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Lattice sub-tilings and frames in LCA groups



Sous-pavages en réseau et trames dans les groupes abéliens, localement compacts

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ABSTRACT

Given a lattice Λ in a locally compact Abelian group G and a measurable subset Ω with finite and positive measure, then the set of characters associated with the dual lattice form a frame for $L^2(\Omega)$ if and only if the distinct translates by Λ of Ω have almost empty intersections. Some consequences of this results are the well-known Fuglede theorem for lattices, as well as a simple characterization for frames of modulates.

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R É S U M É

Soit Λ un réseau. On prouve que les caractères de G associés au réseau dual forment une trame de $L^2(\Omega)$ si et seulement si les différents translats de Ω par Λ sont d'intersection presque vide. Ceci entraîne le théorème bien connu de Fuglede pour les réseaux, ainsi qu'une caractérisation simple des trames de modulation.

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1. Introduction

Let G denote a locally compact and second countable Abelian group (LCA group). A closed subgroup Λ of G is called a *lattice* if it is discrete and co-compact, i.e. the quotient group G/Λ is compact. Recall that, since G is second countable, then any discrete subgroup of G is also countable (see, e.g., [19, Section 12, Example 17]). Assume that G is Abelian, and denote the dual group by \widehat{G} . The dual lattice of Λ is defined as follows:

$$\Lambda^\perp = \{\chi \in \widehat{G} : \langle \chi, \lambda \rangle = 1 \forall \lambda \in \Lambda\}, \quad (1)$$

where $\langle \chi, \lambda \rangle$ indicates the action of character χ on the group element λ .

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We recall that, by the duality theorem between subgroups and quotient groups (see, e.g., [21, Lemma 2.1.2]), the dual lattice Λ^\perp is a subgroup of \widehat{G} which is topologically isomorphic to the dual group of G/Λ , i.e. $\Lambda^\perp \cong \widehat{G/\Lambda}$. Moreover, since G/Λ is compact, the dual lattice Λ^\perp is discrete. Notice also that $\widehat{G/\Lambda^\perp} \cong \widehat{\Lambda}$, which implies that Λ^\perp is co-compact, hence it is a lattice.

Let dg denote a Haar measure on G . For a function f in $L^1(G)$, the Fourier transform of f is defined by

$$\mathcal{F}_G(f)(\chi) = \int_G f(g) \overline{\langle \chi, g \rangle} dg, \quad \chi \in \widehat{G},$$

where $\langle \chi, g \rangle$ denotes the action of the character χ on g . By the inversion theorem [21, Section 1.5.1], a Haar measure $d\chi$ can be chosen on \widehat{G} so that the Fourier transform \mathcal{F}_G is an isometry from $L^2(G)$ onto $L^2(\widehat{G})$. More precisely,

$$\langle \mathcal{F}_G(f), \mathcal{F}_G(g) \rangle_{L^2(\widehat{G}, d\chi)} = \langle f, g \rangle_{L^2(G, dg)} \quad f, g \in L^2(G). \tag{2}$$

For any $\chi \in \widehat{G}$, we define the exponential function e_χ by

$$e_\chi : G \rightarrow \mathbb{C}, \quad e_\chi(g) := \langle \chi, g \rangle.$$

For any measurable subset Ω of G , we let $|\Omega|$ denote the Haar measure of Ω . Throughout this paper, we let $\mathbf{1}_\Omega$ denote the characteristic function of the set Ω . We shall also use the addition symbol ‘+’ for the group action, and 0 for the neutral element, since G is Abelian.

Definition 1 (Sub-tiling). Let $\Omega \subset G$ be a measurable set with finite and positive Haar measure, and let Λ be a lattice subgroup of G . We say that (Ω, Λ) is a *sub-tiling pair* for G if

$$\sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(g - \lambda) \leq 1 \quad \text{a.e. } g \in G. \tag{3}$$

By replacing the inequality with an equality, the definition is that of a *tiling pair*. In this weaker form, it is equivalent to say that the translates of Ω by elements of Λ are a.e. disjoint, i.e. (Ω, Λ) is a sub-tiling pair for G if and only if

$$|\Omega \cap (\Omega + \lambda)| = 0 \quad \forall \lambda \in \Lambda, \lambda \neq 0.$$

Observe also that any sub-tiling set is a subset of a tiling set.

Our main result is the following. Recall that a cross section $Q_\Lambda \subset G$ for a group G and a lattice Λ is a measurable set of representatives of G/Λ .

Theorem 2 (Main result). Let Λ be a lattice in G , let $\Omega \subset G$ be a set with finite and positive measure, and let $Q_\Lambda \subset G$ be a cross section for G/Λ . Then the following are equivalent.

- 1) The pair (Ω, Λ) is sub-tiling for G .
- 2) For a.e. $\chi \in \widehat{G}$ it holds

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\mathbf{1}_\Omega)(\chi + \tilde{\lambda})|^2 = |Q_\Lambda| |\Omega|.$$

- 3) The system of translates $\{\sqrt{|\Omega|}^{-1} \mathbf{1}_\Omega(\cdot - \lambda) : \lambda \in \Lambda\}$ is orthonormal in $L^2(G)$.
- 4) The exponential set $E_{\Lambda^\perp} = \{e_{\tilde{\lambda}} : \tilde{\lambda} \in \Lambda^\perp\}$ is a frame for $L^2(\Omega)$.

Moreover, if any of the above conditions holds, then the frame in point 4) is tight, with constant $|Q_\Lambda|$.

As a first corollary, we can obtain the following result, which was proved by B. Fuglede in the Euclidean setting [6], and in the present setting by S. Pedersen with a different approach [18].

Corollary 3. A set of finite and positive measure Ω tiles G with translations by Λ if and only if the exponential set E_{Λ^\perp} is an orthogonal basis for $L^2(\Omega)$.

Let us now denote with $M : \Lambda^\perp \rightarrow \mathcal{U}(L^2(\Omega))$ the modulations $M_{\tilde{\lambda}} f(g) = e_{\tilde{\lambda}}(g) f(g)$. As a second consequence of Theorem 2, we obtain the following.

Corollary 4. Conditions 1)–4) of Theorem 2 are equivalent to

5) The system of translates $\Psi_{\Lambda^\perp} = \{M_{\tilde{\lambda}}\psi : \tilde{\lambda} \in \Lambda^\perp\}$ is a frame for $L^2(\Omega)$, with frame bounds $0 < A|Q_\Lambda| \leq B|Q_\Lambda| < \infty$, for any $\psi \in L^2(\Omega)$ satisfying

$$0 < A \leq \text{ess inf} |\psi|^2 \leq \text{ess sup} |\psi|^2 \leq B < \infty.$$

The novelty of this paper is that it relates sublattices with frames of exponentials. Moreover, the proof of 2) \Rightarrow 3) in [Theorem 2](#) is shown by using the bracket map of a system of translates ([Corollary 8](#)) that have been introduced to study some properties of translation invariant spaces (see [\[7\]](#) and the references therein).

The setting of LCA groups allows us to prove simultaneously results for a large variety of groups, namely $\mathbb{R}^n, \mathbb{Z}^d, \mathbb{T}^k$, and all finite groups F with discrete topology, as well as the so-called elementary LCA groups $G = \mathbb{R}^n \times \mathbb{Z}^d \times \mathbb{T}^k \times F$. Observe that knowing [Theorem 2](#) and [Corollary 3](#) for every factor group does not immediately provide the corresponding results for G .

The motivation for this paper comes from the problem of studying the relationship between spectrum sets and tiling pairs, whose roots dates back to a 1974 paper of B. Fuglede [\[6\]](#). There he proved that a set $E \subset \mathbb{R}^d, d \geq 1$, of positive Lebesgue measure, tiles \mathbb{R}^d by translations with a lattice Λ if and only if $L^2(E)$ has an orthogonal basis of exponentials indexed by the annihilator of Λ . A more general statement in \mathbb{R}^d , which says that if $E \subset \mathbb{R}^d, d \geq 1$, has positive Lebesgue measure, then $L^2(E)$ has an orthogonal basis of exponentials (not necessary indexed by a lattice) if and only if E tiles \mathbb{R}^d by translations, has been known as the Fuglede Conjecture.

A variety of results were proved, establishing connections between tiling and orthogonal exponential bases. See, for example, [\[17,11,16,14,12\]](#). In 2001, I. Laba proved that the Fuglede conjecture is true for the union of two intervals in the plane [\[15\]](#). In 2003, A. Iosevich, N. Katz and T. Tao [\[9\]](#) proved that the Fuglede conjecture holds for convex planar domains. It was also proved in [\[11\]](#) and [\[17\]](#) that Λ tiles \mathbb{R}^d by the unit cube Q^d if and only if Λ is a spectral set for Q^d . In 2004, T. Tao [\[22\]](#) disproved the Fuglede Conjecture in dimension $d = 5$ and larger, by exhibiting a spectral set in \mathbb{R}^5 , which does not tile the space by translations. In [\[13\]](#), M. Kolountzakis and M. Matolcsi also disproved the reverse implication of the Fuglede Conjecture for dimensions $d = 4$ and higher. In [\[4\]](#) and [\[3\]](#), the dimension of counter-examples was further reduced. In fact, B. Farkas, M. Matolcsi and P. Mora show in [\[3\]](#) that the Fuglede conjecture is false in \mathbb{R}^3 . The general feeling in the field is that sooner or later the counter-examples of both implications will cover all dimensions. However, in [\[10\]](#) the authors showed that the Fuglede Conjecture holds in two-dimensional vector spaces over prime fields. Then, in [\[2\]](#) the authors prove that tiling implies spectral in \mathbb{Z}_p^3, p prime, and that the Fuglede Conjecture is true for \mathbb{Z}_2^3 and \mathbb{Z}_3^3 . Very recently, important developments in LCA groups, with crucial implications on sampling theory, have been developed by E. Agora, J. Antezana and C. Cabrelli in [\[1\]](#), where the authors could obtain a full characterization of Riesz bases for multi-tiling sets in LCA groups.

2. Notations and preliminaries

Let Λ be a lattice in an LCA group G . Denote by $Q_\Lambda \subset G$ a measurable cross section of G/Λ . By definition, a cross section is a set of representatives of all cosets in G/Λ , so that the intersection of Q_Λ with any coset $g + \Lambda$ has only one element. The existence of a Borel measurable cross section is guaranteed by [\[5, Theorem 1\]](#). Moreover, it is evident that (Q_Λ, Λ) is a tiling pair for G , while any tiling set Ω differs from a cross section at most for a zero measure set.

Let $d\dot{g}$ be a normalized Haar measure for G/Λ . Then the relation between the Haar measure on G and the Haar measure for G/Λ is given by *Weil's formula*: for any function $f \in L^1(G)$, the periodization map $\Phi(\dot{g}) = \sum_{\lambda \in \Lambda} f(g + \lambda), \dot{g} \in G/\Lambda$ is well defined almost everywhere in G/Λ , belongs to $L^1(G/\Lambda)$, and

$$\int_G f(g) dg = |Q_\Lambda| \int_{G/\Lambda} \sum_{\lambda \in \Lambda} f(g + \lambda) d\dot{g}. \tag{4}$$

This formula is a special case of [\[20, Theorem 3.4.6\]](#). The constant $|Q_\Lambda|$, called the *lattice size*, appears in [\(4\)](#) because G/Λ is equipped with the normalized Haar measure $d\dot{g}$.

Definition 5 (*Dual integrable representations* [\[7\]](#)). Let G be an LCA group, and let π be a unitary representation of G on a Hilbert space \mathcal{H} . We say that π is *dual integrable* if there exists a sesquilinear map $[\cdot, \cdot]_\pi : \mathcal{H} \times \mathcal{H} \rightarrow L^1(\widehat{G})$, called *bracket map* for π , such that

$$\langle \phi, \pi(g)\psi \rangle_{\mathcal{H}} = \int_{\widehat{G}} [\phi, \psi]_\pi(\chi) e_{-g}(\chi) d\chi \quad \forall g \in G, \quad \forall \phi, \psi \in \mathcal{H}.$$

Example 6. Let Λ be a lattice in an LCA group G . For any $\lambda \in \Lambda$, define $T_\lambda \phi(g) = \phi(g - \lambda)$ on $\phi \in L^2(G)$ and $M_\lambda h(\chi) = e_\lambda(\chi)h(\chi)$ on $h \in L^2(\widehat{G})$. Let us denote with Q_{Λ^\perp} a cross section for the annihilator lattice Λ^\perp . Thus, by Plancherel's formula [\(2\)](#) and Weil's formula [\(4\)](#), we have

$$\begin{aligned} \langle \phi, T_\lambda \psi \rangle_{L^2(G)} &= \langle \mathcal{F}_G(\phi), M_\lambda \mathcal{F}_G(\psi) \rangle_{L^2(\widehat{G})} = \int_{\widehat{G}} \mathcal{F}_G(\phi)(\chi) \overline{\mathcal{F}_G(\psi)(\chi)} e_{-\lambda}(\chi) \, d\chi \\ &= |Q_{\Lambda^\perp}| \int_{\widehat{G}/\Lambda^\perp} \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(\phi)(\dot{\chi} + \tilde{\lambda}) \overline{\mathcal{F}_G(\psi)(\dot{\chi} + \tilde{\lambda})} e_{-\lambda}(\dot{\chi} + \tilde{\lambda}) \, d\dot{\chi} \\ &= |Q_{\Lambda^\perp}| \int_{\widehat{G}/\Lambda^\perp} \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(\phi)(\dot{\chi} + \tilde{\lambda}) \overline{\mathcal{F}_G(\psi)(\dot{\chi} + \tilde{\lambda})} e_{-\lambda}(\dot{\chi}) \, d\dot{\chi}. \end{aligned}$$

Since $\mathcal{F}_G(\phi) \overline{\mathcal{F}_G(\psi)} \in L^1(\widehat{G})$, we have that

$$[\phi, \psi]_T(\dot{\chi}) := |Q_{\Lambda^\perp}| \sum_{\tilde{\lambda} \in \Lambda^\perp} \mathcal{F}_G(\phi)(\dot{\chi} + \tilde{\lambda}) \overline{\mathcal{F}_G(\psi)(\dot{\chi} + \tilde{\lambda})} \quad \text{a.e. } \dot{\chi} \in \widehat{G}/\Lambda^\perp$$

defines a sesquilinear map $[\cdot, \cdot]_T : L^2(G) \times L^2(G) \rightarrow L^1(\widehat{G}/\Lambda^\perp)$, so T is a dual integrable representation of Λ on $\mathcal{H} = L^2(G)$.

A relevant application of dual integrable representations is the possibility to characterize bases of unitary orbits in terms of their associated bracket maps. The following result has been proved in [7, Proposition 5.1].

Theorem 7. *Let G be a countable Abelian group, let π be a dual integrable representation of G on a Hilbert space \mathcal{H} , and let $\phi \in \mathcal{H}$. The system $\{\pi(g)\phi : g \in G\}$ is orthonormal in \mathcal{H} if and only if $[\phi, \phi]_\pi(\chi) = 1$ for almost every $\chi \in \widehat{G}$.*

As a consequence of Theorem 7 and of Example 6, and of the basic fact $|Q_\Lambda| |Q_{\Lambda^\perp}| = 1$ (for completeness, we have provided a proof in the appendix), we have the following result.

Corollary 8. *Let T and Λ be as in Example 6, and let $\phi \in L^2(G)$. Then the system of translates $\{T_\lambda \phi : \lambda \in \Lambda\}$ is an orthonormal system in $L^2(G)$ if and only if*

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\phi)(\chi + \tilde{\lambda})|^2 = |Q_\Lambda| \quad \text{a.e. } \chi \in \widehat{G}.$$

3. Proof of Theorem 2

In this section we shall prove Theorem 2 and its corollaries.

Proof of Theorem 2.

1) \Rightarrow 4) It is well-known [21] that, for any cross section Q_Λ , the exponential set E_{Λ^\perp} is an orthogonal basis for $L^2(Q_\Lambda)$. Thus, for all $f \in L^2(Q_\Lambda)$,

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f, \frac{1}{\sqrt{|Q_\Lambda|}} e_{\tilde{\lambda}} \rangle_{L^2(Q_\Lambda)}|^2 = \|f\|_{L^2(Q_\Lambda)}^2. \tag{5}$$

Since condition (1) says that Ω is contained in some cross section Q_Λ , then the previous identity still holds for all $f \in L^2(\Omega)$. Hence E_{Λ^\perp} is a tight frame for $L^2(\Omega)$ with constant $|Q_\Lambda|$.

4) \Rightarrow 1) Suppose, by contradiction, that Ω is not a subtiling set. Then we claim that for all cross section Q_Λ , there exist $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_2 \neq 0$, such that

$$|(Q_\Lambda + \lambda_1) \cap \Omega \cap (\Omega + \lambda_2)| > 0. \tag{6}$$

If this is true, then let $\Omega_1 = (Q_\Lambda + \lambda_1) \cap \Omega \cap (\Omega + \lambda_2)$, and $\Omega_2 = \Omega_1 - \lambda_2$. Both are subsets of Ω with positive measure and, since $\lambda_2 \neq 0$, they are disjoint because $\Omega_1 \subset Q_\Lambda + \lambda_1$ and $\Omega_2 \subset Q_\Lambda + \lambda_1 - \lambda_2$. Therefore, the function

$$f = \mathbf{1}_{\Omega_1} - \mathbf{1}_{\Omega_2}$$

is nonzero and belongs to $L^2(\Omega)$. Then, for all $\tilde{\lambda} \in \Lambda^\perp$ we have

$$\langle f, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)} = \int_{\Omega_1} e_{\tilde{\lambda}}(g) \, dg - \int_{\Omega_2} e_{\tilde{\lambda}}(g) \, dg = \int_{\Omega_1} (e_{\tilde{\lambda}}(g) - e_{\tilde{\lambda}}(g - \lambda_2)) \, dg = 0.$$

This implies that the system E_{Λ^\perp} can not be a frame for $L^2(\Omega)$.

In order to prove (6), let us proceed by contradiction and suppose that for all $\lambda \in \Lambda$ and all $\lambda^* \in \Lambda$, $\lambda^* \neq 0$, we have

$$|(Q_\Lambda + \lambda) \cap \Omega \cap (\Omega + \lambda^*)| = 0.$$

Now take $\lambda' \in \Lambda$, $\lambda' \neq 0$. By definition of cross section, we have

$$\Omega \cap (\Omega + \lambda') = \bigsqcup_{\lambda \in \Lambda} (Q_\Lambda + \lambda) \cap \Omega \cap (\Omega + \lambda')$$

which implies that $|\Omega \cap (\Omega + \lambda')| = 0$. Hence, Ω would be a subtiling set of G by Λ , which is a contradiction.

1) \Rightarrow 2) Since (5) holds, we can obtain 2) by choosing $f = \overline{e_\chi} \mathbf{1}_\Omega$.

2) \Rightarrow 3) This follows as an application of Corollary 8.

3) \Rightarrow 1) By orthogonality, we have that for all $\lambda \in \Lambda$, $\lambda \neq 0$

$$0 = \langle \mathbf{1}_\Omega, \mathbf{1}_\Omega(\cdot - \lambda) \rangle_{L^2(G)} = |\Omega \cap (\Omega + \lambda)|$$

so Ω is sub-tiling. \square

Proof of Corollary 3. If (Ω, Λ) is a tiling pair, then it is well-known that E_{Λ^\perp} is an orthogonal basis for $L^2(\Omega)$. To prove the converse, assume by contradiction that Ω is not tiling. Then one of the following cases holds:

- i. Ω is a strictly sub-tiling set, i.e. there exists a cross section Q_Λ such that $\Omega \subset Q_\Lambda$ and $|Q_\Lambda \setminus \Omega| > 0$;
- ii. Ω is not a sub-tiling set, so that (6) holds.

For case i., observe that the assumption of E_{Λ^\perp} being an orthogonal basis for $L^2(\Omega)$ implies

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f, \frac{1}{\sqrt{|\Omega|}} e_{\tilde{\lambda}} \rangle_{L^2(\Omega)}|^2 = \|f\|_{L^2(\Omega)}^2 \quad \forall f \in L^2(\Omega).$$

On the other hand, since E_{Λ^\perp} is an orthogonal basis for $L^2(Q_\Lambda)$, we have

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f \mathbf{1}_\Omega, \frac{1}{\sqrt{|Q_\Lambda|}} e_{\tilde{\lambda}} \rangle_{L^2(Q_\Lambda)}|^2 = \|f \mathbf{1}_\Omega\|_{L^2(Q_\Lambda)}^2 \quad \forall f \in L^2(\Omega)$$

so that $|\Omega| = |Q_\Lambda|$, which contradicts i.

For case ii., in Theorem 2 we already proved that E_{Λ^\perp} can not even be a frame. \square

Proof of Corollary 4. Assume 4) holds, i.e. that E_{Λ^\perp} is a tight frame for $L^2(\Omega)$ with constant $|Q_\Lambda|$. Then

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f, M_{\tilde{\lambda}} \psi \rangle_{L^2(\Omega)}|^2 = \sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f \overline{\psi}, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)}|^2 = |Q_\Lambda| \|f \overline{\psi}\|_{L^2(\Omega)}^2 \quad \forall f \in L^2(\Omega).$$

Since $A \|f\|_{L^2(\Omega)}^2 \leq \|f \overline{\psi}\|_{L^2(\Omega)}^2 \leq B \|f\|_{L^2(\Omega)}^2$, this proves 5). Conversely, assume that 5) holds. Then, since $A > 0$, for all $f \in L^2(\Omega)$ we can write

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)}|^2 = \sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f \overline{\psi}, M_{\tilde{\lambda}} \psi \rangle_{L^2(\Omega)}|^2,$$

so that, by the hypotheses on ψ , we get

$$\frac{A}{B} |Q_\Lambda| \|f\|_{L^2(\Omega)}^2 \leq \sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f, e_{\tilde{\lambda}} \rangle_{L^2(\Omega)}|^2 \leq \frac{B}{A} |Q_\Lambda| \|f\|_{L^2(\Omega)}^2 \quad \forall f \in L^2(\Omega).$$

Thus E_{Λ^\perp} is a frame, hence proving 4). Observe that, by Theorem 2, this implies that E_{Λ^\perp} is a tight frame with constant $|Q_\Lambda|$, hence improving the inequalities above. \square

4. Comments on the related work

The statement of Theorem 2 relating subtilings pairs (Ω, Λ) for a lattice Λ and frames of exponentials for $L^2(\Omega)$ is new. However, the proof has similarities with existing proofs of the similar statement for tiling sets by lattices and orthonormal bases of exponentials.

The result in Corollary 3 is proved by B. Fuglede ([6], Lemma 6) for $G = \mathbb{R}^n$ and by S. Pedersen ([18], Theorem 3.6) for LCA groups.

In both papers, [6] and [18], as well as in the present work, the implication 1) \Rightarrow 4) is done in the same way by observing that E_{Λ^\perp} is an orthogonal basis of exponentials of $L^2(Q_\Lambda)$. As for the implication 4) \Rightarrow 1), both papers give a direct proof of the fact that $\Omega \cap (\Omega + \lambda)$ have measure zero for all $\lambda \in \Lambda$ and that also the set $G \setminus \bigcup_{\lambda \in \Lambda} (\Omega + \lambda)$ has measure zero. In the present paper, we give an argument by contradiction assuming that 1) does not hold and exhibiting a non-zero function in $L^2(\Omega)$, which is perpendicular to all $e_{\tilde{\lambda}}, \tilde{\lambda} \in \Lambda^\perp$.

The manuscript [8] by A. Iosevich gives a proof of the equivalence of 1) and 2) in Theorem 2 for a tiling set $\Omega \in \mathbb{R}^n$ by a lattice Λ . It is then stated without proof that 2) and 4) are equivalent by a density argument. The proof of 1) \Leftrightarrow 2) in [8] goes as follows. Consider the functions

$$f(x) \equiv \sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(x + \lambda), \quad x \in \mathbb{R}^n \quad \text{and} \quad H(\xi) \equiv \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_{\mathbb{R}^n}(\mathbf{1}_\Omega)(\xi + \tilde{\lambda})|^2.$$

Using Fourier Analysis, it can be shown that, as a periodic function in $L^2(Q_{\Lambda^\perp})$, the Fourier coefficients of H are

$$\widehat{H}(\lambda) = \frac{|\Omega \cap (\Omega + \lambda)|}{|Q_{\Lambda^\perp}|}, \quad \lambda \in \Lambda, \tag{7}$$

and the Fourier coefficients of f , as a periodic function in $L^2(Q_\Lambda)$, are

$$\widehat{f}(\tilde{\lambda}) = \frac{\mathcal{F}_{\mathbb{R}^n}(\mathbf{1}_\Omega)(\tilde{\lambda})}{|Q_\Lambda|}, \quad \tilde{\lambda} \in \tilde{\Lambda}. \tag{8}$$

Assuming that 1) of Theorem 2 holds for a tiling set Ω , equation (7) shows, using the Fourier inversion theorem, that H is constant with value $|\Omega||Q_\Lambda|$ a.e., since $|Q_\Lambda||Q_{\Lambda^\perp}| = 1$. This shows 2) of Theorem 2. Conversely, assuming that 2) of Theorem 2 holds, equation (7) shows that Ω is a subtiling set of \mathbb{R}^n by Λ . Equation (8) is then used to show that $f(x) = 1$ a.e., which shows 1).

A proof along the lines described above can be designed for LCA groups and subtiles. In the proof given in the present work, we have proved the equivalence of 1), 2) and 3) in Theorem 2 by using the notion of bracket map (see [7] and the references therein) and the characterization of frame sequences of translates of a single function along lattices stated in Theorem 7 and Corollary 8. As in [8], our paper also uses Fourier Analysis to compute Fourier coefficients (see Example 6).

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Appendix A

We provide here a self-contained proof of the following basic result.

Lemma 9. *Let G be an LCA group, let \widehat{G} be its dual group, and let the Haar measures dg on G and $d\chi$ on \widehat{G} be chosen in such a way that the group Fourier transform be an isometry, i.e. such that (2) holds. Let $\Lambda \subset G$ be a lattice subgroup of G , and denote with $Q_\Lambda \subset G$ a cross section for G/Λ , let $\Lambda^\perp \subset \widehat{G}$ be the dual lattice of Λ , and denote with $Q_{\Lambda^\perp} \subset \widehat{G}$ a cross section for $\widehat{G}/\Lambda^\perp$. Then*

$$|Q_\Lambda||Q_{\Lambda^\perp}| = 1$$

i.e. the product of the size of the lattice times the size of the dual lattice, computed with respect to Haar measures satisfying (2), is 1.

Proof. Observe first that, by (2), we have

$$\begin{aligned} |Q_\Lambda| &= \int_G |\mathbf{1}_{Q_\Lambda}(g)|^2 dg = \int_{\widehat{G}} |\mathcal{F}_G(\mathbf{1}_{Q_\Lambda})(\chi)|^2 d\chi \\ &= \int_{Q_{\Lambda^\perp}} \sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\mathbf{1}_{Q_\Lambda})(\chi + \tilde{\lambda})|^2 d\chi, \end{aligned} \tag{9}$$

where the last identity is due to the fact that $(Q_{\Lambda^\perp}, \Lambda^\perp)$ is a tiling pair for \widehat{G} by definition of cross-section. We can thus apply the same argument used to prove point 2) of [Theorem 2](#): since $\{e_{\tilde{\lambda}} : \tilde{\lambda} \in \Lambda^\perp\}$ is an orthogonal basis of $L^2(Q_\Lambda)$, then it satisfies (5), so by choosing $f = \overline{e_\chi} \mathbf{1}_{Q_\Lambda}$ we get

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\mathbf{1}_{Q_\Lambda})(\chi + \tilde{\lambda})|^2 = |Q_\Lambda|^2 \quad \text{a.e. } \chi \in \widehat{G}.$$

The claim then follows by inserting this identity in (9). \square

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