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Asymptotic behavior of solutions of Monge–Ampère equations with general perturbations of boundary values

Xiaobiao Jia*, [a] and Xuemei Li [b]

[a] School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450046, Henan, China.

[b] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, Shaanxi, China.

E-mails: jiaxiaobiao@ncwu.edu.cn (X. Jia), pwflxm@stu.xjtu.edu.cn (X. Li)

Abstract. In this paper, we consider the asymptotic behavior of solutions of Monge–Ampère equations with general boundary value conditions in half spaces, which reveals the accurate effect of boundary value condition on asymptotic behavior and improves the result in [13].

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1. Introduction

The classical Liouville theorem on the Monge–Ampère equation

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^n$$

shows that any classical convex solution is a quadratic polynomial (cf. Jörgens [15] as $n = 2$, Calabi [8] as $n \leq 5$ and Pogorelov [22] as $n \geq 2$). This theorem has also been got via different approaches such as Cheng-Yau [9] and Jost-Xin [16], etc. In [6], Caffarelli proved that above theorem holds for viscosity solutions as well.

The asymptotic behavior under several kinds of perturbations has been studied extensively in the last decades. Caffarelli–Li [7] considered the perturbation of right hand term $f(x)$ only occurring in bounded domains. They proved that u converges to some quadratic polynomial at infinity (if $n \geq 3$), or to some quadratic polynomial plus multiple of $\log|x|$ at infinity (if $n = 2$).

* Corresponding author.

For $n = 2$, the same result has been obtained by using complex variable methods (cf. [10, 11]). Bao-Li-Zhang [5] extended the asymptotic behavior result in [7] and considered the perturbation of right hand term $f(x)$ occurring in $\mathbb{R}^n$. They also deduced that, under proper decay rate of $f(x)$ at infinity, u must converge at infinity with corresponding convergent rate. Similar arguments for other equations have been widely discussed by many researchers such as k-Hessian equations [4, 18], special Lagrangian equations [17, 19, 20], parabolic Monge-Ampère equations [26–28], maximal hypersurfaces equation [12], general fully nonlinear equations [14], etc.

The Liouville theorem on the Monge–Ampère equation

$$\begin{cases} \det D^2 u(x) = 1 & \text{in } \mathbb{R}^n_+, \\ u(x) = \frac{1}{2}|x|^2 & \text{on } \{x_n = 0\}, \end{cases}$$

states that any convex viscosity solution satisfying quadratic growth condition must be a quadratic polynomial(cf. [21,24]). Under fixed perturbation on boundary conditions, the asymptotic behavior on the Monge–Ampère equation in half spaces was considered by Jia–Li [13]. In details, they studied the asymptotic behavior at infinity of convex (viscosity) solution of the following Monge–Ampère equation

$$\begin{cases} \det D^2 u(x) = 1 & \text{in } \mathbb{R}^n_+ \backslash \overline{B}_1^+, \\ u(x) = \frac{1}{2}|x|^2 + d \log \sqrt{x^T Q x} & \text{on } \left\{|x'| > 1, x_n = 0\right\}. \end{cases} \tag{1}$$

where the dimension $n \geq 2$, Q is a $n \times n$ symmetric positive definite matrix and for some $\varrho \in (0,1]$ and non-zero constant d,

- $\varrho|x|^2 \leq x^T Q x \leq \varrho^{-1}|x|^2, \quad \forall\, x \in \mathbb{R}^n$,
- $\frac{1}{2}|x|^2 + d \log \sqrt{x^T Q x}$ is strictly convex on $\{|x'| > 1, x_n = 0\}$.

The principal result stated that if u solves (1) and satisfies the quadratical growth condition

$$\mu|x|^2 \leq u(x) \leq \mu^{-1}|x|^2 \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B_1^+ \tag{2}$$

for some $0 < \mu \leq \frac{1}{2}$, then u tends to a quadratic polynomial plus an implicit function, which can be controlled by $\log|x|$ at infinity. Note here that the existence of an implicit function was caused by the perturbation of the boundary value.

Natural and interesting questions arise here: How about the asymptotic behavior at infinity if the perturbation of the boundary value is worse than $\log|x|$? And does it have better asymptotic behavior at infinity if the perturbation is better than $\log|x|$?

To answer the above questions, in this paper, we mainly study the asymptotic behavior of Monge–Ampère equations with more general boundary conditions as below

$$\begin{cases} \det D^2 u(x) = f(x) & \text{in } \mathbb{R}^n_+, \\ u(x) = \frac{1}{2}|x'|^2 + g(x') & \text{on } \{x_n = 0\}, \end{cases} \tag{3}$$

where the dimension $n \geq 2$, $f(x)$ satisfies

$$support(f - 1) \subset B_1^+(0), \quad 0 < \lambda \leq f(x) \leq \Lambda < \infty \tag{4}$$

for constants λ, Λ, and $g(x') \in C^m(\mathbb{R}^{n-1})$ satisfies

$$\left|D^k g(x')\right| \leq |x'|^{\theta - k} \quad \text{in } \mathbb{R}^{n-1} \backslash B_1 \quad \text{for all } 0 \leq k \leq m \tag{5}$$

for some integer $m \geq 3$, constant $\theta < \min\{\frac{1}{n}, \frac{1}{3}\}$ and

$$\frac{1}{2}|x'|^2 + g(x') \text{ is strictly convex in } \mathbb{R}^{n-1}. \tag{6}$$

Note that, by (5), one can define the Poisson integral

$$P[g](x)=\frac{2x_n}{\omega_n}\int_{R^{n-1}}\frac{g(y')}{|x-y'|}dy'. \tag{7}$$

It is clear that $P[g]$ is harmonic and continuous up to the boundary of $\mathbb{R}^n_+$ (cf. [2,25]).

The main result of this paper is the following.

Theorem 1. *Let $u\in C^0(\mathbb{R}^n_+\backslash B_1^+)$ be a convex viscosity solution of* (3) *with* (2), (4), (5) *and* (6). *Then, for any $\alpha\in(0,1)$, $u\in C^{m-1,\alpha}(\overline{\mathbb{R}}^n_+\backslash\overline{B}_1^+)$ and there exist some invertible upper-triangular matrix T with* $\det T=1$ *and constant $b_n\in\mathbb{R}$ such that*

(i) *if $n=2$,*

$$\left|u(x)-\left(\frac{1}{2}x^TT^TTx+b_nx_n+P[g](Tx)\right)\right|\le C\frac{x_2}{|x|^2}\quad in\ \mathbb{R}^2_+\backslash B_R^+,$$

where C and $R\ge 1$ depend only on μ and θ. Furthermore, for any $1\le k\le m-1$,

$$|x|^{k+1}\left|D^k\left(u(x)-\left(\frac{1}{2}x^TT^TTx+b_2x_2+P[g](Tx)\right)\right)\right|\le C\quad in\ \mathbb{R}^2_+\backslash B_R^+,$$

where C also depends on k.

(ii) *if $n\ge 3$, for any $\delta\in(0,\frac{2-2\theta}{n-1})$ if $\theta\ge-\frac{n-3}{2}$ and for $\delta=1$ if $\theta<-\frac{n-3}{2}$, we have*

$$\left|u(x)-\left(\frac{1}{2}x^TT^TTx+b_nx_n+P[g](Tx)\right)\right|\le C\left(\frac{x_n}{|x|^n}\right)^\delta\quad in\ \mathbb{R}^n_+\backslash B_R^+,$$

where C and $R\ge 1$ depend only on n, μ, δ and θ. Furthermore, for any $1\le k\le m-1$,

$$|x|^{k+(n-1)\delta}\left|D^k\left(u(x)-\left(\frac{1}{2}x^TT^TTx+b_nx_n+P[g](Tx)\right)\right)\right|\le C\quad in\ \mathbb{R}^n_+\backslash B_R^+,$$

where C also depends on k.

Remark 2. In Theorem 1, the case $n\ge 3$ implies that of $n=2$. In fact, as $n=2$, by $\theta<\min\{\frac{1}{n},\frac{1}{3}\}$ in (5), it is easy to see that $\theta<\frac{1}{2}=-\frac{n-3}{2}$.

Remark 3. Theorem 1 still holds for the normalized Monge–Ampère equation

$$\begin{cases}\det D^2u(x)=1 & \text{in } \mathbb{R}^n_+\backslash B_1^+,\\ u(x)=\frac{1}{2}|x'|^2+g(x') & \text{on } \{x_n=0,|x'|>1\},\end{cases}$$

where $g(x')\in C^m(\mathbb{R}^{n-1}\backslash B_1^+)$ satisfies (5) and

$$\frac{1}{2}|x'|^2+g(x') \text{ is strictly convex in } \mathbb{R}^{n-1}\backslash B_1^+.$$

In fact, by the same arguments in [13], one can construct some new function v such that $v=u$ in $\mathbb{R}^n_+\backslash B_1^+$ and v satisfies Theorem 1.

This paper mainly improves the analysis on asymptotic behavior of solutions of Monge–Ampère equations with different perturbations of boundary value conditions. Our approach includes two steps: *Rough estimate*, in which we plan to show that after proper transformation, $u-\frac{1}{2}|x|^2=O(|x|^{5/3})$ at infinity, and *Accurate estimate*, in which the precise asymptotic behavior including optimal decay rate will be given. Note that the Poisson integral method settled the asymptotic term and improved the method in [13].

Throughout this paper, we always say C and other constants are universal, which means that they depend only on n, μ and θ. And universal constants may change from line to line.

2. Proof of Theorem 1

Firstly, one can easily obtain a proof of the regularity result $u \in C^{m-1,\alpha}(\overline{\mathbb{R}}^n_+ \backslash \overline{B}_1^+)$ by the similar arguments in [13, Lemma 3.1]. So we omit it here.

In the following, we show our main result in two steps:

(1) *Rough estimate:* to show that after proper translation, u is near to some quadratic polynomial at infinity. Here we allow that u can be controlled by this quadratic polynomial plus some function, which does'nt converge to zero at infinity.

(2) *Accurate estimate:* to find out the most suitable function, such that after subtracting this function and the quadratic polynomial mentioned in the last step, u converges to zero with an exact decay rate at infinity.

2.1. *Rough estimate*

In this subsection, we mainly adopt the method in [13]. To minimize condition of θ as soon as possible, we should do better estimate, even if we call this rough estimate.

For any $M > \mu^{-1}$, let

$$\widehat{u}(x) = \frac{1}{M} u\left(M^{1/2} x\right), \quad x \in \mathcal{O} := \frac{1}{M^{1/2}} S_M(u). \tag{8}$$

where $S_M(u) = \{x \in \overline{\mathbb{R}}^n_+ : u(x) < M\}$. By the quadratic growth condition (2),

$$\mu^{1/2} \overline{B}_1^+ \subset \mathcal{O} \subset \mu^{-1/2} \overline{B}_1^+. \tag{9}$$

Obviously, $\widehat{u}$ solves

$$\begin{cases} \det D^2 \widehat{u}(x) = 1 & \text{in } \mathcal{O}, \\ \widehat{u}(x) = \frac{1}{2}|x'|^2 + \frac{1}{M} g\left(M^{1/2} x'\right) & \text{on } \partial\mathcal{O} \cap \{x_n = 0\}, \\ \widehat{u}(x) = 1 & \text{on } \partial\mathcal{O} \cap \{x_n > 0\}. \end{cases} \tag{10}$$

By the existence results (cf. [1, 3]), there exists a unique viscosity solution of the following Dirichlet problem

$$\begin{cases} \det D^2 \xi = 1 & \text{in } \mathcal{O}, \\ \xi = \frac{1}{2}|x'|^2 & \text{on } \partial\mathcal{O} \cap \{x_n = 0\}, \\ \xi = 1 - \frac{1}{M} P[g]\left(M^{1/2} x'\right) & \text{on } \partial\mathcal{O} \cap \{x_n > 0\}, \end{cases} \tag{11}$$

where $P[g]$ is the Poisson integral given by (7). By [23, Theorem 6.4], there exists universal $c_0 > 0$ such that

$$|D\xi(x)| \le c_0^{-1}, \quad c_0 I \le \left[D^2 \xi(x)\right] \le c_0^{-1} I, \quad \left|D^3 \xi(x)\right| \le c_0^{-1} \quad \text{in } \overline{B}^+_{c_0}. \tag{12}$$

Then one can show the following lemmas:

Lemma 4. *There exists universal $C > 0$ such that for any $M \ge \max\{\mu^{-1}, c_0^{-2}\}$,*

$$|\widehat{u} - \xi| \le C M^{-1/2} \quad \text{in } \overline{\mathcal{O}} \backslash B^+_{M^{-1/2}}.$$

Proof. In view of the definitions of $\widehat{u}$ and ξ and (5), one can deduce that

$$|\widehat{u} - \xi| = \frac{1}{M} \left|P[g]\left(M^{1/2} x'\right)\right| \le C M^{-1+\frac{\theta}{2}} \quad \text{on } \partial\mathcal{O}. \tag{13}$$

By the quadratical growth condition (2), we have

$$\mu M^{-1} \le \widehat{u} \le \mu^{-1} M^{-1} \quad \text{on } \partial B^+_{M^{-1/2}} \cap \{x_n > 0\}. \tag{14}$$

By (12), we have

$$|\xi(x)| \le 2 c_0^{-1} M^{-1/2} \quad \text{in } \overline{B}^+_{M^{-1/2}}. \tag{15}$$

By the virtue of (13), (14) and (15), we get

$$|\widehat{u}-\xi| \le CM^{-1/2} \quad \text{on } \partial\left(\mathcal{O}\backslash B^+_{M^{-1/2}}\right).$$

It together with the comparison principle completes the proof of Lemma 4. □

Lemma 5. *There exists universal $C>0$ such that for any $M \ge \max\{\mu^{-2}, c_0^{-2}\}$,*

$$|D\xi(0)| \le CM^{-1/4}.$$

Proof. By the definition of ξ and (5), it only needs to verify $|D_n\xi(0)| \le CM^{-1/4}$.

By (12), for any $\widetilde{x} = (0, \widetilde{x}_n) \in \overline{B}^+_{c_0}$, there exists $\vartheta \in (0,1)$ such that

$$\xi(0,\widetilde{x}_n) = \xi(0) + D_n\xi(0)\cdot \widetilde{x}_n + \frac{1}{2}D_{nn}\xi(\vartheta\widetilde{x})\,\widetilde{x}_n^2.$$

It follows that there exists universal $C>0$ such that

$$|D_n\xi(0)| \le \frac{|\xi(0,\widetilde{x}_n)| + C\widetilde{x}_n^2}{\widetilde{x}_n}. \tag{16}$$

Now one can choose $\widetilde{x}_n$ such that $\widehat{u}(0,\widetilde{x}_n) = M^{-1/2}$. It follows from (2) that

$$M^{-1/4}\mu^{1/2} \le \widetilde{x}_n \le M^{-1/4}\mu^{-1/2}.$$

It together with Lemma 4 and (16) implies that

$$|D_n\xi(0)| \le \frac{|\widehat{u}(0,\widetilde{x}_n)| + CM^{-1/2} + C\widetilde{x}_n^2}{\widetilde{x}_n} \le CM^{-1/4},$$

where $C>0$ is universal. It completes the proof. □

In order to obtain the *Rough estimate* and find out the weakest θ as soon as possible, we should precisely describe the property of cross section of the rescaled function u.

Let

$$E_M = \left\{x \in \overline{\mathbb{R}}^n_+ : \; x^T D^2\xi(0)x \le 1\right\}. \tag{17}$$

Lemma 6. *There exist large universal k_0 and $\widetilde{C}$ such that for all $k \ge k_0$, $M = 2^{\frac{4}{3}k}$ and $M' \in [2^{k-1}, 2^k]$,*

$$\left(\frac{2M'}{M} - \widetilde{C}2^{-\frac{1}{2}k}\right)^{1/2} E_M \subset \frac{S_{M'}(u)}{M^{1/2}} \subset \left(\frac{2M'}{M} + \widetilde{C}2^{-\frac{1}{2}k}\right)^{1/2} E_M. \tag{18}$$

Proof. By the definition of $\widehat{u}$, we have

$$\frac{S_{M'}(u)}{M^{1/2}} = \left\{\widehat{u} < \frac{M'}{M}\right\}.$$

It then follows from (15) and Lemma 4 that

$$\left\{\xi < \frac{M'}{M} - \frac{C}{M^{1/2}}\right\} \subset \frac{1}{M^{1/2}} S_{M'}(u) \subset \left\{\xi < \frac{M'}{M} + \frac{C}{M^{1/2}}\right\}. \tag{19}$$

By (12), for any $x \in \overline{B}^+_{c_0}$, we have

$$\left|\xi(x) - \xi(0) - D\xi(0)\cdot x - \frac{1}{2}x^T D^2\xi(0)x\right| \le C|x|^3. \tag{20}$$

Now we prove (18). Firstly, we show the first relation of (18). For any $x \in (\frac{2M'}{M} - \widetilde{C}2^{-\frac{1}{2}k})^{1/2}E_M$, by (17), we have

$$\frac{1}{2}x^T D^2\xi(0)x \le \frac{M'}{M} - \widetilde{C}2^{-\frac{1}{2}k} \le \frac{M'}{M}. \tag{21}$$

It together with (12) implies that $|x| \le C\left(\frac{M'}{M}\right)^{1/2}$ for some universal C. Then from Lemma 5, (20) and (21), we get

$$\begin{aligned}\xi(x) &\le \xi(0) + D\xi(0)\cdot x + \frac{1}{2}x^T D^2\xi(0)x + c_0^{-1}|x|^3\\ &\le CM^{-1/4}\left(\frac{M'}{M}\right)^{1/2} + \frac{M'}{M} - \widetilde{C}2^{-\frac{1}{2}k} + C\left(\frac{M'}{M}\right)^{3/2}.\end{aligned}$$

One can choose large universal $k_0 > 0$ and $\widetilde{C} > 0$ such that

$$CM^{-1/4}\left(\frac{M'}{M}\right)^{1/2} - \widetilde{C}2^{-\frac{1}{2}k} + C\left(\frac{M'}{M}\right)^{3/2} < -\frac{C}{M^{1/2}}$$

for any $k \ge k_0$, which yields that

$$\left(\frac{2M'}{M} - \widetilde{C}2^{-\frac{1}{2}k}\right)^{1/2} E_M \subset \left\{\xi < \frac{M'}{M} - \frac{C}{M^{1/2}}\right\}.$$

The first inclusion of (18) then follows from (19).

Next we show the second inclusion of (18). For any $x \in \frac{1}{M^{1/2}}S_{M'}(u)$, (2) implies

$$|x| \le \mu^{-1/2}\left(\frac{M'}{M}\right)^{1/2}.$$

This together with Lemma 5, (19) and (20) implies

$$\begin{aligned}\frac{1}{2}x^T D^2\xi(0)x &\le \xi(x) - \xi(0) - D\xi(0)\cdot x + c_0^{-1}|x|^3\\ &\le \frac{M'}{M} + CM^{-1/2} + CM^{-1/4}\left(\frac{M'}{M}\right)^{1/2} + C\left(\frac{M'}{M}\right)^{3/2}.\end{aligned}$$

Choosing larger universal k_0 and $\widetilde{C}$ again such that for any $k \ge k_0$,

$$\frac{1}{2}x^T D^2\xi(0)x \le \frac{2M'}{M} + \widetilde{C}2^{-\frac{1}{2}k},$$

we have

$$\left\{\xi < \frac{M'}{M} - \frac{C}{M^{1/2}}\right\} \subset \left(\frac{2M'}{M} + \widetilde{C}2^{-\frac{1}{2}k}\right)^{1/2} E_M,$$

and hence the second inclusion of (18) follows from (19). □

Along the same arguments in [13, Lemma 3.5-3.7], one can deduce the following lemma, which ends the *Rough estimate* and brings us some crucial derivatives estimates. See [13] for its proof in details.

Lemma 7. *There exists a real invertible bounded upper-triangular matrix T with $\det T = 1$ such that if $v(x) = u(y)$ and $y = T^{-1}x$, then*

$$\begin{cases}\det D^2 v = 1 & \text{in } \mathbb{R}^n_+\backslash\overline{TB_1^+},\\ v(x) = \frac{1}{2}|x'|^2 + g(x') & \text{on } \{x_n = 0\}\end{cases} \tag{22}$$

and for any $1 \le k \le m-1$,

$$\left|D^k\left(v(x) - \frac{1}{2}|x|^2\right)\right| \le C|x|^{\frac{5}{3}-k} \quad \text{in } \overline{\mathbb{R}}^n_+\backslash B^+_{R_0},$$

where $R_0 \ge 1$ is universal, and $C > 0$ depends only on n, μ and k.

2.2. *Accurate estimate*

In this section, we show the accurate asymptotic behavior of solutions at infinity by using the *Rough estimate* obtained in Lemma 7. The asymptotic behavior of solutions at infinity of linear elliptic equations in half spaces, established in [13, Theorem 2.4], will be employed on many occasions.

Lemma 8. *Let v be given by Lemma 7. Then*

(i) *for $n=2$, there exists some constant b_2 such that*

$$\left|v(x)-\frac{1}{2}|x|^2-b_2x_2-P[g](x)\right|\le C\frac{x_2}{|x|^2}\quad in\ \overline{\mathbb{R}}^2_+\backslash B^+_R, \tag{23}$$

where $P[g]$ is given by (7), and $C>0$ and $R>1$ are universal. Furthermore, for any $1\le k\le m-1$,

$$|x|^{k+1}\left|D^k\left(v(x)-\frac{1}{2}|x|^2-b_2x_2-P[g](x)\right)\right|\le C\quad in\ \overline{\mathbb{R}}^2_+\backslash B^+_R, \tag{24}$$

where C also depends on k.

(ii) *for $n\ge 3$, there exists some constant b_n such that*

$$\left|v(x)-\frac{1}{2}|x|^2-b_nx_n-P[g](x)\right|\le C\left(\frac{x_n}{|x|^n}\right)^{\delta}\quad in\ \overline{\mathbb{R}}^n_+\backslash B^+_R, \tag{25}$$

where $\delta\in(0,\frac{2-2\theta}{n-1})$ if $\theta\ge-\frac{n-3}{2}$ and $\delta=1$ if $\theta<-\frac{n-3}{2}$, $P[g]$ is given by (7), and $C>0$ and $R>1$ also depend on δ. Furthermore, for any $1\le k\le m-1$,

$$|x|^{k+(n-1)\delta}\left|D^k\left(v(x)-\frac{1}{2}|x|^2-b_nx_n-P[g](x)\right)\right|\le C\quad in\ \overline{\mathbb{R}}^n_+\backslash B^+_R, \tag{26}$$

where C also depends on k.

Proof. By Lemma 7, there exists universal $R_1>1$ such that

$$\begin{cases}\det D^2v(x)=1 & \text{in } \mathbb{R}^n_+\backslash\overline{B}^+_{R_1},\\ v(x)=\frac{1}{2}|x'|^2+g(x') & \text{on } \{x_n=0\}.\end{cases}$$

Let $V(x)=v(x)-\frac{1}{2}|x|^2$. By Lemma 7, there exists universal $C>0$ such that

$$|DV(x)|\le C|x|^{\frac{2}{3}}\quad\text{and}\quad\left|D^2V(x)\right|\le C|x|^{-\frac{1}{3}}\quad\text{in } \overline{\mathbb{R}}^n_+\backslash B^+_{R_1}. \tag{27}$$

In view of $\ln\det(I_n+D^2V)=\ln\det I_n=0$, we get

$$\begin{cases}a_{ij}(x)D_{ij}V(x)=0 & \text{in } \mathbb{R}^n_+\backslash\overline{B}^+_{R_1},\\ V(x)=g(x') & \text{on } \{x_n=0\},\end{cases} \tag{28}$$

where $a_{ij}(x)=\int_0^1[sD^2V+I_n]^{ij}(x)ds$.

Differentiating $\ln\det(I_n+D^2V)=0$ with respect to x_l, $l=1,\cdots,n-1$, we have

$$\begin{cases}\widetilde{a}_{ij}(x)D_{ij}V_l(x)=0 & \text{in } \mathbb{R}^n_+\backslash\overline{B}^+_{R_1},\\ V_l(x)=g_l(x') & \text{on } \{x_n=0\},\end{cases}$$

where $\widetilde{a}_{ij}(x)=[D^2V+I_n]^{ij}(x)$, $V_l=D_lV$ and $g_l=D_lg$. This implies that

$$\begin{cases}\widetilde{a}_{ij}D_{ij}\left(V_l-P[g_l]\right)=\widetilde{f}_l & \text{in } \mathbb{R}^n_+\backslash\overline{B}^+_{R_1},\\ V_l-P[g_l]=0 & \text{on } \{x_n=0\},\end{cases} \tag{29}$$

where

$$\widetilde{f}_l:=-\widetilde{a}_{ij}(x)D_{ij}P[g_l]=O\left(|x|^{\theta-3}\right)\quad\text{as } |x|\to\infty.$$

By (27), we have

$$\left|a_{ij}(x)-\delta_{ij}\right|+\left|\widetilde{a}_{ij}(x)-\delta_{ij}\right|\le C|x|^{-\frac{1}{3}}\quad\text{in } \overline{\mathbb{R}}^n_+\backslash B^+_{R_1} \tag{30}$$

and for any $l=1,\cdots,n-1$,

$$\left|D\left(V_l-P[g_l]\right)\right|\to 0\quad\text{as } |x|\to\infty. \tag{31}$$

By (29),(30), (31) and [13, Theorem 2.4] with $t=1-\theta\le n-1$, we have

$$\left|V_l-P[g_l]\right|\le C\left(\frac{x_n}{|x|^n}\right)^{\delta}\quad\text{in } \overline{\mathbb{R}}^n_+\backslash B^+_R, \tag{32}$$

where $\delta \in (0, \min\{1, \frac{1-\theta}{n-1}\})$, $R \geq R_1$ and $C > 0$ are universal.

For any $x \in \{x_n = 0, |x| \geq R+1\}$, by the Schauder estimates, we get

$$\left|D\left(V_l(x) - P[g_l]\right)\right| \leq C\left(\left\|V_l - P[g_l]\right\|_{L^\infty(B_1^+(x))} + \|\widetilde{f}_l\|_{C^{0,1}\left(\overline{B}_1^+(x)\right)}\right) \leq C\left(|x|^{-n\delta} + |x|^{\theta-3}\right).$$

Choosing $\delta \in (\frac{1}{n}, \min\{1, \frac{1-\theta}{n-1}\})$ such that $-n\delta < -1$, we have that for any $l = 1, \cdots, n-1$,

$$\left|V_{ln}(x', 0)\right| \leq C|x'|^{\max\{-n\delta, \theta-2\}}, \quad x \in \left\{|x'| \geq R, x_n = 0\right\}.$$

Since $\max\{-n\delta, \theta-2\} < -1$, there exists some constant b_n such that

$$V_n(x', 0) \to b_n \quad \text{as } |x'| \to \infty.$$

Let $M = \max\{V_n(x) : x \in (\partial B_{R_1} \cap \{x_n \geq 0\}) \cup \{x_n = 0, |x'| \geq R\}\}$. Since for any $\varepsilon > 0$,

$$\widetilde{a}_{ij}(x) D_{ij} V_n(x) = \widetilde{a}_{ij}(x) D_{ij}(M + \varepsilon x_n) = 0 \quad \text{in } \mathbb{R}^n_+ \setminus \overline{B}^+_R,$$

by (27) and the comparison principle, for any $\varepsilon > 0$ small,

$$|V_n| \leq M + \varepsilon x_n \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R.$$

By the arbitrariness of ε, we have

$$|V_n| \leq M \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R.$$

Combining this inequality with

$$\widetilde{a}_{ij}(x) D_{ij} V_n(x) = 0 \quad \text{in } \mathbb{R}^n_+ \setminus \overline{B}^+_{R_1}$$

and [13, Theorem 2.4], we deduce

$$V_n(x) \to b_n \quad \text{as } |x| \to \infty. \tag{33}$$

Applying (28) and (30), we have

$$\begin{cases} a_{ij}(x) D_{ij}\left(V - b_n x_n - P[g]\right) = \widehat{f}(x) & \text{in } \mathbb{R}^n_+ \setminus \overline{B}^+_{R_1}, \\ V - b_n x_n - P[g] = 0 & \text{on } \{x_n = 0\}, \end{cases} \tag{34}$$

where

$$\begin{aligned} \widehat{f}(x) &:= -a_{ij}(x) D_{ij} P[g](x) = \Delta P[g] + \left(a_{ij}(x) - \delta_{ij}\right) D_{ij} P[g] \\ &= O\left(|x|^{-\frac{7}{3}+\theta}\right) \quad \text{as } |x| \to \infty. \end{aligned}$$

Then there exists small $\tau > 0$ such that

$$\left|\widehat{f}\right| \leq |x|^{-2-\tau} \quad \text{as } |x| \to \infty. \tag{35}$$

By (32) and (33), we have

$$\left|D\left(V - b_n x_n - P[g]\right)\right| \to 0 \quad \text{as } |x| \to \infty.$$

Coupling the above estimate, (30), (34), (35) and [13, Theorem 2.4], we get for any $\delta' \in (0, \min\{1, \frac{\tau}{n-1}\})$, there exist $R \geq R_1$ and C depending only on n, μ and δ' such that

$$\left|V - b_n x_n - P[g]\right| \leq C\left(\frac{x_n}{|x|^n}\right)^{\delta'} \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R. \tag{36}$$

Next we improve (36). Indeed, (36) implies that

$$|V - b_n x_n| \leq C\left|P[g]\right| \leq C|x|^\theta \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R.$$

It then follows from Lemma 7 that

$$\left|D^2(V - b_n x_n)\right| \leq C|x|^{\theta-2} \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R.$$

By the definition of a_{ij}, we also have

$$\left|a_{ij}(x)-\delta_{ij}\right| \le C|x|^{\theta-2} \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R.$$

It then follows from the definition of $\widehat{f}$ that

$$\widehat{f}(x) = O\left(|x|^{-4+2\theta}\right) \quad \text{as } |x| \to \infty.$$

When $n=2$, by [13, Theorem 2.4] with $t = 2-2\theta > 1$, we get

$$\left|V - b_2 x_2 - P[g]\right| \le C\frac{x_2}{|x|^2} \quad \text{in } \overline{\mathbb{R}}^2_+ \setminus B^+_R,$$

which establishes (23).

When $n \ge 3$, by [13, Theorem 2.4], we have that for any $\delta \in (0, \min\{1, \frac{2-2\theta}{n-1}\})$,

$$\left|V - b_n x_n - P[g]\right| \le C\left(\frac{x_n}{|x|^n}\right)^\delta \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R$$

for some larger C and R depending also on δ. Especially, if $2-2\theta > n-1$ (i.e. $\theta < -\frac{n-1}{2}$), by [13, Theorem 2.4], we have

$$\left|V - b_n x_n - P[g]\right| \le C\frac{x_n}{|x|^n} \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus B^+_R,$$

which establishes (25).

Finally, we show (24) and (26).

For any $x \in \overline{\mathbb{R}}^n_+ \setminus B^+_{2R}$, let $r = |x|$. For any $y \in \mathscr{B}_2 := \{B_2(0) : x + \frac{r}{4}y \in \mathbb{R}^n_+ \setminus \overline{B}^+_r\}$, define

$$\overline{V}(y) = \left(\frac{4}{r}\right)^2 \left\{V\left(x+\frac{r}{4}y\right) - b_n\left(x_n + \frac{r}{4}y_n\right) - P[g]\left(x + \frac{r}{4}y\right)\right\}.$$

Then by (34), $\overline{V}$ solves

$$\overline{a}_{ij}(y)\overline{V}_{ij}(y) = \overline{f}(y) \quad \text{in } \mathscr{B}_2,$$

where

$$\overline{a}_{ij}(y) = a_{ij}\left(x + \frac{r}{4}y\right) \quad \text{and} \quad \overline{f}(y) = \widehat{f}\left(x+\frac{r}{4}y\right).$$

One can easily deduce that for any $0 \le k \le m-3$ and $\alpha \in (0,1)$, $\overline{f}$ satisfies

$$\left\|\overline{f}(y)\right\|_{C^{k,\alpha}\left(\overline{\mathscr{B}_2}\right)} \le Cr^{-4+2\theta},$$

and

$$\left\|\overline{V}\right\|_{L^\infty\left(\overline{\mathscr{B}_2}\right)} \le \begin{cases} Cr^{-3} & \text{if } n=2, \\ Cr^{-2-\delta(n-1)} & \text{if } n \ge 3, \end{cases}$$

where $\delta \in (0, \min\{1, \frac{2-2\theta}{n-1}\})$. By the Schauder estimates, we have for any $0 \le k \le m-1$,

$$\begin{aligned} \left|D^k\overline{V}(y)\right| &\le C\left(\left\|\overline{f}(y)\right\|_{C^{k,\alpha}\left(\overline{\mathscr{B}_2}\right)} + \|\overline{V}\|_{L^\infty\left(\overline{\mathscr{B}_2}\right)}\right) \\ &\le \begin{cases} Cr^{-4+2\theta} + Cr^{-3} & \text{if } n=2, \\ Cr^{-4+2\theta} + Cr^{-2-\delta(n-1)} & \text{if } n \ge 3, \end{cases} \\ &\le \begin{cases} Cr^{-3} & \text{if } n=2, \\ Cr^{-2-\delta(n-1)} & \text{if } n \ge 3, \end{cases} \quad \text{in } y \in \mathscr{B}_1 := \left\{B_1(0) : x + \frac{r}{4}y \in \overline{\mathbb{R}}^n_+ \setminus B^+_r\right\} \end{aligned}$$

for any $C > 0$ depending only on n, μ, δ and k. Then we arrive at the desired estimates (24) and (26). □

Finally, Lemma 8 implies Theorem 1.

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