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Algebraic geometry

A Tannakian classification of torsors on the projective line

Une classification tannakienne des torseurs sur la droite projective

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ABSTRACT

In this small note, we present a Tannakian proof of the theorem of Grothendieck–Harder on the classification of torsors under a reductive group on the projective line over a field.

RÉSUMÉ

Nous présentons dans cette courte Note une démonstration tannakienne du théorème de Grothendieck–Harder sur la classification des torseurs pour un groupe réductif, sur la droite projective définie sur un corps.

1. Introduction

Let k be a field, let G/k be a reductive group and let $\mathbb{P}^1_k$ be the projective line over k. In this small note we present a Tannakian proof of the classification of G-torsors on $\mathbb{P}^1_k$, thereby reproving known results of A. Grothendieck [11] and G. Harder [15, Satz 3.4.] (over arbitrary fields). To state our main theorem, we denote by

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m))$$

the set of isomorphism classes of exact tensor functors

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m).$$

Theorem 1.1 *(cf. Theorem 3.3, Proposition 3.4). There exists a canonical bijection*

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \cong H^1_{\text{ét}}(\mathbb{P}^1_k, G).$$

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In particular, there exists a canonical bijection

$$\mathrm{Hom}(\mathbb{G}_m, G)/G(k) \cong H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G).$$

If $A \subseteq G$ denotes a maximal split torus, then

$$\mathrm{Hom}(\mathbb{G}_m, G)/G(k) \cong X_*(A)^+$$

is in bijection with the set of dominant cocharacters of $A \subseteq G$ (for the choice of some minimal parabolic of G), which gives a very concrete description of the set $H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G)$ (cf. Corollary 3.5). Our proof of Theorem 1.1, which originated in questions about torsors over the Fargues–Fontaine curve (cf. [1]), is based on the Tannakian description of G-torsors (cf. Lemma 3.1), the Tannakian theory of filtered fiber functors (cf. [19]), the canonicity of the Harder–Narasimhan filtration (cf. Lemma 2.2) and, most importantly, the well-known understanding of the category $\mathrm{Bun}_{\mathbb{P}^1_k}$ of vector bundles on $\mathbb{P}^1_k$ (cf. Theorem 2.1). In particular, we use crucially the fact that

$$H^1_{\text{ét}}(\mathbb{P}^1_k, \mathcal{E}) = 0$$

for $\mathcal{E}$ a semistable vector bundle on $\mathbb{P}^1_k$ of slope > 0.

In a last section, we mention applications of Theorem 1.1 to the computation of the Brauer group of $\mathbb{P}^1_k$ (avoiding Tsen's theorem) and to the Birkhoff–Grothendieck decomposition of $G(k((t)))$.

2. Vector bundles on $\mathbb{P}^1_k$

Let k be an arbitrary field. We recall, in a more canonical form, the classification of vector bundles on the projective line $\mathbb{P}^1_k$ due to A. Grothendieck (cf. [11]). Let

$$\mathrm{Rep}_k(\mathbb{G}_m)$$

be the category of finite-dimensional representations of the multiplicative group $\mathbb{G}_m$ over k. More concretely, the category $\mathrm{Rep}_k(\mathbb{G}_m)$ is equivalent to the Tannakian category of finite-dimensional $\mathbb{Z}$-graded vector spaces over k.

Over $\mathbb{P}^1_k$ there is the canonical $\mathbb{G}_m$-torsor

$$\eta\colon \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k,\ (x_0, x_1) \mapsto [x_0 : x_1],$$

also called the "Hopf bundle". Given a representation $V \in \mathrm{Rep}_k(\mathbb{G}_m)$, the contracted product

$$\mathcal{E}(V) := \mathbb{A}^2_k \setminus \{0\} \times^{\mathbb{G}_m} V \to \mathbb{P}^1_k$$

defines a (geometric) vector bundle over $\mathbb{P}^1_k$. The well-known classification of the category

$$\mathrm{Bun}_{\mathbb{P}^1_k}$$

of vector bundles on $\mathbb{P}^1_k$ can now be phrased in the following way.

Theorem 2.1. *The functor*

$$\mathcal{E}(-)\colon \mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

is an exact, faithful tensor functor inducing a bijection on isomorphism classes.

However, the functor $\mathcal{E}(-)$ is not an equivalence. For example, the category $\mathrm{Rep}_k(\mathbb{G}_m)$ is abelian, while $\mathrm{Bun}_{\mathbb{P}^1_k}$ is not. Specifically this is caused by non-zero morphisms of semistable vector bundles of different slopes. We recall that, for X, a smooth projective curve over k the slope $\mu(\mathcal{E}) \in \mathbb{Q} \cup \{\infty\}$ of a vector bundle $\mathcal{E}$ of rank r on X is defined by

$$\mu(\mathcal{E}) = \frac{\deg(\Lambda^r \mathcal{E})}{r}$$

and that $\mathcal{E}$ is called semistable, if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for every subbundle $0 \neq \mathcal{F} \subseteq \mathcal{E}$. It can be checked that for some fixed $\mu \in \mathbb{Q}$ the category Bun^{μ}_X of semistable vector bundles on X of slope μ or ∞ is abelian and that each vector bundle $\mathcal{E}$ admits a canonical filtration, the so-called "Harder–Narasimhan filtration",

$$0 = \mathcal{E}_n \subseteq \mathcal{E}_{n-1} \subseteq \ldots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 := \mathcal{E}$$

such that each graded piece $\mathcal{E}_i/\mathcal{E}_{i+1}$ is a semistable vector bundle of some slope μ_i and $\mu_n \geq \mu_{n+1} \geq \ldots \geq \mu_0$ (cf. [16, Section 1.3]). In the case of $X = \mathbb{P}^1_k$, these results have a very concrete form. Namely, a vector bundle $\mathcal{E}$ is semistable if and only if

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1_k}(n)$$

is isomorphic to a direct some of copies of the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(n)$ with $n = \mu(\mathcal{E})$. The Harder–Narasimhan filtration of a vector bundle $\mathcal{E}(V)$ with $V \in \mathrm{Rep}_k(\mathbb{G}_m)$ can therefore be described as follows. Write

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

with $\mathbb{G}_m$ acting on V_i by the character[1]

$$\mathbb{G}_m \to \mathbb{G}_m,\ z \mapsto z^{-i}$$

and set

$$\mathrm{fil}^i(V) := \bigoplus_{j \geq i} V_j$$

for $i \in \mathbb{Z}$. Then the Harder–Narasimhan filtration of $\mathcal{E} := \mathcal{E}(V)$ is given by

$$\ldots \subseteq \mathrm{HN}^{i+1}(\mathcal{E}) \subseteq \mathrm{HN}^i(\mathcal{E}) \subseteq \ldots \subseteq \mathcal{E}$$

where

$$\mathrm{HN}^i(\mathcal{E}) := \mathcal{E}(\mathrm{fil}^i(V)).$$

Let us denote by

$$\mathrm{FilBun}_{\mathbb{P}^1_k}$$

the category of filtered vector bundles on $\mathbb{P}^1_k$, i.e. the category of vector bundles $\mathcal{E}$ on $\mathbb{P}^1_k$ together with a separated and exhaustive decreasing filtration $\mathrm{Fil}^\bullet(\mathcal{E})$ by locally direct summands $\mathrm{Fil}^i(\mathcal{E}) \subseteq \mathcal{E}$ (cf. [19, Chapter 4]). The category $\mathrm{FilBun}_{\mathbb{P}^1_k}$ has a natural exact structure by considering sequences

$$0 \to (\mathcal{E}, \mathrm{Fil}^\bullet(\mathcal{E})) \to (\mathcal{E}', \mathrm{Fil}^\bullet(\mathcal{E}')) \to (\mathcal{E}'', \mathrm{Fil}^\bullet(\mathcal{E}'')) \to 0$$

of filtered vector bundles such that the restriction to each Fil^i remains exact.

Lemma 2.2. *Sending a vector bundle $\mathcal{E}$ to the filtered vector bundle $\mathcal{E}$ with the Harder–Narasimhan filtration* $\mathrm{HN}^\bullet(\mathcal{E})$ *defines a fully faithful tensor functor*

$$\mathrm{HN}\colon \mathrm{Bun}_{\mathbb{P}^1_k} \to \mathrm{FilBun}_{\mathbb{P}^1_k}$$

into the exact tensor category of filtered vector bundles on $\mathbb{P}^1_k$.

Proof. This is clear from the description of the Harder–Narasimhan filtration. □

We remark that the functor HN is *not* exact as one sees for example by looking at the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_k}(-1) \to \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k} \to \mathcal{O}_{\mathbb{P}^1_k}(1) \to 0$$

on $\mathbb{P}^1_k$.

Sending a filtered vector bundle $(\mathcal{E}, F^\bullet)$ to the associated graded vector bundle

$$\mathrm{gr}(\mathcal{E}) := \bigoplus_{i \in \mathbb{Z}} F^i\mathcal{E}/F^{i+1}\mathcal{E}$$

[1] The sign is explained by the fact that the standard representation $z \mapsto z$ of $\mathbb{G}_m$ is sent by $\mathcal{E}(-)$ to $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ and not to $\mathcal{O}_{\mathbb{P}^1_k}(1)$.

defines an exact tensor functor

$$\mathrm{gr}\colon \mathrm{FilBun}_{\mathbb{P}^1_k} \to \mathrm{GrBun}_{\mathbb{P}^1_k}$$

(cf. [19, Chapter 4]).

The following lemma is immediate from Theorem 2.1, Lemma 2.2 and the fact that

$$H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong k.$$

Lemma 2.3. *The composite functor*

$$\mathrm{Rep}_k(\mathbb{G}_m) \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{gr}} \mathrm{GrBun}_{\mathbb{P}^1_k}$$

is an equivalence of exact categories from $\mathrm{Rep}_k(\mathbb{G}_m)$ *onto its essential image, which consists of graded vector bundles*

$$\mathcal{E} = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}^i$$

such that each $\mathcal{E}^i$ *is semistable of slope* i.

3. Torsors over $\mathbb{P}^1_k$

Let G/k be an arbitrary reductive group. In this section, we want to classify G-torsors on $\mathbb{P}^1_k$ for the étale topology. For this, we keep the notation from the last section. In particular, there is the functor

$$\mathcal{E}(-)\colon \mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

from Theorem 2.1.

In order to apply the formulations from the previous section, we need a more bundle theoretic interpretation of G-torsors (for the étale topology). This is achieved by the Tannakian formalism (cf. [6]).

Lemma 3.1. *Let* S *be a scheme over* k. *Sending a* G*-torsor* $\mathcal{P}$ *over* S *to the exact tensor functor*

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S,\ V \mapsto \mathcal{P} \times^G (V \otimes_k \mathcal{O}_S)$$

defines an equivalence from the groupoid of G*-torsors to the groupoid of exact tensor functors from* $\mathrm{Rep}_k(G)$ *to* Bun_S. *The inverse equivalence sends an exact tensor functor* $\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S$ *the* G*-torsor* $\mathrm{Isom}^{\otimes}(\omega_{\mathrm{can}}, \omega)$ *of isomorphisms of* ω *to the canonical fiber functor* $\omega_{\mathrm{can}}\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S$, $V \mapsto V \otimes_k \mathcal{O}_S$.

In fact, for a general affine group scheme over k, one has to use the fpqc-topology in Lemma 3.1. However, as G is assumed to be reductive, thus in particular smooth, a theorem of Grothendieck (cf. [12, Theorem 11.7]) allows us to reduce to the étale topology.

Composing an exact tensor functor

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

with the Harder–Narasimhan functor

$$\mathrm{HN}\colon \mathrm{Bun}_{\mathbb{P}^1_k} \to \mathrm{FilBun}_{\mathbb{P}^1_k}$$

defines a, a priori not necessarily exact, tensor functor

$$\mathrm{HN} \circ \omega\colon \mathrm{Rep}_k(G) \to \mathrm{FilBun}_{\mathbb{P}^1_k}.$$

But using Haboush's theorem reductivity of G actually implies that the composition $\mathrm{HN} \circ \omega$ is still exact.

Lemma 3.2. *Let*

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

be an exact tensor functor. Then the composition

$$\mathrm{HN} \circ \omega\colon \mathrm{Rep}_k(G) \to \mathrm{FilBun}_{\mathbb{P}^1_k}$$

is still exact.

Proof. The crucial observation is that the functors

$$\omega,\ \mathrm{gr}\circ\mathrm{HN}$$

are compatible with duals, and exterior resp. symmetric products. This is clear for ω as ω is assumed to be exact and follows from Lemma 2.3 for the functor $\mathrm{gr}\circ\mathrm{HN}$. In fact, for a representation $V\in\mathrm{Rep}_k(\mathbb{G}_m)$ with associated vector bundle

$$\mathcal{E}:=\mathcal{E}(V)$$

we can conclude

$$\Lambda^r(\mathcal{E})\cong\mathcal{E}(\Lambda^r(V))\text{ resp. }\mathrm{Sym}^r(\mathcal{E})\cong\mathcal{E}(\mathrm{Sym}^r(V))$$

by exactness of the functor $\mathcal{E}(-)$. But by Lemma 2.3

$$\mathrm{gr}\circ\mathrm{HN}\circ\mathcal{E}(-)$$

is an exact tensor equivalence of $\mathrm{Rep}_k(\mathbb{G}_m)$ with a subcategory of $\mathrm{GrBun}_{\mathbb{P}^1_k}$, which implies the stated compatibility with exterior and symmetric powers. Using this, the proof can proceed similarly to [5, Theorem 5.3.1]. We note that for a representation V of G there is a canonical isomorphism

$$\mathrm{Sym}^r(V^\vee)\cong\mathrm{TS}_r(V)^\vee$$

from the r-th symmetric power $\mathrm{Sym}^r(V^\vee)$ of the dual of V to the dual of the module

$$\mathrm{TS}_r(V)=(V^{\otimes r})^{S_r}\subseteq V^{\otimes r}$$

of symmetric tensors. In particular, G-invariant homogenous polynomials on V define G-invariant linear forms on $\mathrm{TS}_r(V)$.

Let now $0\to V\xrightarrow{f}V'\xrightarrow{g}V''\to 0$ be an exact sequence in $\mathrm{Rep}_k(G)$. We have to check that the sequence

$$0\to\tilde\omega(V)\xrightarrow{\tilde\omega(f)}\tilde\omega(V')\xrightarrow{\tilde\omega(g)}\tilde\omega(V'')\to 0$$

with

$$\tilde\omega:=\mathrm{gr}\circ\mathrm{HN}\circ\omega$$

is still exact. We claim that $\tilde\omega(f)$ is injective. This can be checked after taking the exterior power $\Lambda^{\dim V}$ of f because $\tilde\omega$ commutes with exterior powers. In particular, to prove injectivity, we can reduce the claim for general f to the case $\dim V=1$. Tensoring with the dual of V reduces further to the case where V is moreover trivial. By Haboush's theorem (cf. [14]), there exists an $r>0$ and a G-invariant homogenous polynomial $f\in\mathrm{Sym}^r(V'^\vee)$ such that $f_{|V}\neq 0$. Using the above isomorphism $\mathrm{Sym}^r(V^\vee)\cong\mathrm{TS}_r(V)^\vee$, this shows that there exists an $r>0$ such that the morphism

$$V\cong\mathrm{TS}_r(V)\xrightarrow{\mathrm{TS}_r(f)}\mathrm{TS}_r(V')$$

splits. This implies that $\tilde\omega(\mathrm{TS}_r(f))$ splits and thus that $\tilde\omega(f)$ is in particular injective because $\tilde\omega$ commutes with the symmetric tensors TS_r as it commutes with symmetric powers and duals.

Dualizing yields that $\tilde\omega(g)$ is surjective at the generic point of $\mathbb{P}^1_k$. However, the sequence

$$0\to\tilde\omega(V)\xrightarrow{\tilde\omega(f)}\tilde\omega(V')\xrightarrow{\tilde\omega(g)}\tilde\omega(V'')\to 0$$

lies in the essential image of the functor $\mathrm{Rep}_k(\mathbb{G}_m)\to\mathrm{GrBun}_{\mathbb{P}^1_k}$ from Lemma 2.3. In particular, we see that the cokernel of $\tilde\omega(g)$ cannot have torsion, i.e. that it is zero. Finally, exactness in the middle of the sequence follows because

$$\mathrm{rk}(\tilde\omega(V'))=\mathrm{rk}(V')=\mathrm{rk}(V)+\mathrm{rk}(V'')=\mathrm{rk}(\tilde\omega(V))+\mathrm{rk}(\tilde\omega(V'')).$$

This finishes the proof. □

We briefly recall some results about filtered fiber functors on $\mathrm{Rep}_k(G)$ (cf. [19] and [4]). By definition, a filtered fiber functor for $\mathrm{Rep}_k(G)$ over a k-scheme S is an exact tensor functor

$$\omega\colon\mathrm{Rep}_k(G)\to\mathrm{FilBun}_S$$

into the exact tensor category of filtered vector bundles (with filtration by locally direct summands) on S. Associated with each filtered fiber functor ω is an exact tensor functor

$$\mathrm{gr}\circ\omega\colon \mathrm{Rep}_k(G)\to \mathrm{GrBun}_S,$$

i.e. a graded fiber functor, by mapping a filtered vector bundle to its associated graded. A splitting γ of a filtered fiber functor ω is a graded fiber functor

$$\gamma : \mathrm{Rep}_k(G)\to \mathrm{GrBun}_S$$

together with an isomorphism

$$\omega\cong \mathrm{fil}\circ\gamma$$

where the exact tensor functor

$$\mathrm{fil}\colon \mathrm{GrBun}_S\to \mathrm{FilBun}_S$$

sends a graded vector bundle

$$\mathcal{E}=\bigoplus_{i\in\mathbb{Z}}\mathcal{E}^i$$

to the filtered vector bundle $(\mathcal{E},\mathrm{fil}^\bullet\mathcal{E})$ with filtration

$$\mathrm{fil}^i\mathcal{E}=\bigoplus_{j\geq i}\mathcal{E}^j.$$

For a scheme $f: S'\to S$ over S let $\omega_{S'}$ be the base change of the filtered fiber functor ω to S', i.e. $\omega_{S'}$ is defined as the composition

$$\mathrm{Rep}_k(G)\xrightarrow{\omega}\mathrm{FilBun}_S\xrightarrow{f^*}\mathrm{FilBun}_{S'},$$

which is again a filtered fiber functor. For a filtered fiber functor ω, the presheaf

$$\mathrm{Spl}(\omega)(S'):=\{\text{set of splittings of }\omega_{S'}\}/\cong$$

of splittings of ω up to isomorphism (where the isomorphism respects the given isomorphisms $\omega\cong\mathrm{fil}\circ\gamma$) on the category of S-schemes is represented by an fpqc-torsor for the affine and faithfully flat group scheme

$$U(\omega):=\mathrm{Ker}(\mathrm{Aut}^{\otimes}(\omega)\to\mathrm{Aut}^{\otimes}(\mathrm{gr}\circ\omega))$$

over S (cf. [19, Lemma 4.20]). In particular, every filtered fiber functor

$$\omega\colon \mathrm{Rep}_k(G)\to\mathrm{FilBun}_S$$

admits a splitting fpqc-locally on S. The group scheme $U(\omega)$ is unipotent (cf. [19, Theorem 4.40]) and has an explicit decreasing filtration by normal subgroups

$$U(\omega)=U_1(\omega)\supseteq\ldots\supseteq U_i(\omega)\supseteq\ldots$$

for $i\geq 1$, which has moreover the property that for $i\geq 1$ the quotient

$$\mathrm{gr}^iU(\omega):=U_i(\omega)/U_{i+1}(\omega)$$

is abelian and isomorphic to

$$\mathrm{gr}^iU(\omega)\cong\mathrm{Lie}(\mathrm{gr}^iU(\omega))\cong\mathrm{gr}^i\omega(\mathrm{Lie}(G)),\ i\geq 1.$$

We can now give a proof of our main theorem about the classification of G-torsors on $\mathbb{P}^1_k$. We denote for a scheme S over k by

$$\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G),\mathrm{Bun}_S)$$

the groupoid of exact tensor functors $\omega\colon \mathrm{Rep}_k(G)\to\mathrm{Bun}_S$ and by

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G),\mathrm{Bun}_S)$$

its set of isomorphism classes. Similarly, we use the notations

$$\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G),\mathrm{Rep}_k(\mathbb{G}_m))$$

resp.

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m))$$

for the groupoid resp. the isomorphism classes of exact tensor functors

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m).$$

Theorem 3.3. *Let G be a reductive group over k. Then the composition with $\mathcal{E}(-)$ defines faithful functor*

$$\Phi\colon \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_{\mathbb{P}^1_k})$$

which induces a bijection

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \cong H^1_{\text{ét}}(\mathbb{P}^1_k, G)$$

on isomorphism classes.

Proof. By Lemma 2.3 the composition

$$\mathrm{Rep}_k(\mathbb{G}_m) \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{gr}} \mathrm{GrBun}_{\mathbb{P}^1_k}$$

is an equivalence onto its essential image. In particular, the functor

$$\Phi\colon \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_{\mathbb{P}^1_k})$$

is faithful and induces an injection on isomorphism classes. Thus, we have to prove that every exact tensor functor

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

factors as

$$\omega \cong \mathcal{E}(-) \circ \omega'$$

for some exact tensor functor

$$\omega'\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m).$$

Let $\tilde{\omega} := \mathrm{HN} \circ \omega$ be the functor

$$\tilde{\omega}\colon \mathrm{Rep}_k(G) \xrightarrow{\omega} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_{\mathbb{P}^1_k}.$$

By Lemma 3.2, the functor $\tilde{\omega}$ is still exact, i.e. a filtered fiber functor in the terminology of [19], and we can use the results recalled above. We get a $U(\tilde{\omega})$-torsor

$$\mathrm{Spl}(\tilde{\omega})$$

of splittings of $\tilde{\omega}$. But for the filtration

$$U(\tilde{\omega}) \supseteq U_2(\tilde{\omega}) \supseteq \ldots$$

the graded quotients

$$\mathrm{gr}^i U(\tilde{\omega}) \cong \mathrm{gr}^i \tilde{\omega}(\mathrm{Lie}(G))$$

are semistable vector bundles of slope $i \geq 1$. Hence,

$$H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{gr}^i U(\tilde{\omega})) = 0$$

because

$$\mathrm{gr}^i U(\tilde{\omega}) \cong \mathcal{O}_{\mathbb{P}^1_k}(i)^{\oplus n}$$

by Theorem 2.1. We can conclude that

$$H^1_{\text{ét}}(\mathbb{P}^1_k, U(\tilde{\omega})) = 1,$$

hence the $U(\tilde{\omega})$-torsor

$$\mathrm{Spl}(\tilde{\omega})$$

is in fact trivial, i.e. there exists a splitting

$$\gamma : \mathrm{Rep}_k(G) \to \mathrm{GrBun}_{\mathbb{P}^1_k}$$

of $\tilde{\omega}$ already over $\mathbb{P}^1_k$. As

$$\gamma \cong \mathrm{gr} \circ \tilde{\omega}$$

the functor γ takes its image in the full subcategory

$$\{\, \mathcal{E} = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}^i \in \mathrm{GrBun}_{\mathbb{P}^1} \mid \mathcal{E}^i \text{ semistable of slope } i\},$$

which by Lemma 2.3 is equivalent to the category $\mathrm{Rep}_k\mathbb{G}_m$ of representations of $\mathbb{G}_m$. Thus there exists an exact tensor functor

$$\omega' : \mathrm{Rep}_k(G) \to \mathrm{Rep}_k\mathbb{G}_m$$

such that

$$\omega \cong \mathcal{E}(-) \circ \omega',$$

by simply setting

$$\omega' := \mathcal{E}_{\mathrm{gr}}(-)^{-1} \circ \mathrm{gr} \circ \tilde{\omega}$$

where

$$\mathcal{E}_{\mathrm{gr}}(-) : \mathrm{Rep}_k\mathbb{G}_m \to \{\, \mathcal{E} = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}^i \in \mathrm{GrBun}_{\mathbb{P}^1} \mid \mathcal{E}^i \text{ semistable of slope } i\},$$

is the equivalence of Lemma 2.3. □

Let

$$\omega^{\mathbb{G}_m}_{\mathrm{can}} : \mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{Vec}_k,\ V \mapsto V$$

be the canonical fiber functor of $\mathrm{Rep}_k(\mathbb{G}_m)$ over k. Composing with $\omega^{\mathbb{G}_m}_{\mathrm{can}}$ defines a morphism

$$\Phi : \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Vec}_k)$$

of groupoids, where the right-hand side denotes the groupoid of exact tensor functors

$$\mathrm{Rep}_k(G) \to \mathrm{Vec}_k,$$

which by Lemma 3.1 identifies with the groupoid of G-torsors on $\mathrm{Spec}(k)$. Geometrically, the morphism Φ can be identified on isomorphisms classes with the map

$$i_x^* : H^1_{\text{ét}}(\mathbb{P}^1_k, G) \to H^1_{\text{ét}}(\mathrm{Spec}(k), G)$$

restricting a G-torsor over $\mathbb{P}^1_k$ to a G-torsor over $\mathrm{Spec}(k)$ along a k-rational point $x \in \mathbb{P}^1_k(k)$.

Moreover, there is a canonical map

$$\Psi : \mathrm{Hom}(\mathbb{G}_m, G)/G(k) \to H^1_{\text{ét}}(\mathbb{P}^1_k, G)$$

by sending a cocharacter $\chi : \mathbb{G}_m \to G$ to the G-torsor

$$\eta \times^{\mathbb{G}_m} G$$

where $\eta : \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$ is the Hopf bundle. We note that each G-torsor obtained this way is automatically Zariski-locally on $\mathbb{P}^1_k$ trivial.

Proposition 3.4. *The map* Ψ *is injective and identifies* $\mathrm{Hom}(\mathbb{G}_m, G)/G(k))$ *with the subset* $H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G) \subseteq H^1_{\text{ét}}(\mathbb{P}^1_k, G)$*. Moreover, for every k-rational point* $x \in \mathbb{P}^1_k(k)$*, the sequence*

$$1 \to H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G) \to H^1_{\text{ét}}(\mathbb{P}^1_k, G) \xrightarrow{i_x^*} H^1_{\text{ét}}(\mathrm{Spec}(k), G) \to 1$$

is exact and

$$H^1_{\text{ét}}(\mathbb{P}^1_k, G) \cong \coprod_H H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, H)$$

where the disjoint union is taken over all pure inner forms H of G over k (up to isomorphy).

Proof. The last statement follows from the first by replacing G by H (note that $H^1_{\text{ét}}(\mathbb{P}^1_k, G) \cong H^1_{\text{ét}}(\mathbb{P}^1_k, H)$ for a pure inner form H of G). By the Tannakian formalism, the quotient $\mathrm{Hom}(\mathbb{G}_m, G)/G(k)$ embeds into the isomorphism classes of exact, tensor functors $\mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m)$. Thus we have to prove two things. First, that (up to isomorphism) every Zariski-locally trivial G-torsor on $\mathbb{P}^1_k$ lies in the image of Ψ and that a G-torsor on $\mathbb{P}^1_k$ is Zariski-locally trivial if and only if its image in $H^1_{\text{ét}}(\mathrm{Spec}(k), G)$ is trivial. Let $\mathcal{P}$ be a G-torsor over $\mathbb{P}^1_k$ whose image is trivial in $H^1_{\text{ét}}(\mathrm{Spec}(k), G)$. We know from Theorem 3.3 that $\mathcal{P}$ is associated with some exact tensor functor

$$\omega' \colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k \mathbb{G}_m.$$

More precisely, $\mathcal{P}$ corresponds under Lemma 3.1 to the exact tensor functor $\omega := \mathcal{E}(-) \circ \omega' \colon \mathrm{Rep}_k G \to \mathrm{Bun}_{\mathbb{P}^1_k}$. If $i_x^*\mathcal{P}$ is trivial, then $i_x^* \circ \omega$ is isomorphic to the trivial fiber functor $\omega_0 \colon \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$. Also, the composition

$$\mathrm{Rep}_k(\mathbb{G}_m) \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{i_x^*} \mathrm{Vec}_k$$

is isomorphic to the trivial fiber functor on $\mathrm{Rep}_k(\mathbb{G}_m)$. Thus, we can conclude that ω' preserves, up to isomorphism, the respective trivial fiber functors on $\mathrm{Rep}_k(G)$ and $\mathrm{Rep}_k(\mathbb{G}_m)$. Thus, by the Tannakian formalism, ω' is induced, up to isomorphism, from some cocharacter $\chi \colon \mathbb{G}_m \to G$. This proves that $\mathcal{P}$ lies in the image of Ψ, which implies both desired claims. □

The classification results of Grothendieck and Harder on torsors on $\mathbb{P}^1_k$ (cf. [11] resp. [15]) are most concretely stated in the following form.

Corollary 3.5. *Let k be a field and let G/k be a reductive group with maximal split subtorus* $A \subseteq G$*. Then there exist canonical bijections*

$$X_*(A)^+ \cong \mathrm{Hom}(\mathbb{G}_m, G)/G(k) \cong H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G),$$

where $X_*(A)^+$ *denotes the set of dominant cocharacters of* $A \subseteq G$ *(for the choice of some minimal parabolic).*

Proof. By Proposition 3.4 it suffices to show

$$X_*(A)^+ \cong \mathrm{Hom}(\mathbb{G}_m, G)/G(k).$$

First, we claim that the canonical map

$$\mathrm{Hom}(\mathbb{G}_m, A)/N_G(A)(k) \to \mathrm{Hom}(\mathbb{G}_m, G)/G(k)$$

is a bijection. Surjectivity follows because the image of every cocharacter of G is contained in some maximal k-split torus and all maximal k-split tori in G are conjugated over k (cf. [3, Theorem 4.21]). Injectivity follows from (cf. [3, Corollary 4.22]). Namely, if $\chi, \chi' \colon \mathbb{G}_m \to A$ are two cocharacters that are conjugated by $g \in G(k)$, i.e. $\chi'(-) = g\chi(-)g^{-1}$, then (cf. [3, Corollary 4.22]) implies that there exists $h \in N_G(A)(k)$ such that $h\chi(-)h^{-1} = \chi'(-)$. But the orbits under $N_G(A)(k)$ on $X_*(A)$ are the orbits under the Weyl group $W_k(A) := (N_G(A)(k)/Z_G(A)(k)$ of the relative root system of G with respect to A (cf. [3, Théorème 5.3]) and the choice of a minimal parabolic defines a unique Weyl chamber in $X_*(A)$ (cf. [3, Corollary 5.9]). Then

$$X_*(A)/W_k(A) \cong X_*(A)^+$$

follows because the Weyl group permutes the Weyl chambers in $X_*(A)^+$ simply transitively. □

A description of $H^1_{\text{ét}}(\mathbb{P}^1_k, G)$, similar to the one of us, can be found in [10].

Of course, it is an interesting question to try to extend the method in this paper to arbitrary smooth projective curves X over k. Let us resume the main points of our argument for $X = \mathbb{P}^1_k$ in Theorem 3.3. These are:

1) for any exact tensor functor $\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_X$, the composition

$$\mathrm{Rep}_k(G) \xrightarrow{\omega} \mathrm{Bun}_X \xrightarrow{\mathrm{HN}} \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_X$$

is an exact tensor functor[2];

2) the category

$$\mathcal{T}_X := \{\mathcal{E} = \bigoplus_{\lambda\in\mathbb{Q}} \mathcal{E}^\lambda \mid \mathcal{E}^\lambda \text{ is semistable of slope } \lambda\}$$

is equivalent to $\mathrm{Rep}_k(\mathbb{G}_m)$;

3) for every semistable vector bundle $\mathcal{E}$ on X with positive slopes the group $H^1_{\text{ét}}(X, \mathcal{E})$ vanishes.

Point 1) may fail in general as the tensor product of semistable vector bundles on a general X may no longer be semistable (implying that $\mathrm{HN}(-)$ is not a tensor functor in this case), but however it is true for X of genus 0 or 1 and k arbitrary or X arbitrary and $\mathrm{char}(k) = 0$. On the other hand, 3) is satisfied only if the genus of X is 0 or 1. Thus let us assume that X is of genus 0 or 1. Then the argument in Lemma 3.2 goes through and 1) would be satisfied as well. Moreover, the category $\mathcal{T}_X$ is then Tannakian and, in particular, isomorphic to the category of representations of some Galois gerbe G_X over k (cf. [17, §2] for the notion of a Galois gerbe). If $X \neq \mathbb{P}^1_k$ is of genus 0, i.e. a Brauer–Severi curve, and $k = \mathbb{R}$ one might guess (cf. [9, Proposition 5.1]) that G_X is isomorphic to the Weil group of $\mathbb{R}$. The analog of Theorem 3.3 should yield the classification in [9, Proposition 5.1]. If k is algebraically closed of characteristic 0 and X an elliptic curve, then using Atiyah's classification of vector bundles on elliptic curves Philipp Reichenbach has shown that G_X fits into a non-split extension

$$1 \to \mathbb{D}_{\mathbb{Q}} \to G_X \to \mathbb{D}_{\mathrm{Pic}^0_X(k)} \times \mathbb{G}_a \to 1.$$

Here for M an abelian group, $\mathbb{D}_M$ denotes the multiplicative group scheme over k with character group M and $\mathrm{Pic}^0_X(k)$ the k-rational points of the Jacobian Pic^0_X of X.

4. Applications

In this section, we present some applications of the classification of torsors (following (cf. [8]), which discusses analogous applications to the Fargues–Fontaine curve).

The first application is the computation of the Brauer group of $\mathbb{P}^1_k$. For this, we recall the theorem of Steinberg (cf. [18, Chapter 3.2.3]). If k is a field of cohomological dimension $\mathrm{cd}(k) \leq 1$, then Steinberg's theorem states that

$$H^1_{\text{ét}}(\mathrm{Spec}(k), G) = 1$$

for every smooth connected affine algebraic group G/k. In particular, the Brauer group

$$\mathrm{Br}(k) = 0$$

of such fields vanishes. For example, separably closed or finite fields are of cohomological dimension ≤ 1.

Theorem 4.1. *If k is of cohomological dimension $\mathrm{cd}(k) \leq 1$, then the Brauer group*

$$\mathrm{Br}(\mathbb{P}^1_k) \cong H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m) = 0$$

vanishes.

Proof. By [13, Corollary 2.2.] there is an isomorphism

$$\mathrm{Br}(\mathbb{P}^1_k) \cong H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$$

of the Brauer group $\mathrm{Br}(\mathbb{P}^1_k)$ parametrizing equivalence classes of Azumaya algebras over $\mathcal{O}_{\mathbb{P}^1_k}$ with the cohomological Brauer group $H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$. It suffices to show that for every $n \geq 0$ the canonical map

$$H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{PGL}_n) \to H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$$

arising as a boundary map of the short exact sequence

[2] We include the $\mathbb{Q}$ as for a general X the Harder–Narasimhan filtration is indexed by $\mathbb{Q}$ and not by $\mathbb{Z}$.

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

is trivial. Because k is of cohomological dimension ≤ 1, there exists using Steinberg's theorem in the case $G = \mathrm{GL}_n$ or $G = \mathrm{PGL}_n$ and Theorem 3.3 together with Proposition 3.4, a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{GL}_n) & \longrightarrow & H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{PGL}_n) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(\mathbb{G}_m, \mathrm{GL}_n)/\mathrm{GL}_n(k) & \longrightarrow & \mathrm{Hom}(\mathbb{G}_m, \mathrm{PGL}_n)/\mathrm{PGL}_n(k). \end{array}$$

It suffices to show that the top horizontal arrow, or equivalently the lower horizontal arrow, is surjective. But every cocharacter

$$\chi : \mathbb{G}_m \to \mathrm{PGL}_n$$

can be lifted to GL_n because for the standard torus $T \cong \mathbb{G}_m^n \subseteq \mathrm{GL}_n$ there is a split exact sequence

$$0 \to X_*(\mathbb{G}_m) \to X_*(T) \to X_*(T/\mathbb{G}_m) \to 0$$

on cocharacter groups where $T/\mathbb{G}_m$ is a maximal torus of PGL_n. □

For a general field k, i.e. k not necessarily of cohomological dimension ≤ 1, the Brauer group of $\mathbb{P}^1_k$ is given by

$$\mathrm{Br}(\mathrm{Spec}(k)) \cong \mathrm{Br}(\mathbb{P}^1_k)$$

as can be calculated from Theorem 4.1 using the spectral sequence

$$E_2^{pq} = H^p(\mathrm{Gal}(\bar{k}/k), H^q_{\text{ét}}(\mathbb{P}^1_{\bar{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$$

where $\bar{k}$ denotes a separable closure of k.

The next application we give is to the uniformization of G-torsors.

Theorem 4.2. *Let k be a field and let G be reductive group over k. If $x \in \mathbb{P}^1_k(k)$ is k-rational point, then every G-torsor*

$$\mathcal{P} \in H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G)$$

which is locally trivial for the Zariski topology becomes trivial on $\mathbb{P}^1_k \setminus \{x\}$.

Proof. By Proposition 3.4, we know that every such G-torsor $\mathcal{P}$ is isomorphic to the pushout

$$\mathcal{P} \cong \eta \times^{\mathbb{G}_m} G$$

along a cocharacter

$$\chi : \mathbb{G}_m \to G$$

of the canonical $\mathbb{G}_m$-torsor

$$\eta : \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$$

corresponding to the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ on $\mathbb{P}^1_k$. But

$$\mathcal{O}_{\mathbb{P}^1_k}(-1)_{|\mathbb{P}^1_k \setminus \{x\}}$$

is trivial because $\mathbb{P}^1_k \setminus \{x\} \cong \mathbb{A}^1_k$. This shows the claim. □

Finally, we reprove the Birkhoff–Grothendieck decomposition of $G(k((t))$ for a reductive group G over k (cf. [7, Lemma 4]).

Theorem 4.3. *Let $A \subseteq G$ be a maximal split torus in G. Then there exists a canonical bijection*

$$X_*(A)^+ \cong G(k[t^{-1}]) \backslash G(k((t))) / G(k[[t]]),$$

where $X_*(A)^+$ denotes the set of dominant cocharacters of $A \subseteq G$.

Proof. Let $x \in \mathbb{P}^1_k(k)$ be a k-rational point. By Beauville–Laszlo [2] and Lemma 3.1, there is an injective map

$$\gamma : G(k[t^{-1}])\backslash G(k((t)))/G(k[[t]]) \to H^1_{\text{ét}}(\mathbb{P}^1_k, G)$$

by gluing the trivial G-torsor on $\mathbb{P}^1_k \setminus \{x\}$ with the trivial G-torsor on the formal completion

$$\text{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1_k,x})$$

along an isomorphism on $\text{Spec}(\text{Frac}(\widehat{\mathcal{O}}_{\mathbb{P}^1_k,x}))$. Note that $\widehat{\mathcal{O}}_{\mathbb{P}^1_k,x} \cong k[[t]]$. From Proposition 3.4, we can conclude that the G-torsors obtained in this way are actually locally trivial for the Zariski topology. By Theorem 4.2, we can conversely see that the image of γ contains the set $H^1_{\text{Zar}}(\mathbb{P}^1_k, G)$. Using Proposition 3.4, we can conclude that

$$G(k[t^{-1}])\backslash G(k((t)))/G(k[[t]]) \cong H^1_{\text{Zar}}(\mathbb{P}^1_k, G) \cong X_*(A)^+. \qquad \square$$

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