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Dynamical systems / *Systèmes dynamiques*

On the existence of generalised syzygies in the planar three-body problem

Sur l'existence des syzygies généralisées dans le problème plan des trois corps

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Abstract. We consider the Newtonian planar three-body problem. One defines a generalised syzygy as a configuration where the three bodies or their velocities become collinear. Assuming that the motion is bounded and collision-free, we provide a simple sufficient condition for the existence of such configurations. Our proof is elementary and uses only basic tools from the Sturm–Liouville theory.

Résumé. Nous considérons le problème plan des trois corps. On définit une syzygie généralisée comme une configuration où les trois corps ou leurs vitesses deviennent colinéaires. En supposant que le mouvement est borné et sans collision, nous fournissons une condition suffisante pour l'existence de telles configurations. Nos principaux outils sont élémentaires et basés sur la théorie de Sturm–Liouville.

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Considérons trois corps P_1, P_2, P_3 dans le plan avec des masses $m_1 > 0$, $m_2 > 0$, $m_3 > 0$ qui s'attirent conformément à la loi de Newton. En supposant que le centre de gravité est fixe et placé à l'origine on obtient les équations du mouvement sous la forme suivante

$$\ddot{z}_1 = m_2 \frac{z_{21}}{|z_{21}|^3} - m_3 \frac{z_{13}}{|z_{13}|^3}, \quad \ddot{z}_2 = m_3 \frac{z_{32}}{|z_{32}|^3} - m_1 \frac{z_{21}}{|z_{21}|^3}, \quad \ddot{z}_3 = m_1 \frac{z_{13}}{|z_{13}|^3} - m_2 \frac{z_{32}}{|z_{32}|^3}, \tag{1}$$

avec $z_k = x_k + i y_k \in \mathbb{C}$, $k = 1,2,3$ et $z_{kl} = z_k - z_l$.

Nous disons que les trois corps P_1, P_2, P_3 forment une *syzygie généralisée* (éclipse) à l'instant $t_0 \in I$, si l'un des triplets $(z_1, z_2, z_3)(t_0)$ (positions) ou $(\dot{z}_1, \dot{z}_2, \dot{z}_3)(t_0)$ (vitesses) devient colinéaire.

En considérant deux corps quelconques, par exemple P_1 et P_2, on dit que pour $t = 0$ la configuration de P_1, P_2, P_3 est *antisymétrique*, si les aires orientées de deux parallélogrammes

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engendrés par les vecteurs $(z_1(0), z_2(0))$ et $(\dot{z}_1(0), \dot{z}_2(0))$ sont toutes les deux non nulles et de signes opposés. Nous montrons ensuite, que toute solution définie sur $\mathbb{R}^+$ bornée dont la configuration à l'instant $t = 0$ est antisymétrique, atteint nécessairement une syzygie généralisée en un temps qui n'excède pas la valeur $T = \pi\alpha^{3/2}/\sqrt{2M}$, $M = \sum_i m_i$ où α est la distance maximale entre les corps. L'idée de la démonstration est de se ramener à l'étude de l'équation de Riccati $\dot{C} + C^2 = A$ où $C = \dot{X}X^{-1}$ et X est la matrice 2×2 formée par les coordonnées de P_1 et P_2. La matrice A correspond alors à l'endomorphisme réduit de Wintner–Conley [1] dont la trace est toujours négative. Les positions des singularités de C sont à leur tour contrôlées grâce à la théorie de Sturm–Liouville. En remarquant qu'une syzygie généralisée correspond soit à l'annulation de $\det(C)$ soit à une explosion en temps fini de C, nous pouvons conclure.

1. Introduction

Let P_1, P_2, P_3 be the three points in the plain with strictly positive masses m_1, m_2, m_3 and euclidean coordinates $(x_i, y_i) \in \mathbb{R}^2$, $i = 1,2,3$. The Newtonian three-body problem [7] can be written as follows

$$\ddot{z}_1 = m_2\frac{z_{21}}{|z_{21}|^3} - m_3\frac{z_{13}}{|z_{13}|^3}, \quad \ddot{z}_2 = m_3\frac{z_{32}}{|z_{32}|^3} - m_1\frac{z_{21}}{|z_{21}|^3}, \quad \ddot{z}_3 = m_1\frac{z_{13}}{|z_{13}|^3} - m_2\frac{z_{32}}{|z_{32}|^3}, \tag{2}$$

where $z_k = x_k + iy_k \in \mathbb{C}$, $k = 1,2,3$ and $z_{kl} = z_k - z_l$. Assuming that the total linear momentum (which is preserved) is zero, one can always admit that

$$\sum_k m_k\dot{z}_k = 0, \quad \sum_k m_k z_k = 0, \tag{3}$$

by placing the center of mass of the bodies at the origin.

Let $t \mapsto z_i(t)$, $i = 1,2,3$ be any collision-free solution of equations (2) defined for $t \in I = [0, a)$, $a > 0$ and determined by a set of initial conditions $(z_i(0), \dot{z}_i(0))$, $i = 1,2,3$.

Definition 1. *We say that the three bodies P_1, P_2, P_3 form a generalised syzygy (eclipse) at the moment $t_0 \in I$ if one of the complex triplets $(z_1, z_2, z_3)(t_0)$ (positions) or $(\dot{z}_1, \dot{z}_2, \dot{z}_3)(t_0)$ (velocities) is collinear.*

Definition 2. *Considering any two bodies, for example P_1 and P_2, one says that for $t = 0$ the configuration of P_1, P_2, P_3 is* antisymmetric, *if the oriented areas of the two parallelograms spanned by the vectors $(z_1(0), z_2(0))$ and $(\dot{z}_1(0), \dot{z}_2(0))$ are both nonzero and have the opposites signs. This condition is equivalent algebraically to*

$$\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}(0)\cdot\det\begin{bmatrix} \dot{x}_1 & \dot{y}_1 \\ \dot{x}_2 & \dot{y}_2 \end{bmatrix}(0) = \left(x_1y_2 - y_1x_2\right)(0)\cdot\left(\dot{x}_1\dot{y}_2 - \dot{y}_1\dot{x}_2\right)(0) < 0. \tag{4}$$

Remark 3. It is elementary to show, with the help of formulas (3) and (4), that the property to be antisymmetric, does not depend on the choice of the pair of the bodies: once verified for two bodies P_i, P_j, $i \neq j$ the condition (4) will be also satisfied for the two remaining choices of pairs.

We can formulate now our main result:

Theorem 4. *Let $t \mapsto (z_1(t), z_2(t), z_3(t))$ be a solution of the three-body problem* (2) *defined for $t \geq 0$ and satisfying the following conditions:*

(a) *It is collision-free and uniformly bounded, i.e. $\exists\ \alpha > 0$, $|z_{ij}(t)| \leq \alpha$, $\forall\ i \neq j$, $\forall\ t \geq 0$*
(b) *The initial configuration at $t = 0$ is antisymmetric*

Then, there exists $0 < t_0 \leq T$ such that the three bodies form a generalised syzygy at the moment $t = t_0$ where

$$T = \frac{\pi\alpha^{3/2}}{\sqrt{2M}}, \quad M = \sum_i m_i. \tag{5}$$

Montgomery in his work [5], using the different approach, has shown that, with the exception of Lagrange's solution, every solution to the zero angular momentum, Newtonian three-body problem, suffers a syzygy - i.e. collinear positions at some instant. Our result is free from the angular momentum hypothesis, but the generalised syzygies condition is weaker than the one used in [5] since it now includes the possibility of collinear velocities.

2. The proof of the Main Result

We start by introducing the new variables $w_i = m_i z_i$, $i = 1,2,3$ so that the relations (3) imply

$$\sum_i w_i = 0. \tag{6}$$

Writing $w_k = X_k + iY_k$, $X_k = m_k x_k$, $Y_k = m_k y_k$, $k = 1,2,3$ and using the previous equality, one derives from equations (2) the following 2×2 matrix equation

$$\ddot{X} = AX, \quad X = \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}, \quad A = \begin{bmatrix} -m_2\rho_3 - m_{13}\rho_2 & m_1\left(\rho_3-\rho_2\right) \\ m_2\left(\rho_3-\rho_1\right) & -m_1\rho_3 - m_{32}\rho_1 \end{bmatrix}, \tag{7}$$

where $\rho_1 = 1/|z_{32}|^3$, $\rho_2 = 1/|z_{13}|^3$, $\rho_3 = 1/|z_{21}|^3$, $m_{ij} = m_i + m_j$.

So, X is now a particular solution of a second-order linear differential equation if we consider matrix A as a known function of time.

The matrix A is related to the Wintner–Conley endomorphism encoding the forces. Its higher dimension version was investigated in [1,3] in the study of n-body configurations and the same equation (7) was used (up to reduction, transposition and some scaling factor).

Remark 5. It can be easily verified that both definitions of the generalised syzygy (Definition 1) and antisymmetric configuration (Definition 2) are invariant when replacing x_i, y_i by X_i, Y_i and $\dot{x}_i, \dot{y}_i$ by $\dot{X}_i, \dot{Y}_i$ for $i = 1,2,3$.

The change of variable $C = \dot{X}X^{-1}$ transforms the linear equation (7) into the matrix Riccati equation

$$\dot{C} + C^2 = A. \tag{8}$$

Since the configuration of the bodies is antisymmetric for $t = 0$ (see Definition 1), by continuity, the matrix $X(t)$ is invertible and C is well defined with $\det(C(t)) < 0$ for $t \geq 0$ sufficiently close to 0. The main idea of the proof is that the first generalised syzygy happens for $t = t_0 > 0$ if $\det(C(t_0)) = 0$ or C has a blow-up (singularity) at $t = t_0$. Indeed, according to the hypothesis of Theorem 4, the function X is smooth for all $t \geq 0$. So, $C = \dot{X}X^{-1}$ can have a singularity at $t = t_0 > 0$ only if $\det(X(t_0)) = 0$ i.e. the bodies are collinear. If $\det(C(t_0)) = 0$, then $\det(\dot{X}(t_0)) = 0$ and the velocities of the bodies become collinear.

Taking the trace of the both sides of the equation (8), one derives the following scalar Riccati equation

$$\dot{\omega} + \omega^2 = \phi = \mathrm{Tr}(A) + 2\det(C), \quad \omega = \mathrm{Tr}(C). \tag{9}$$

Since the mutual distances $|z_{ij}|$ are bounded by $\alpha > 0$, using the formula for A given in (7), it is elementary to show that

$$\mathrm{Tr}(A(t)) \leq -2\frac{m_1+m_2+m_3}{\alpha^3} = -\theta^2, \quad \forall\ t \geq 0. \tag{10}$$

Now, if we assume that the motion of the bodies starts from an antisymmetric configuration at $t = 0$ and that the first generalised syzygy happens at $t = t_0 > 0$ (if not, we put $t_0 = +\infty$), one has $\det(C(t)) < 0$ and $t \mapsto C(t)$ is a smooth function for all $t \in [0, t_0)$. Therefore, according to (9), (10):

$$\dot{\omega} + \omega^2 = \phi, \quad \phi(t) \leq -\theta^2, \quad \forall\ t \in [0, t_0). \tag{11}$$

Linearising the above equation via the change $\omega = \dot{x}/x$, one transforms it to the following form:

$$\ddot{x} = \phi x, \quad \phi(t) \le -\theta^2, \quad t \in [0, t_0). \tag{12}$$

The well know fact from the Sturm–Liouville theory (see [2]) states that x admits a zero between any two consecutive zeros of any nontrivial solution y of the linear equation $\ddot{y} = -\theta^2 y$. But the last one has the general solution given by $y(t) = B\cos(\theta t + K)$, $B, K \in \mathbb{R}$. Thus, the blow-up of $\omega = \dot{x}/x$ happens necessarily for some $0 < t_0 \le T$ with T given by the formula (5). Thus, we have shown that the generalised syzygy happens in the interval $(0, T]$ and the proof of Theorem 4 is finished. □

3. Conclusion and numerical results

As can be seen from the proof of Theorem 4, the same result holds for solutions not necessary bounded and collision-free on the whole $\mathbb{R}^+$ axis, but only during the sufficiently long period of time preceding the first generalised syzygy. We also believe that our result can be generalised to the case of $d+1$ bodies in d-dimensional space with $d \ge 3$. In particular, as shown in [6], in this situation, the any zero angular momentum bounded solution suffers infinitely many "coplanar" instants were $d+1$ masses all lie on a single affine hyperplane. Finally, knowledge of the constant angular momentum and energy, when applied to the equation (8), can bring undoubtedly more precise information about the presence of syzygies (generalised or not).

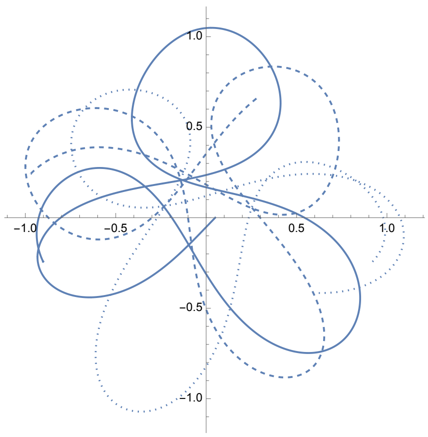

Figure 1. Trajectories of the perturbed version of the figure eight solution taken from [4]: $m_1 = m_2 = m_3 = 1$, $x_1(0) = 0.97000436 - 0.05$, $x_2(0) = -0.97000436$, $y_1(0) = -0.24308753$, $y_2(0) = 0.24308753$, $\dot{x}_1(0) = 0.93240737/2 - 0.05$, $\dot{x}_2(0) = 0.93240737/2$, $\dot{y}_1(0) = 0.8647314/2$, $\dot{y}_2(0) = 0.8647314/2 + 0.04$, $0 \le t \le 10$

In order to illustrate numerically our Theorem 4, we choose the perturbed version of the figure eight solution whose initial conditions were provided in [4]. It is bounded and collision-free in time interval $[0, 10]$. Figure 1 demonstrates the corresponding trajectories of the three bodies. We plot the quantities $\Delta_1 = x_1 y_2 - x_2 y_1$ and $\Delta_2 = \dot{x}_1 \dot{y}_2 - \dot{x}_2 \dot{y}_1$ (determinants in (2)) as functions of time $t \in [0, 10]$ (Figure 3). Then each intersection of the graphs of $\Delta_{1,2}$ with the horizontal axis corresponds to a generalised syzygy. The formula (5) gives the value $T = 4.18$ were we estimate α as $\alpha = 2.2$ (see Figure 2).

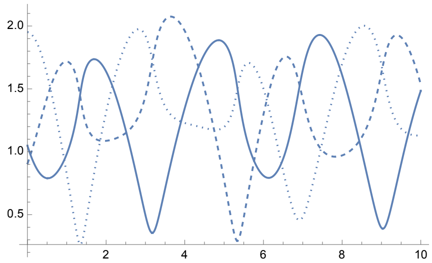

Figure 2. Mutual distances $|z_{ij}|$ of the bodies for $0 \le t \le 10$

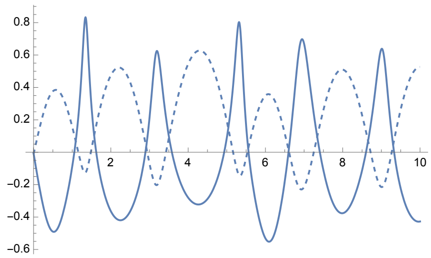

Figure 3. Determinants Δ_1 (dashed line) and Δ_2 for $0 \le t \le 10$

As can be seen in Figure 3, one observes, for example, an antisymmetric configuration of the bodies at $t = 2$ and generalised syzygies occurring before the momentum $t = 4$. This is in perfect agreement with the estimate given in Theorem 4.

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