C. R. Acad. Sci. Paris, Ser. I 356 (2018) 125–128

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory

Multiplicative functions additive on generalized pentagonal numbers

Les fonctions multiplicatives qui sont additives sur les nombres pentagonaux généralisés

Byungchan Kim [a], Ji Young Kim [b], Chong Gyu Lee [c], Poo-Sung Park [d]

[a] *School of Liberal Arts, Seoul National University of Science and Technology, 232 Gongneung-ro, Nowon-gu, Seoul 01811, Republic of Korea*
[b] *Department of Mathematical Sciences, Seoul National University, 1 Gwanak-ro, Gwanak-Gu, Seoul 08826, Republic of Korea*
[c] *Department of Mathematics, Soongsil University, 369 Sangdo-ro, Dongjak-gu, Seoul 06978, Republic of Korea*
[d] *Department of Mathematics Education, Kyungnam University, Changwon-si, Gyeongsangnam-do 51767, Republic of Korea*

ARTICLE INFO

Article history:
Received 15 October 2017
Accepted after revision 22 December 2017
Available online 9 January 2018

Presented by the Editorial Board

ABSTRACT

We prove that the set GP of all nonzero generalized pentagonal numbers is an additive uniqueness set; if a multiplicative function f satisfies the equation

$$f(a+b) = f(a) + f(b),$$

for all $a, b \in GP$, then f is the identity function.

RÉSUMÉ

Nous prouvons que l'ensemble GP de tous les nombres pentagonaux généralisés non nuls est un ensemble d'unicité additive ; si une fonction multiplicative f satisfait l'équation

$$f(a+b) = f(a) + f(b),$$

pour tous $a, b \in GP$, alors f est la fonction identité.

1. Introduction

An arithmetic function $f : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. In 1992, Spiro proved that if a multiplicative function f satisfies $f(p_0) \neq 0$ for some prime p_0 and

E-mail addresses: bkim4@seoultech.ac.kr (B. Kim), jykim98@snu.ac.kr, jykim.math@gmail.com (J.Y. Kim), cglee@ssu.ac.kr (C.G. Lee), pspark@kyungnam.ac.kr (P.-S. Park).

https://doi.org/10.1016/j.crma.2017.12.011
1631-073X/

$$f(p+q) = f(p) + f(q) \text{ for all primes } p \text{ and } q,$$

then f is the identity function [9]. More generally, Spiro asked which subset E of $\mathbb{Z}^+$ could determine an arithmetic function f uniquely in $\mathcal{S}$ under conditions

$$f(a+b) = f(a) + f(b) \text{ for all } a, b \in E,$$

where $\mathcal{S}$ is a set of arithmetic functions. Such a set E is called an *additive uniqueness set* for $\mathcal{S}$ following Spiro's theme.

After Spiro's work, this interesting subject has been studied and extended in many directions (see [1], [2], [3], [4], [5], [6], [7], and [8], for example). In particular, Chung and Phong [3] showed that the set of all triangular numbers is an additive uniqueness set for multiplicative functions, while Chung [2] showed that the set of square numbers is not an additive uniqueness set for multiplicative functions.

So it is natural to examine pentagonal numbers. The nonzero *generalized pentagonal numbers* are the integers obtained by the formula

$$P_n = \frac{n(3n-1)}{2},$$

with $n = \pm 1, \pm 2, \pm 3, \ldots$. Let GP be the set of nonzero generalized pentagonal numbers;

$$GP = \{1, 2, 5, 7, 12, 15, 22, 26, 35, \ldots\}.$$

In this article, we prove that the set GP is an additive uniqueness set for multiplicative functions.

Theorem 1.1. *If a multiplicative function $f : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ satisfies*

$$f(a+b) = f(a) + f(b),$$

for arbitrary generalized pentagonal numbers a, b, then f is the identity function.

2. Proof of Theorem 1.1

We will prove the Theorem using induction on n. We assume that $f(k) = k$ for all $k < n$. Since f is multiplicative, it suffices to prove the case that n is a prime power. For notational convenience, we let $P_m^\epsilon = \frac{m(3m+\epsilon)}{2}$ where $\epsilon \in \{-1, +1\}$.

Proposition 2.1. *P_n^ϵ is a product of two coprime numbers.*

Proof. If n is even, then

$$P_n^\epsilon = \frac{n}{2} \cdot (3n+\epsilon),$$

where $\gcd(\frac{n}{2}, 3n+\epsilon) \le \gcd(n, 3n+\epsilon) = \gcd(n, \epsilon) = 1$.

When n is odd,

$$P_n^\epsilon = n \cdot \frac{3n+\epsilon}{2},$$

where $\gcd(n, \frac{3n+\epsilon}{2}) \le \gcd(n, 3n+\epsilon) = 1$. $\square$

Lemma 2.2. *Let $p \neq 5$ be a prime and let $r \in \mathbb{Z}^+$. Then there are $a, b \in GP$ and $\lambda \in \mathbb{Z}^+$ such that*

$$\lambda p^r = a + b,$$

where $\gcd(\lambda, p) = 1$ with $\lambda < p^r$. Moreover, a and b are products of coprime numbers which are smaller than p^r. Furthermore, the same statement is true for $p = 5$ with $r > 1$.

Proof. We split the proof into four cases: $p = 2$, $p = 3$, $p = 5$, and $p \ge 7$.

Case $p = 2$**:** Since $2^r \equiv \epsilon \pmod 3$, we can let $2^r = 3m + \epsilon$ for a positive odd integer m. Then

$$P_m^\epsilon + P_m^\epsilon = \frac{m(3m+\epsilon)}{2} + \frac{m(3m+\epsilon)}{2} = m(3m+\epsilon) = m \cdot 2^r,$$

where the largest factor $\frac{3m+\epsilon}{2}$ of P_m^ϵ is smaller than 2^r. By letting $a = P_m^\epsilon$, $b = P_m^\epsilon$ and $\lambda = m = \frac{2^r - \epsilon}{3}$, we get a and b whose factors are smaller than 2^r and $\gcd(\lambda, 2) = 1$. Hence, the $p = 2$ case follows.

Case $p=3$**:** When r is even, we let $r=2\ell$ for some $\ell\in\mathbb{Z}^+$. Since $3^2\equiv -1 \pmod{10}$, we may assume that $3^{2\ell}=10m+\epsilon$ for some $m\in\mathbb{Z}^+$. Let $n=6m+\epsilon$. Then

$$\begin{aligned} P_n^{-\epsilon}+P_{2m}^{\epsilon} &= \frac{(6m+\epsilon)(18m+2\epsilon)}{2}+\frac{2m(6m+\epsilon)}{2} \\ &= (6m+\epsilon)(10m+\epsilon)=(6m+\epsilon)\cdot 3^{2\ell}=(6m+\epsilon)\cdot 3^r, \end{aligned}$$

where $\max(6m+\epsilon, 9m+\epsilon, m, 6m+\epsilon)<10m+\epsilon=3^r$. As $\gcd(6m+\epsilon,3)=1$, by setting $a=P_n^{-\epsilon}$, $b=P_{2m}^{\epsilon}$ and $\lambda=6m+\epsilon$, we have the desirable result.

When r is odd, we let $r=2\ell-1$ for some $\ell\in\mathbb{Z}^+$. For $n=3^{\ell-1}$, we find that

$$P_n^{\epsilon}+P_n^{-\epsilon}=\frac{n(3n+\epsilon)}{2}+\frac{n(3n-\epsilon)}{2}=3n^2=1\cdot 3^{2\ell-1}=1\cdot 3^r,$$

where $\max(n, \frac{3n\pm\epsilon}{2})<3n^2=3^r$. By choosing $a=P_n^{\epsilon}$, $b=P_n^{-\epsilon}$ and $\lambda=1$ for this case, we complete the proof of the $p=3$ case.

Case $p=5$**:** When $r=2k$, we set $n=5^k$. Then, we find that

$$P_n^{-1}+P_n^{+1}=\frac{n(3n-1)}{2}+\frac{n(3n+1)}{2}=3n^2=3\cdot 5^r,$$

where $\max(n, \frac{3n\pm1}{2})<n^2=5^r$. Thus, we can set $a=P_n^{-1}$, $b=P_n^{+1}$ and $\lambda=3$ for this case.

If $r=4k+3$, we set $5^r=13n+8$ for an odd integer n. Then

$$\begin{aligned} P_{n+1}^{-1}+P_{8n+5}^{-1} &= \frac{(n+1)(3n+2)}{2}+(8n+5)(12n+7) \\ &= \frac{3}{2}(5n+3)(13n+8)=\frac{3}{2}(5n+3)\cdot 5^r, \end{aligned}$$

where $\max(\frac{n+1}{2}, 3n+2, 8n+5, 12n+7)<13n+8=5^r$. Since $\frac{3}{2}(5n+3)<13n+8=5^r$ and $\gcd(5n+3,5)=1$, we can set $a=P_{n+1}^{-1}$, $b=P_{8n+5}^{-1}$, and $\lambda=\frac{3}{2}(5n+3)$ for this case.

Finally, if $r=4k+1>1$, then, $5^r=39n+5$ such that n is a multiple of 10. We observe that

$$\begin{aligned} P_{15n+2}^{-1}+P_{23n+3}^{+1} &= \frac{5(15n+2)(9n+1)}{2}+\frac{(23n+3)(69n+10)}{2} \\ &= (29n+4)(39n+5)=(29n+4)\cdot 5^r, \end{aligned}$$

where $\max(\frac{15n+2}{2}, 9n+1, 23n+3, \frac{69n+10}{2})<39n+5=5^r$. Since n is a multiple of 5, we see that $\gcd(29n+4,5)=1$. Therefore, by setting $a=P_{15n+2}^{-1}$, $b=P_{23n+3}^{+1}$ and $\lambda=29n+4$, we obtain the desirable result.

Case $p\ge 7$**:** Since $p\ge 7$, $p^r\equiv\epsilon \pmod{10}$ or $p^r\equiv 3\epsilon \pmod{10}$.

(1) $p^r\equiv\epsilon \pmod{10}$: Let $p^r=10m+\epsilon$ and $n=6m+\epsilon$. We see that $\gcd(6m+\epsilon,10m+\epsilon)=1$ and $\max(6m+\epsilon, m, 9m+\epsilon)<10m+\epsilon=p^r$. Thus, the equality

$$\begin{aligned} P_n^{-\epsilon}+P_{2m}^{\epsilon} &= \frac{(6m+\epsilon)(18m+2\epsilon)}{2}+\frac{2m(6m+\epsilon)}{2} \\ &= (6m+\epsilon)(10m+\epsilon)=(6m+\epsilon)\cdot p^r \end{aligned}$$

implies that the desirable result follows by setting $a=P_n^{-\epsilon}$, $b=P_{2m}^{\epsilon}$, and $\lambda=6m+\epsilon$.

(2) $p^r\equiv 3\epsilon \pmod{10}$: Since $p\ne 3$, $p^r\equiv 7\epsilon \pmod{30}$ or $p^r\equiv 13\epsilon \pmod{30}$.

When $p^r\equiv 7\epsilon \pmod{30}$, let $p^r=30m+7\epsilon$ and let $n=4m+\epsilon$. Then we obtain that

$$\begin{aligned} P_n^{\epsilon}+P_{2n}^{-\epsilon} &= \frac{n(3n+\epsilon)}{2}+\frac{2n(6n-\epsilon)}{2} \\ &= \frac{n(15n-\epsilon)}{2}=(4m+\epsilon)(30m+7\epsilon)=(4m+\epsilon)\cdot p^r. \end{aligned}$$

We see that $\max(n, \frac{3n+\epsilon}{2}, 6n-\epsilon)=24m+5\epsilon<30m+7\epsilon=p^r$ and $\gcd(4m+\epsilon, 30m+7\epsilon)=1$. Thus we can choose $a=P_n^{\epsilon}$, $b=P_{2n}^{-\epsilon}$ and $\lambda=4m+\epsilon$ to conclude the case.

When $p^r \equiv 13\epsilon \pmod{30}$, let $p^r = 30m + 13\epsilon$ and let $n = 2m + \epsilon$. Then we observe that

$$P_n^{-\epsilon} + P_{3n}^{-\epsilon} = \frac{n(3n-\epsilon)}{2} + \frac{3n(9n-\epsilon)}{2}$$
$$= n(15n - 2\epsilon) = (2m+\epsilon)(30m+13\epsilon) = (2m+\epsilon)\cdot p^r,$$

where $\max(n, \frac{3n-\epsilon}{2}, 3n, \frac{9n-\epsilon}{2}) = 9m + 4\epsilon < 30m + 13\epsilon = p^r$ and $\gcd(2m+\epsilon, 30m+13\epsilon) = 1$. By letting $a = P_n^{-\epsilon}$, $b = P_{3n}^{-\epsilon}$ and $\lambda = 2m + \epsilon$, we obtain the desirable result. □

Proof of Theorem 1.1. We will show $f(n) = n$ for any positive integer n and will use the induction on n.

(1) By the multiplicative property of f, we get $f(1) = 1$.
(2) By the additive property of f on GP, we get

$$f(2) = f(1) + f(1) = 2,\ f(3) = f(1) + f(2) = 3,\ f(4) = f(1) + f(3) = 4.$$

(3) Because $f(1) + f(5) = f(6) = f(2)f(3)$, we get $f(5) = 5$.
(4) Let n be an integer larger than 5. Suppose that $f(k) = k$ for all $k < n$. The multiplicativity of f and the factorization of $n = \prod_{i=1}^{\ell} p_i^{e_i}$ says

$$f(n) = \prod_{i=1}^{\ell} f(p_i^{e_i}).$$

If $\ell \geq 2$, then $p_i^{e_i} < n$ for all i and hence the induction hypothesis guarantees that $f(p_i^{e_i}) = p_i^{e_i}$. So $f(n) = n$.
If $\ell = 1$, then $n = p^e$ for some prime p and a positive integer e. Lemma 2.2 says

$$\lambda \cdot p^e = a + b,$$

where a and b are generalized pentagonal numbers of which coprime factors are smaller than p^e, $\lambda < p^e$ and $\gcd(\lambda, p^e) = 1$. Thus the multiplicativity and additivity on GP of f implies that

$$f(\lambda)\cdot f(n) = f(a) + f(b),$$

where $f(\lambda) = \lambda$, $f(a) = a$ and $f(b) = b$ by the induction hypothesis. So we get the desirable result. □

3. A concluding remark

One might ask whether the set $\{\frac{n(3n-1)}{2} : n \in \mathbb{Z}^+\}$ of nonzero ordinary pentagonal numbers is an additive uniqueness set for multiplicative functions. As it has less possible additive combinations available, it is a much harder problem. Moreover, it is connected to a deep general Catalan's conjecture to find integer solutions r and s for $p^r - q^s = k$, where p and q are distinct primes and k is a positive integer. Our work on ordinary pentagonal numbers is on progress and we hope we can address this case soon.

Acknowledgements

The authors are grateful to the referee for his or her valuable comments which improve the exposition of this paper. In particular, Lemma 2.2 is based on the referee's suggestion. The authors are also thankful to the workshop "Trends in Number Theory", where this project was initiated. The authors were supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1A09917344, NRF-2017R1D1A1B03028905, NRF-2016R1D1A1B01009208) and by the Ministry of Science and ICT (NRF-2017R1A2B1010761).

References

[1] B. Bašić, Characterization of arithmetic functions that preserve the sum-of-squares operation, Acta Math. Sin. 30 (2014) 689–695.
[2] P.V. Chung, Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$, Math. Slovaca 46 (1996) 165–171.
[3] P.V. Chung, B.M. Phong, Additive uniqueness sets for multiplicative functions, Publ. Math. (Debr.) 55 (1999) 237–243.
[4] A. Dubickas, P. Šarka, On multiplicative functions which are additive on sums of primes, Aequ. Math. 86 (2013) 81–89.
[5] J.-H. Fang, A characterization of the identity function with equation $f(p+q+r) = f(p) + f(q) + f(r)$, Combinatorica 31 (2011) 697–701.
[6] P.-S. Park, Multiplicative functions commutable with sums of squares, Int. J. Number Theory (2018), https://doi.org/10.1142/S179304211850029X, in press.
[7] P.-S. Park, On k-additive uniqueness of the set of squares for multiplicative functions, Aequ. Math. (2018), https://doi.org/10.1007/s00010-017-0517-4, in press.
[8] P.-S. Park, On multiplicative functions which are additive on almost primes, Publ. Math. (Debr.) (2018), in press.
[9] C.A. Spiro, Additive uniqueness sets for arithmetic functions, J. Number Theory 42 (1992) 232–246.