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On sofic groupoids and their full groups

*Sur les groupoïdes sofiques et leurs groupes pleins*Luiz Cordeiro¹

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ABSTRACT

We prove that the class of sofic groupoids is stable under several measure-theoretic constructions. In particular, we show that virtually sofic groupoids are sofic. We answer a question of Conley, Kechris, and Tucker-Drob by proving that an aperiodic pmp groupoid is sofic if and only if its full group is metrically sofic.

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R É S U M É

Nous démontrons dans cette note que plusieurs constructions de théorie de la mesure préservent la classe des groupoïdes sofiques. En particulier, nous montrons qu'un sous-groupoïde virtuellement sofique est sofique. Nous répondons aussi à une question de Conley, Kechris et Tucker-Drob en démontrant que, pour qu'un groupoïde apériodique muni d'une mesure de probabilité invariante soit sofique, il est nécessaire et suffisant que son groupe plein soit métriquement sofique.

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1. Introduction

The notion of soficity for groups was introduced by Gromov [11] in his work on symbolic dynamics. In 2010, Elek and Lippner [7] introduced soficity for equivalence relations in the same spirit as Gromov's original definition, i.e. an equivalence relation R , induced by some action of the free group \mathbb{F}_∞ , is sofic if the Schreier graph of the \mathbb{F}_∞ -space X can be approximated, in a suitable sense, by Schreier graphs of finite \mathbb{F}_∞ -spaces.

Alternative definitions by Ozawa [16] and Păunescu [17] describe soficity at the level of the so-called full semigroup of R , which can be immediately generalized to groupoids. We will describe general elementary techniques to deal with (abstract) sofic groupoids.

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1.1. Probability measure-preserving groupoids and full semigroups

We will follow the notations of [5]: given a groupoid G , the source and range maps will be, respectively, $s(g) = g^{-1}g$ and $r(g) = gg^{-1}$ for $g \in G$, and the unit space of G will be denoted $G^{(0)}$.

A *discrete measurable groupoid* is a groupoid G endowed with a standard Borel space structure such that the product and inversion maps are Borel, and such that $s^{-1}(x)$ is countable for every $x \in G^{(0)}$.

The *Borel full semigroup* of a discrete measurable groupoid G is the set $[[G]]_B$ of Borel subsets $\alpha \subseteq G$ such that the restrictions $s|_\alpha$ and $r|_\alpha$ of the source and range maps are injections, and thus Borel isomorphisms onto their respective images ([13, Theorem 15.2]).

$[[G]]_B$ is an inverse monoid with the usual product and inverse of sets, namely

$$\alpha\beta = \{ab : (a, b) \in (\alpha \times \beta) \cap G^{(2)}\}, \quad \alpha^{-1} = \{a^{-1} : a \in \alpha\}$$

and $G^{(0)}$ is the unit of $[[G]]_B$, which we will instead denote by $G^{(0)} = 1$ when no confusion arises.

A *probability measure-preserving (pmp) groupoid* is a discrete measurable groupoid G with a Borel probability measure μ on $G^{(0)}$ satisfying $\mu(s(\alpha)) = \mu(r(\alpha))$ for all $\alpha \in [[G]]_B$. We write (G, μ) for a pmp groupoid when we need the measure μ to be explicit. The measure μ induces a pseudometric d_μ on $[[G]]_B$ via

$$d_\mu(\alpha, \beta) = \mu(s(\alpha\Delta\beta)) = \mu(r(\alpha\Delta\beta)).$$

The *trace* of $\alpha \in [[G]]_B$ is defined as $\text{tr}(\alpha) = \mu(\alpha \cap G^{(0)})$. In fact, the trace and the pseudometric above, along with the semigroup operation, determine each other: for example, the unit 1 of $[[G]]_B$ is the only element of trace 1, and

$$d_\mu(\alpha, \beta) = \text{tr}(\alpha^{-1}\alpha) + \text{tr}(\beta^{-1}\beta) - \text{tr}(\alpha^{-1}\alpha\beta^{-1}\beta) - \text{tr}(\beta^{-1}\alpha)$$

and similarly one can write the trace in terms of the pseudometric d_μ .

The (measured) *full semigroup* of a pmp groupoid (G, μ) is the quotient metric space $[[G]]$ (or $[[G]]_\mu$ to make μ explicit) of $[[G]]_B$ under the pseudometric d_μ . In fact, $[[G]]$ is an inverse semigroup, with the quotient operation endowed from $[[G]]_B$, and the trace map $\text{tr} : [[G]]_B \rightarrow \mathbb{R}$ factors through a map on $[[G]]$. We will not distinguish $[[G]]_B$ and $[[G]]$ unless strictly necessary.

The *Borel full group* $[G]_B$ of a discrete measurable groupoid G is the set of those $\alpha \in [[G]]_B$ with $s(\alpha) = r(\alpha) = G^{(0)}$, and, when G is pmp, the image of $[G]_B$ in $[[G]]$, denoted $[G]$ or $[G]_\mu$, is called the (measured) *full group* of G .

Definition 1.1. A subset A of a pmp groupoid (G, μ) is called *null* if $\mu(s(A)) = 0$ (equivalently, $\mu(r(A)) = 0$), and *conull* if its complement $G \setminus A$ is null. A property of the points of G is said to hold a.e. (almost everywhere) if it holds on a conull subset.

Example 1.2. Let R be a countable Borel equivalence relation on a standard probability space (X, μ) , and suppose that μ is invariant (see [8]). We can see R as a pmp groupoid as follows: the product is defined by $(x, y)(y, z) = (x, z)$. The unit space of R is the diagonal $\{(x, x) : x \in X\}$, which we identify with X and endow with the probability measure μ . The Borel full semigroup of R can be identified with the semigroup of partial Borel isomorphisms $f : A \rightarrow B$, $A, B \subseteq X$, for which $(f(x), x) \in R$ for all $x \in A$, by associating such f with the inverse of its graph, $\{(f(x), x) : x \in X\}$. The pmp groupoids that are isomorphic (in the measure-theoretic sense) to one constructed in this way are called *principal groupoids*.

Example 1.3. Let Y be a finite set and Y^2 the largest equivalence relation on Y , endowed with the usual (discrete) Borel structure. The only probability measure on Y that makes Y^2 pmp is the normalized counting measure: $\mu_\#(A) = |A|/|Y|$. We denote the associated metric by $d_\#$ and call it the *normalized Hamming distance*.

Note that if Y and Z are finite sets, then the map $[[Y^2]] \ni \alpha \mapsto \alpha \times (Z^2)^{(0)} \in [[Y^2 \times Z^2]]$ is a trace-preserving embedding. The map $(y_1, y_2, z_1, z_2) \mapsto (y_1, z_1, y_2, z_2)$ is a measure-preserving isomorphism between the groupoids $Y^2 \times Z^2$ and $(Y \times Z)^2$, which induces a trace-preserving isomorphism between the respective two full semigroups. Therefore, if Y and Z are finite sets, there are a finite set W and trace-preserving embeddings from $[[Y^2]]$ and $[[Z^2]]$ into $[[W^2]]$.

Definition 1.4. A *sofic approximation* of a pmp groupoid G is a sequence of maps $\pi = \{\pi_k : [[G]] \rightarrow [[Y_k^2]]\}$, where Y_k are finite sets, such that for all $\alpha, \beta \in [[G]]$,

- (i) $\lim_{k \rightarrow \infty} \text{tr}(\pi_k(\alpha)) = \text{tr}(\alpha)$;
- (ii) $\lim_{k \rightarrow \infty} d_\#(\pi_k(\alpha\beta), \pi_k(\alpha)\pi_k(\beta)) = 0$.

A pmp groupoid G is *sofic* if it admits a sofic approximation.

We would like to note that this definition of sofic approximation differs from that of Bowen [2], where additional properties are required of the sequence π in order to study the entropy of sofic groupoids. However, the existence of a sofic approximation in the stronger sense of [2] is actually equivalent to the existence of a sofic approximation as in the definition above.

2. Permanence properties

Our first permanence result deals with products of pmp groupoids.

Theorem 2.1. *Two pmp groupoids (G, μ) and (H, ν) are sofic if and only if $(G \times H, \mu \times \nu)$ is sofic.*

Sketch of proof. The main point of the proof is to show that the subsemigroup M of $[[G \times H]]$ containing elements that can be written as $\bigcup_{i=1}^n \alpha_i \times \beta_i$, such that $\alpha_i \in [[G]]$, $\beta_i \in [[H]]$

- (i) if $s(\alpha_i) \cap s(\alpha_j)$ and $s(\beta_i) \cap s(\beta_j)$ are both nonempty then $i = j$;
- (ii) if $r(\alpha_i) \cap r(\alpha_j)$ and $r(\beta_i) \cap r(\beta_j)$ are both nonempty then $i = j$,

is dense in $[[G \times H]]$. Common measure-theoretic arguments allow us to approximate any $\phi \in [[G \times H]]$ by subsets of the form $\bigcup_{i=1}^n \alpha_i \times \beta_i$, $\alpha_i \in [[G]]$, $\beta_i \in [[H]]$, and $\alpha_i \times \beta_i$ pairwise disjoint; however, (i) and (ii) are not necessarily satisfied, so to deal with this, one substitutes $\alpha_i \times \beta_i \setminus \left(\bigcup_{j \neq i} \alpha_j \times \beta_j \right)$ for $\alpha_i \times \beta_i$, which deals with (i), and perform a similar procedure for (ii).

After that, take sofic approximations $\pi = \{\pi_k : [[G]] \rightarrow [[Y_k^2]]\}$ and $\theta = \{\theta_k : [[H]] \rightarrow [[Z_k^2]]\}$ and define $\pi \otimes \theta = \{(\pi \otimes \theta)_k\}$ as $(\pi \otimes \theta)_k \left(\bigcup \alpha_i \times \beta_i \right) = \bigcup \pi_k(\alpha_i) \times \theta_k(\beta_i)$. This is a subset of $Y_k^2 \times Z_k^2$, but not necessarily an element of its full semigroup, since π_k and θ_k do not necessarily preserve properties (i) and (ii). However, these are preserved in the limit, so we can modify $(\pi \otimes \theta)_k$ appropriately on small sets and assume $(\phi \otimes \theta)_k(\alpha) \in [[Y_k^2 \times Z_k^2]] \simeq [[(Y_k \times Z_k)^2]]$ for all $\alpha \in M$. Since M is dense in $[[G \times H]]$, we can extend $\pi \otimes \theta$ to a sofic approximation of $G \times H$. \square

Now we deal with extensions of groupoids, and for this we need to use a notion of *finite index*.

Definition 2.2. A subgroupoid $H \subseteq G$ has *finite index* in G if there exist $\psi_1, \dots, \psi_n \in [G]$ such that $\{\psi_i H : i = 1 \dots n\}$ is a partition of G (up to null sets). We call ψ_1, \dots, ψ_n *left transversals* of H in G . Note that if H has finite index in G , then $H^{(0)} = G^{(0)}$ (up to null sets).

Remark. In [15], Kida introduced and studied the *index map* of a subgroupoid H of a pmp groupoid G with $H^{(0)} = G^{(0)}$ as follows: for all $x \in G^{(0)}$, define an equivalence relation \sim_x on $r^{-1}(x)$ by setting $g \sim_x h \iff g^{-1}h \in H$ for all $g, h \in r^{-1}(x)$. The index $I(x)$ at x is defined as the number of \sim_x -equivalence classes. If H is ergodic, then H has finite index in G if and only if the index map is finite a.e. In this case, the index map is a.e. equal to the cardinality of any set of left transversals of H in G (this follows from Theorem 2.3 below).

If G is a groupoid and $x \in G^{(0)}$, define the *isotropy group* $G_x^x = s^{-1}(x) \cap r^{-1}(x)$, which is in fact a group with the operation inherited from G , and the *orbit relation* $(r, s)(G) = \{(r(g), s(g)) : g \in G\}$ on the unit space $G^{(0)}$.

A common question is how much of the structure of a pmp groupoid G can be described by the structures of $(r, s)(G)$ and the isotropy groups G_x^x ; for example, G is amenable if and only if $(r, s)(G)$ is amenable and a.e. isotropy group G_x^x is amenable (see [1, Theorem 4.2.7]). This naturally leads to the question: if H is a subgroupoid of a pmp group G with $H^{(0)} = G^{(0)}$, how does a finite index of $H \subseteq G$ relate to the index of the relations $(r, s)(H) \subseteq (r, s)(G)$, as defined in [9], and to the index of the isotropy groups $H_x^x \subseteq G_x^x$?

Theorem 2.3. *Suppose that H is an ergodic subgroupoid of a pmp groupoid G with $H^{(0)} = G^{(0)}$. Then H has finite index in G if and only if $(r, s)(H)$ has finite index in $(r, s)(G)$ and H_x^x has finite index in G_x^x for μ -a.e. $x \in G^{(0)}$.*

Proof. We prove only the harder implication: if the latter condition is satisfied, take invertible choice functions ψ_1, \dots, ψ_n for $(r, s)(H) \subseteq (r, s)(G)$ [9, Lemma 1.3], and an application of the Lusin–Novikov Theorem [13, Theorem 18.10] yields $\phi_1, \dots, \phi_n \in [G]$ such that $(r, s)(\phi_i) = \{(x, \psi_i(x)) : x \in G^{(0)}\}$. The index $[G_x^x : H_x^x]$ is $(r, s)(H)$ -invariant, so it is equal a.e. to a number m . Using the selection theorem for periodic relations [13, Theorem 12.16], one can take elements $\theta_1, \dots, \theta_m \in [G]$ such that $s(g) = r(g)$ for all $g \in \theta_j$, and such that for a.e. $x \in G^{(0)}$, $\{\theta_j H_x^x\}$ is a partition of G_x^x . Then $\phi_i \theta_j$ are left transversal for H in G . \square

Theorem 2.4. *Suppose that (G, μ) is a pmp groupoid and $H \subseteq G$ is a subgroupoid of finite index (in particular, $H^{(0)} = G^{(0)}$). If (H, μ) is sofic, so is (G, μ) .*

Sketch of proof. Suppose that ψ_1, \dots, ψ_N are left transversals for $H \subseteq G$. For each $\alpha \in [[G]]$, let $\alpha_{i,j} = \psi_i^{-1} \alpha \psi_j \cap H$, and note that $\alpha_{i,j} \in [[H]]$. Let $\pi = \{\pi_k : [[H]] \rightarrow [[Y_k^2]]\}$ be a sofic approximation of H . Define $\phi_k(\alpha) = \bigcup_{i,j} \pi(\alpha_{i,j}) \times \{(i, j)\}$ so that up to modifications on small sets (as in the proof of 2.1), we can assume $\phi_k(\alpha) \in [[Y_k^2 \times \{1, \dots, N\}^2]]$. This gives us a sequence of maps $\phi = \{\phi_k : [[G]] \rightarrow [[Y_k^2 \times \{1, \dots, N\}^2]]\}$.

To check that $\Xi = \{\Xi_k\}$ is a sofic approximation, use its definition and the fact that $\bigcup_j \alpha_{i,j} \beta_{j,l} = (\alpha\beta)_{i,l}$ to show that it asymptotically preserves products, and, to show that Ξ is asymptotically trace-preserving, note that $\text{tr} \Xi(\alpha) = \frac{1}{N} \sum_{i=1}^N \text{tr}(\alpha_{i,i})$. Since $\alpha_{i,i} \cap G^{(0)} = \psi_i^{-1} \alpha \psi_i \cap G^{(0)}$, $\psi_i \in [G]$ and G is pmp, one obtains $\text{tr}(\alpha_{i,i}) = \text{tr}(\alpha)$ and we are done. \square

In the next theorem, we will deal mostly with distinct measures on a given discrete measurable groupoid, and how soficity is preserved under some measure-theoretic constructions. We will simply say that a measure μ on a discrete measurable groupoid G is sofic if (G, μ) is sofic.

If H is a non-null subgroupoid of a pmp groupoid G , denote by μ_H the normalized measure on $H^{(0)}$, $\mu_H(A) = \mu(A)/\mu(H^{(0)})$ for $A \subseteq H^{(0)}$, which makes H a pmp groupoid in its own right.

Theorem 2.5. *Let G be a discrete measurable groupoid.*

- (1) *A strong limit² of sofic measures is sofic as well.*
- (2) *A convex combination of sofic measures is sofic.*
- (3) *If μ has a disintegration $\mu = \int_{G^{(0)}} p_x \, d\nu(x)$, where ν -a.e. p_x is a sofic measure, then (G, μ) is sofic.*
- (4) *If (G, μ) is sofic and H is a non-null subgroupoid of G , then (H, μ_H) is sofic.*
- (5) *If $\nu \ll \mu$, where (G, ν) is pmp, and μ is sofic, then ν is sofic.*
- (6) *If $\{H_n\}$ is a countable Borel partition of G by non-null subgroupoids, then G is sofic if and only if each H_n is sofic.*

Hence, by (4) and Theorem 2.4, a pmp groupoid is virtually sofic (i.e. it contains a finite-index sofic subgroupoid) if and only if it is sofic. As a particular case of (3), if a.e. ergodic component of (G, μ) is sofic, so is (G, μ) , but at present we do not know if the converse statement is true.

Let us introduce some notation which will be useful in the proof of the theorem above.

Definition 2.6. If $\{(G_i, \mu_i)\}_i$ is a sequence (finite or infinite) of pmp groupoids and $\{t_i\}_i$ is a sequence of nonnegative numbers with $\sum_i t_i = 1$, we define the *convex combination groupoid* (G, μ) as follows: as a measurable groupoid, G is the coproduct $G = \coprod G_i$, i.e. G is the disjoint union of all G_i , the operation on G is the smallest one extending those of each G_i , and the Borel structure of G is generated by those of each G_i . Each measure μ_i can be regarded as a measure on $G^{(0)}$ by the formula $\mu_i(A) = \mu_i(A \cap G_i^{(0)})$, $A \subseteq G^{(0)}$, and so we endow G with the convex combination of those measures: $\mu = \sum_i t_i \mu_i$. We will use the notation $G = \sum_i t_i G_i$ in this case.

Proof of 2.5. (1) is clear, since soficity is an approximation property for the measure.

- (2) The technique is similar to that employed in [18]: suppose first that ν, ρ are sofic measures and $\mu = t\nu + (1-t)\rho$, where $t = p/q$ is rational, $p, q \in \mathbb{Z}$, $0 < p < q$. We can take the sofic approximations $\pi = \{\pi_k : [[G]]_\nu \rightarrow [[Y_k^2]]\}$ and $\theta = \{\theta_k : [[G]]_\rho \rightarrow [[Y_k^2]]\}$ of (G, μ) and (G, ν) , **on the same sets** Y_k (see Example 1.3), and define a sofic approximation $t\pi \oplus (1-t)\theta = \{\xi_k : [[G]]_\mu \rightarrow [[Y_k^2 \times \{1, \dots, q\}^2]]\}$ by $\xi_k(\alpha) = \pi_k(\alpha) \times \{1, \dots, p\} \cup \theta_k(\alpha) \times \{p+1, \dots, q\}$. The general case follows from (1).
- (3) This follows from the previous items, since standard arguments show that in this case μ is a strong limit of convex combinations of sofic p_x .
- (4) This is trivial if $H^{(0)} = G^{(0)}$, since any sofic approximation of G restricts to a sofic approximation of H . On the other hand, if G is a convex combination $G = tH + (1-t)K$, then $H^{(0)}$ and $K^{(0)}$ are disjoint idempotents whose sum of traces is 1. Sofic approximations preserve this information (in the limit), which allows us to decompose, up to negligible errors, a sofic approximation ξ of G as a convex combination of sofic approximations of H and K , similarly to how ξ was constructed in item (1). In the general case, one uses the fact that the convex combination groupoid $\mu(H)H + (1-\mu(H))(G^{(0)} \setminus H^{(0)})G(G^{(0)} \setminus H^{(0)})$ is contained in G and has full unit space.
- (5) By doing appropriate approximations, one may assume that the derivative $f = d\nu/d\mu$ is simple, say $f = \sum_i f_i 1_{X_i}$ where 1_X denotes the characteristic function of $X \subseteq G^{(0)}$. In this case, $\nu = \sum_i \left(\frac{f_i}{\sum_j f_j}\right) \mu_i$, where $\mu_i(A) = \mu(A \cap X_i)/\mu(X_i)$. The result then follows from (1), (2), and (4).
- (6) Apply items (4) and (2) with the fact that $G = \sum_j \mu(H_j^{(0)})H_j$. \square

Let us make a small detour on the structure of finite groupoids in order to obtain a new description of soficity.

² Recall that a net $\{\mu_i\}_i$ of probability measures on a measurable space (X, \mathcal{A}) converges strongly to a measure μ on (X, \mathcal{A}) if $\mu_i(A) \rightarrow \mu(A)$ for all $A \in \mathcal{A}$.

Proposition 2.7. Every finite pmp groupoid (G, μ) is a convex combination of groupoids of the form $Y^2 \times \Gamma$, where Y is a finite set and Γ is a finite group. (In particular, every finite groupoid is a coproduct of those of the form $Y^2 \times \Gamma$.)

Sketch of proof. Letting Y_1, \dots, Y_n be the equivalence classes of the orbit relation $(r, s)(G)$, we obtain $G = \sum \mu(Y_i)(Y_iGY_i)$, where each Y_iGY_i is endowed with the normalized counting measure on $(Y_iGY_i)^{(0)} \simeq Y_i$. For a fixed i , choose an arbitrary point $x \in Y_i$, and for each $y \in Y_i$ choose $h(y) \in G$ with $s(h(y)) = x, r(h(y)) = y$. The map

$$Y_i^2 \times G_x^x \rightarrow Y_iGY_i, \quad (y_1, y_2, \gamma) \mapsto h(y_1)\gamma h(y_2)^{-1}$$

is a measure-preserving groupoid isomorphism. \square

We can apply the previous results in order to show that finite groupoids are sofic: by Proposition 2.7 and Theorem 2.5(6), it suffices to show that groupoids of the form $Y^2 \times \Gamma$, where Y is a finite set and Γ is a finite group, are sofic. Of course, Y^2 is sofic, and either Theorem 2.4 or 2.1 imply that $Y^2 \times \Gamma$ is sofic as well, so we obtain the following corollary.

Corollary 2.8. Every finite groupoid is sofic. Therefore, we may modify the definition of sofic approximation 1.4 by considering arbitrary finite pmp groupoids instead of those of the form Y^2 , and obtain the same notion.

3. Soficity and the full group

Given a pmp groupoid G , the unit space $G^{(0)}$ is a subgroupoid, and the full semigroup $[[G^{(0)}]]$ coincides with the measure algebra $\text{MAlg}(G^{(0)})$, with product given by intersection. In fact, $\text{MAlg}(G^{(0)})$ coincides with the set of idempotents of $[[G]]$, and every $\alpha \in [[G]]$ can be decomposed as $\alpha = \tilde{\alpha}A$, where $\tilde{\alpha} \in [G]$ and $A \in \text{MAlg}(G^{(0)})$ (which corresponds to the fact that every partial isometry in a finite von Neumann algebra can be extended to a unitary). Given $\alpha \in [[G]]$, define $\text{supp}(\alpha) = s(\alpha \setminus G^{(0)})$.

A well-known theorem of Dye [4] states that an aperiodic equivalence relation is determined by its full group, and from that one may ask how the properties of the relation correspond to the properties of the full group; for example, in [10, Theorem 5.7], the authors prove that the amenability of an ergodic equivalence relation is equivalent to the amenability of its full group (as a topological group). We prove that a pmp groupoid G is sofic if and only if $[G]$ is *metrically sofic* (initially defined as *sofic metric groups* in [12]), which solves a question posed by Conley, Kechris and Tucker-Drob in [3] in this case. This can also be seen as a partial answer to the following question asked in Vladimir Pestov’s reviewer’s report for [7]: is a pmp equivalence relation R sofic if and only if its full group $[R]$ is sofic? (One direction was answered positively in [6]: If R is a sofic relation, then $[R]$ is a sofic group.)

Definition 3.1. A separable group Γ with an bi-invariant metric d is *metrically sofic* if it admits a sequence of maps $\pi = \{\pi_k : \Gamma \rightarrow [[Y_k^2]]\}$, called a *sofic approximation* of Γ , satisfying, for all $\alpha, \beta \in \Gamma$,

- (i) $\lim_{k \rightarrow \infty} d(\pi_k(\alpha), \pi_k(\beta)) = d(\alpha, \beta)$;
- (ii) $\lim_{k \rightarrow \infty} d_{\#}(\pi_k(\alpha\beta), \pi_k(\alpha)\pi_k(\beta)) = 0$.

For example, if $\pi = \{\pi_k : [[G]] \rightarrow [[Y_k^2]]\}$ is a sofic approximation of a pmp groupoid G , then $\pi|_{[G]} = \{\pi_k|_{[G]} : [G] \rightarrow [[Y_k^2]]\}$ is a sofic approximation of $[G]$ (with respect to the metric d_{μ}).

One of the main points of Dye’s proof is to recover $\text{MAlg}(G^{(0)})$ from $[G]$ by looking at the classes of elements of $[G]$ whose support is a given $A \in \text{MAlg}(G^{(0)})$, and this is also an important point in our proof. In fact, the metric of $[G]$ allows us to recover the order of $\text{MAlg}(G^{(0)})$ by the following lemma.

Lemma 3.2. Let (G, μ) be a pmp groupoid and $\alpha, \beta \in [G]$ then $\text{supp } \alpha \cap \text{supp } \beta = \emptyset$ if and only if $d_{\mu}(\alpha, \beta) = d_{\mu}(1, \alpha) + d_{\mu}(1, \beta)$.

Definition 3.3. A pmp groupoid G is *aperiodic* if $|s^{-1}(x)| = \infty$ for a.e. $x \in G^{(0)}$.

Just as in the case of equivalence relations, if G is aperiodic, then there are sufficiently many elements in $[G]$ so as to recover $\text{MAlg}(G^{(0)})$, by noting that, for a.e. $x \in G^{(0)}$, at least one of the sets $r(s^{-1}(x))$ or G_x^x is infinite. The former can be dealt with as in [14, Lemma 4.10], and the latter by the Lusin–Novikov Theorem. We obtain:

Lemma 3.4. Suppose that G is an aperiodic pmp groupoid. Then, for all $A \in \text{MAlg}(G^{(0)})$, there exists $\alpha \in [G]$ such that $\text{supp } \alpha = A$.

Theorem 3.5. An aperiodic pmp groupoid G is sofic if and only if the full group $[G]$ is metrically sofic.

Sketch of proof. If G is sofic then $[G]$ is metrically sofic by the comment after Definition 3.1.

Conversely, let $\theta = \{\theta_k : [G] \rightarrow [[Y_k^2]]\}$ be a sofic approximation of $[G]$. Lemma 3.4 allows us to define a sequence of maps $\phi = \{\phi_k : \text{MAlg}(G^{(0)}) \rightarrow \text{MAlg}(Y_k)\}$ by $\phi_k(A) = \text{supp } \theta_k(\alpha)$, where $\alpha \in [G]$ is any element with $\text{supp } \alpha = A$. This way, $\phi(A)$ is independent of the choice of α up to null sets, which are disconsidered in sofic approximations. The pair (θ, ϕ) is asymptotically covariant, in the sense that, for all $\alpha \in [G]$ and $A \in \text{MAlg}(X)$, the distance between $\phi_k(r(\alpha A))$ and $\theta_k(\alpha)(\phi(A))$ converges to zero. After verifying that ϕ is moreover asymptotically order and measure-preserving, we simply define a sofic approximation $\pi = \{\pi_k\}$ of G by $\pi(\alpha) = \theta(\tilde{\alpha})\theta(s(\alpha))$, where $\tilde{\alpha} \in [G]$ is chosen so that $\alpha \subseteq \tilde{\alpha}$. \square

By using similar techniques, Theorem 3.5 can be extended to all pmp groupoids G such that $|s^{-1}(x)| \geq 2$ for a.e. $x \in G^{(0)}$.

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