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Existence of periodic solutions for a class of damped vibration problems

Existence de solutions périodiques pour une classe de problèmes de vibration amortie

Zhiyong Wang [a], Jihui Zhang [b]

[a] *Department of Mathematics, Nanjing University of Information Science & Technology, Nanjing 210044, PR China*
[b] *Institute of Mathematics, School of Mathematics Sciences, Nanjing Normal University, Nanjing 210023, PR China*

A R T I C L E   I N F O

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A B S T R A C T

In this paper, we are concerned with the existence of periodic solutions for a class of damped vibration problems. By introducing some new kinds of superquadratic and asymptotically quadratic conditions, and making use of the generalized mountain pass theorem in critical point theory, we propose a unified approach when the potential function $F(t,x)$ exhibits either an asymptotically quadratic or a superquadratic behavior at infinity, and establish some sufficient conditions on periodic solutions, which extend and improve some recent results in the literature, even without damped vibration term.

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R É S U M É

Nous nous intéressons ici à l'existence de solutions périodiques pour une classe de problèmes de vibration amortie. Nous introduisons de nouvelles conditions de quadraticité asymptotique et de super-quadraticité, et nous utilisons un théorème du col généralisé de la théorie des points critiques. Ainsi, nous proposons une approche unifiée lorsque la fonction potentiel $F(t,x)$ présente un comportement quadratique asymptotique ou super-quadratique à l'infini, et nous établissons des conditions suffisantes pour l'existence de solutions périodiques, ce qui étend et améliore plusieurs résultats récents, même en l'absence du terme de vibration amortie.

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E-mail addresses: mathswzhy1979@gmail.com (Z. Wang), zhangjihui@njnu.edu.cn (J. Zhang).

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1. Introduction and main results

Consider the damped vibration problem

$$\begin{cases} \ddot{u}(t) + q(t)\dot{u} + \nabla F(t, u(t)) = 0, \text{ a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $q \in L^1(0, T; \mathbb{R})$, $\int_0^T q(t)\mathrm{d}t = 0$ and $F : [0, T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^{\mathbb{N}}$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \quad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$.

When $q(t) \equiv 0$, (1.1) reduces to the following classical second order non-autonomous Hamiltonian systems

$$\begin{cases} \ddot{u}(t) + \nabla F(t, u(t)) = 0, \text{ a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.2}$$

Using the variational methods, many existence results are obtained under suitable conditions, we refer the reader to [2–5, 7–11,13,15–17,19–21,23] and the references therein. In 1978, Rabinowitz [10] obtained periodic solutions under the following well-known Ambrosetti–Rabinowitz condition (AR-condition): there exist $\mu > 2$ and $L_1 > 0$ such that

$$0 < \mu F(t, x) \le (\nabla F(t, x), x) \quad \forall |x| \ge L_1 \text{ and for a.e. } t \in [0, T], \tag{AR-condition}$$

where $(\cdot, \cdot)$ is the usual inner product of $\mathbb{R}^{\mathbb{N}}$. Since then, this condition has been used widely to deal with the existence of periodic solutions to problem (1.2), see [3,8,11] and the references therein.

Recently, many authors have devoted to weaken the AR-condition, some existence and multiplicity of results on periodic solutions to problem (1.2) have also been obtained under weaker conditions, see [4,7,9,13,15,16,19–24]. Particularly, in 2002, Fei [4] studied the existence of periodic solutions to problem (1.2) under non-quadratic conditions and established the following result.

Theorem A. *Suppose that F satisfies assumption (A) and the following conditions:*

(S_1) $F(t, x) \ge 0 \quad \forall (t, x) \in [0, T] \times \mathbb{R}^{\mathbb{N}}$;
(S_2) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0$ *uniformly for a.e.* $t \in [0, T]$;
(S_3) $\lim_{|x|\to +\infty} \frac{F(t,x)}{|x|^2} = +\infty$ *uniformly for a.e.* $t \in [0, T]$;
(S_4) $\limsup_{|x|\to +\infty} \frac{F(t,x)}{|x|^r} \le M < +\infty$ *uniformly for some* $M > 0$ *and a.e.* $t \in [0, T]$;
(S_5) $\liminf_{|x|\to +\infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\mu} \ge \varrho > 0$ *uniformly for some* $\varrho > 0$ *and a.e.* $t \in [0, T]$,

where $r > 2$ and $\mu \ge r - 1$. Then problem (1.2) has at least one non-constant periodic solution.

Subsequently, Tao and Tang [15] extended Theorem A and got the following theorem.

Theorem B. *Suppose that F satisfies assumptions (A), (S_1), (S_4), (S_5) with $r > 2$ and $\mu > r - 2$, and the following conditions:*

(S_2^*) $\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^2} < \frac{1}{2}\omega^2$ *uniformly for a.e.* $t \in [0, T]$;
(S_3^*) $\liminf_{|x|\to +\infty} \frac{F(t,x)}{|x|^2} > \frac{1}{2}\omega^2$ *uniformly for a.e.* $t \in [0, T]$,

where $\omega = 2\pi/T$. Then problem (1.2) has at least one non-constant periodic solution.

For the asymptotically quadratic case, applying generalized mountain pass theorem and some techniques of analysis, Ma and Zhang [7] have proved that problem (1.2) has at least one non-constant periodic solution. Concretely speaking, they proved the following theorems.

Theorem C. *Suppose that F satisfies assumptions (A), (S_1), (S_2^*), (S_3^*) and the following conditions:*

(S_4^*) $\limsup_{|x|\to +\infty} \frac{F(t,x)}{|x|^2} \le M < +\infty$ *uniformly for some* $M > 0$ *and a.e.* $t \in [0, T]$;

(S_6) *there exists* $\gamma \in L^1(0, T; \mathbb{R}^+)$ *such that*

$$(\nabla F(t,x), x) - 2F(t,x) \geq \gamma(t) \quad \text{for all } x \in \mathbb{R}^{\mathbb{N}} \text{ and a.e. } t \in [0,T];$$

(S_7) $\lim_{|x|\to+\infty}[(\nabla F(t,x), x) - 2F(t,x)] = +\infty$ *uniformly for a.e.* $t \in [0,T]$.

Then problem (1.2) *has at least one non-constant periodic solution.*

Theorem D. *Suppose that* F *satisfies assumptions* (A), (S_1), (S_2^*)–(S_4^*) *and the following conditions:*

(S_6^*) *there exists* $\gamma \in L^1(0, T; \mathbb{R}^+)$ *such that*

$$(\nabla F(t,x), x) - 2F(t,x) \leq \gamma(t) \quad \text{for all } x \in \mathbb{R}^{\mathbb{N}} \text{ and a.e. } t \in [0,T];$$

(S_7^*) $\lim_{|x|\to+\infty}[(\nabla F(t,x), x) - 2F(t,x)] = -\infty$ *uniformly for a.e.* $t \in [0,T]$.

Then problem (1.2) *has at least one non-constant periodic solution.*

Motivated by the results of [4,7,15,17–19], in present paper, on the one hand, employing the generalized mountain pass theorem, we will focus on the existence of non-constant periodic solutions to a more general damped vibration problem (1.1) under some new kinds of superquadratic and asymptotically quadratic conditions. On the other hand, it is worth noticing that the different techniques are usually used to ensure the compact conditions for the asymptotically quadratic case and superquadratic case, just like the methods of [7,22,23]. Here, in this paper, we will propose a unified approach when the potential function $F(t,x)$ exhibits either an asymptotically quadratic or a superquadratic behavior. We stress that our results are all new even without damped vibration term. For the sake of convenience, we set

$$Q(t) = \int_0^t q(s)\mathrm{d}s, \quad A_1 = \max_{t\in[0,T]} \mathrm{e}^{Q(t)}, \quad A_2 = \min_{t\in[0,T]} \mathrm{e}^{Q(t)}.$$

Now, we can state our main results.

Theorem 1.1. *Suppose that* F *satisfies assumption* (A) *and the following conditions:*

(F_1) $\int_0^T \mathrm{e}^{Q(t)} F(t,x)\mathrm{d}t \geq 0 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^{\mathbb{N}}$;

(F_2) $\limsup_{|x|\to 0} \mathrm{e}^{Q(t)} \frac{F(t,x)}{|x|^2} < \frac{1}{2} A_2 \omega^2$ *uniformly for a.e.* $t \in [0,T]$;

(F_3) $\liminf_{|x|\to+\infty} \mathrm{e}^{Q(t)} \frac{F(t,x)}{|x|^2} > \frac{1}{2} A_1 \omega^2$ *uniformly for a.e.* $t \in [0,T]$;

(F_4) $\limsup_{|x|\to+\infty} \mathrm{e}^{Q(t)} \frac{F(t,x)}{|x|^r} \leq M < +\infty$ *uniformly for some* $M > 0$ *and a.e.* $t \in [0,T]$, *where* $r > 2$;

(F_5) *there exist* $M_1 > 0$, $\mu > r-2$ *and* $k_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\lim_{|x|\to+\infty} k_1(|x|)|x|^{\mu+2-r} = +\infty$, $\lim_{|x|\to+\infty} k_1^{r/\mu}(|x|)|x|^2 = +\infty$ *and* $k_1(z)$ *is non-increasing in* z *for all* $z \in \mathbb{R}^+$, *such that*

$$\mathrm{e}^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \geq k_1(|x|)|x|^{\mu} \quad \forall x \in \mathbb{R}^{\mathbb{N}},\ |x| \geq M_1 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has at least one non-constant periodic solution.*

Remark 1.2. In contrast to Theorem A and Theorem B, the main contributions of Theorem 1.1 are in three aspects. In the first place, we consider more general damped vibration systems (1.1) than second-order non-autonomous Hamiltonian systems (1.2). There is one more point that I should touch on, that (F_1) is weaker than (S_1) even if $q(t) \equiv 0$. Last but not the least, condition (F_5) covers the case of assumption (S_5) with $\mu > r-2$. In fact, we only need to put $k_1(|x|) = \varrho > 0$, $q(t) = 0$ and M_1 large enough. Meanwhile, we emphasis that $k_1(z)$ permits to be zero at infinity, which means that (F_5) is much more general than (S_5). Therefore, Theorem 1.1 significantly unifies and generalizes upon Theorem A and Theorem B.

Theorem 1.3. *Suppose that* F *satisfies assumptions* (A), (F_1)–(F_3) *and the following condition:*

(F_6) *there exist* $M_2 > 0$, $\theta \geq 1$, $k_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\lim_{|x|\to+\infty} k_2(|x|) = +\infty$ *and* $k_2(z)/z^{2\theta}$ *is non-increasing in* z *for all* $z \in \mathbb{R}^+$, *such that*

$$\mathrm{e}^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \geq k_2(|x|) \left(\frac{F(t,x)}{|x|^2}\right)^{\theta} \quad \forall x \in \mathbb{R}^{\mathbb{N}},\ |x| \geq M_2 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has at least one non-constant periodic solution.*

Remark 1.4. (a) Even if $q(t) \equiv 0$, condition (F_6) still seems a new superquadratic growth condition, which generalizes conditions (S_4) and (S_5) with $\mu > r-2$ when $(H1)$ holds. We will follow two steps to demonstrate this claim.

Step 1. We confirm that (H_1), (S_4) and (S_5) could imply $\mu \le r$. It follows from (S_4) that there exists $d_1 > 0$ such that

$$F(t,x) \le M|x|^r \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \ge d_1 \text{ and for a.e. } t \in [0,T]. \tag{1.3}$$

By (S_5), we can choose $d_2 > 0$ such that

$$(\nabla F(t,x), x) - 2F(t,x) \ge \varrho |x|^{\mu} \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \ge d_2 \text{ and for a.e. } t \in [0,T]. \tag{1.4}$$

Let $d_3 := \max\{d_1, d_2\}$; taking account of (1.3), (1.4) and (H_1), we infer that

$$\begin{aligned} M|x|^r &\ge F(t,x) \\ &= \int_0^1 \frac{1}{s}(\nabla F(t,sx), sx)\mathrm{d}s + F(t,0) \\ &\ge \int_0^1 \frac{1}{s}[\varrho|sx|^{\mu} + 2F(t,sx)]\mathrm{d}s \\ &\ge \frac{1}{\mu}\varrho|x|^{\mu} \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \ge d_3 \text{ and for a.e. } t \in [0,T], \end{aligned}$$

which implies that $\mu \le r$.

Step 2. We claim that (H_1), (S_4) and (S_5) with $\mu > r-2$ could imply (F_6) with $q(t) = 0$. In fact, let $M_1 := \max\{d_1, d_2, d_3\}$, by (1.3), (H_1) and (S_5) with $\mu > r-2$, one has

$$(\nabla F(t,x), x) - 2F(t,x) \ge \varrho|x|^{\mu+2-r}\frac{|x|^r}{|x|^2} \ge \frac{\varrho}{M}|x|^{\mu+2-r}\frac{F(t,x)}{|x|^2}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$, $|x| \ge M_1$, a.e. $t \in [0,T]$. Take $\theta = 1$, $k_2(|x|) = \varrho|x|^{\mu+2-r}/M$, noticing $\mu > r-2$, then $\lim_{|x|\to+\infty} k_2(|x|) = +\infty$, and $k_2(|x|)/|x|^2 = \varrho|x|^{\mu-r}/M$ is non-increasing on $\mathbb{R}^+$ by Step 1. Therefore, (F_6) with $q(t) = 0$ holds.

(b) From (a), it is not difficult to see that Theorem 1.3 greatly extends Theorem A and Theorem B. Here, we should point out that Zhang and Tang in [24] have introduced the following new non-quadratic condition:

(ZT) there exist $M_2 > 0, \xi > 0, \eta > 0$ and $\nu \in [0,2)$ such that

$$\left(2 + \frac{1}{\xi + \eta|x|^{\nu}}\right)F(t,x) \le (\nabla F(t,x), x) \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \ge M_2 \text{ and for a.e. } t \in [0,T].$$

It is clear that this new non-quadratic condition (ZT) is a special case of assumption (F_6) with $k_2(|x|) = \frac{|x|^2}{\xi+\eta|x|^{\nu}}$, $q(t) = 0$ and $\theta = 1$.

Theorem 1.5. *Suppose that F satisfies assumptions (A), (F_1)–(F_3) and the following condition:*

(F_7) *there exist $M_3 > 0$, $\sigma \ge 1$, $k_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_3(|x|) = +\infty$ and $k_3(z)/z^{\sigma}$ is non-increasing in z for all $z \in \mathbb{R}^+$ such that*

$$\mathrm{e}^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \ge k_3(|x|)\left(\frac{|\nabla F(t,x)|}{|x|}\right)^{\sigma} \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \ge M_3 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) has at least one non-constant periodic solution.

Remark 1.6. When $q(t) \equiv 0$ and $\sigma > 1$, condition (F_7) was originally due to [19]. In [19], using the new saddle point theorem established by Schechter [12], the authors have investigated the existence of T-periodic solutions to problem (1.2) when the potential function $F(t,x)$ is either locally in t asymptotically quadratic or locally in t superquadratic; meanwhile, Theorem 1.5 will be proved with the aid of the generalized mountain pass theorem, and we will obtain the non-constant periodic solution under different conditions from that of [19]. Thus, Theorem 1.5 is a new result.

From Theorem 1.1, Theorem 1.3 and Theorem 1.5, for the asymptotically quadratic case, we have the following results.

Corollary 1.7. *Suppose that F satisfies assumptions (A), (F_1)–(F_3) and the following conditions:*

(F_4^*) $\limsup_{|x|\to+\infty} e^{Q(t)} \frac{F(t,x)}{|x|^2} \le M < +\infty$ *uniformly for some $M > 0$ and a.e. $t \in [0, T]$;*
(F_5^*) *there exist $M_1 > 0$, $\mu > 0$, $k_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_1(|x|)|x|^\mu = +\infty$, $\lim_{|x|\to+\infty} k_1^{2/\mu}(|x|)|x|^2 = +\infty$ and $k_1(z)$ is non-increasing in z for all $z \in \mathbb{R}^+$, such that*

$$e^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \ge k_1(|x|)|x|^\mu \quad \forall x \in \mathbb{R}^N,\ |x| \ge M_1 \text{ and for a.e. } t \in [0, T].$$

Then problem (1.1) has at least one non-constant periodic solution.

Remark 1.8. Corollary 1.7 itself is a meaningful outcome. Comparing Corollary 1.7 with Theorem C, we obtain the same conclusion under assumption (F_5^*), which is slightly stronger than condition (S_7), while Corollary 1.7 does not require assumption (S_6). So, Corollary 1.7 can be viewed as a useful complement to Theorem C.

Corollary 1.9. *Suppose that F satisfies assumptions (A), (F_1)–(F_3), (F_4^*) and the following condition:*

(F_5^{**}) *there exist $M_1 > 0$, $\mu > 0$, $k_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_1(|x|)|x|^\mu = +\infty$, $\lim_{|x|\to+\infty} k_1^{2/\mu}(|x|)|x|^2 = +\infty$ and $k_1(z)$ is non-increasing in z for all $z \in \mathbb{R}^+$, such that*

$$e^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \le -k_1(|x|)|x|^\mu \quad \forall x \in \mathbb{R}^N,\ |x| \ge M_1 \text{ and for a.e. } t \in [0, T].$$

Then problem (1.1) has at least one non-constant periodic solution.

Corollary 1.10. *Suppose that F satisfies assumptions (A), (F_1)–(F_3), (F_4^*) and the following condition:*

(F_6^*) *there exist $M_2 > 0$, $\theta \ge 1$, $k_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_2(|x|) = +\infty$ and $k_2(z)/z^{2\theta}$ is non-increasing in z for all $z \in \mathbb{R}^+$ such that*

$$e^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \ge k_2(|x|) \quad \forall x \in \mathbb{R}^N,\ |x| \ge M_2 \text{ and for a.e. } t \in [0, T].$$

Then problem (1.1) has at least one non-constant periodic solution.

Corollary 1.11. *Suppose that F satisfies assumptions (A), (F_1)–(F_3), (F_4^*) and the following condition:*

(F_6^{**}) *there exist $M_2 > 0$, $\theta \ge 1$, $k_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_2(|x|) = +\infty$ and $k_2(z)/z^{2\theta}$ is non-increasing in z for all $z \in \mathbb{R}^+$ such that*

$$e^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \le -k_2(|x|) \quad \forall x \in \mathbb{R}^N,\ |x| \ge M_2 \text{ and for a.e. } t \in [0, T].$$

Then problem (1.1) has at least one non-constant periodic solution.

Corollary 1.12. *Suppose that F satisfies assumptions (A), (F_1)–(F_3) and the following conditions:*

(F_4^{**}) $\limsup_{|x|\to+\infty} e^{Q(t)} \frac{|\nabla F(t,x)|}{|x|} \le M < +\infty$ *uniformly for some $M > 0$ and a.e. $t \in [0, T]$;*
(F_7^*) *there exist $M_3 > 0$, $\sigma \ge 1$, $k_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_3(|x|) = +\infty$ and $k_3(z)/z^\sigma$ is non-increasing in z for all $z \in \mathbb{R}^+$ such that*

$$e^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \ge k_3(|x|) \quad \forall x \in \mathbb{R}^N,\ |x| \ge M_3 \text{ and for a.e. } t \in [0, T].$$

Then problem (1.1) has at least one non-constant periodic solution.

Corollary 1.13. *Suppose that F satisfies assumptions (A), (F_1), (F_2), (F_4^{**}) and the following conditions:*

(F_3^*) $\liminf_{|x|\to+\infty} e^{Q(t)} \frac{(\nabla F(t,x), x)}{|x|^2} > A_1\omega^2$ *uniformly for a.e. $t \in [0, T]$;*
(F_7^{**}) *there exist $M_3 > 0$, $\sigma \ge 1$, $k_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to+\infty} k_3(|x|) = +\infty$ and $k_3(z)/z^\sigma$ is non-increasing in z for all $z \in \mathbb{R}^+$ such that*

$$e^{Q(t)}[(\nabla F(t,x), x) - 2F(t,x)] \le -k_3(|x|) \quad \forall x \in \mathbb{R}^N,\ |x| \ge M_3 \text{ and for a.e. } t \in [0, T].$$

Then problem (1.1) has at least one non-constant periodic solution.

The remainder of this paper is organized as follows. In Section 2, some necessary notations and preliminaries are presented. In Section 3, we firstly observe that although the energy functional of problem (1.1) may possess an unbounded (PS) sequence (see Definition 2.1 below), we can prove that all (C) sequences (see Definition 2.1 below) of this functional are bounded (see Lemma 3.1 and Lemma 3.3 below), then we adopt the route of [15] to prove our main results by the generalized mountain pass theorem in [11]. Finally, in Section 4, we will give some examples to illustrate our results.

2. Preliminaries

Let

$$H_T^1 := \left\{ u : [0,T] \to \mathbb{R}^{\mathbb{N}} \,|\, u \text{ is absolutely continuous }, u(0)=u(T), \dot{u} \in L^2(0,T;\mathbb{R}^{\mathbb{N}}) \right\}$$

be a Hilbert space with the inner product

$$(u,v) := \int_0^T (\dot{u}(t), \dot{v}(t))\mathrm{d}t + \int_0^T (u(t), v(t))\mathrm{d}t \qquad \forall u, v \in H_T^1.$$

The corresponding norm is

$$\|u\| := \left(\int_0^T |\dot{u}(t)|^2 \mathrm{d}t + \int_0^T |u(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \qquad \forall u, v \in H_T^1.$$

For $u \in H_T^1$, let $\bar{u} := \frac{1}{T}\int_0^T u(t)\mathrm{d}t$, $\tilde{u}(t) := u(t) - \bar{u}$ and $\widetilde{H}_T^1$ be the subspace of H_T^1 given by $\widetilde{H}_T^1 := \{u \in H_T^1 \,|\, \bar{u} = 0\}$. Then one has

$$\|\widetilde{u}\|_\infty^2 \le \frac{T}{12}\|\dot{u}\|_{L^2}^2, \quad \text{(Sobolev's inequality)}$$

and

$$\|\widetilde{u}\|_{L^2}^2 \le \frac{T^2}{4\pi^2}\|\dot{u}\|_{L^2}^2, \quad \text{(Wirtinger's inequality)}$$

where

$$\|u\|_{L^2} := \left(\int_0^T |u(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \quad \text{and} \quad \|\widetilde{u}\|_\infty := \max_{t\in[0,T]} |\widetilde{u}(t)|.$$

Since the embedding of H_T^1 into $C(0,T;\mathbb{R}^{\mathbb{N}})$ is compact, there exists $d > 0$ such that

$$\|u\|_\infty \le d\|u\| \tag{2.1}$$

for all $u \in H_T^1$.

Consider the functional $\varphi : H_T^1 \to \mathbb{R}$ defined by

$$\varphi(u) := \frac{1}{2}\int_0^T \mathrm{e}^{Q(t)}|\dot{u}(t)|^2 \mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)} F(t, u(t))\mathrm{d}t. \tag{2.2}$$

Then φ is continuously differentiable on H_T^1 (see [8]). Moreover,

$$(\varphi'(u), v) = \int_0^T \mathrm{e}^{Q(t)}(\dot{u}(t), \dot{v}(t))\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}(\nabla F(t, u(t)), v(t))\mathrm{d}t \tag{2.3}$$

for any u, $v \in H_T^1$. It is well known that the periodic solutions to problem (1.1) correspond to the critical points of φ (see [8]).

Definition 2.1. Let E be a real Banach space, we say that $\{u_n\}$ in E is a Palais–Smale sequence ((PS) sequence) for φ if $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. The functional $\varphi \in C^1(E,\mathbb{R})$ satisfies the Palais–Smale condition ((PS) condition) if any Palais–Smale sequence contains a convergent subsequence.

Definition 2.2. Let E be a real Banach space, we say that $\{u_n\}$ in E is a Cerami sequence ((C) sequence) for φ if $\varphi(u_n)$ is bounded and $\varphi'(u_n)(1+\|u_n\|)\to 0$ as $n\to+\infty$. The functional $\varphi\in C^1(E,\mathbb{R})$ satisfies the Cerami condition ((C) condition) if any Cerami sequence contains a convergent subsequence.

We shall use the following generalized mountain pass theorem to prove our results.

Theorem 2.3. *Let E be a real Banach space with $E=V\oplus X$, where V is finite dimensional. Suppose $\varphi\in C^1(E,\mathbb{R})$ satisfies the (PS) condition, and*

(i) *there exist $\rho,\alpha>0$ such that $\varphi|_{\partial B_\rho\cap X}\geq\alpha$, where $B_\rho:=\{u\in E|\,\|u\|\leq\rho\}$, ∂B_ρ denotes the boundary of B_ρ;*

(ii) *there exist $e\in\partial B_1\cap X$ and $s_0>\rho$ such that if $Q\equiv(\bar{B}_{s_0}\cap V)\oplus\{se|\,0\leq s\leq s_0\}$, then $\varphi|_{\partial Q}\leq 0$.*

Then φ possesses a critical value $c\geq\alpha$ which can be characterized as

$$c:=\inf_{h\in\Gamma}\max_{u\in Q}\varphi(h(u)),$$

where $\Gamma:=\{h\in C(\bar{Q},E)|\,h=id \text{ on } \partial Q\}$, here, id denotes the identity operator.

Remark 2.4. As shown in [1], a deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition, and it turns out that the generalized mountain pass theorem holds true under condition (C).

3. Proofs of main results

In this section, we start with some compactness conditions, which play crucial roles in establishing our results. For the sake of convenience, in the following, we will denote various positive constants as C_i, $i=1,2,3,\cdots$.

Lemma 3.1. *Assume that (A), (F_3)–(F_5) hold, then the functional φ satisfies condition (C).*

Proof. Assume that $\{u_n\}$ is a (C) sequence of φ, then one has

$$\varphi(u_n)\leq C_1,\quad \|\varphi'(u_n)\|_{(H_T^1)^*}(1+\|u_n\|)\leq C_1, \tag{3.1}$$

where $(H_T^1)^*$ is the dual space of H_T^1.

To begin with, by (F_4), there exists $M_4>0$ such that

$$e^{Q(t)}F(t,x)\leq M|x|^r$$

for all $|x|\geq M_4$ and a.e. $t\in[0,T]$, which jointly with assumption (A) that

$$e^{Q(t)}F(t,x)\leq M|x|^r+A_1h_1(t) \tag{3.2}$$

for all $x\in\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in[0,T]$, where $h_1(t):=\max_{|x|\leq M_4}a(|x|)b(t)\geq 0$. It follows from (2.2), (3.1) and (3.2) that

$$\begin{aligned} C_1\geq\varphi(u_n)&=\frac{1}{2}\int_0^T e^{Q(t)}|\dot{u}_n(t)|^2\mathrm{d}t-\int_0^T e^{Q(t)}F(t,u_n(t))\mathrm{d}t\\ &\geq\frac{1}{2}A_2\|\dot{u}_n\|_{L^2}^2-M\int_0^T|u_n(t)|^r\mathrm{d}t-A_1\int_0^T h_1(t)\mathrm{d}t. \end{aligned} \tag{3.3}$$

On the other hand, by (F_5), one has

$$e^{Q(t)}[(\nabla F(t,x),x)-2F(t,x)]\geq k_1(|x|)|x|^\mu$$

for all $|x|\geq M_1$ and a.e. $t\in[0,T]$, which combining with assumption (A) yields

$$e^{Q(t)}[(\nabla F(t,x),x)-2F(t,x)]\geq k_1(|x|)|x|^\mu-A_1h_2(t) \tag{3.4}$$

for all $x\in\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in[0,T]$, where $h_2(t):=(2+M_1)\max_{|x|\leq M_1}a(|x|)b(t)\geq 0$. It follows from (2.2), (2.3), (3.1) and (3.4) that

$$\begin{aligned}
3C_1 &\geq 2\varphi(u_n) - (\varphi'(u_n), u_n) \\
&= \int_0^T e^{Q(t)}[(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt \\
&\geq \int_0^T k_1(|u_n|)|u_n|^{\mu}dt - A_1 \int_0^T h_2(t)dt
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we have

$$\int_0^T k_1(|u_n|)|u_n|^{\mu}dt \leq C_2 \tag{3.5}$$

for all $n \in \mathbb{N}$.

Next, we have to discuss two cases:

Case 1. $\mu > r$. By (3.5), Hölder inequality and the properties of $k_1(z)$, one has

$$\begin{aligned}
\int_0^T |u_n|^r dt &\leq T^{\frac{\mu-r}{\mu}} \left(\int_0^T |u_n|^{\mu} dt \right)^{\frac{r}{\mu}} \\
&= T^{\frac{\mu-r}{\mu}} \left[\int_0^T \frac{1}{k_1(|u_n|)} k_1(|u_n|)|u_n|^{\mu}dt \right]^{\frac{r}{\mu}} \\
&\leq T^{\frac{\mu-r}{\mu}} \left[\int_0^T \frac{1}{k_1(\|u_n\|_{\infty})} k_1(|u_n|)|u_n|^{\mu}dt \right]^{\frac{r}{\mu}} \\
&\leq T^{\frac{\mu-r}{\mu}} \left[\frac{1}{k_1(d\|u_n\|)} \right]^{\frac{r}{\mu}} \left[\int_0^T k_1(|u_n|)|u_n|^{\mu}dt \right]^{\frac{r}{\mu}} \\
&\leq \frac{C_3}{k_1^{\frac{r}{\mu}}(d\|u_n\|)}.
\end{aligned} \tag{3.6}$$

Then, from (3.3) and (3.6), we have

$$C_1 \geq \varphi(u_n) \geq \frac{1}{2} A_2 \|\dot{u}_n\|_{L^2}^2 - \frac{MC_3}{k_1^{\frac{r}{\mu}}(d\|u_n\|)} - A_1 \int_0^T h_1(t)dt. \tag{3.7}$$

Case 2. $\mu \leq r$. By (2.1), (3.5) and the properties of $k_1(z)$, we deduce that

$$\begin{aligned}
\int_0^T |u_n|^r dt &\leq \|u_n\|_{\infty}^{r-\mu} \int_0^T |u_n|^{\mu}dt \\
&\leq d^{r-\mu}\|u_n\|^{r-\mu} \int_0^T \frac{1}{k_1(|u_n|)} k_1(|u_n|)|u_n|^{\mu}dt \\
&\leq d^{r-\mu}\|u_n\|^{r-\mu} \frac{C_2}{k_1(d\|u_n\|)} \\
&= \frac{C_4}{k_1(d\|u_n\|)} \|u_n\|^{r-\mu},
\end{aligned}$$

using (3.3), which implies

$$C_1 \geq \varphi(u_n) \geq \frac{1}{2}A_2\|\dot{u}_n\|_{L^2}^2 - \frac{MC_4}{k_1(d\|u_n\|)}\|u_n\|^{r-\mu} - A_1\int_0^T h_1(t)\mathrm{d}t. \tag{3.8}$$

Finally, we claim that $\{u_n\}$ is bounded; otherwise, going if necessary to a subsequence, we assume that $\|u_n\| \to +\infty$ as $n \to +\infty$. Set $v_n := \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded in H_T^1. Hence, there exists a subsequence, again denoted by $\{v_n\}$, such that

$$v_n \rightharpoonup v_0 \quad \text{weakly in } H_T^1, \tag{3.9}$$

$$v_n \to v_0 \quad \text{strongly in } C(0,T;\mathbb{R}^{\mathbb{N}}). \tag{3.10}$$

Dividing both sides of (3.7) and (3.8) by $\|u_n\|^2$ respectively, using the properties of $k_1(z)$, we can find that

$$\|\dot{v}_n\|_{L^2} \to 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

Hence, (3.11) always holds true whenever $\mu > r$ or $\mu \leq r$. So, it follows from (3.10) and (3.11) that

$$v_n \to \bar{v}_0 \quad \text{as } n \to +\infty,$$

which implies that

$$v_0 = \bar{v}_0 \text{ and } T|\bar{v}_0|^2 = \|\bar{v}_0\|^2 = 1.$$

Consequently,

$$|u_n(t)| \to +\infty \text{ as } n \to +\infty \text{ uniformly for a.e. } t \in [0,T]. \tag{3.12}$$

It follows from (F_3) that there exists $M_5 > 0$ such that

$$F(t,x) \geq 0 \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \geq M_5 \text{ and for a.e. } t \in [0,T], \tag{3.13}$$

which implies that

$$F(t,x) \geq -h_3(t) \quad \forall x \in \mathbb{R}^{\mathbb{N}}, \text{ and for a.e. } t \in [0,T]$$

by assumption (A), where $h_3(t) := \max_{|x|\leq M_5} a(|x|)b(t) \geq 0$. Denote $\Omega_n := \{t \in [0,T] \,|\, |u_n(t) \geq M_5\}$, by (3.12), we have $\mathrm{meas}(\Omega_n) > 0$. So, from (F_3), (3.12), (3.13) and Fatou's Lemma, we get

$$\begin{aligned}
\liminf_{n\to+\infty}\frac{\int_0^T \mathrm{e}^{Q(t)}F(t,u_n(t))\mathrm{d}t}{\|u_n\|^2} &= \liminf_{n\to+\infty}\left[\frac{\int_{\Omega_n}\mathrm{e}^{Q(t)}F(t,u_n(t))\mathrm{d}t}{\|u_n\|^2} + \frac{\int_{[0,T]\setminus\Omega_n}\mathrm{e}^{Q(t)}F(t,u_n(t))\mathrm{d}t}{\|u_n\|^2}\right] \\
&\geq \liminf_{n\to+\infty}\left[\frac{\int_{\Omega_n}\mathrm{e}^{Q(t)}F(t,u_n(t))\mathrm{d}t}{\|u_n\|^2} - \frac{A_2\int_0^T h_3(t)\mathrm{d}t}{\|u_n\|^2}\right] \\
&\geq \int_{\Omega_n}\liminf_{n\to+\infty}\frac{\mathrm{e}^{Q(t)}F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2\mathrm{d}t \\
&= \int_{\Omega_n}\liminf_{|u_n(t)|\to+\infty}\frac{\mathrm{e}^{Q(t)}F(t,u_n(t))}{|u_n(t)|^2}|\bar{v}_0(t)|^2\mathrm{d}t \\
&> 0.
\end{aligned} \tag{3.14}$$

However, by (3.1) and (3.11), we have

$$\liminf_{n\to+\infty}\frac{\int_0^T \mathrm{e}^{Q(t)}F(t,u_n)\mathrm{d}t}{\|u_n\|^2} = 0,$$

which contradicts (3.14). Thus, $\{u_n\}$ is bounded in H_T^1.

Since H_T^1 is a reflexive Banach space, there exist $u \in H_T^1$ and a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } H_T^1. \tag{3.15}$$

By Proposition 1.2 in [8], we see that

$$u_n \to u \quad \text{strongly in } C(0,T;\mathbb{R}^{\mathbb{N}}). \tag{3.16}$$

Then,

$$\int_0^T |u_n(t)-u(t)|^2 \mathrm{d}t \to 0 \quad \text{as } n\to+\infty. \tag{3.17}$$

It follows from (3.16) and assumption (A) that

$$\int_0^T \mathrm{e}^{Q(t)}(\nabla F(t,u_n(t))-\nabla F(t,u(t)), u_n(t)-u(t))\mathrm{d}t \to 0 \tag{3.18}$$

as $n\to+\infty$. In view of (3.15) and $\varphi'(u_n)\to 0$, we obtain

$$(\varphi'(u_n)-\varphi'(u), u_n-u)\to 0 \quad \text{as } n\to+\infty. \tag{3.19}$$

In addition, by (2.3), we have that

$$\begin{aligned}(\varphi'(u_n)-\varphi'(u), u_n-u) &= \int_0^T \mathrm{e}^{Q(t)}(\dot u_n(t)-\dot u(t), \dot u_n(t)-\dot u(t))\mathrm{d}t \\ &\quad - \int_0^T \mathrm{e}^{Q(t)}(\nabla F(t,u_n(t))-\nabla F(t,u(t)), u_n(t)-u(t))\mathrm{d}t.\end{aligned} \tag{3.20}$$

Thus, in light of (3.18)–(3.20), one arrives

$$0\le A_2\int_0^T |\dot u_n(t)-\dot u(t)|^2\mathrm{d}t \le \int_0^T \mathrm{e}^{Q(t)}|\dot u_n(t)-\dot u(t)|^2\mathrm{d}t \to 0$$

as $n\to+\infty$. Therefore,

$$\int_0^T |\dot u_n(t)-\dot u(t)|^2\mathrm{d}t\to 0 \quad \text{as } n\to+\infty,$$

which combining (3.17) yields that

$$\|u_n-u\| = \left(\int_0^T |u_n(t)-u(t)|^2\mathrm{d}t + \int_0^T |\dot u_n(t)-\dot u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \to 0 \quad \text{as } n\to+\infty.$$

That is, $\{u_n\}$ strongly converges to u on H_T^1. Hence, $\{u_n\}$ possesses a strong convergent subsequence, which means that φ satisfies condition (C). □

Lemma 3.2. *Assume that (A), (F_3) and (F_6) hold, then the functional φ satisfies condition (C).*

Proof. From the arguments of Lemma 3.1, we only need to prove that $\{u_n\}$ is bounded in H_T^1. It follows from (F_6) that

$$\mathrm{e}^{Q(t)}[(\nabla F(t,x),x)-2F(t,x)] \ge k_2(|x|)\left(\frac{F(t,x)}{|x|^2}\right)^{\theta} \tag{3.21}$$

for all $x\ge M_2$ and a.e. $t\in[0,T]$. Bearing in mind that (3.13), let $M_6:=\max\{M_2,M_5\}$ and denote $\Omega_n^*:=\{t\in[0,T]\,|\,|u_n(t)|\ge M_6\}$. By (2.1)–(2.3), (3.1), (3.21) and the properties of $k_2(z)$, we obtain

$$
\begin{aligned}
3C_1 &\geq 2\varphi(u_n) - (\varphi'(u_n), u_n) \\
&= \int_0^T \mathrm{e}^{Q(t)}[(\nabla F(t,u_n), u_n(t)) - 2F(t,u_n)]\mathrm{d}t \\
&= \int_{\Omega_n^*} \mathrm{e}^{Q(t)}[(\nabla F(t,u_n), u_n(t)) - 2F(t,u_n)]\mathrm{d}t + \int_{[0,T]\setminus\Omega_n^*} \mathrm{e}^{Q(t)}[(\nabla F(t,u_n), u_n(t)) - 2F(t,u_n)]\mathrm{d}t \\
&\geq \int_{\Omega_n^*} k_2(|u_n|)\left(\frac{F(t,u_n)}{|u_n|^2}\right)^{\theta}\mathrm{d}t - A_1\int_0^T h_4(t)\mathrm{d}t \\
&\geq \int_{\Omega_n^*} k_2(d\|u_n\|)\frac{F^{\theta}(t,u_n)}{d^{2\theta}\|u_n\|^{2\theta}}\mathrm{d}t - A_1\int_0^T h_4(t)\mathrm{d}t,
\end{aligned}
$$

where $h_4(t) := (2+M_6)\max_{|x|\leq M_6} a(|x|)b(t) \geq 0$. Hence, we get that

$$
\int_{\Omega_n^*} F^{\theta}(t,u_n(t))\mathrm{d}t \leq \frac{C_5}{k_2(d\|u_n\|)}\|u_n\|^{2\theta},
$$

noticing assumption (A) and (3.13), suggests that

$$
\begin{aligned}
\int_0^T |F(t,u_n(t))|^{\theta}\mathrm{d}t &= \int_{\Omega_n^*} F^{\theta}(t,u_n(t))\mathrm{d}t + \int_{[0,T]\setminus\Omega_n^*} |F(t,u_n(t))|^{\theta}\mathrm{d}t \\
&\leq \frac{C_5}{k_2(d\|u_n\|)}\|u_n\|^{2\theta} + \int_0^T h_5^{\theta}(t)\mathrm{d}t
\end{aligned}
\tag{3.22}
$$

for all $n \in \mathbb{N}$, where $h_5(t) := \max_{|x|\leq M_6} a(|x|)b(t) \geq 0$. Furthermore, by (3.1), (3.22) and Hölder inequality, one has

$$
\begin{aligned}
C_1 \geq \varphi(u_n) &= \frac{1}{2}\int_0^T \mathrm{e}^{Q(t)}|\dot{u}_n(t)|^2\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}F(t,u_n(t))\mathrm{d}t \\
&\geq \frac{1}{2}A_2\|\dot{u}_n\|_{L^2}^2 - A_1C_6\left(\int_0^T |F(t,u_n(t))|^{\theta}\mathrm{d}t\right)^{\frac{1}{\theta}} \\
&\geq \frac{1}{2}A_2\|\dot{u}_n\|_{L^2}^2 - A_1C_6\left(\frac{C_5}{k_2(d\|u_n\|)}\|u_n\|^{2\theta} + \int_0^T h_5^{\theta}(t)\mathrm{d}t\right)^{\frac{1}{\theta}} \\
&\geq \frac{1}{2}A_2\|\dot{u}_n\|_{L^2}^2 - \frac{C_7}{k_2^{\frac{1}{\theta}}(d\|u_n\|)}\|u_n\|^2 - C_8
\end{aligned}
\tag{3.23}
$$

for all $n \in \mathbb{N}$.

Now we claim $\{u_n\}$ is bounded, otherwise, going if necessary to a subsequence, we can assume that $\|u_n\| \to +\infty$ as $n \to +\infty$. With the same manner of Lemma 3.1, dividing both sides of (3.23) by $\|u_n\|^2$, using the properties of $k_2(z)$, we conclude that $|u_n(t)| \to +\infty$ as $n \to +\infty$ uniformly for a.e. $t \in [0,T]$. From (F_3), assumption (A) and Fatou's Lemma, we infer that

$$
\liminf_{n\to+\infty} \frac{\int_0^T \mathrm{e}^{Q(t)}F(t,u_n)\mathrm{d}t}{\|u_n\|^2} > 0. \tag{3.24}
$$

On the other hand, note that (3.1) and (3.11), one has

$$
\liminf_{n\to+\infty} \frac{\int_0^T \mathrm{e}^{Q(t)}F(t,u_n)\mathrm{d}t}{\|u_n\|^2} = 0,
$$

which contradicts (3.24). Then, $\{u_n\}$ is bounded in H_T^1. Therefore, φ satisfies condition (C). □

Lemma 3.3. *Assume that* (A), (F_3) *and* (F_7) *hold, then the functional* φ *satisfies condition (C).*

Proof. It follows from (3.1), (F_7) and the properties of $k_3(|x|)$ that

$$\begin{aligned} 3C_1 &\geq 2\varphi(u_n) - (\varphi'(u_n), u_n) \\ &= \int_0^T e^{Q(t)}[(\nabla F(t,u_n), u_n) - 2F(t,u_n)]dt \\ &\geq \int_0^T k_3(|u_n|)\frac{|\nabla F(t,u_n)|^\sigma}{|u_n|^\sigma}dt - A_1\int_0^T h_6(t)dt \\ &\geq \int_0^T k_3(d\|u_n\|)\frac{|\nabla F(t,u_n)|^\sigma}{d^\sigma\|u_n\|^\sigma}dt - A_1\int_0^T h_6(t)dt \end{aligned} \tag{3.25}$$

for all $n \in \mathbb{N}$, where $h_6(t) := (2+M_3)\max_{|x|\leq M_3} a(|x|)b(t) \geq 0$. As a consequence,

$$\int_0^T |\nabla F(t,u_n)|^\sigma dt \leq \frac{C_9}{k_3(d\|u_n\|)}\|u_n\|^\sigma. \tag{3.26}$$

Since H_T^1 could be embedded into $L^p(0,T;\mathbb{R}^\mathbb{N})$ for $1 \leq p \leq +\infty$, hence there exists $\tau_p > 0$ such that

$$\|u\|_{L^p} \leq \tau_p\|u\| \quad \forall u \in H_T^1, \tag{3.27}$$

which together with (2.3), (3.1), (3.26) and Hölder inequality, we infer that

$$\begin{aligned} C_1 &\geq (\varphi'(u_n), u_n) \\ &= \int_0^T e^{Q(t)}|\dot{u}_n|^2dt - \int_0^T e^{Q(t)}(\nabla F(t,u_n), u_n)dt \\ &\geq A_2\|\dot{u}_n\|_{L^2}^2 - A_1\left(\int_0^T |\nabla F(t,u_n)|^\sigma dt\right)^{\frac{1}{\sigma}}\|u_n\|_{L^{\sigma'}} \\ &\geq A_2\|\dot{u}_n\|_{L^2}^2 - A_1\tau_{\sigma'}\left(\frac{C_9}{k_3(d\|u_n\|)}\right)^{\frac{1}{\sigma}}\|u_n\|^2 \end{aligned} \tag{3.28}$$

for all $n \in \mathbb{N}$, where $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$.

Finally, we claim that $\{u_n\}$ is bounded; otherwise, going if necessary to a subsequence, we assume that $\|u_n\| \to +\infty$ as $n \to +\infty$. In the same way as in the proof of Lemma 3.1, multiplying both sides of (3.28) by $\|u_n\|^{-2}$, using the properties of $k_3(z)$, we can obtain that $|u_n(t)| \to +\infty$ as $n \to +\infty$ uniformly for a.e. $t \in [0,T]$, and then from (F_3), assumption (A) and Fatou's Lemma, we have

$$\liminf_{n\to+\infty}\frac{\int_0^T e^{Q(t)}F(t,u_n)dt}{\|u_n\|^2} > 0. \tag{3.29}$$

On the other hand, thanks to (3.1) and (3.11), implies that

$$\liminf_{n\to+\infty}\frac{\int_0^T e^{Q(t)}F(t,u_n)dt}{\|u_n\|^2} = 0,$$

which contradicts (3.29). Thus, $\{u_n\}$ is bounded in H_T^1. Using the same arguments as in Lemma 3.1, we can get that φ satisfies condition (C). □

Now we are in a position to prove our main results. We only give the proofs of Theorem 1.1, Theorem 1.3, Theorem 1.5, Corollary 1.7, Corollary 1.10, Corollary 1.12, and Corollary 1.13; the other results can be proved similarly.

Proof of Theorem 1.1. Let $H_T^1 := \widetilde{H}_T^1 \oplus \mathbb{R}^{\mathbb{N}}$, where $\dim \mathbb{R}^{\mathbb{N}} < +\infty$. From Lemma 3.1, we know that φ satisfies condition (C). By virtue of Theorem 2.3 and Remark 2.4, we only need to verify the assertions:

(φ_1) $\inf_{u\in S} \varphi(u) > 0$;

(φ_2) $\sup_{u\in Q} \varphi(u) < +\infty$, $\sup_{u\in\partial Q} \varphi(u) \le 0$,

where $S := \widetilde{H}_T^1 \cap \partial B_\rho$, $Q := \{x \in \mathbb{R}^{\mathbb{N}} \mid |x| \le s_0\} \oplus \{se \mid 0 \le s \le s_0, e(t) \in \widetilde{H}_T^1\}$ and $\rho < s_0$.

Firstly, from (F_2), we can find $\varepsilon_1 := \frac{1}{2}\left(\frac{1}{2}A_2\omega^2 - \limsup_{|x|\to 0} \mathrm{e}^{Q(t)}\frac{F(t,x)}{|x|^2}\right) > 0$ and $\delta > 0$ such that

$$\mathrm{e}^{Q(t)}F(t,x) \le \left(\frac{1}{2}A_2 - \varepsilon_1\right)\omega^2|x|^2 \quad \forall |x| \le \delta \text{ and for a.e. } t \in [0,T].$$

For $u \in \widetilde{H}_T^1$ with $\|u\| \le 12\delta/T$, by Sobolev's inequality, we have $\|u\|_\infty \le \delta$. Consequently, using Wirtinger's inequality, one has

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T \mathrm{e}^{Q(t)}|\dot{u}(t)|^2\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}F(t,u(t))\mathrm{d}t \\ &\ge \frac{1}{2}A_2\|\dot{u}\|_{L^2}^2 - \left(\frac{1}{2}A_2 - \varepsilon_1\right)\omega^2\int_0^T |u(t)|^2\mathrm{d}t \\ &\ge \varepsilon_1\|\dot{u}\|_{L^2}^2 \\ &\ge \varepsilon_1 C_{10}\|u\|^2.\end{aligned}$$

Let $\rho \in (0, 12\delta/T)$, then

$$\inf_{u\in S}\varphi(u) \ge \varepsilon_1 C_{10}\rho^2 > 0,$$

which implies (φ_1) holds.

Finally, we check (φ_2). Let $\varepsilon_2 := \frac{1}{2}\left(\liminf_{|x|\to+\infty} \mathrm{e}^{Q(t)}\frac{F(t,x)}{|x|^2} - \frac{1}{2}A_1\omega^2\right)$, from (F_3), we could get $\varepsilon_2 > 0$, and we can choose $M_7 > 0$ such that

$$\mathrm{e}^{Q(t)}F(t,x) \ge \left(\frac{1}{2}A_1 + \varepsilon_2\right)\omega^2|x|^2 \quad \forall |x| \ge M_7 \text{ and for a.e. } t \in [0,T].$$

Therefore, for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$, by assumption (A), we obtain

$$\mathrm{e}^{Q(t)}F(t,x) \ge \left(\frac{1}{2}A_1 + \varepsilon_2\right)\omega^2|x|^2 - A_1 h_7(t),$$

where $h_7(t) := \max_{|x|\le M_7} a(|x|)b(t) \ge 0$. Let $\bar{H}_T^1 := \mathrm{span}\{e\} \oplus \mathbb{R}^{\mathbb{N}}$ with $e := (\sin(\omega t), 0, \ldots, 0) \in \widetilde{H}_T^1$. Observe that, for $x + se \in \widetilde{H}_T^1$,

$$\begin{aligned}\varphi(x+se) &= \frac{1}{2}\int_0^T \mathrm{e}^{Q(t)}|s\dot{e}|^2\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}F(t,x+se)\mathrm{d}t \\ &\le \frac{1}{2}A_1\omega^2 s^2\int_0^T \cos^2\omega t\mathrm{d}t - \left(\frac{1}{2}A_1 + \varepsilon_2\right)\omega^2\int_0^T |x+se|^2\mathrm{d}t + M^* \\ &\le -\varepsilon_2\omega^2 s^2\int_0^T \sin^2\omega t\mathrm{d}t - \frac{1}{2}A_1\omega^2\int_0^T |x|^2\mathrm{d}t + M^* \\ &= -\frac{1}{2}\varepsilon_2\omega^2 s^2 T - \frac{1}{2}A_1\omega^2|x|^2 T + M^*,\end{aligned} \tag{3.30}$$

where $M^* := A_1\int_0^T h_7(t)\mathrm{d}t$. Let $Q = \{x \in \mathbb{R}^{\mathbb{N}} \mid |x| \le R_1\} \oplus \{se \mid 0 \le s \le R_2\}$ with $R_2 > \rho$ and $R_1 > 0$. Define

$$Q_1 := \{x \in \mathbb{R}^{\mathbb{N}} \mid |x| \le R_1\},$$
$$Q_2 := \{x + R_2 e \mid x \in \mathbb{R}^{\mathbb{N}},\ |x| \le R_1\},$$
$$Q_3 := \{x + se \mid x \in \mathbb{R}^{\mathbb{N}},\ |x| = R_1,\ 0 \le s \le R_2\}.$$

It is easy to check that $\partial Q = Q_1 \cup Q_2 \cup Q_3$.

(1) By (F_1), we have

$$\varphi(x) = -\int_0^T e^{Q(t)} F(t,x)\mathrm{d}t \le 0$$

for all $x \in \mathbb{R}^{\mathbb{N}}$. Hence, one has $\varphi(u) \le 0$ for all $u \in Q_1$.

(2) Let $R_2 \ge \sqrt{\frac{2M^*}{\varepsilon_2 \omega^2 T}}$, from (3.30), we get

$$\begin{aligned}\varphi(x + R_2 e) &\le -\frac{1}{2}\varepsilon_2 \omega^2 R_2^2 T - \frac{1}{2}A_1 \omega^2 |x|^2 T + M^* \\ &\le -\frac{1}{2}\varepsilon_2 \omega^2 R_2^2 T + M^* \\ &\le 0,\end{aligned}$$

which implies $\varphi(u) \le 0$ for all $u \in Q_2$.

(3) Let $R_1 \ge \sqrt{\frac{2M^*}{A_1 \omega^2 T}}$, from (3.30), we obtain

$$\begin{aligned}\varphi(x + se) &\le -\frac{1}{2}\varepsilon_2 \omega^2 s^2 T - \frac{1}{2}A_1 \omega^2 R_1^2 T + M^* \\ &\le -\frac{1}{2}A_1 \omega^2 R_1^2 T + M^* \\ &\le 0,\end{aligned}$$

which implies $\varphi(u) \le 0$ for all $u \in Q_3$.

From the above discussions in (1)–(3), put $s_0 := \max\{R_1, R_2\} > \rho > 0$, we can infer that $\sup_{u \in \partial Q} \varphi(u) \le 0$. Furthermore, by (3.30), we have $\sup_{x+se \in Q} \varphi(x + se) \le M^* < +\infty$, that is, (φ_2) holds. From Theorem 2.3, we know that φ possesses a critical point $u(t)$ whose critical value c satisfies $c \ge \alpha > 0$. By (F_1) and (2.2), we can see that $u(t)$ is non-constant. Hence, problem (1.1) has at least one non-constant periodic solution in H_T^1. □

Proof of Theorem 1.3. From Lemma 3.2, using the same arguments as in Theorem 1.1, we see that problem (1.1) has at least one non-constant periodic solution in H_T^1. □

Proof of Theorem 1.5. Clearly, with the aid of Lemma 3.3 and the arguments of Theorem 1.1, we can easily get that problem (1.1) has at least one non-constant periodic solution in H_T^1. □

Proof of Corollary 1.7. Take $r = 2$, by Theorem 1.1, we can obtain that problem (1.1) has at least one non-constant periodic solution in H_T^1 immediately. □

Proof of Corollary 1.10. From (F_3) and (F_4^*), we conclude that $F(t,x)$ at infinity is positive for all $x \in \mathbb{R}^{\mathbb{N}}$, a.e. $t \in [0,T]$ and asymptotically quadratic, thus (F_6) is equivalent to (F_6^*). By Theorem 1.3, we have that problem (1.1) has at least one non-constant periodic solution in H_T^1. □

Proof of Corollary 1.12. Applying (F_3), (F_4^{**}) and (F_7^*), we have

$$M \ge \frac{e^{Q(t)}|\nabla F(t,x)|}{|x|} \ge \frac{e^{Q(t)}(\nabla F(t,x), x)}{|x|^2} \ge \frac{2e^{Q(t)}F(t,x)}{|x|^2} > A_1 \omega^2 \tag{3.31}$$

for $|x|$ large enough and a.e. $t \in [0,T]$. From (3.31), we deduce that $|\nabla F(t,x)|$ at infinity is positive for all $x \in \mathbb{R}^{\mathbb{N}}$, a.e. $t \in [0,T]$ and asymptotically linear, which means that (F_7) and (F_7^*) are equivalent. Then, by Theorem 1.5, one has that problem (1.1) has at least one non-constant periodic solution in H_T^1. □

Proof of Corollary 1.13. By (F_3^*) and (F_7^{**}), one has

$$\frac{e^{Q(t)}F(t,x)}{|x|^2} \geq \frac{e^{Q(t)}(\nabla F(t,x),x)}{2|x|^2} > \frac{1}{2}A_1\omega^2$$

for $|x|$ large enough and a.e. $t \in [0,T]$, which implies (F_3) holds. Moreover, utilizing (F_3^*) and (F_4^{**}), we have:

$$M \geq \frac{e^{Q(t)}|\nabla F(t,x)|}{|x|} \geq \frac{e^{Q(t)}(\nabla F(t,x),x)}{|x|^2} > A_1\omega^2.$$

Therefore, we know that $|\nabla F(t,x)|$ at infinity is also positive for all $x \in \mathbb{R}^{\mathbb{N}}$, a.e. $t \in [0,T]$ and asymptotically linear. Using the similar arguments of Corollary 1.12, we can deduce that problem (1.1) has at least one non-constant periodic solution in H_T^1. □

4. Examples

In this section, we will give two examples to illustrate the fact that our results could deal with both the superquadratic and asymptotical quadratic cases. To do this, for simplicity, we only address problem (1.2), that is, in what follows, we are concerned with problem (1.1) without damped vibration term.

Example 4.1 *(Superquadratic case at infinity).* Let $D := 4 - (4\ln 5 + \sin 4 - \ln^2 5) > 0$. Consider

$$F(t,x) = g_1(t)h(x) \qquad \forall x \in \mathbb{R}^{\mathbb{N}} \text{ and for a.e. } t \in [0,2\pi],$$

where $g_1(t) \in C(0,2\pi;\mathbb{R}^+)$, $\inf_{t\in[0,2\pi]} g_1(t) > 0$ and

$$h(x) = \begin{cases} \frac{1}{4}|x|^4, & |x| \leq 2, \\ |x|^2\ln(1+|x|^2) + \sin|x|^2 - \ln^2(1+|x|^2) + D & |x| > 2. \end{cases}$$

Then, we have

$$\liminf_{|x|\to+\infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\lambda} = 0 \quad \text{uniformly for a.e. } t \in [0,T] \text{ and all } \lambda > 0,$$

which implies that $F(t,x)$ does not satisfy the results of Theorem A, Theorem B and the conclusions of [6,15,16,22,23]. But, put $k_2(|x|) = \ln(1+|x|^2), \theta = 1, q(t) = 0, T = 2\pi$, a direct computation shows that $F(t,x)$ satisfies all conditions of Theorem 1.3. Hence, problem (1.2) has at least one non-constant periodic solution.

Example 4.2 *(Asymptotically quadratic case at infinity).* Consider

$$F(t,x) = g_2(t)[|x|^2 - \ln(1+|x|^2)] \qquad \forall x \in \mathbb{R}^{\mathbb{N}} \text{ and for a.e. } t \in [0,2\pi],$$

where $g_2(t) \in C(0,2\pi;\mathbb{R}^+)$, $\inf_{t\in[0,2\pi]} g_2(t) > 1/2$. Then we can also choose $k_2(|x|) = \ln(1+|x|^2), \theta = 1, q(t) = 0, T = 2\pi$; it is easy to check that $F(t,x)$ satisfies all conditions of Theorem 1.3 (or Corollary 1.10). So, problem (1.2) has at least one non-constant periodic solution.

Remark 4.3. In [14], Tang and Wu have introduced a class of new superquadratic condition:

(TW) there exist $a > 0$ and $L_2 > 0$ such that

$$(\nabla F(t,x),x) - 2F(t,x) \geq \frac{a}{|x|^2}F(t,x) \quad \forall x \in \mathbb{R}^{\mathbb{N}}, |x| \geq L_2 \text{ and for a.e. } t \in [0,T].$$

We should mention that condition (TW) is weaker than (F_6) with $q(t) = 0$. But the result of [14] is essential to rely on the assumption (S_3). In other words, the main theorem in [14] only can handle the superquadratic potential functions and cannot treat the asymptotically quadratic potential functions like Example 4.2. At the same time, we note that $F(t,x)$ in Example 4.2 also satisfies all conditions of Theorem C, however, Theorem C only work in the asymptotically quadratic case rather than in the superquadratic case. Here, Theorem 1.3 can deal with both superquadratic and asymptotical quadratic cases. In this sense, Theorem 1.3 is a new result.

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