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On maximizing the fundamental frequency of the complement of an obstacle

Sur la maximisation de la fréquence fondamentale du complément d'un obstacle

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A R T I C L E I N F O

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A B S T R A C T

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying a Hayman-type asymmetry condition, and let D be an arbitrary bounded domain referred to as an "obstacle". We are interested in the behavior of the first Dirichlet eigenvalue $\lambda_1(\Omega \setminus (x+D))$.
First, we prove an upper bound on $\lambda_1(\Omega \setminus (x+D))$ in terms of the distance of the set $x+D$ to the set of maximum points x_0 of the first Dirichlet ground state $\phi_{\lambda_1} > 0$ of Ω. In short, a direct corollary is that if

$$\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x+D)) \tag{1}$$

is large enough in terms of $\lambda_1(\Omega)$, then all maximizer sets $x+D$ of μ_Ω are close to each maximum point x_0 of ϕ_{λ_1}.
Second, we discuss the distribution of $\phi_{\lambda_1(\Omega)}$ and the possibility to inscribe wavelength balls at a given point in Ω.
Finally, we specify our observations to convex obstacles D and show that if μ_Ω is sufficiently large with respect to $\lambda_1(\Omega)$, then all maximizers $x+D$ of μ_Ω contain all maximum points x_0 of $\phi_{\lambda_1(\Omega)}$.

R É S U M É

Soit $\Omega \subset \mathbf{R}^n$ un domaine borné satisfaisant une condition de type Hayman asymétrique et soit D un domaine borné arbitraire, dénommé « obstacle ». Nous nous intéressons au comportement de la première valeur propre de Dirichlet $\lambda_1(\Omega \setminus (x+D))$.
Nous établissons, dans un premier temps, une borne supérieure pour cette valeur propre en termes de la distance de l'ensemble $x+D$ à l'ensemble des points x_0 où la fonction propre du premier état de base de Dirichlet $\phi_{\lambda_1} > 0$ de Ω atteint son maximum. En bref, un corollaire immédiat est que, si

$$\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x+D))$$

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est suffisamment grand en fonction de $\lambda_1(\Omega)$, alors tous les ensembles maximisant $x+D$ de μ_Ω sont proches de chaque point x_0 où ϕ_{λ_1} est maximum.
Ensuite, nous discutons la distribution de $\phi_{\lambda_1(\Omega)}$ et la possibilité d'inscrire des boules de longueur d'onde en un point donné de Ω.
Enfin, nous appliquons nos observations aux obstacles convexes D, et nous montrons que, si μ_Ω est suffisamment grand par rapport à $\lambda_1(\Omega)$, alors tous les ensembles maximisant $x+D$ de μ_Ω contiennent tous les points x_0 où $\phi_{\lambda_1(\Omega)}$ est maximum.

1. Introduction and background

We consider the natural problem (seemingly first posed by Davies) of placing an obstacle in a domain so as to maximize the fundamental frequency of the complement of the obstacle. To be more precise, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let D be another bounded domain referred to as an "obstacle". The problem is to determine the optimal translate $x+D$ so that the fundamental Dirichlet Laplacian eigenvalue $\lambda_1(\Omega \setminus (x+D))$ is maximized/minimized.

In case the obstacle D is a ball, physical intuition suggests that for sufficiently regular domains and sufficiently small balls, Ω, $\lambda_1(\Omega \setminus B_r(x))$ will be maximized when $x = x_0$, a point of maximum of the ground state Dirichlet eigenfunction ϕ_{λ_1} of Ω. Heuristically, such maximum points x_0 seem to be situated deeply in Ω, hence removing a ball around x_0 should be an optimal way of truncating the lowest possible frequency. Our methods give equally good results for Schrödinger operators on a large class of bounded domains sitting inside Riemannian manifolds (see the remarks at the end of Section 2).

The following well-known result of Harrell–Kröger–Kurata treats the case when Ω satisfies convexity and symmetry conditions.

Theorem 1.1 *([11]). Let Ω be a convex domain in $\mathbb{R}^n$ and B a ball contained in Ω. Assume that Ω is symmetric with respect to some hyperplane H. Then,*

(a) at the maximizing position, B is centered on H, and
(b) at the minimizing position, B touches the boundary of Ω.

The last result of Harrell–Kröger–Kurata seems to work under a rather strong symmetry assumption. We also recall that the proof of Harrell–Kröger–Kurata proceeds via a moving planes method, which essentially measures the derivative of $\lambda_1(\Omega \setminus B)$ when B is shifted in a normal direction to the hyperplane (also see p. 58 of [13]). See also related work in [4], [14].

There does not seem to be any result in the literature treating domains without symmetry or convexity properties.

In our note, we consider bounded domains $\Omega \subset \mathbb{R}^n$ that satisfy an asymmetry assumption in the following sense.

Definition 1.2. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the asymmetry assumption with coefficient α (or Ω is α-asymmetric) if for all $x \in \partial\Omega$, and all $r_0 > 0$,

$$\frac{|B_{r_0}(x) \setminus \Omega|}{|B_{r_0}(x)|} \geq \alpha. \tag{2}$$

This condition seems to have been introduced in [12]. Further, the α-asymmetry property was utilized by D. Mangoubi in order to obtain inradius bounds for Laplacian nodal domains (cf. [16]) as nodal domains are asymmetric with $\alpha = \frac{C}{\lambda^{(n-1)/2}}$.

From our perspective, the notion of asymmetry is useful as it basically rules out narrow "spikes" (i.e. with relatively small volume) entering deeply into Ω. For example, let us also observe that convex domains trivially satisfy our asymmetry assumption with coefficient $\alpha = \frac{1}{2}$.

2. The basic estimate for general obstacles

With the above in mind, we consider any bounded α-asymmetric domain $\Omega \subset \mathbb{R}^n$ and a bounded obstacle domain D. We denote the first positive Dirichlet eigenvalue and eigenfunction of Ω by λ_1 and $\phi_{\lambda_1(\Omega)}$ respectively and let

$$M := \{x \in \Omega \mid \phi_{\lambda_1}(x) = \|\phi_{\lambda_1(\Omega)}\|_{L^\infty(\Omega)}\} \tag{3}$$

be the set of maximum points of $\phi_{\lambda_1(\Omega)}$.

Let us also put

$$\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x+D)). \tag{4}$$

Finally, for a given translate $x+D$ of the obstacle, let us set

$$\rho_x := \max_{y\in M} d(y, x+D), \tag{5}$$

measuring the maximum distance from a maximum point of $\phi_{\lambda_1(\Omega)}$ to the translate $x+D$.

Our main estimate is the following.

Theorem 2.1. *Let us fix a translate $(x+D)$ and assume that $\rho_x > 0$. Then*

$$\lambda_1(\Omega \setminus (x+D)) \leq \beta(\rho_x)\lambda_1(\Omega), \tag{6}$$

where β is a continuous decreasing function defined as

$$\beta(\rho) = \begin{cases} \beta_0 = \beta_0(n,\alpha), & \rho\sqrt{\lambda_1(\Omega)} > r_0 := r_0(n,\alpha), \\ \frac{c_0}{\rho^2\lambda_1(\Omega)}, & \rho\sqrt{\lambda_1(\Omega)} \leq r_0, \quad c_0 = c_0(n), \end{cases} \tag{7}$$

where $\beta_0 r_0 = c_0$.

We remark that, in particular, if ρ_x is of sub-wavelength order (i.e. $\lesssim \frac{1}{\sqrt{\lambda_1(\Omega)}}$), then $\lambda_1(\Omega \setminus (x+D)) \lesssim \frac{1}{\rho_x^2}$. If the obstacle D is convex, we can say more (see Theorem 4.1 below).

Proof of Theorem 2.1. The proof essentially exploits the fact that there are "almost inscribed" wavelength balls centered at maximum points of $\phi_{\lambda_1(\Omega)}$. To make this statement precise, we recall the following theorem from [6], which works for **all domains** in compact Riemannian manifolds of dimension $n \geq 3$ (planar domains are known to have wavelength inradius from the work of Hayman ([12])).

Theorem 2.2. *Let $\dim M \geq 3$, $\epsilon_0 > 0$ be fixed, Ω a domain inside M, and $x_0 \in \Omega$ be such that $|\varphi_\lambda(x_0)| = max_\Omega|\varphi_\lambda|$, where φ_λ is the ground-state Dirichlet eigenfunction of Ω. There exists $r_0 = r_0(\epsilon_0)$, such that*

$$\frac{|B_{r_0} \cap \Omega|}{|B_{r_0}|} \geq 1 - \epsilon_0, \tag{8}$$

where B_{r_0} denotes $B\left(x_0, \frac{r_0}{\sqrt{\lambda_1}}\right)$.

We note that the existence of such an "almost-inscribed" wavelength ball was first established by Lieb (see [15]), and followed by further contributions from Maz'ya–Shubin (see [18]). The latter brings to light the importance of small or "negligible capacities" in quantifying the "almost-inscribed"-ness (see in particular Theorem 1.1 and Subsection 5.1 of [18]). The main contribution of Theorem 2.2 is the specification of the location of the "almost-inscribed" wavelength ball. For completeness, recall that Theorem 2.2 relies on two main ingredients – namely, the Feynman–Kac formula and certain capacity estimates related to hitting probabilities of Brownian motion. We first establish that a Brownian particle starting at any max point of the ground-state eigenfunction has low probability of hitting the boundary of the domain; more precisely, such a probability is bounded above by $1 - \mathrm{e}^t$ at time scales $\sim \frac{t}{\lambda_1(\Omega)}$. On the other hand, by reducing t and r and keeping $\frac{t}{r^2} =$ constant, we are able to show that the particle has *comparatively* high probability of escaping a ball of radius $\sim \frac{r}{\sqrt{\lambda_1(\Omega)}}$ around the max point, which tells us that the "size" of the ball $B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}})$ outside the domain Ω is fairly small. This gives us a comparison of "sizes" of $B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}})$ and $B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}}) \setminus \Omega$ in terms of probability. Using the fact that the heat kernel is the transition density for Brownian motion, in [10] Grigor'yan and Saloff-Coste are able to estimate the hitting probabilities of relatively compact sets $K \subset M$ by a Brownian particle, in terms of pointwise heat kernel bounds on M and capacity of K. In our setting, we wish to use their results on the set $K := B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}}) \setminus \Omega$. Using in particular Remark 4.1 of [10], and isocapacitary inequalities due to Maz'ya (see [17], Section 2.2.3), we are able to translate a comparison of size in terms of probability into a comparison of size in terms of capacity (which fits nicely with the insights of [18]) and then in terms of volume, respectively. We refer to [6] for more details (see also [19] for an extension to Schrödinger operators along similar lines). We also note that it follows from the proof that in Theorem 2.2, r_0 can be taken as $r_0 = \epsilon_0^{\frac{n-2}{2n}}$, which is slightly better than the scaling in [15]. This has applications to the inner radius problem of nodal domains of Laplace eigenfunctions, see [5], [7] for more details.

Now, it is clear that under the α-asymmetry assumption, there exists an $r_0 := r_0(\alpha, n)$, such that around each maximum point $x_0 \in \Omega$ of $\phi_{\lambda_1(\Omega)}$ one can find a fully inscribed ball $B_{r_0/\sqrt{\lambda_1(\Omega)}}(x_0) \subseteq \Omega$. By the definition of ρ_x, it follows that we can find a maximum point $x_0 \in (\Omega \setminus (x+D))$ and an inscribed ball $B_{\rho_0}(x_0)$ where

$$\rho_0 := \min\left(\frac{r_0}{\sqrt{\lambda_1(\Omega)}}, \rho_x\right). \tag{9}$$

As the first eigenvalue is monotonic with respect to inclusion, we see that

$$\lambda_1(\Omega \setminus (x+D)) \leq \lambda_1(B_{\rho_0}(x_0)) = \frac{C}{\rho_0^2}, \tag{10}$$

where $C = C(n)$ is a universal constant.

Expressing the right-hand side of the last inequality in terms of $\lambda_1(\Omega)$, we define the function $\beta(\rho)$ as above.

This concludes the proof. □

Here, we have considered the obstacle problem in the case of Euclidean spaces, on reasonably well-behaved domains, and for the operator $-\Delta + \lambda_1(\Omega)$, as that seems to be the primary case of interest. However, we also include some remarks outlining some straightforward generalizations.

Remark 2.3. It is clear that removing capacity zero sets from α-asymmetric domains considered in Definition 1.2 will lead to the same conclusions. Indeed, in this situation, we will not be dealing with fully inscribed balls as above; instead, we will have balls whose first eigenvalue is comparable to the one of an inscribed one.

Remark 2.4. Also, in the setting of curved spaces, one has absolutely similar results for $\Omega \subseteq M$, where (M, g) is a smooth compact Riemannian manifold, if we allow the constants to depend on the dimension, asymmetry, and the metric g.

Remark 2.5. Lastly, it is clear that the results of [19] allow us to extend our discussion here from operators of the form $-\Delta + \lambda_1(\Omega)$ to Schrödinger operators of the form $-\Delta + V$, where V is bounded above. The conclusions are analogous with $\lambda_1(\Omega)$ replaced by $\|V\|_{L^\infty}$ and the proofs are identical.

Now, as an immediate implication of Theorem 2.1, we have the following corollary.

Corollary 2.6. *Suppose that $\mu_\Omega = C_0\lambda_1(\Omega)$, where $C_0 > \frac{c_0}{r_0^2}$ is a given fixed constant and c_0, r_0 are the constants in Theorem 2.1. Then, for a maximizer $\bar{x} + D$ of μ_Ω we have*

$$\rho_{\bar{x}} \leq \beta^{-1}(C_0). \tag{11}$$

In particular, if C_0 is large,

$$\rho_{\bar{x}} \lesssim \frac{1}{\sqrt{C_0\lambda_1(\Omega)}}. \tag{12}$$

In other words, the above corollary can be interpreted as follows: either μ_Ω is comparable to $\lambda_1(\Omega)$, or the maximum points of $\phi_{\lambda_1(\Omega)}$ are near the maximizer sets $\bar{x} + D$ of μ_Ω.

We note that the localization in the Corollary above gets better when C_0 is large. By Faber–Krahn's inequality, straightforward examples with large C_0 are domains Ω for which $|\Omega \setminus (x+D)|$ is sufficiently small for some x.

Particularly, for bounded convex domains in $\mathbb{R}^n$, by a theorem of Brascamp–Lieb (see Section 6 of [1] in particular), the level sets of $\phi_{\lambda_1(\Omega)}$ are convex. Since $\phi_{\lambda_1(\Omega)}$ is real analytic and it can be assumed positive on $\Omega \setminus \partial\Omega$ without loss of generality, this means that it has a unique point of maximum. So, in this setting, our result heuristically says that if the removal of a ball B_r has a "significant effect" on the vibration of $\Omega \setminus B_r$, then B_r must be centered quite close to the max point of the ground-state Dirichlet eigenfunction ϕ_{λ_1} of the domain Ω, where the bound on ρ_x gives the quantitative relation between the "effect" and the order of "closeness". In a sense, this can be seen to be complementary to Corollary II.3 of [11].

3. Inscribed balls and distribution of $\phi_{\lambda_1(\Omega)}$

Further, we specify our results to the obstacle being a ball D. We point out a few statements related to the connection between the distribution of $\phi_{\lambda_1(\Omega)}$ and the possibility to inscribe a large ball at a given point x in Ω.

First, by Theorem 2.2 above, we immediately have the following observation.

Proposition 3.1. *Let Ω be α-asymmetric and let* $\operatorname{inrad}(\Omega)$ *denote the inner radius of Ω. If x_0 is a point of maximum of $\phi_{\lambda_1(\Omega)}$, then there exists an inscribed ball $B_{C\,\operatorname{inrad}(\Omega)}(x_0) \subseteq \Omega$, where $C = C(n, \alpha)$.*

Proof of Proposition 3.1. We observe that by the results of [16], α-asymmetric domains Ω satisfy

$$\frac{C_1(\alpha, n)}{\sqrt{\lambda_1(\Omega)}} \leq \operatorname{inrad}(\Omega) \leq \frac{C_2(n)}{\sqrt{\lambda_1(\Omega)}}. \tag{13}$$

Now, it follows from our Theorem 2.2 (see [6]) that there exists an inscribed wavelength ball at the max point x_0, which concludes the proof. □

In particular, the last proposition applies for convex domains. We mention in this connection that localization results for maximum points of $\phi_{\lambda_1(\Omega)}$ in case Ω in planar convex domains can be found in the work of Grieser–Jerison (see [9]).

On the other hand, it is natural to ask how large is $\phi_{\lambda_1(\Omega)}$ at points admitting a large inscribed ball. For reasonably nicely behaved domains, we have the following:

Corollary 3.2. *Let Ω be a $C^{2,\beta}$-regular α-asymmetric domain and let $\phi_{\lambda_1(\Omega)}$ be normalized so that $\|\phi_{\lambda_1(\Omega)}\|_{L^\infty(\Omega)} = 1$. Suppose that for $\tilde{x} \in \Omega$ there exists a maximal inscribed ball $B_r(\tilde{x}) \subseteq \Omega$ where $r := c\,\operatorname{inrad}(\Omega)$ for some $0 < c \leq 1$, such that $\frac{|\Omega \setminus B_r(\tilde{x})|}{|\Omega|}$ is sufficiently small. Then*

$$\phi_\lambda(\tilde{x}) > C, \tag{14}$$

where $C = C(|\Omega|, \partial\Omega, c, n)$.

Analogously, one can show a similar statement by demanding that $|B_r(\tilde{x}) \cap \Omega|$ is sufficiently large in comparison to $|\Omega|$.

Proof of Corollary 3.2. Let us first suppose that

$$|\Omega| = \kappa r^n, \quad \kappa > \omega_n, \tag{15}$$

where ω_n is the volume of a ball of radius 1. We use the Faber–Krahn inequality to obtain

$$\begin{aligned}\lambda_1(\Omega \setminus B_r(\tilde{x})) &\geq \frac{C}{|\Omega \setminus B_r(\tilde{x})|^{2/n}} = \frac{C}{(|\Omega| - \omega_n r^n)^{2/n}} = \frac{C}{(\kappa - \omega_n)^{2/n} r^2} = \\ &= \frac{C}{(\kappa - \omega_n)^{2/n}(c\,\operatorname{inrad}(\Omega))^2} \geq \frac{C C_2(n)}{c^2(\kappa - \omega_n)^{2/n}} \lambda_1(\Omega) =: \tilde{C}_0 \lambda_1(\Omega).\end{aligned} \tag{16}$$

By assumption, $\tilde{C}_0$ is sufficiently large, i.e., in particular, $\tilde{C}_0 > \frac{c_0}{r_0^2}$, so we may apply Corollary 2.6 to obtain that

$$\rho_{\tilde{x}} \leq \beta^{-1}(\tilde{C}_0) = \sqrt{\frac{c_0}{\tilde{C}_0 \lambda_1(\Omega)}}. \tag{17}$$

On the other hand, the Schauder a priori estimates up to the boundary for $\phi_{\lambda_1(\Omega)}$ (see [8], Theorem 6.6) yield the existence of $\gamma = \gamma(\Omega, n)$, such that

$$\|\nabla \phi_{\lambda_1(\Omega)}\|_{L^\infty(\Omega)} \leq \gamma(\Omega, n)\sqrt{\lambda_1(\Omega)}. \tag{18}$$

As by assumption $\phi_{\lambda_1(\Omega)}(x_0) = 1$ and $\tilde{C}_0$ is sufficiently large, then

$$\phi_{\lambda_1(\Omega)}(\tilde{x}) \geq C = C(c_0, \tilde{C}_0, \gamma), \tag{19}$$

which concludes the claim. □

4. Relation between maximum points and convex obstacles

Note that Theorem 2.1 holds for arbitrary obstacles and gives a bound on the distance ρ_x to maximum points of $\phi_{\lambda_1(\Omega)}$. However, it is desirable to deduce that $\rho_x = 0$, i.e. maximizers actually contain the maximum points of $\phi_{\lambda_1(\Omega)}$.

From Proposition 3.1 and Theorem 2.1, we deduce the following:

Theorem 4.1. *Let D be a convex obstacle, and $\bar{x} + D$ maximize $\lambda_1(\Omega \setminus (x + D))$. Then there exists a constant $C_0 = C_0(\alpha, n)$ such that if $\lambda_1(\Omega \setminus (\bar{x} + D)) \geq C\lambda_1(\Omega)$ for some $C \geq C_0$, then $\rho_{\bar{x}} = 0$.*

In other words, either $\mu_\Omega \sim \lambda_1(\Omega)$ or $\rho_{\bar{x}} = 0$.

Proof. To the contrary let us suppose that $\rho_{\bar{x}} = d(\bar{x} + D, x_0) > 0$ where x_0 is a maximum point of $\phi_{\lambda_1(\Omega)}$ and $\lambda_1(\Omega \setminus (\bar{x} + D)) \geq C\lambda_1(\Omega)$ for an arbitrary large $C > 0$.

We apply the statement of Proposition 3.1 and deduce that there is a wavelength inscribed ball B at x_0. As D is a convex domain, we can find a wavelength half-ball $B^{1/2} \subset \Omega \setminus (\bar{x} + D)$ containing x_0. By the assumption and eigenvalue monotonicity with respect to inclusion:

$$C\lambda_1(\Omega) \leq \lambda_1(\Omega \setminus (\bar{x} + D)) \leq \lambda_1(B^{1/2}) \leq \frac{C_1}{(\mathrm{inrad}(\Omega))^2} = C_2\lambda_1(\Omega), \tag{20}$$

where $C_2 = C_2(n, \alpha)$. Taking C sufficiently large, we get a contradiction. □

It is clear that for explicit applications, particularly in the case of convex domains, Theorem 4.1 is dependent on a precise knowledge of the location of the maximum point of $\phi_{\lambda_1(\Omega)}$. Localization of the maximum point of $\phi_{\lambda_1(\Omega)}$ (or more generally, the "hot spot") is a problem that is far from being settled. Here we take the space to augment Theorem 4.1 with the recent results of [2].

First we recall the definition of the "heart" of a convex body Ω. The following intuitive definition appears in [3], and it is equivalent to the (more technical) definition presented in [2].

Definition 4.2. Let P be a hyperplane in $\mathbb{R}^n$ that intersects Ω so that $\Omega \setminus P$ is the union of two components located on either side of P. The domain Ω is said to have the interior reflection property with respect to P if the reflection through P of one of these subsets, denoted Ω_s, is contained in Ω, and in that case P is called a hyperplane of interior reflection for Ω. When Ω is convex, the heart of Ω, denoted by $\heartsuit(\Omega)$, is defined as the intersection of all such $\Omega \setminus \Omega_s$ with respect to the hyperplanes of interior reflection of Ω.

The following result is contained in Proposition 4.1 of [2].

Proposition 4.3 *([2]). The unique maximum point x_0 of $\phi_{\lambda_1(\Omega)}$ is contained in $\heartsuit(\Omega)$. Furthermore, x_0 is contained in the interior of $\heartsuit(\Omega)$, if the latter is non-empty.*

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