional to real powers of the independent variable; moreover, LAGRANGE ([40], Pt. I, Chapter I) even attempted to prove that, generally, the powers occuring are positive integers, while fractional or negative powers could occur only in cases corresponding to isolated, special values of the argument.

Thus, a function, defined in the beginning of Volume 1 of EULER's *Introductio* as any analytic expression, is later declared to be (in our terminology) a function analytic everywhere except, perhaps, at isolated special points in the vicinity of which it could be represented by a generalized power series (see also § 10).

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[16] In essence, such an interpretation of analytic representability is similar to the conception held by J. GREGORY (see Footnote 11).

[17] As LAGRANGE says ([40], p. 15):

On appelle fonction d'une ou de plusieurs quantités, toute expression de calcul dans laquelle ces quantités entrent d'une manière quelconque, mêlées ou non d'autres quantités qu'on regarde comme ayant des valeurs données et invariables, tandis que les quantités de la fonction peuvent recevoir toutes les valeurs possibles. Ainsi, dans les fonctions on ne considère que les quantités qu'on suppose variables, sans aucun égard aux constantes qui peuvent y être mêlées.

9. Continuous and Discontinuous (mixed) Functions in Euler's Sense; the Controversy About the Vibrating String

Actually studied in Volume 1 of the *Introductio* are only analytic functions. However, EULER knew that functions of a different kind also exist. This fact is noticed in the beginning of Volume 2 of the *Introductio*, which is devoted mainly to the theory of plane curves. Just as some curved line corresponds to any function of x, so also curved lines are represented by functions of x, says EULER, continuing ([41], p. 11) thus:

> *From such an idea about curved lines at once follows their division into continuous and discontinuous or mixed ones.*
>
> *(Ex hac linearum curvarum idea statim sequitur earum divisio in continuas et discontinuas seu mixtas.)*

This terminology, which for EULER had a special sense, unusual to us, was used right up to the time when BOLZANO (in 1817) and CAUCHY (in 1821) attached the now generally adopted meaning to the expressions *continuous* and *discontinuous;* sometimes it was used even later than that.

In EULER's sense continuity meant invariability, immutability of the law—of the equation determining the function over all the domain of values of the independent variable, while discontinuity of a function meant a change of the analytical law, an existence of different laws on two or more intervals of this domain. *Discontinuous* curves, explained EULER, are composed from *continuous* parts, being exactly for this reason called *mixed* or *irregular (irregulares);* also, he sometimes called such curves *mechanical (mechanices).* In geometry, according to EULER, mainly *continuous* (*i.e.* analytic) curves are studied.

Discontinuous, or *mixed* functions and curves of Volume 2 of the *Introductio* correspond to our piecewise analytic functions; thus their inclusion into mathematical analysis offered no essential extension of the concept of function.[18]

[18] According to an opinion recently expressed by I. GRATTAN-GUINNESS ([42], pp. 6–7) EULER's term *continuous* is synonymous with our "differentiable" while his "discontinuous" corresponds to our "continuous". On the other hand, A. SPEISER [42a] had written, "By a *continuous function* EULER, like LEIBNIZ before him, means a function specified by an analytic law, precisely as are those now called analytic functions. They have the property of being determined in their entire range by an arbitrarily small piece ..." TRUESDELL [42b, pp. XLI–XLIII], accepting SPEISER's statement, contended that the context of partial differential equations, in which EULER introduced his discontinuous functions, made it plain that he regarded those functions as failing to be differentiable only at isolated points. He wrote, "EULER's physical universe ... is piecewise smooth, still indeed "continuous" though in lesser degree than the LEIBNITZIAN." Later [51, pp. 243, 247–248, 296–297, 419] TRUESDELL adduced evidence to show that in the context of the vibrating string EULER meant by "function" (not necessarily continuous in his sense) what we should now call a continuous function with piecewise continuous slope and curvature. However, see Footnote 22a for EULER's use of functions discontinuous in the modern sense.

Leaving aside the problem of EULER's having identified analytical expressions with analytic functions (in essence an illegitimate thing to do), I remark that EULER's functions, whether continuous or discontinuous (mixed) in any of his senses of those words, can have discontinuities in the modern sense at isolated points.

In his late works EULER, as I shall show, took a broader point of view as regards *discontinuous* functions; see below.

However, not later than the very year in which the *Introductio* was published (we recall that the manuscript of this work was completed in 1744), EULER understood that the class of *discontinuous* functions (curves), far from being exhausted by *mixed* functions (curves), should be essentially extended. As noticed by A.I. MARKUSHEVICH ([43], pp. 108–109) EULER had seen the necessity for such an extension even in 1744, during his work on the *Methodus inveniendi lineas maximi minimive proprietates gaudentes* (*E*.65) when he compared extremal curves – solutions of variational problems – with curves differing infinitely little from them in the vicinity of one, or of a few, isolated points.

Nevertheless, the main impulse for further development of the concept of function came from EULER's work on mathematical physics, beginning with the celebrated problem concerning infinitely small vibrations of a finite homogeneous string fixed at both ends[19]. The first mathematical interpretation of this problem, speculations about which go back to GALILEO, was offered by TAYLOR (in 1715), though the first decisive step toward the theory was made by D'ALEMBERT in a memoir communicated to the *Acad. Roy. Sci. et Belles-Lettr. Berlin* at the end of 1746 and published in its *Histoire* in 1749 [45].

D'ALEMBERT expressed the conditions of this problem by equations equivalent to a partial differential equation

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$$

(which appeared in an explicit form in EULER's memoir *E*. 213, published in 1755) and proved that the general solution of the problem could be represented by a sum of two arbitrary functions

$$y=\varphi(x+at)+\psi(x-at),$$

which, because of the boundary conditions, reduces to

$$y=\varphi(at+x)-\varphi(at-x).$$

In each particular case the functions appearing in the general solution are determined by the initial form of the string (and the initial velocities of its points). Of course, these initial conditions could be various, but D'ALEMBERT rigidly restricted the class of admitted initial forms of the string, holding that without such restrictions no solution of the problem by mathematical analysis would be possible. Among restrictions imposed by D'ALEMBERT particularly interesting is the assumption that the initial form of the string must be represented over all its extent by one and the same equation, *i.e.* that in EULER's sense the string is *continuous*.

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[19] GRATTAN-GUINNESS' statement ([42], p. 6) to the effect that the distinction between *continuous* and *discontinuous* functions made by EULER in Vol. 2 of his *Introductio* was occasioned by his study of the problem of the string is rather doubtful. So far as I know, the only correction to the manuscript of this volume, which already was in the hands of its Swiss publisher, M. BOUSQUET, and the edition of which was supervised by J. CASTILLON, was sent by EULER on December 15, 1744, through G. CRAMER ([44], N°. 462–464). The printing of the *Introductio*, as evidenced by a letter of CRAMER to EULER dated August 13, 1746 ([44], N°. 467), had begun during the winter of 1746–1747, while on April 8, 1748, CASTILLON informed EULER (*ibidem*, N°. 369) that it had been completed.

EULER soon responded to D'ALEMBERT's memoir, with which he had become acquainted soon after its communication, by presenting, on May 16, 1748, his own memoir, *De vibratione chordarum exercitatio*, *E.* 119, published in *Nova Acta Eruditorum* in 1749 (French version: *Sur la vibration des cordes*, *E.* 140, published in 1750 ([46], pp. 50–77) by the same *Acad. Roy. Berlin*).

Highly valuing D'ALEMBERT's method as a whole, EULER disagreed with him as to the nature of functions admitted in the initial conditions (and, consequently, in the solution of the problem). Guided by physical considerations and profound mathematical intuition, even in stating the problem he wrote ([46], p. 64):

> *...la première vibration dépend de notre bon plaisir, puisqu'on peut, avant de lâcher la corde, lui donner une figure quelconque; ce qui fait que le mouvement vibratoire de la même corde peut varier à l'infini, suivant qu'on donne à la corde telle ou telle figure au commencement du mouvement.*

Repeating this assertion in the research itself, which in its first part rather resembles that of D'ALEMBERT, EULER (p. 72) considers a

> *courbe anguiforme, soit régulière, contenue dans une certaine équation, soit irrégulière ou mécanique,*

i.e. with no restrictions to be imposed on the form of the string. In one particular case he produces a solution corresponding to the *continuous* initial form represented by a trigonometric series

$$y = \alpha \sin \frac{\pi x}{l} + \beta \sin \frac{2\pi x}{l} + \gamma \sin \frac{3\pi x}{l} + \cdots, \qquad (*)$$

the string being fixed at the end points $x=0$ and $x=l$.

D'ALEMBERT did not agree with EULER. Thus began the long controversy about the nature of functions to be allowed in the initial conditions and in the integrals of partial differential equations, which continued to appear in an ever increasing number in the theory of elasticity, hydrodynamics, aerodynamics, and differential geometry.

Soon the controversy gained a new dimension with the entry of a new participant, D. BERNOULLI, whose contribution was published in 1755. Developing the principle of superposition of modes, introduced by him in his earlier studies, BERNOULLI maintained that both the arbitrary initial form of the string and its subsequent vibrations could be represented by an infinite series of terms including sines of multiple angles. According to BERNOULLI an appropriate choice of coefficients makes such series (*) as general as power series; however, the method of calculating "FOURIER coefficients" remained unknown to him.

EULER, who shortly before had offered, in one special case, a solution in the form of a series (*), excluded any possibility of representing in such a form arbitrary *mixed* functions or extensive classes of *continuous* functions, *e.g.*, algebraic ones. (See his *Remarques sur les mémoires précédents de M. Bernoulli* (*E.* 213), published in 1755; *Eclaircissements sur le mouvement des cordes vibrantes* (*E.* 317), published in 1766; *Sur le mouvement d'une corde qui au commencement n'a été ébranlée que dans une partie* (*E.* 339), published in 1767 ([46], pp. 237, 385, 430–431).)[20]

D'ALEMBERT rejected D. BERNOULLI's solution also. However the controversy did not end. It was taken up by LAGRANGE (in 1759–1762), and, somewhat later, by other prominent mathematicians (MONGE, LAPLACE, ARBOGAST, FOURIER and others).

This controversy, a most detailed history of which up to 1788 is presented by C. TRUESDELL [51], was of utmost importance both for the progress of mathematical physics and for the methodological development of the foundations of mathematical analysis. From the point of view of my subject, it is essential that, from the very beginning of his study of the problem of the string, EULER laid down the thesis that in its solution curves of an arbitrary form should be admitted, *i.e.* curves which do not belong to the class of *mixed* functions and, generally (in EULER's opinion), do not comply with any analytical law.

In more detail EULER developed his views on this subject in his *De usu functionum discontinuarum in analysi* (*E.* 322) forwarded to the Petersburg Academy in the spring of 1763 and published in 1767 ([52], pp. 74–91). In this memoir, *continuous* functions are defined, in terms of geometrical images, by assuming not only that the relation between coordinates of all points of any such curve is determined by one and the same law or equation but also that (pp. 75–76)

> *all the parts of the* [continuous] *curve are firmly connected with each other in such a way as to make impossible any change in them without disturbing the connection of continuity.*
>
> *(omnes curvae partes ita vinculo arctissimo inter se cohaerent, ut nulla in illis mutatio salvo continuitatis nexu locum invenire possit).*

EULER emphasizes that what he means is not the connectedness, or continuity, of the course, or run, of the curve *(continuitas tractu)*, but, exclusively, the single-

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20 EULER supposed that a function *continuous* on some interval is defined by one and only one expression over all this interval (as will be said below). Thus, according to EULER, an odd, periodic sum of a sine series could not represent either any algebraic function or, as a rule, transcendent functions. Later on, in his memoir *Disquisitio ulterior super seriebus secundum multiplae cuisdam anguli progredientibus* (*E.* 704), forwarded to the Petersburg Academy of sciences on (June 9) May 29, 1777, and published posthumously in 1798 ([47], pp. 333–355), EULER deduced formulae for the "FOURIER coefficients" on the interval $[0, \pi]$. However, he took no further part in the controversy about representability of functions by trigonometric series.

Somewhat earlier (in 1772, see [48]) D. BERNOULLI, starting from other reasoning, developed the function

$$y=\frac{\pi}{2}-\frac{x}{2} \qquad (*)$$

into a sine series, noticing correctly that the development holds in the interval $(0, 2\pi)$ and also describing quite strictly the behavior of the series both at the ends of, and beyond, this interval. He also considered a few more examples.

It is remarkable that the same development of the functions (*) had been known to EULER who, notwithstanding this contradiction of his own opinion, included it both in his letter to GOLDBACH dated July 4, 1744 ([49], p. 195) and in his *Institutiones calculi differentialis* (*E.*212) published in 1755 ([50], pt. 2, § 92) without mentioning that the development holds for $0<x<2\pi$ only. This is not the only occasion on which EULER knew examples which did not comply with his conceptions but which he may have considered to be insignificant exceptions from the general rule.

ness of the corresponding analytical law. Thus, the two conjugate branches of a hyperbola constitute one *continuous* curve. This main property of *continuous* lines which, for EULER, directly followed from his conception of *continuity*, could be expressed otherwise: any small part of a *continuous* line (function) uniquely determines this line as a whole (see Footnote 18).

Long ago I. YU. TIMCHENCKO ([53], p. 482) noticed that

> to the extent that EULER identified analytical expressions with functions representable by TAYLOR series the property of "continuity" corresponds to the property of uniqueness of analytic functions in WEIERSTRASS' sense[21].

As to discontinuous curves, EULER ([52], p. 76) defines them as

> *all curves not determined by any definite equation, of the kind wont to be traced by a free stroke of the hand.*
>
> *(omnes enim lineae curvae per nullam certam aequationem determinatae, cuiusmodi libero manus tractu delineari solent).*

Again, this *discontinuity* does not apply to the course of the curve; *discontinuous* are also such lines as extend continuously *(etiamsi continuo procedant)* in the sense of connectedness. If we disregard the empirical fact that ideal geometrical figures cannot be traced, *discontinuous* functions thus correspond to our arbitrary piecewise continuous functions with piecewise continuous derivatives of both the first and the second order (*cf.* [51], p. 247)[22]. Without this last condition, implied by the geometrical description though not formulated explicitly, the *discontinuity* becomes absolutely arbitrary so that no part of a *discontinuous* curve need be *continuous*, *i.e.* analytically representable and thus, according to EULER, analytic.

The breadth of EULER's new conception is also confirmed by his mentioning, immediately after giving a description of the whole class of *discontinuous*, or *mechanical*, curves, that *(ibidem)* to this class

> *should be attributed also lines usually called mixed (Atque huc etiam referri convenit lineas vulgo mixtas vocatas)*

as, *e.g.*, the boundary of a polygon (an example repeatedly considered during the controversy about the string) *etc.*

In the subsequent part of his memoir EULER studies the role of different kinds of functions in mathematics. In traditional branches of both mathematical analysis and higher geometry *continuous* functions are studied, the case being somewhat different in that newly discovered and as yet little developed field of integral calculus, the integration of equations containing differentials of functions of two or more variables.

Just as arbitrary constant quantities appear in the integrals of ordinary differential equations, so solutions of that essentially new kind of equations contain

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[21] The uniqueness of the development of a function (of a real variable) into a TAYLOR series under the assumption that such a series does exist had been established by C. MACLAURIN ([54], Vol. **2**, pp. 610–611).

[22] In the problem of the string, also supposed is its continuity (connectedness) over all the interval of its vibration.

discontinuous functions, absolutely indefinite and dependent upon our discretion *(ab arbitrio nostro)* ([52], p. 86). EULER supposed that exactly this circumstance constitutes the main feature (and main power) of integrating partial differential equations, which presents a most extensive sphere of further research. To partial differential equations EULER devoted somewhat later almost the whole of Volume *3* of his *Institutiones calculi integralis* (*E.* 385), published in 1770, once again vigorously emphasizing the usefulness of *discontinuous* functions ([55], §§ 37 and 299).

10. Euler's General Definition of a Function

Since, according to EULER, *discontinuous* functions generally are not analytically representable, the definition of a function given in Volume *1* of the *Introductio* and somewhat modified in its Volume *2* became too narrow. So as to formulate another definition comprising all known classes of relation EULER turned to a notion which was always present albeit not explicitly expressed in any method of introducing functions: the general notion of correspondence between pairs of elements, each belonging to its own set of values of variable quantities. This notion, unconnected with any definite analytical expression, had been used more than once in reasonings implicitly contained in Volume *1* of the *Introductio*, especially in its Chapters 2 and 3, the first of which opens with following phrase ([39], p. 32):

> *Functions are transmuted into other forms either by introducing another variable quantity instead of the initially used or* [even] *while retaining the same variable quantity.*
>
> *(Functiones in alias formas transmutantur vel loco quantitatis variabilis aliam introducendo vel eandem quantitatem variabilem retinendo).*

Examples given in the same passage illustrate how one and the same variable quantity can be represented in various forms. Thus, a function of z, $u=2-3z+z^2$ is the same as $u=(1-z)(2-z)$, and $v=a^4-4a^3z+6a^2z^2-4az^3+z^4$ is transmuted into a more simple function of y, $v=y^4$, by a substitution $a-z=y$, while an irrational function of z,

$$w=\sqrt{a^2+z^2},$$

becomes a rational function of y,

$$w=\frac{a^2+y^2}{2y},$$

after a substitution

$$z=\frac{a^2-y^2}{2y}.$$

It is obvious that any such two (or more) analytical expressions possess a common property, *viz* they establish in different form the same correspondence between two sets of numerical values of the variable z and the corresponding function u or v or w.

Now this idea of relationship must needs be given in as universal and as abstract form as possible, and exactly this did EULER do when he formulated his new definition of a function in the preface to his *Institutiones calculi differentialis* (published in 1755) ([50], p. 4):

> *If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former quantities are called functions of the latter. This denomination is of broadest nature and comprises every method by means of which one quantity could be determined by others. If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way or are determined by it are called functions of it.*
>
> *(Quae autem quantitates hoc modo ab aliis pendent, ut his mutatis etiam ipsae mutationes subeant, eae harum functiones appellari solent; quae denominatio latissime patet atque omnes modos, quibus una quantitas per alias determinari potest, in se complectitur. Si igitur x denotet quantitatem variabilem, omnes quantitates, quae utcunque ab x pendent seu per eam determinantur, eius functiones vocantur.)*

However, in the book itself, devoted to the differential calculus, only analytic functions are considered, a circumstance which enabled EULER to manage without explicit use of the concept of the limit of a function (only once mentioned in the preface), basing himself on a peculiar "calculus of zeros" [56].

EULER's concept of function exerted a great positive influence on the whole subsequent development of mathematics. First of all, of essential importance, was the isolation of the class of *continuous* functions, *i.e.* analytic functions representable by power series, and the discovery of the main properties peculiar to this class, of which up to now I have mentioned only uniqueness (characteristic, as was found out only in the 20^{th} century, of the even more general class of quasi-analytic functions).

Besides this property, EULER (also to some extent, D'ALEMBERT) determined other essential properties of analytic functions. Thus, he showed (in 1755, published in 1778) that analytic functions map a sphere conformally on to a plane, preserving similarity of infinitely small figures; the expression itself *(projectio conformis)* is due to F. SCHUBERT, who used it in 1789, after EULER's death. EULER was the first to use complex quantities in calculations of definite integrals and, in connection with this, deduced (in 1777, published in 1797), using general analytical considerations, the so-called CAUCHY-RIEMANN equations, which D'ALEMBERT had derived in 1752 in the course of his hydrodynamical researches. Thus the general theory of analytic functions of the 19^{th} century, with its three directions developed by CAUCHY, RIEMANN, and WEIERSTRASS, was rooted in the works of EULER and D'ALEMBERT.

Not less important for the subsequent development of mathematical analysis was the introduction of arbitrary *discontinuous* functions and the study of a number of problems concerning relations between intrinsic properties of one or another class of functions of a real variable and the nature of the mathematical apparatus used to represent those functions.

Notwithstanding the prolonged and persistent opposition of D'ALEMBERT, who sometimes pointed out really weak or insufficiently founded details in EULER's conception (special difficulties were connected with the problems of discontinuity, in our sense of the word, of the slope and curvature of the initial form of the string), this conception gradually became more and more widely disseminated. The first to come out in favor of EULER was LAGRANGE (in 1759–1762) in his works on the propagation of sound and on vibration of strings; though he turned his coat for some time to D'ALEMBERT's side, he returned later on (in 1788) to his previous stand.

With some reservations or specifications EULER's point of view was supported later by many other mathematicians, including G. MONGE, P.S. LAPLACE, M.J. CONDORCET and L. ARBOGAST. Even D'ALEMBERT, during his last years, changed his opinion and allowed in the solutions of partial differential equations of any order *discontinuous* functions the derivatives of which up to the same order possess no saltus (*Sur les fonctions discontinues*, 1780). Actually D'ALEMBERT used the concept of left-derivative and right-derivative at a point [57].

It should be noticed in this connection that these discussions revealed the need for a more distinct separation of continuous from discontinuous functions (in our sense), as was indeed effected by L. ARBOGAST in a work [58] to which the Petersburg Academy of Sciences in 1790 awarded the prize for its competition of 1787 regarding the nature of arbitrary functions to be admitted in solving partial differential equations.

ARBOGAST thought it possible (though not in the problem of the string, in which the continuity of the curve is conditioned by its very nature) to use not only functions with discontinuous derivatives but also functions discontinuous at isolated points[22a]; these he called ([58], p. 11) *fonctions discontigües*

> *parce que toutes leur parties ne tiennent pas, ou ne sont pas contigües les unes aux autres.*

However, ARBOGAST offered no analytical definition of continuity (or discontinuity). Mathematicians of the 18^{th} century had not felt need for such a definition; if necessary, they described the main property of continuity verbally.

Thus, for example, explaining methods of approximate calculation of definite integrals in Volume *1* of his *Institutiones calculi integralis* (*E.* 342), published in 1768, EULER wrote ([59], §§ 297 and 300) that the calculation of $\int X\,dx$ would be

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[22a] As this paper was going to press, C. TRUESDELL called to my attention EULER's paper E 340, *Eclaircissements plus détaillés sur la génération et la propagation du son, et sur la formation de l'écho*, *Opera omnia*, ser. III, Vol. 1, ed. E. BERNOULLI, R. BERNOULLI, F. RUDIO, A. SPEISER, 1926. In this paper, which was presented to the Berlin academy on 19 and 26 September 1765 and was published in 1767, EULER considers the wave equation in the context of aerial disturbances. There, in contrast with the problem of the vibrating string, the physical problem does not require solutions continuous in the modern sense. To study solutions of the functional equation EULER regards as equivalent to or perhaps a replacement for the partial differential equation, he introduces functions that have the value 0 at all points except one. He remarks that since these pulse functions form what is called now a (non-enumerable) basis for the set of all functions, use of them as initial values for a wave function makes it possible to describe concisely and in geometric terms the entire theory of propagation and reflection of plane waves. It is interesting to note also that EULER effects these solutions by diagrams in which the pulse functions are represented. The matter is explained at length by TRUESDELL [42b, pp. LXI–LXII].

the more accurate the smaller are the assumed increments of the independent variable x provided the increments of the integrand X were also small. Also verbally EULER describes (§ 304) the behavior of a discontinuous function $X = \frac{1}{\sqrt{1-x^2}}$ in the vicinity of the point $x=1$, noticing that any small increment in x gives rise to an extremely large change of the function X, but he did not use the term *continuity*.[23]

Discontinuities in the derivatives of solutions of partial differential equations present great difficulties, to overcome which proved possible only at the much higher level of mathematical analysis as reached in the second half of the 19th century (*cf.* [51], pp. 286–297). In our time a further, extremely broad and most important development of this problem grew from functional analysis. What I mean here is the theory of generalized solutions of partial differential equations (in particular, of the wave equation) developed mainly at the hands of S. L. SOBOLEV (1936) and L. SCHWARTZ (1945). These generalized functions (SOBOLEV) or distributions (SCHWARTZ) are linear functionals which need not be differentiable in the usual sense but possess generalized derivatives.

Just as the modern theory of summation of series showed the essential correctness of EULER's views on the importance and use of divergent series, so also the theory of generalized functions illustrates strikingly EULER's profound intuition and perspicacity as regards discontinuous functions.

However, in neither case did the general state of mathematical analysis in the 18th century allow EULER either to establish his ideas accurately (from the point of view of subsequent generations) or to formulate exact definitions, or, also, to save him from errors, some of which were noticed even by his junior contemporaries.

11. Criticism of the Concept of "Mixed" Functions; Charles (1780) and Fourier (1807–1821)

The first of EULER's ideas to be criticized was his isolation of the class of *mixed functions*. Soon after his death it was shown that functions which were introduced by different analytic expressions in different regions of some finite (or, sometimes, infinite) interval could be represented also by one and the same equation. The first examples of such functions were offered by J. CHARLES in his *Fragment sur les fonctions discontinues*, 1780 [60].

Much later, a no lesser person than CAUCHY himself considered it worthwhile to devote a paper expressly to this problem, the *Mémoire sur les fonctions continues*

[23] It should be noticed that in the case under consideration EULER essentially interprets the definite integral as being the limit of the sum $\sum X(x_k)\Delta x_k$; he himself supposed ([59], § 302) that the integration could be carried out as accurately as needed, adding, however, that an absolute exactness is attainable only if all the Δx_k be infinitely small, *i.e.* equal to zero.

Such a conception of a definite integral, originating with LEIBNIZ and revitalized in a more precise formulation due to CAUCHY, differed from the basic definition adopted by EULER and his contemporaries, according to which the integral was understood to be a function the differential of which is equal to $X\,dx$, the definite integral being equal to the difference between the values of the primitive function (a terme due to LAGRANGE) at the upper and lower limits of integration.

[61] (published in 1844). His simplest example was a function

$$y=\begin{cases} =x, & x\geqq 0, \\ -x, & x<0, \end{cases}$$

thus *discontinuous* but simultaneously representable by a single equation $y=\sqrt{x^2}$ for every $-\infty<x<+\infty$ and thus *continuous*. Thus the discrimination between *mixed* and *continuous* functions proved itself theoretically untenable.

Much more important, though, was the criticism of this same notion of *mixed functions* within the framework of the theory of trigonometric series. As we have seen (see § 9) in two of his memoirs (*E.* 317 and 339) EULER flatly denied that it was possible to represent the initial figure of the string, as defined on two parts of a given finite interval by two different equations, by a series of terms containing sines of multiple arcs.

In the beginning of the 19$^{\text{th}}$ century, FOURIER refuted this assertion in his works on the theory of propagation of heat, which also gave rise to the general theory of trigonometric series. Even in 1805, in a fragment recently published by I. GRATTAN-GUINNESS ([62], p. 183), FOURIER wrote:

> *Il résulte de mes recherches sur cet objet que les fonctions arbitraires même discontinues peuvent toujours être représentées par les développements en sinus ou cosinus d'arcs multiples, et que les intégrales* [of the partial differential equations] *qui contiennent ces développements sont précisement aussi générales que celles où entrent les fonctions arbitraires d'arcs multiples. Conclusion que le célèbre Euler a toujours repoussée.*

FOURIER goes on to present a few examples illustrated by graphs. He developed his reasoning in more detail in his *Théorie de la propagation de la chaleur dans les solides*, forwarded to the *Institut de France* on December 21, 1807, but published only recently, again by GRATTAN-GUINNESS (see [62]), and, afterwards, in his basic *Théorie analytique de la chaleur* in 1822 [63].

Conclusions reached by FOURIER in 1807 startled mathematicians of older generations, and LAGRANGE, for one, distrusted them; on the other hand, after 1822 they received an enthusiastic welcome from young mathematicians.

Brought up in traditions of the 18$^{\text{th}}$ century, FOURIER himself supposed that a trigonometric series may be used so as to represent any *mixed* function and offered no statisfactory analysis of the problem of representing functions by such a series. However, once the problem had been stated, in the next few years it became the subject of special studies based on the new general conception of the calculus, the elements of which had been systematically developed by CAUCHY in his *Cours d'analyse ... 1re partie: analyse algébrique*, 1821 [64] and *Résumé des leçons ... sur le calcul infinitésimal*, 1823 [65].

The coefficients of FOURIER series of any given function $f(x)$ being equal to integrals of the products $f(x)\cos nx$ and $f(x)\sin nx$, the class of such series was gradually broadened as more and more general definitions of the integral were formulated. Also, the concepts of convergence and of summation of series gradually acquired new content.

12. Digression: The Analytical Representation of Functions

Here we shall not go into the details of numerous researches devoted to sufficient conditions for representing functions by FOURIER series. I shall mention only that from the conditions presented by P. LEJEUNE-DIRICHLET in 1829–1837 [66] it followed that any bounded function if piecewise continuous and piecewise monotone over a given interval could be developed into a FOURIER series convergent to that function. This meant that an arbitrary curve traced over a given interval by a free stroke of the hand (*i.e.*, any arbitrary *discontinuous*, in EULER's sense, and bounded function) could be represented by a single analytical law, thus changing it into a *continuous* one. Of course, not every function continuous over a given interval is representable by its FOURIER series which, in this interval, may diverge at infinitely many points.

Whether a given function be representable analytically or not depends on the methods of analytical expression admitted. In Volume 1 of his *Introductio* EULER declared the most general form of analytical expression to be a power series generated by a denumerable number (a modern term) of additions and multiplications of the variable x and a denumerable set of constants, an additional limiting process being allowed[24].

Later on EULER definitely expressed his confidence in the fact that his *discontinuous* functions are not, generally speaking, analytic, explaining in addition (*e.g.*, in his *Eclaircissements sur le mouvement des cordes vibrantes*, *E.* 317 ([46], p. 385)) that

> *on regarderoit fort mal à propos toutes les courbes comme renfermées dans cette équation parabolique*
>
> $$y = A + Bx + Cx^2 + Dx^3 + etc$$
>
> *quoi qu'on puisse faire passer cette courbe par une infinité de points donnés.*

And right he was: CAUCHY proved that even a function infinitely differentiable at a given point could still fail to be analytic at that point. His example,

$$F(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

published in 1823 in his *Résumé des leçons ... sur le calcul infinitésimal* [65], has become classical[25]. Moreover, as was shown by A. PRINGSHEIM (in 1893) there are infinitely differentiable functions not analytic over any interval.

If the fund of algebraic expressions be extended, the realm of analytically representable functions broadens most extraordinarily. Thus WEIERSTRASS showed that any function continuous over a closed interval could be represented in that interval by a sum of uniformly convergent series of integer polynomials (published 1885). Furthermore, even discontinuous functions of a very complex

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[24] As pointed out above (Footnote 11), such a construction recalls the idea of J. GREGORY.

[25] This function could be written down by a single analytical expression, *viz*, by a sum of everywhere convergent series of exponential functions of special form.

nature, the classification of which was developed by R. BAIRE (in 1898–1899), can be represented by sums of convergent series and multiple series of polynomials. H. LEBESGUE called analytically representable any function that could be constructed by a denumerable set of additions, multiplications and limiting processes, carried out according to a definite law regarding the independent variable and a denumerable set of constant quantities.

BAIRE's classification (as established in 1905 by LEBESGUE) embraces all such functions which are also measurable in E. BOREL's sense. This law of construction LEBESGUE named *une expression analytique* (cf. [10]).

13. Euler's General Definition Recognized: Condorcet (1778), Lacroix (1797), Fourier (1821), Lobatchevsky (1834), Dirichlet (1837)

Thus the division into *continuous* functions and *discontinuous* functions (*mixed* ones included) failed to retain its place in mathematics[26]; on the other hand, the general definition of a function due to EULER (see § 10) gradually gained wider and wider recognition and use. It seems that the first to appraise correctly the importance of this new definition was CONDORCET, who developed EULER's conception in an unpublished *Traité du calcul integral*, an unfinished manuscript of which, transmitted to the Paris Academy of sciences in 1778–1782, is preserved at the Library of the *Institut de France*, complete with proofsheets.[27]

As projected by its author this book should have had five parts, only two of which were actually written. The first of these parts, entitled *De fonctions analytiques*, opens with an explanation of what is understood to be an analytic function (see [67], p. 134):

> *Je suppose que j'aie un certain nombre de quantités x, y, z, ..., F, et que pour chaque valeur déterminée de x, y, z, ... etc., F ait une ou plusieurs valeurs déterminées qui y répondent; je dis que F est une fonction de x, y, z, ...*

Offering a few examples of explicit and implicit functions, introduced by means of equations, CONDORCET continues:

> *Enfin, si je sais que lorsque x, y, z seront déterminées, F le sera aussi, quand même je ne connoîtrois ni la manière d'exprimer F en x, y, z, ni la forme de l'équation entre F et x, y, z; je saurai que F est fonction de x, y, z.*

Finally, three kinds of functions are distinguished:

(1) Functions the form of which is known (we should say, explicit functions);

(2) Functions introduced by unsolved equations between F and x, y, z (implicit functions); and

(3) Functions given only by certain conditions (*e.g.* by differential equations).

[26] I am leaving aside the function $y=(-1)^x$ considered in 1727–1728 in the correspondence of EULER with JOH. BERNOULLI ([44], No. 190–192) and, also, in Vol. 2 of EULER's *Introductio* ([41], § 517). This function, which is expressed by an equation and thus is in this sense *continuous*, assumes real values only for such values of x as are irreducible fractions with odd denominators. In Vol. 2 of the *Introductio* EULER showed that this function, which he called paradoxical, is represented, as we should say now, by two everywhere dense sets of isolated points belonging to the straight lines $y=1$ and $y=-1$.

[27] Not to be confused with CONDORCET's earlier book of the same title (Paris, 1765).

Some mechanical examples are given so as to illustrate the third kind, to which also are attributed

> *des fonctions qui ne sont connues que parce qu'on sait en général qu'une certaine quantité sera déterminée lorsque d'autres quantités le seront.*

Again, examples of some physical phenomena the mathematical description of which is unknown are given.

As is seen, CONDORCET was the first to use the term *fonction analytique* for describing functions of arbitrary nature, the adjective *analytique* implying above all functions considered in mathematical analysis. Continuing his exposition, CONDORCET attempts to derive a TAYLOR series formally for an arbitrary function, almost in the way LAGRANGE had done, a little earlier, in his memoir, *Sur une nouvelle espèce de calcul relatif à la différentiation et à l'intégration des quantités variables*, published in 1774 ([68], pp. 441–476). It seems, however, that the term *fonction analytique* is due primarily to CONDORCET.

Although CONDORCET's unfinished *Traité* never saw the light of day, its printed pages had been read by a number of mathematicians at Paris, as S.F. LACROIX indicated in the preface to his course of mathematical analysis in three volumes [69]. Moreover, in defining a function LACROIX followed EULER and CONDORCET. Noticing that at first a function of some quantity had been understood to be any of its powers (the same inaccuracy was commited by LAGRANGE in the beginning of his *Théorie des fonctions analytiques* [40]) then, also, any other algebraic expressions, LACROIX continues (p. 1):

> *Enfin de nouvelles idées, amenées par le progrès de l'analyse, ont donné lieu à la définition suivante des fonctions.* [The definition itself, emphasized by the author, follows immediately.] *Toute quantité dont la valeur dépend d'une ou de plusieurs autres quantités, est dite fonction de ces dernières, soit qu'on sache ou qu'on ignore par quelles opérations il faut passer pour remonter de celles-ci à la première.*

LACROIX's *Traité*, being widely known, contributed greatly to the dissemination of the new concept of function. It is true that in many other books and manuals of that time the old definition of a function as being an analytical expression was still used. As I have mentioned (see Footnote 17), this was the case with LAGRANGE's *Théorie des fonctions analytiques*, first published in 1797, the second, revised and supplemented edition of which appeared in 1813. Essentially the same interpretation of the concept of function was implied also in CAUCHY's *Analyse algébrique* (in 1821) although in the definition itself the term *analytical expression* is not used[28].

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[28] CAUCHY's definition is ([64], Chap. I, § I):

Lorsque des quantités variables sont tellement liées entre elles que, la valeur de l'une d'elles étant donnée, on puisse en conclure les valeurs de toutes les autres, on conçoit d'ordinaire ces diverses quantités exprimées au moyen de l'une d'entre elles, qui prend alors le nom de variable indépendente et les autres quantités exprimées au moyen de la variable indépendente sont ce qu'on appelle des fonctions de cette variable.

Contrary to the opinion of M. KLINE ([70], p. 950) who holds *that an analytical expression is not required* by CAUCHY and to the opinion of F. A. MEDVEDEV ([71], p. 238), I suppose that CAUCHY

But then, rather soon, EULER's general definition was accepted by three scholars of the highest calibre, in all three cases in connection with their researches on the theory of trigonometric series. First of all, one finds such a definition in FOURIER's *Théorie analytique de la chaleur*, published in 1821 ([63], p. 500):

> *En général, la fonction $f(x)$ représente une suite de valeurs ou ordonnées, dont chacune est arbitraire.*

Immediately FOURIER repeats himself, maintaining that he does not suppose these ordinates to be subject to a common law, they succeed each other in any manner whatever, and each ordinate could be considered as given individually. I shall touch upon the sense implied by FOURIER (and other mathematicians) in speaking about *arbitrary* nature of a functional dependence below.

Following this laconic definition by FOURIER, whose work at once acquired wide fame, LOBATCHEVSKY and DIRICHLET published much more wordy definitions. In his article, Об исчезании тригонометрических строк *(On the disappearance [convergence] of trigonometric series)*, in 1834, LOBATCHEVSKY wrote ([72], p. 43):

> *General conception demands that a function of x be called a number which is given for each x and which changes gradually together with x. The value of the function could be given either by an analytical expression, or by a condition which offers a means for testing all numbers and selecting one of them; or, lastly, the dependence may exist but remain unknown.*

> (Обшее понятие требует, чтобы функцией от x называть число, которое дается для каждого x и вместе с x постепенно изменяется. Значение функции может быть дано или аналитическим выражением, или условием, которое подает средство испытывать все числа и выбирать одно из них; или наконец зависимость может существовать и оставаться неизвестной.)

Then, declaring that, though no contradictory examples were yet known, the alleged possibility of representing any function analytically is no more than an arbitrary assumption, LOBATCHEVSKY concludes (p. 44):

> *It seems impossible to doubt either the truth that everything in the world could be expressed by numbers or the correctness [of the judgement] that any change and relation in it is represented by an analytic function. Meanwhile the broad view of the theory allows the existence of dependence only in the sense that numbers, in connection with one another, be regarded as though given together. For this reason Lagrange, in his Calcul des fonctions,*[29] *with which he wished to replace the differential calculus, damaged the generality of the concept as much as he thought to gain in the strictness of judgement.*

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actually thought here only of analytically expressed functions. This is implied both by his formulation, in which he twice mentions that on *conçoit d'ordinaire* that the functions are *exprimées au moyen de la variable indépendante*, and by his separation (following the definition) of explicit and implicit functions, the latter being characterized by the fact that the equations which they and the independent variable should satisfy are not solved *algebraically*.

[29] In his *Leçons sur le calcul des fonctions* (1801, 2nd ed. 1806 [73]) LAGRANGE offered the same definition of a function as given by him earlier (in 1797) in his *Théorie des fonctions analytiques* (see Footnote 17).

(Кажется нельзя сомневаться ни в истине того, что все в мире может быть представлено числами; ни в справедливости того, что всякая в нем перемена и отношение выражается аналитической функцией. Между тем обширный взгляд теории допускает существование зависимости только в том смысле, чтобы числа, одни с другими в связи, принимать как бы данными вместе. Лагранж в своем вычислении функций (*Calcul des fonctions*), которыми хотел заменить дифференциальное, столько же, следовательно, повредил обширности понятия, сколько думал выиграть в строгости суждения.)

The tendency to include in the concept of function also such hypothetical dependences as might turn out to be not analytically representable is thus expressed absolutely distinctly. But then, because LOBATCHEVSKY's term *gradually* means continuously in CAUCHY's sense, LOBATCHEVSKY's definition taken literally somewhat unexpectedly concerns continuous functions only.

The same is true concerning the definition that DIRICHLET offered in 1837 in his memoir *Über die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinusreihen*, from which I shall now quote the whole relevant passage ([66], pp. 135–136):

Man denke sich unter a und b zwei feste Werthe und unter x eine veränderliche Grösse, welche nach und nach alle zwischen a und b liegenden Werthe annehmen soll. Entspricht nun jedem x ein einziges, endliches y, und zwar so, dass, während x das Intervall von a bis b stetig durchläuft, y=f(x) sich ebenfalls allmählich verändert, so heisst y eine stetige oder continuirliche Function von x für dieses Intervall. Es ist dabei gar nicht nöthig, dass y in diesem ganzen Intervalle nach demselben Gesetze von x abhängig sei, ja man braucht nicht einmal an eine durch mathematische Operationen ausdrückbare Abhängigkeit zu denken. Geometrisch darstellt, d.h. x und y als Abszisse und Ordinate gedacht, erscheint eine stetige Function als eine zusammenhängende Curve von der jeder zwischen a und b enthaltenen Abszisse nur ein Punkt entspricht. Diese Definition schreibt den einzelnen Theilen der Curve kein gemeinsames Gesetz vor; man kann sich dieselbe aus den verschiedenartigsten Theilen zusammengesetzt oder ganz gesetzlos gezeichnet denken. Es geht hieraus hervor, dass eine solche Function für ein Intervall als vollständig bestimmt nur dann anzusehen ist, wenn sie entweder für den ganzen Umfang desselben graphisch gegeben ist, oder mathematischen, für die einzelnen Theile desselben geltenden Gesetzen unterworfen wird. So lange man über eine Function nur für einen Theil des Intervalls bestimmt hat, bleibt die Art ihrer Fortsetzung für das übrige Intervall ganz Willkür überlassen.

In essence the definitions of LOBATCHEVSKY and DIRICHLET are identical, the only difference being that DIRICHLET thought it necessary to add a geometrical explanation. Their positively general nature concerning continuous functions and the possibility of their being directly generalized so as to include discontinuous functions are absolutely evident.

Since the authors considered discontinuous functions, their restricting their definitions to functions continuous in CAUCHY's sense seems the more surprising: functions (or derivatives) with isolated points of discontinuity are explicitly included in the sufficient conditions for representing a function by FOURIER series as established by LOBATCHEVSKY and DIRICHLET themselves. And to DIRICHLET we are also indebted for his celebrated example of a function discontinuous at each point of the interval $0 \leqq x \leqq 1$:

$$f(x) = \begin{cases} 0 & \text{for rational values of } x, \\ 1 & \text{for irrational values of } x. \end{cases}$$

Why did both these scholars think it expedient to restrict their definitions with continuous functions? The most natural explanation of this circumstance has been offered by MEDVEDEV ([71] pp. 242–243): the class of functions just isolated, functions continuous in CAUCHY's sense, immediately became extremely important, and it was precisely this class which it was necessary to free from the restriction of analytical representation the more so since even later some scholars, for example V. YA. BUNYAKOVSKY ([74], p. 246) and G. G. STOKES ([75], p. 240), identified continuity in the sense of CAUCHY with *continuity* in EULER'S sense.

H. BURKHARDT noticed that only in 1841 did A. COURNOT formulate a definition of a function with the degree of generality that came to be commonly attributed to DIRICHLET and, later on, to both DIRICHLET and LOBATCHEVSKY[30]. The attribution to DIRICHLET is due to HANKEL, whose work was published in 1870. COURNOT'S *Théorie des fonctions*, t. I (Paris, 1841) having proved unavailable, I shall quote his words as given by BURKHARDT ([76], p. 968):

> *Nous concevons qu'une grandeur peut dépendre d'une autre, sans que cette dépendance soit de nature à pouvoir être exprimée par une combinaison des signes de l'algèbre.*

Somewhat further COURNOT *(ibidem)* suggested that it is possible to

> *imaginer une théorie qui aurait pour objet la discussion des propriétés générales des fonctions.*

14. Hankel on Functionality

It is obvious that, as just mentioned, a concept of function of no lesser generality was really due both to LOBATCHEVSKY and to DIRICHLET. However neither COURNOT'S book nor LOBATCHEVSKY'S article enjoyed in those times any wide popularity, as is evidenced by H. HANKEL'S *Untersuchungen über die unendlich oft oszillierenden und unstetigen Funktionen* ([26], publ. 1870). Having presented a concise historical essay, HANKEL then offers introductory remarks on the concept of function, formulating the following definition ([26], p. 49):

> *Eine Funktion heißt y von x, wenn jedem Werte der veränderlichen Größe x innerhalb eines gewissen Intervalles ein bestimmter Wert von y entspricht; gleichviel, ob y in dem ganzen Intervalle nach demselben Gesetze von x abhängt oder nicht; ob die Abhängigkeit durch mathematische Operationen ausgedrückt werden kann oder nicht.*

HANKEL goes on to add *(ibidem)* that he will call this definition by DIRICHLET'S name

[30] In his commentaries on the work of LOBATCHEVSKY, G. L. LUNZ ([72], pp. 15–16) interpreted the quoted definition as relating to any function. According to LUNZ, the word *gradually* is used here by LOBATCHEVSKY on a par with *consecutively* rather than with *continuously* (in CAUCHY'S sense). As pointed out by MEDVEDEV ([71], pp. 235–236), this interpretation is rather doubtful.

> *weil sie* [this definition] *seinen Arbeiten über die Fourierschen Reihen, welche die Unhaltbarheit jenes älteren Begriffes zweifellos dargetan haben, zugrunde liegt.*

This old concept HANKEL also calls EULERsche *Auffassung* (p. 48), recalling the *continuous* and *discontinuous* functions in the *Introductio.*

Then, on p. 53, HANKEL qualifies his definition saying that the *bestimmter Wert von y* does not include the case of infinite discontinuity and offers a new definition almost coinciding with part of the original preceding the semicolon. Exactly in this or in a similar form the general definition of a function was included in courses in mathematical analysis at the end of the 19^{th} century and in the 20^{th}.

It should be noticed that HANKEL formulated his definition prudently: Not reproducing DIRICHLET's definition, he restricted himself to remarking that his own definition actually is the cornerstone of DIRICHLET's *Arbeiten über die Fourierschen Reihen.*

Having contributed so much to the study of discontinuous functions, HANKEL could hardly have failed to notice that the definition of DIRICHLET himself had to do with continuous functions, a circumstance pointed out only in our time, by A. CHURCH [77], A. OSTROWSKY [78], and other authors.

15. The Historical Role of Euler's General Definition

Thus, it seems that for HANKEL the main point was the *spirit* of DIRICHLET's definition rather than its literal formulation. On the other hand, in contrasting DIRICHLET's definition with *die Eulersche Auffassung* HANKEL was positively mistaken.

As shown above (see § 10), EULER's concept of function actually underwent essential evolution, and if one or another name is to be connected with the definition of a function in one-to-one correspondence, that name should be EULER's; EULER it was whose concept, described in 1755, was developed by many scholars, LOBATCHEVSKY and DIRICHLET included.

A special consideration of the arbitrary nature of functional relations and of their *analytical representability* is warranted.

First, different notions about the degree of arbitrariness and about the kind of behavior of the functions used are characteristic of different times and different generations of mathematicians. Though EULER, LACROIX, or FOURIER never came across such functions as the discontinuous function due to DIRICHLET[31] mentioned above (see § 13), their concept of a function as being an arbitrary correspondence was for *their* time as general as was DIRICHLET's concept for *his* time. And, for that matter, *Dirichlet* himself did not imagine such functions as came to be introduced in the times of G. CANTOR, BAIRE, BOREL and LEBESGUE.

Second, as has been said (see § 12), the problem of analytical representability of functions came out to be much more complex than had been supposed by mathematicians right up to the beginning of the 20^{th} century. Circumvention of analytical representability was thought necessary during a long period beginning

[31] In this connection it is nevertheless instructive to remember EULER's *paradoxical* function $y=(-1)^x$ (see Footnote 26).

with EULER and ending with DIRICHLET and COURNOT. But then, it was gradually found out that more and more extensive classes of functions, at first those complying with DIRICHLET's conditions in the theory of FOURIER series, then continuous functions and even those of more general nature, are representable by means of one or another analytical method.

U. DINI, in his *Fondamenti per la teorica delle funzioni di variabili reali*, published in 1878 (German edition, 1892), quite appropriately inquired ([79], p. 49)

> *ob bei Aufrechterhaltung der ganzen in der Definition enthaltenen Allgemeinheit es stets möglich sein wird, in einem gewissen Intervall eine Funktion y von x für alle Werte der Variabelen in diesem Intervall durch eine oder mehrere, endliche oder unendliche Reihen von Rechnungsoperationen, die man mit der Variabelen vornimmt, analytisch auszudrücken oder nicht.*

Also, added DINI, the current level of mathematical knowledge being taken into account, a quite satisfactory answer to his question is just impossible.

As noticed above (see § 12) LEBESGUE, in 1905, gave a positive answer to this question concerning all measurable functions, simultaneously offering an example of a function not representable analytically in his sense.

I am compelled to leave aside the related problem of the legitimacy of BAIRE's and LEBESGUE's constructions, later subjected to criticism from the point of view of "effectivism", "contructivism" and other directions of the foundations of mathematics.

If rejection of analytical representability turns out to be in a sense illusory, of what importance then is EULER's definition of 1755? Also, of what importance are all the definitions growing from it? The weak side of EULER's definition did not escape the attention of HANKEL who, for one, regarded it as a *reine Nominaldefinition* ([26], p. 49), pointing out that functions defined so universally possess no common property whatsoever.

The proper answer to the question just posed is given by the development itself of the theory of functions. As time went on, the class of functions considered, growing broader and broader, underwent essential changes. Analytical expressions composed by means of comparatively simple calculating operations having been almost the only subject of study during approximately two centuries, they never lost their importance. But then, with the course of time, it became necessary to study different classes of functions (continuous, differentiable, with finite variation, pointwise discontinuous, measurable, *etc.*) introduced by means of one or another basic property which defines the whole structure of a given class independently of whether the functions of this class are analytically representable. As formulated by N. N. LUZIN in his book Интеграл и тригонометрический ряд *(Integral and trigonometric series)* published in 1915 ([80], p. 50),

> *the main difference between methods of studying functions within the framework of mathematical analysis and* [alternatively] *theory of functions is that classical analysis deduces properties of any function starting from the properties of those analytical expressions and formulae by which this function is defined, while the theory of functions determines the properties of function starting from that property which a priori distinguishes the class of functions considered.*

(Основная разница в методе изучения функций анализом и теорией функций состоит в том, что классический анализ извлекает свойства функции из свойств тех аналитических выражений и формул, которыми она определена, тогда как теория функций действительного переменного выводит свойства функции из того свойства, которое *a priori* характеризует рассматриваемый класс функций.)

It is also important to notice that, within the theory of functions, verbal descriptions of the behavior of functions over one or another set of values of the independent variable become generally used.

As mentioned above, modern mathematical logic discovered essential difficulties inherent in the universal, hence nonalgorithmic definition of a function. Even in 1927, H. WEYL maintained, quite correctly, that ([81], p. 8)

Niemand kann erklären, was eine Funktion ist. Aber: "Eine Funktion f ist gegeben, wenn auf irgendeine bestimmte gesetzmässige Weise jeder reelen Zahl a eine Zahl b zugeordnet ist ... Man sagt dann, b sei der Wert der Funktion f für den Argumentwert a".

Thus, two differently defined functions are considered identical if, for all possible values of *a*, the corresponding values of *b* coincide. Opinions of mathematicians about the sense of the words *auf irgendeine bestimmte gesetzmässige Weise* (emphasized by me, not by WEYL) differ. However, EULER's general *(nominal)* definition of a function, which became necessary as early as the middle of the 18th century, has been successfully used – to borrow an expression uttered on another occasion – as *ein Medium freien Werdens* – for more and more complex constructions in the theory of functions and, also, has opened up new horizons in the development of many branches of mathematical analysis and its applications. Even the difficulties inherent in this definition served a positive role in the statement and study of a number of problems in foundations of mathematics and mathematical logics.

Addendum

When this article was almost complete I received the *Tagungsbericht, Problemgeschichte der Mathematik 22.9. bis 28.9.1974, Mathematisches Forschungsinstitut Oberwolfach*, BRD.

From this source I learn that the central subject of the Conference was the development of the concept of function, to which almost half the reports were devoted. The first report delivered by Dr. KARIN REICH was a summary of the original version of this article (see *Acknowledgment*): *Bericht über einen Aufsatz von A.P. Juschkewitsch zur Geschichte des Funktionsbegriffs*. Other reports on the subject were those by C.J. SCRIBA, E.M. BRUINS, C.O. SELENIUS, I. SCHNEIDER, O. VOLK, I. GRATTAN-GUINNESS and H. GERICKE. Participants in the concluding discussion were H. GERICKE, G. HIRSCH, J.J.M. BOS and others.

The summaries of the reports published in the *Tagungsbericht* are too concise to be taken into account here, and I may only hope that the reports themselves will be published. Also, I regret that one source, mentioned in the report of SCRIBA, *viz*, S. BOCHNER, *The rise of functions* (Rice Univ. Studies **56** (1970), No. 2, 3–21 (1971)), remains unknown to me.

Acknowledgement. This article is a considerably enlarged and revised version of my earlier work, О развитии понятия функции published in Russian in the Историко-математические исследования, XVII, 1966, 123–151. I am pleased to express my gratitude to O.B. SHEYNIN who undertook the translation of the new version into English. I am also indebted to SHEYNIN for a few remarks which he made during his work.

Translator's note. In translating the above pages, I offer my mite to acquaint the scientific community at large with a work by the effective, if not formal, doyen of Soviet historians of mathematics, whose seventieth birthday was celebrated recently; to whom I owe such reputation as I do enjoy; and to whom, finally, I am honored to dedicate this translation. Knowing beforehand that the article is intended for this *Archive*, I have all the more tried my best to maintain a high linguistic standard, a problem really difficult for a person not practising spoken English. Professor YOUSCHKEVITCH has essentially facilitated my work by supplying translations of many terms and, also, by his readiness to accept insignificant changes while preserving the general meaning of his text. Professor C. TRUESDELL corrected the translation to such an extent as in effect to prove the inadequacy of my efforts.

O.B.S.

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