C. R. Acad. Sci. Paris, Ser. I 356 (2018) 963–966

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebraic geometry/Topology

Milnor and Tjurina numbers for a hypersurface germ with isolated singularity

Nombres de Milnor et Tjurina pour les germes d'hypersurfaces à singularité isolée

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ARTICLE INFO

Article history:
Received 17 May 2018
Accepted 6 July 2018
Available online 19 July 2018

Presented by Claire Voisin

ABSTRACT

Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Let μ and τ be the corresponding Milnor and Tjurina numbers. We show that $\frac{\mu}{\tau} \leq n$. As an application, we give a lower bound for the Tjurina number in terms of n and the multiplicity of f at the origin.

RÉSUMÉ

Soit $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ un germe de fonction analytique au voisinage de l'origine avec une seule singularité isolée. Soient μ et τ les nombres de Milnor et Tjurina correspondants. Nous montrons que $\frac{\mu}{\tau} \leq n$. Comme application, nous donnons une minoration du nombre de Tjurina en fonction de n et de la multiplicité de f à l'origine.

1. Main result

Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Set $X = f^{-1}(0)$. Let $S = \mathbb{C}\{x_1, \ldots, x_n\}$ denote the formal power series ring. Set $J_f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ as the Jacobian ideal. Then the Milnor and Tjurina algebras are defined as

$$M_f = S/J_f, \text{ and } T_f = S/(J_f, f).$$

Since X has isolated singularities, M_f and T_f are finite dimensional $\mathbb{C}$-vector spaces. The corresponding dimension μ and τ are called the Milnor and Tjurina numbers, respectively. It is clear that $\mu \geq \tau$.

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https://doi.org/10.1016/j.crma.2018.07.004
1631-073X/

Consider the following long exact sequence of $\mathbb{C}$-algebras:

$$0 \to \mathrm{Ker}(f) \to M_f \xrightarrow{f} M_f \to T_f \to 0 \tag{1}$$

where the middle map is multiplication by f, and $\mathrm{Ker}(f)$ is the kernel of this map. Then $\dim_{\mathbb{C}} \mathrm{Ker}(f) = \tau$.

Recall a well-known result given by J. Briançon and H. Skoda in [1],

$$f^n \in J_f,$$

which shows that $f^n = 0$ in M_f, i.e. $(f^{n-1}) \subset \mathrm{Ker}(f)$. Here (f^{n-1}) is the ideal in M_f generated by f^{n-1}. The following theorem is a direct application of this result.

Theorem 1.1. *Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Then,*

$$\frac{\mu}{\tau} \le n.$$

Moreover, $\frac{\mu}{\tau} = n$, if and only if, $\mathrm{Ker}(f) = (f^{n-1})$.

Proof. Since $f^n = 0$ in M_f, we have the following finite decreasing filtration:

$$M_f \supset (f) \supset (f^2) \supset \cdots \supset (f^{n-1}) \supset (f^n) = 0$$

where (f^i) is the ideal in M_f generated by f^i.

Consider the following long exact sequence:

$$0 \to \mathrm{Ker}(f) \cap (f^i) \to (f^i) \xrightarrow{f} (f^i) \to (f^i)/(f^{i+1}) \to 0 \tag{2}$$

where the middle map is multiplication by f. Then,

$$\dim_{\mathbb{C}}\{(f^i)/(f^{i+1})\} = \dim_{\mathbb{C}}\{\mathrm{Ker}(f) \cap (f^i)\} \le \dim_{\mathbb{C}} \mathrm{Ker}(f) = \tau.$$

Therefore,

$$\mu = \dim_{\mathbb{C}} M_f = \dim_{\mathbb{C}} T_f + \sum_{i=1}^{n-1} \dim_{\mathbb{C}}\{(f^i)/(f^{i+1})\} \le n \cdot \tau.$$

$\frac{\mu}{\tau} = n$ if and only if, for any $1 \le i \le n-1$, $\mathrm{Ker}(f) \cap (f^i) = \mathrm{Ker}(f)$, i.e. $\mathrm{Ker}(f) \subset (f^i)$. On the other hand, $(f^{n-1}) \subset \mathrm{Ker}(f)$. Hence, $\mathrm{Ker}(f) = (f^{n-1})$. □

K. Saito showed ([8]) that $\frac{\mu}{\tau} = 1$ holds, if and only if, f is weighted homogeneous, i.e. analytically equivalent to such a polynomial. It leads to the following natural question.

Question 1.2. Is this upper bound of $\frac{\mu}{\tau}$ optimal? When can the optimal upper bound be obtained?

Remark 1.3. Recently, A. Dimca and G.-M. Greuel showed ([3, Theorem 1.1]) that the upper bound $\frac{\mu}{\tau} \le 2$ can never be achieved for the isolated plane curve singularity case unless f is smooth at the origin. Moreover, they gave ([3, Example 4.1]) a sequence of isolated plane curve singularity with the ratio $\frac{\mu}{\tau}$ strictly increasing towards 4/3. In particular, the singularities can be chosen to be all either irreducible, or consisting of smooth branches with distinct tangents. Based on these computations, they asked ([3, Question 4.2]) whether

$$\frac{\mu}{\tau} < 4/3$$

for any isolated plane curve singularity.

Example 1.4. It is clear that $\frac{\mu}{\tau} > n-1$ implies that $f^{n-1} \notin J_f$.

Consider the function germ:

$$f = (x_1 \cdots x_n)^2 + x_1^{2n+2} + \cdots + x_n^{2n+2},$$

which defines an isolated singularity at the origin. B. Malgrange showed ([7]) that the monodromy on the $(n-1)$-th cohomology of the Milnor fibre has a Jordan block with size n. Coupled with the theorem by J. Scherk ([9, Theorem]), it gives us that $f^{n-1} \notin J_f$. It can be checked with the software SINGULAR that $\frac{\mu}{\tau} < 1.5$ for $n \leq 7$, which is far away from our upper bound n.

2. Applications

Theorem 1.1 implies a well-known result in complex singularity theory, which states that the Milnor number of an analytic function germ is finite (or non-zero) if and only if the Tjurina number is so (see [4, Lemma 2.3, Lemma 2.44]).

2.1. A lower bound for the Tjurina number

First we recall a well-known lower bound for μ in terms of n and the multiplicity m of f at the origin. The following description can be found in [5].

The sectional Milnor numbers associated with the germ X are introduced by Teissier [10]. The i-th sectional Milnor number of the germ X, denoted μ^i, is the Milnor number of the intersection of X with a general i-dimensional plane passing through the origin (it does not depend on the choice of the generic planes). Then $\mu = \mu^n$. The Minkowski inequality for mixed multiplicities says that the sectional Milnor numbers always form a log-convex sequence [11]. In other words, we have

$$\frac{\mu^n}{\mu^{n-1}} \geq \frac{\mu^{n-1}}{\mu^{n-2}} \geq \cdots \geq \frac{\mu^1}{\mu^0},$$

where $\mu^0 = 1$ and $\mu^1 = m - 1$. Then

$$\mu \geq (m-1)^n. \tag{3}$$

Moreover, the equality holds if and only if f is a semi-homogeneous function (i.e. $f = f_m + g$, where f_m is a homogeneous polynomial of degree m defining an isolated singularity at the origin and g consists of terms of degree at least $m+1$) after a biholomorphic change of coordinates. For a detailed proof, see [13, Proposition 3.1].

The next corollary is a direct consequence of Theorem 1.1 and (3).

Corollary 2.1. *Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Then,*

$$\tau \geq \frac{(m-1)^n}{n}.$$

It is clear that, even for the homogeneous polynomial case, this lower bound can never be obtained when $n > 1$. In fact, in this case, $\tau = \mu = (m-1)^n > \frac{(m-1)^n}{n}$.

2.2. Another lower bound for the Tjurina number

Another lower bound for μ is given by A. G. Kushnirenko using the Newton number ([6]). Let Γ be the boundary of the Newton polyhedron of f, i.e. Γ is a polyhedron of dimension $n-1$ in $\mathbb{N}^n$ (where $\mathbb{N} = \{0, 1, 2, \cdots\}$) determined in the usual way by the non-zero coefficients in f. Then f is said to be convenient if Γ meets each of the coordinate axes of $\mathbb{R}^n$. Let S be the union of all line segments in $\mathbb{R}^n$ joining the origin to points of Γ. For a convenient f, the Newton number $\nu(f)$ is defined as:

$$\nu = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}1!V_1 + (-1)^n,$$

where V_n is the n-dimensional volume of S and for $1 \leq q \leq n-1$, V_q is the sum of the q-dimensional volumes of the intersection of S with the coordinate planes of dimension q. A. G. Kushnirenko showed that, if f is convenient, then,

$$\mu \geq \nu.$$

Moreover, $\mu = \nu$ holds, if f is non-degenerate. (For the definition of non-degenerate, see [6, Definition 1.19].) Again, this gives us a corresponding lower bound for the Tjurina number.

Corollary 2.2. *Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity, which is convenient. Then,*

$$\tau \geq \frac{\nu}{n},$$

where ν is the Newton number.

Question 2.3. Are the lower bounds of τ in Corollary 2.1 and 2.2 optimal? When can the optimal lower bounds be obtained?

For some special class of polynomials, the bound for the ratio $\frac{\mu}{\tau}$ can be improved. For example, A. Dimca showed that $f^2 \in J_f$ for semi-weighted homogeneous polynomials ([2, Example 3.5]), hence $\frac{\mu}{\tau} \leq 2$ and $\tau \geq \frac{(m-1)^n}{2}$ in this case.

Example 2.4. Choose $f = x^m + y^m + z^m + g$, where g has degree at least $m+1$. Then $\mu = (m-1)^3$. It is shown in [12, Example 4.7] that $\tau_{\min} = (2m-3)(m+1)(m-1)/3$, when g varies.

Acknowledgements

The author is grateful to Alexandru Dimca for useful discussions. The author is partially supported by Nero Budur's research project G0B2115N from the Research Foundation of Flanders.

References

[1] J. Briançon, H. Skoda, Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbb{C}^n$, C. R. Acad. Sci. Paris, Ser. A 278 (1974) 949–951 (in French).
[2] A. Dimca, Differential forms and hypersurface singularities, Singularity theory and its applications, Part I, in: Coventry, 1988/1989, in: Lecture Notes in Mathematics, vol. 1462, Springer, Berlin, 1991, pp. 122–153.
[3] A. Dimca, G.-M. Greuel, On 1-forms on isolated complete intersection curve singularities, arXiv:1709.01451v2.
[4] G.-M. Greuel, C. Lossen, E. Shustin, Introduction to Singularities and Deformations, Springer Monographs in Mathematics, Springer, Berlin, 2007.
[5] J. Huh, Milnor numbers of projective hypersurfaces with isolated singularities, Duke Math. J. 163 (2014) 1525–1548.
[6] A.G. Kouchnirenko, Polyèdres de Newton et nombres de Milnor, Invent. Math. 32 (1) (1976) 1–31.
[7] B. Malgrange, Letter to the editors, Invent. Math. 20 (1973) 171–172.
[8] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen, Invent. Math. 14 (1971) 123–142 (in German).
[9] J. Scherk, On the monodromy theorem for isolated hypersurface singularities, Invent. Math. 58 (3) (1980) 289–301.
[10] B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, Singularités à Cargèse. Asterisque, vol. 7 et 8, Société mathématique de France, Paris, 1973, pp. 285–362.
[11] B. Teissier, Sur une inégalité à la Minkowski pour les multiplicités, Ann. of Math. (2) 106 (1) (1977) 38–44.
[12] J. Wahl, A characterization of quasihomogeneous Gorenstein surface singularities, Compos. Math. 55 (3) (1985) 269–288.
[13] S.S.-T. Yau, H. Zuo, Complete characterization of isolated homogeneous hypersurface singularities, Pac. J. Math. 273 (1) (2015) 213–224.