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Compatible Hamiltonian operators for the Krichever–Novikov equation

Opérateurs hamiltoniens compatibles pour l'équation de Krichever–Novikov

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ABSTRACT

It has been proved by Sokolov that Krichever–Novikov equation's hierarchy is hamiltonian for the Hamiltonian operator $H_0 = u_x \partial^{-1} u_x$ and possesses two weakly non-local recursion operators of degrees 4 and 6, L_4 and L_6. We show here that H_0, $L_4 H_0$ and $L_6 H_0$ are compatible Hamiltonians operators for which the Krichever–Novikov equation's hierarchy is hamiltonian.

RÉSUMÉ

Il a été démontré par Sokolov que la hiérarchie de l'équation de Krichever–Novikov est hamiltonienne pour l'opérateur hamiltonien $H_0 = u_x \partial^{-1} u_x$ et possède deux opérateurs de récursion faiblement non locaux de degrés 4 et 6, L_4 et L_6. Nous montrons ici que H_0, $L_4 H_0$ et $L_6 H_0$ sont des opérateurs hamiltoniens compatibles pour lesquels la hiérarchie de l'équation de Krichever–Novikov est hamiltonienne.

In the study of finite-gap solutions of KP, an integrable $(1+1)$-dimensional PDE was discovered, the Krichever–Novikov equation. One of its forms (equivalent to the original one in [7]) is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{P(u)}{u_1}, \tag{1}$$

where $u = u(t, x)$, $u_n = (\frac{\mathrm{d}}{\mathrm{d}x})^n(u)$, and P is a polynomial of degree at most 4. Let $\mathcal{V} = \mathbb{C}[u, u_1^{\pm}, u_2, \ldots]$ and $\mathcal{K}$ be the fraction field of $\mathcal{V}$. Let us denote $\frac{\mathrm{d}}{\mathrm{d}x}$ by ∂. The *differential order* d_F of a function $F \in \mathcal{V}$ is the highest integer n such that $\frac{\partial F}{\partial u_n} \neq 0$.

One of the attributes of equation (1) is to be part of an infinite hierarchy of compatible evolution PDEs of odd differential orders

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$$\frac{\mathrm{d}u}{\mathrm{d}t_i} = G_i \in \mathcal{V}, i \geq 0, \tag{2}$$

where G_i has differential order $(2i+1)$. One says that $F, G \in \mathcal{V}$ are *compatible*, or *symmetries* of one another, if

$$\{F, G\} := X_F(G) - X_G(F) = 0, \tag{3}$$

where X_F denotes the derivation of $\mathcal{V}$ induced by the evolution equation $u_t = F$, that is

$$X_F = \sum_{n \geq 0} F^{(n)} \frac{\partial}{\partial u_n}. \tag{4}$$

(3) endows $\mathcal{V}$ with a Lie algebra bracket, and the G_i's span an infinite-dimensional abelian subalgebra of $(\mathcal{V}, \{.,.\})$, which we will denote by $\mathcal{S}$. The first four equations in the hierarchy are:

$$\begin{aligned} G_0 &= u_1, \\ G_1 &= u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{P(u)}{u_1}, \\ G_2 &= u^{(5)} - 5\frac{u_4 u_2}{u_1} - \frac{5}{2}\frac{u_3^2}{u_1} + \frac{25}{2}\frac{u_3 u_2^2}{u_1^2} - \frac{45}{8}\frac{u_2^4}{u_1^3} - \frac{5}{3}P\frac{u_3}{u_1^2} + \frac{25}{6}P\frac{u_2^2}{u_1^3} - \frac{5}{3}P_u\frac{u_2}{u_1} - \frac{5}{18}\frac{P^2}{u_1^3} + \frac{5}{9}u_1 P_{uu}, \end{aligned} \tag{5}$$

$$\begin{aligned} G_3 = u_7 &- 7\frac{u_2 u_6}{u_1} - \frac{7}{6}\frac{u_5}{u_1^2}(2P + 12u_3u_1 - 27u_2^2) - \frac{21}{2}\frac{u_4^2}{u_1} + \frac{21}{2}\frac{u_4}{u_1^3}(2P - 11u_2^2) \\ &- \frac{7}{3}\frac{u_4}{u_1^2}(2P_u u_1 - 51u_2u_3) + \frac{49}{2}\frac{u_3^3}{u_1^2} + \frac{7}{12}\frac{u_3^2}{u_1^3}(22P - 417u_2^2) + \frac{2499}{8}\frac{u_2^4}{u_1^4}u_3 \\ &\frac{91}{3}P_u\frac{u_2}{u_1^2}u_3 - \frac{595}{6}P\frac{u_2^2}{u_1^4}u_3 - \frac{35}{18}\frac{u_3}{u_1^4}(2P_{uu}u_1^4 - P^2) - \frac{1575}{16}\frac{u_2^6}{u_1^5} + \frac{1813}{24}\frac{u_2^4}{u_1^5}P \\ &- \frac{203}{6}\frac{u_2^3}{u_1^3}P_u + \frac{49}{36}\frac{u_2^2}{u_1^5}(6P_{uu}u_1^4 - 5P^2) - \frac{7}{9}\frac{u_2}{u_1^3}(2P_{uuu}u_1^4 - 5PP_u) + \frac{7}{54}\frac{P^3}{u_1^5} \\ &- \frac{7}{9}P_{uu}\frac{P}{u_1} + \frac{7}{9}P_{uuuu}u_1^3 - \frac{7}{18}\frac{P_u^2}{u_1}. \end{aligned} \tag{6}$$

It is known ([6,10]) that all integrable hierarchies admit a pseudodifferential operator $L \in \mathcal{V}((\partial^{-1}))$ satisfying

$$X_F(L) = [D_F, L] \tag{7}$$

for all F in the hierarchy, where D_F denotes the *Fréchet derivative* of F:

$$D_F = \sum_n \frac{\partial F}{\partial u_n}\partial^n \in \mathcal{V}[\partial]. \tag{8}$$

A pseudodifferential operator satisfying (7) is called a *recursion operator* (for F). In [3], two rational recursions operators for (1) were found, of orders 4 and 6:

$$L_4 = H_1 H_0^{-1}, \quad L_6 = H_2 H_0^{-1}, \tag{9}$$

where

$$\begin{aligned} H_0 =& u_1\partial^{-1}u_1, \\ H_1 =& \frac{1}{2}(u_1^2\partial^3 + \partial^3 u_1^2) + (2u_3u_1 - \frac{9}{2}u_2^2 - \frac{2}{3}P)\partial + \partial(2u_3u_1 - \frac{9}{2}u_2^2 - \frac{2}{3}P) \\ &+ G_1\partial^{-1}G_1 + u_1\partial^{-1}G_2 + G_2\partial^{-1}u_1, \\ H_2 =& \frac{1}{2}(u_1^2\partial^5 + \partial^5 u_1^2) + (3u_3u_1 - \frac{19}{2}u_2^2 - P)\partial^3 + \partial^3(3u_3u_1 - \frac{19}{2}u_2^2 - P) \\ &+ (u_5u_1 - 9u_3u_2 + \frac{19}{2}u_3^2 - \frac{2}{3}\frac{u_3}{u_1}(5P - 39u_2^2) + \frac{u_2^2}{u_1^2}(5P - 9u_2^2) + \frac{2}{3}\frac{P^2}{u_1^2} + u_1^2P_{uu})\partial \\ &+ \partial(u_5u_1 - 9u_3u_2 + \frac{19}{2}u_3^2 - \frac{2}{3}\frac{u_3}{u_1}(5P - 39u_2^2) + \frac{u_2^2}{u_1^2}(5P - 9u_2^2) + \frac{2}{3}\frac{P^2}{u_1^2} + u_1^2P_{uu}) \\ &+ G_1\partial^{-1}G_2 + G_2\partial^{-1}G_1 + u_1\partial^{-1}G_3 + G_3\partial^{-1}u_1. \end{aligned} \tag{10}$$

Moreover, L_4 and L_6 are both *weakly non-local*, i.e. of the form

$$E(\partial) \in \mathcal{V}[\partial] + \sum_i p_i \partial^{-1} \frac{\delta \rho_i}{\delta u}, \tag{11}$$

where the ρ_i's are conserved densities of (1). Recall that the *variational derivative* $\frac{\delta}{\delta u}$ is defined as follows:

$$\frac{\delta F}{\delta u} = D_F^*(1) = \sum_n (-\partial)^n (\frac{\partial F}{\partial u_n}). \tag{12}$$

In [11], Sokolov showed that the space of symmetries of (1), $\mathcal{S}$, is preserved by L_4. The same argument applies to L_6, which was found later. He also establishes that the hierarchy of the Krichever–Novikov equation is *hamiltonian* for H_0: there exists a sequence $\phi_i \in \mathcal{V}$ such that

$$G_i = H_0(\frac{\delta \phi_i}{\delta u}) \text{ for all } i \geq 0. \tag{13}$$

A *Hamiltonian operator* $H = AB^{-1} \in \mathcal{V}(\partial)$ with A and B right coprime is a skewadjoint rational differential operator inducing a non-local Poisson lambda bracket, which is equivalent to the following identity (see equation (6.13) in [4])

$$\begin{aligned} &A^*(D_{B(F)}(A(G)) + D^*_{A(G)}(B(F)) - D_{B(G)}(A(F)) + D^*_{B(G)}(A(F))) \\ &= B^*(D_{A(G)}(A(F)) - D_{A(F)}(A(G))) \end{aligned} \tag{14}$$

for all $F, G \in \mathcal{V}$.

Lemma 1. *Let $L \in \mathcal{V}(\partial)$ be a skewadjoint rational operator. If there exists an infinite-dimensional (over $\mathbb{C}$) subspace $\mathcal{W} \subset \mathcal{V}$ such that $B(W) \subset \frac{\delta}{\delta u}\mathcal{V}$ and such that for all $G \in \mathcal{W}$, $E = A(G)$ satisfies*

$$X_E(L) = D_E L + L D_E^*, \tag{15}$$

then L is a Hamiltonian operator. Conversely, if L is a Hamiltonian operator and $G \in \mathcal{V}$, then $D_{B(G)} = D^*_{B(G)}$ if and only if $A(G)$ satisfies equation (15).

Proof. Let us first give an equivalent form of (15) involving only differential operators.

$$\begin{aligned} (1.15) &\iff X_E(A) - D_E A = AB^{-1}(X_E(B) + D_E^* B) \\ &\iff X_E(A) - D_E A = -B^{*-1} A^*(X_E(B) + D_E^* B) \\ &\iff A^*(X_E(B) + D_E^* B) = B^*(D_E A - X_E(A)) \\ &\iff A^*(X_E + D_E^*) B = B^*(D_E - X_E) A. \\ &\iff A^*(D_{B(F)}(E) + D_E^*(B(F))) = B^*(D_E(A(F)) - D_{A(F)}(E)) \quad \forall F \in \mathcal{V}. \end{aligned} \tag{16}$$

Comparing the last line of (16) with (14), it is clear that if H is Hamiltonian, then $E = A(G)$ satisfies equation (15) is and only if $D_{B(G)}$ is self-adjoint. It is also clear that if $A(G)$ satisfies (15) and $D_{B(G)}$ is self-adjoint, then (F, G) satisfies (14) for any $F \in \mathcal{V}$. Therefore, if we consider $\mathcal{W} \subset \mathcal{V}$ infinite-dimensional subspace of $\mathcal{V}$ such that $A(\mathcal{W})$ satisfies (15) and $B(\mathcal{W}) \subset \frac{\delta}{\delta u}\mathcal{V}$, we deduce that (14) is satisfied for any $(F, G) \in \mathcal{V} \times \mathcal{W}$. To conclude, we note that (14) can be rewritten as an identity of bidifferential operator, i.e. it amounts to say that some expression of the form $\sum m_{ij} F^{(i)} G^{(j)}$, where $m_{ij} \in \mathcal{V}$ is trivial, i.e. $m_{ij} = 0$ for all i, j. Namely, (14) is equivalent to

$$\begin{aligned} &A^*(X_{A(G)}(B)(F) - X_{A(F)}(B)(G) + (D_A)^*_G(B(F)) + (D_B)^*_G(A(F))) \\ &= B^*(X_{A(F)}(A)(G) - X_{A(G)}(A)(F)), \end{aligned} \tag{17}$$

where given a differential operator P, an element $F \in \mathcal{V}$, the differential operator $(D_P)_F$ is defined by

$$(D_P)_F(G) = X_G(P)(F) \quad \forall G \in \mathcal{V}. \tag{18}$$

If a bidifferential operator vanishes on $\mathcal{V} \times \mathcal{W}$, it must be identically 0, since $\mathcal{W}$ is infinite dimensional. Hence, L is an Hamiltonian operator. □

Lemma 2. *Let $L = CD^{-1}$ be a rational operator and $(F_n)_{n \geq 0}$ a sequence spanning an infinite-dimensional subspace of $\mathcal{K}$ satisfying $C(F_n) = D(F_{n+1}) \in \mathcal{V}$ for all $n \geq 0$. Assume that L is recursion for all the $D(F_n)$'s and that the $D(F_n)$'s are hamiltonian for some Hamiltonian operator $H \in \mathcal{V}(\partial)$. Then, provided LH is skew-adjoint, LH is a Hamiltonian operator for which all the $D(F_n)$'s are hamiltonian ($n \geq 1$).*

Proof. By Lemma 1, H satisfies equation (15) for all $D(F_n), n \geq 0$, hence so does LH (L is recursion for $D(F_n)$ for all $n \geq 0$). To conclude using Lemma 1, one needs to check that $D(F_n) = LH(\frac{\delta \rho_n}{\delta u})$ for some $\rho_n \in \mathcal{V}$ for all $n \geq 1$. Let $P, Q \in \mathcal{V}[\partial]$ be right coprime differential operators such that $LH = PQ^{-1}$. Let A, B be right coprime differential operators such that $H = AB^{-1}$. $D(F_n)$ is hamiltonian for H for all $n \geq 0$, meaning that there exist two sequences in $\mathcal{V}$, $(\phi_n)_{n\geq 0}$ and $(\rho_n)_{n\geq 0}$, such that $\frac{\delta \rho_n}{\delta u} = B(\phi_n)$ and $D(F_n) = A(\phi_n)$ for all $n \geq 0$. In the language of [2], $\frac{\delta \rho_n}{\delta u}$ and $C(F_n)$ are $CD^{-1}AB^{-1}$ associated, hence (quote result) there exists ψ_n such that $C(F_n) = P(\psi_n)$ and $Q(\psi_n) = \frac{\delta \rho_n}{\delta u}$ for all $n \geq 0$. Therefore, by Lemma 1.1, LH is a Hamiltonian operator for which $(C(F_n))_{n\geq 0}$ are hamiltonian. □

Theorem. *H_0, H_1 and H_2 are compatible Hamiltonian operators.*

Proof. Let $\alpha, \beta, \gamma \in \mathbb{C}$ and let $L_{\alpha,\beta,\gamma} = (\alpha H_0 + \beta H_1 + \gamma H_2)H_0^{-1}$. $L_{\alpha,\beta,\gamma}$ is a recursion operator for the whole Krichever–Novikov hierarchy $\mathcal{S}$. Moreover, it maps $\mathcal{S}$ to itself as was proved in [11], meaning that if $L_{\alpha,\beta,\gamma} = AB^{-1}$ with A, B right coprime and $G \in \mathcal{S}$, then $G = B(F)$ for some $F \in \mathcal{K}$ and $A(F) \in \mathcal{S}$. The theorem follows from Lemma 2. □

Remark 3. It follows from Lemma 1 that $H = H_2 H_1^{-1} H_0$ is a Hamiltonian operator of degree 1. However, it is not weakly non-local. More generally, all the $(H_2 H_1^{-1})^n H_0$, for $n \in \mathbb{Z}$ are pairwise compatible Hamiltonian operators. It is known since the work of Magri ([8], see also [5]) that from a pair of compatible Hamiltonian operators, one can construct infinitely many.

Remark 4. Every Hamiltonian operator $K = AB^{-1}$ over $\mathcal{V}$, where A and B are right coprime induces a Lie algebra bracket on the space of functionals $\mathcal{F}(K) := \{\int f \in \mathcal{V}/\partial\mathcal{V} | \frac{\delta f}{\delta u} \in ImB\}$, (well-)defined by $\{\int f, \int g\} = \int \frac{\delta f}{\delta u} AB^{-1}(\frac{\delta g}{\delta u})$ (see section 7.2 in [4]). Note that $\mathcal{F}(H_0) = \mathcal{V}/\partial\mathcal{V}$ but that $\mathcal{F}(H_1)$ and $\mathcal{F}(H_2)$ consist only of the conserved densities of the Krichever–Novikov equation.

We recall that if a rational differential operator $L = AB^{-1}$, with $A, B \in \mathcal{V}[\partial]$ right coprime generates an infinite dimensional abelian subspace of $(\mathcal{V}, \{.,.\})$, in the sense that there exist $(F_n)_{n\geq 0} \in \mathcal{K}$ such that $A(F_n) = B(F_{n+1})$ for all $n \geq 0$ and such that the $B(F_n)$'s span an infinite-dimensional abelian subspace of $(\mathcal{V}, \{.,.\})$, then for all $\lambda \in \mathbb{C}$, $Im(A + \lambda B)$ must be a sub Lie algebra of $(\mathcal{V}, \{.,.\})$ (see [1]). The recursion operators $L_{\alpha,\beta,\gamma}$ satisfy this condition.

Note that weakly non-local Hamiltonian operators were introduced in [9], where the authors study the complete set of weakly non-local Hamiltonian operators for both the KdV and the NLS hierarchies.

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