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Partial differential equations/Numerical analysis

On non-linear Stokes problems with viscosity depending on the distance to the wall ☆

Sur l'étude de problèmes de Stokes non linéaires avec une viscosité qui dépend de la distance à la paroi

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A R T I C L E   I N F O

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A B S T R A C T

In fluid mechanics, the RANS modeling (Reynolds Averaged Navier–Stokes equation) assumes that the period of the averaged solutions to the Navier–Stokes equations is several orders of magnitude larger than the turbulent fluctuations. A type of simple model often used by engineers is a mixing-length model called "Smagorinsky modeling". In this paper, we present some theoretical and numerical results on a mixing-length model in which the eddy viscosity is depending on the strain tensor and on the distance to the wall of the fluid flow domain. In particular, we show that the so-called von Karman model becomes an ill-posed problem when the laminar viscosity tends to zero.

R É S U M É

En mécanique des fluides, la décomposition de Reynolds appliquée aux équations de Navier–Stokes suppose que la période des solutions moyennées est bien plus grande que les fluctuations turbulentes locales. Un modèle de longueur de mélange populaire chez les ingénieurs est appelé « modèle de Smagorinsky ». Dans cette courte note, nous présentons quelques résultats théoriques et numériques sur un modèle de longueur de mélange dont la viscosité turbulente dépend à la fois du tenseur des déformations et de la distance à la paroi la plus proche. En particulier, nous montrons que le modèle dit de von Karman devient mal posé lorsque la viscosité laminaire tend vers zéro.

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1. Introduction

If $\boldsymbol{u}$ and p are the velocity and the pressure of an incompressible viscous fluid of density ρ, submitted to a force $\boldsymbol{f}$, flowing in a cavity $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$, the Navier–Stokes equations on Ω take the form

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div}(2\mu \boldsymbol{\epsilon}(\boldsymbol{u})) + \nabla p = \boldsymbol{f}, \tag{1}$$

$$\operatorname{div}(\boldsymbol{u}) = 0, \tag{2}$$

with $\boldsymbol{u} = 0$ on $\partial\Omega$, where here $\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$ and $\boldsymbol{f}$ is a given smooth force field.

In Smagorinsky's models [8], the viscosity μ depends on $\boldsymbol{\epsilon}(\boldsymbol{u})$ and on a length scale, which is usually function of the discretization mesh size and the distance $d_{\partial\Omega}(\boldsymbol{x})$ of a point $\boldsymbol{x} \in \Omega$ to the boundary $\partial\Omega$, to reflect the fact that turbulence near walls is different. In this paper, we treat Smagorinsky models in which the viscosity takes the form

$$\mu = \rho(\nu_0 + l^{2-\alpha} \varkappa^{\alpha} d^{\alpha}_{\partial\Omega} |\boldsymbol{\epsilon}(\boldsymbol{u})|), \tag{3}$$

where $\nu_0 > 0$ corresponds to a laminar kinematic viscosity, $|\boldsymbol{\epsilon}| = (\Sigma^3_{i,j=1} \boldsymbol{\epsilon}^2_{i,j})^{1/2}$, $\varkappa = 0.41$ is the von Kármán constant, α is a positive constant and l is a characteristic length of Ω.

The cases with $\alpha = 0$ can be treated in usual Sobolev spaces and their analysis can be found in some papers (Baranger, [1]). At the opposite, the cases with $\alpha > 0$ have to be treated in weighted Sobolev spaces and present several difficulties.

In particular, a very popular model for treating a fluid flow in-between two plates is the von Kármán model corresponding to $\alpha = 2$. We show that this model becomes ill posed when the kinematic laminar viscosity ν_0 is vanishing. Moreover, for $\alpha = 2$, Korn's inequality is probably missing (see Remark 3 below). The well-posed character of the problem is an open problem.

2. Main results

In this short note, we start by presenting some results proven in [7] concerning the analysis of a stationary Stokes problem with a viscosity given by (3). To do this, let us consider the stationary Stokes problem with renormalized p and $\mathbf{f}$ by ρ $(p := p/\rho, \mathbf{f} := \mathbf{f}/\rho)$:

$$-2\,\operatorname{div}[(\nu_0 + l^{2-\alpha} \varkappa^{\alpha} d^{\alpha}_{\partial\Omega} |\boldsymbol{\epsilon}(\boldsymbol{u})|)\cdot \boldsymbol{\epsilon}(\boldsymbol{u})] + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega, \tag{4}$$

$$\operatorname{div}(\boldsymbol{u}) = 0, \quad \text{in } \Omega, \tag{5}$$

$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega. \tag{6}$$

Here, Ω is a bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$, $\boldsymbol{f} \in L^2(\Omega)^3$ and $0 \le \alpha < 2$. In the following, $L^3_{d^\alpha}(\Omega)$ will denote the set of measurable functions $g : \Omega \to \mathbb{R}$ such that $\int_\Omega |g|^3 d^{\alpha}_{\partial\Omega} \mathrm{d}x < \infty$ and $W^{1,3}_{d^\alpha,0}(\Omega)$ will be the set of functions of $L^3_{d^\alpha}(\Omega)$ whose first derivatives are in $L^3_{d^\alpha}(\Omega)$ with vanishing trace on $\partial\Omega$ (see [6]). The space $W^{1,3}_{d^\alpha}(\Omega)$ is provided with the norm $\|g\|_{W^{1,3}_{d^\alpha}(\Omega)} = (\int_\Omega (|g|^3 + |\nabla g|^3) d^{\alpha}_{\partial\Omega}(x) \mathrm{d}x)^{\frac{1}{3}}$. In order to obtain a natural formulation of Equation (4), we have to define the reflexive Banach space $\boldsymbol{X} = H^1_0(\Omega)^3 \cap W^{1,3}_{d^\alpha,0}(\Omega)^3$ provided with the norm $\|\cdot\|_{H^1(\Omega)} + \|\cdot\|_{W^{1,3}_{d^\alpha}(\Omega)}$. Moreover, we denote by $\boldsymbol{X}_{\mathrm{div}}$ the space $\boldsymbol{X}_{\mathrm{div}} := \{\boldsymbol{u} \in \boldsymbol{X} : \operatorname{div}(\boldsymbol{v}) = 0\}$.

Multiplying (4) by a test function $\boldsymbol{v}$, integrating by parts on Ω and taking into account (5) and (6), we can show that $\boldsymbol{u}$ is solution to the following problem: find $\boldsymbol{u} \in \boldsymbol{X}_{\mathrm{div}}$ satisfying

$$\int_\Omega 2(\nu_0 + l^{2-\alpha} \varkappa^{\alpha} d^{\alpha}_{\partial\Omega} |\boldsymbol{\epsilon}(\boldsymbol{u})|)\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}) \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \mathrm{d}x, \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{\mathrm{div}}. \tag{7}$$

Remark 1. When Ω is a polyhedral domain, it is easy to show that $W^{1,3}_{d^\alpha,0}(\Omega) \subset H^1_0(\Omega)$ with continuous injection as $0 \le \alpha < \frac{1}{2}$, implying that $\boldsymbol{X} = W^{1,3}_{d^\alpha,0}(\Omega)^3$. However, when $\alpha \ge \frac{1}{2}$, $W^{1,3}_{d^\alpha,0}(\Omega)$ is no more contained in $H^1_0(\Omega)$ and conversely. It is the reason for which $\boldsymbol{X}$ is defined with $H^1_0(\Omega)^3 \cap W^{1,3}_{d^\alpha,0}(\Omega)^3$.

By considering the real function $A : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ defined by

$$A(s, x) = \nu_0 \frac{s^2}{2} + l^{2-\alpha} \varkappa^{\alpha} d^{\alpha}_{\partial\Omega}(x) \frac{s^3}{3}, \tag{8}$$

we can easily show that the functional $J : \boldsymbol{X}_{\mathrm{div}} \to \mathbb{R}$ given by

$$J(\boldsymbol{v}) = \int_\Omega [2A(|\boldsymbol{\epsilon}(\boldsymbol{v})|, x) - \boldsymbol{f}\cdot\boldsymbol{v}]\mathrm{d}x \tag{9}$$

is $C^1(\boldsymbol{X}_{\mathrm{div}})$ ([6,3]) and its derivative DJ in $\boldsymbol{u} \in \boldsymbol{X}_{\mathrm{div}}$ takes on the form

$$\mathrm{D}J(\boldsymbol{u})\,\boldsymbol{v} = \int_\Omega [2(\nu_0 + l^{2-\alpha}\varkappa^\alpha d^\alpha_{\partial\Omega}|\boldsymbol{\epsilon}(\boldsymbol{u})|)\boldsymbol{\epsilon}(\boldsymbol{u}):\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{f}\cdot\boldsymbol{v}]\mathrm{d}x, \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{\mathrm{div}}. \tag{10}$$

By comparison with (7), we conclude that the Stokes Problem (10) can be interpreted as Euler Equation $\mathrm{D}J(\boldsymbol{u}) = 0$ for the minimization of the functional J in a "saddle point" formulation with the constraint $\mathrm{div}(\boldsymbol{u}) = 0$.

Using standard Korn's inequality in $H^1_0(\Omega)^3$ and Korn's inequality in the weighted Sobolev space $W^{1,3}_{d^\alpha,0}(\Omega)^3$ with $0 < \alpha < 2$ ([5]) together with the compactness of the embedding of $W^{1,3}_{d^\alpha,0}(\Omega)^3$ in $L^3_{d^\alpha}(\Omega)$ ([4]), we can prove that the functional J is coercive on $\boldsymbol{X}_{\mathrm{div}}$ (see [2] for the definition). After remarking that J is strictly convex, we can prove the main result of this paper:

Theorem 2.1. *Under hypotheses $0 \le \alpha < 2$ and $\boldsymbol{f} \in L^2(\Omega)^3$, there exists a unique $\boldsymbol{u} \in \boldsymbol{X}_{\mathrm{div}}$ satisfying $J(\boldsymbol{u}) \le J(\boldsymbol{v})$, for every $\boldsymbol{v}$ in $\boldsymbol{X}_{\mathrm{div}}$. Moreover $\boldsymbol{u}$ is the unique solution to (7).*

In order to obtain a weak solution $(\boldsymbol{u}, p)$ of equations (4)–(6), it is natural to look for p in the space $Y = L^2_0 \oplus L^{3/2}_{d^{-\alpha/2},0}$ (index zero means that the integral is vanishing) by using an inf–sup condition with the spaces $\boldsymbol{X}$–Y. Actually this is an open problem when $W^{1,3}_{d^\alpha,0}(\Omega)$ is not embedded in $H^1_0(\Omega)$.

However, we can prove the following relations using inf–sup conditions $L^2_0 - H^1_0$ and $L^{3/2}_{d^{-\alpha/2},0} - W^{1,3}_{d^\alpha,0}(\Omega)$:

Lemma 1. *There exists a unique p_1 in L^2_0 and a unique p_2 in $L^{3/2}_{d^{-\alpha/2},0}$ satisfying*

$$\int_\Omega p_1\cdot\mathrm{div}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega 2\nu_0\boldsymbol{\epsilon}(\boldsymbol{u}):\boldsymbol{\epsilon}(\boldsymbol{v})\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}x, \quad \forall\, \boldsymbol{v} \in H^1_0(\Omega)^3, \tag{11}$$

$$\int_\Omega p_2\cdot\mathrm{div}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega 2l^{2-\alpha}\varkappa^\alpha \mathrm{d}^\alpha_{\partial\Omega}|\boldsymbol{\epsilon}(\boldsymbol{u})|\,\boldsymbol{\epsilon}(\boldsymbol{u}):\boldsymbol{\epsilon}(\boldsymbol{v})\mathrm{d}x, \ \forall\, \boldsymbol{v} \in W^{1,3}_{d^\alpha,0}(\Omega)^3. \tag{12}$$

We thus immediately obtain the following existence result:

Theorem 2.2. *For $0 \le \alpha < 2$, the couple $(\boldsymbol{u}, p = p_1 + p_2) \in \boldsymbol{X} \times Y$ is solution to the non-linear Stokes problem:*

$$\int_\Omega 2(\nu_0 + l^{2-\alpha}\varkappa^\alpha d^\alpha_{\partial\Omega}|\boldsymbol{\epsilon}(\boldsymbol{u})|)\boldsymbol{\epsilon}(\boldsymbol{u}):\boldsymbol{\epsilon}(\boldsymbol{v}) - \int_\Omega p\,\mathrm{div}(\boldsymbol{v})\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x} \ \ \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{13}$$

$$\int_\Omega q\,\mathrm{div}(\boldsymbol{u})\mathrm{d}\boldsymbol{x} = 0 \quad \forall q \in Y. \tag{14}$$

The problem of the uniqueness of pressure p is open when $W^{1,3}_{d^\alpha,0}(\Omega)$ is not included in $H^1_0(\Omega)$. This is the case when α is close to 2.

Remark 2. For $0 \le \alpha < 2$, we can take $\nu_0 = 0$ and minimize the functional J on $\boldsymbol{X}_{\mathrm{div}} = \{\boldsymbol{v} \in W^{1,3}_{d^\alpha,0}(\Omega)^3 : \mathrm{div}(\boldsymbol{v}) = 0\}$. Then we have existence and uniqueness of the solution $(\boldsymbol{u}, p = p_2)$ of the non-linear Stokes problem. This result remains true when ν_0 is strictly positive and α is such that $W^{1,3}_{d^\alpha,0}(\Omega) \subset H^1_0(\Omega)$. Indeed, in this case $L^{3/2}_{d^{-\alpha/2},0}(\Omega) \subset L^2_0(\Omega)$ and $Y = L^{3/2}_{d^{-\alpha/2},0}(\Omega)$.

Remark 3. For $\alpha = 2$ (von Karman model), there is no trace of $\boldsymbol{u}$ on $\partial\Omega$ when $\boldsymbol{u} \in W^{1,3}_{d^2}(\Omega)^3$. An example is given by $g(x) = \ln(|\ln(x)|)$, $x \in (0, \frac{1}{2})$. So we cannot take $\nu_0 = 0$ when imposing $\boldsymbol{u} = \boldsymbol{0}$ on the boundary. When $\nu_0 = 0$, the von Kármán model is ill-posed.

Remark 4. A consequence of Remark 3 is that when using a numerical method applied to the von Kármán model with ν_0 small with respect to the numerical viscosity, the obtained results depend on the mesh of the method.

Table 1
Numerical resolution of the non-linear stationary Stokes problem (4)–(6) (top) and the stationary Navier–Stokes problem (down). The domain is a rectangular parallelepiped $\Omega = [0,1] \times [0,1] \times [0,0.1]$ with N nodes on each side for a total of $5N^3$ tetrahedra. The force is given by $\boldsymbol{f} = (0.3 * (y-0.5)^2, 0.3 * (x-0.5)^2, 0)$ and we set $l = 0.1$.

		$\alpha = 0$						$\alpha = 2$					
		$\nu_0 = 1e^{-5}$			$\nu_0 = 1e^{-7}$			$\nu_0 = 1e^{-5}$			$\nu_0 = 1e^{-7}$		
		ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T
N	20	4.05e−4	4.84e−2	12	4.08e−4	4.86e−2	12	1.79e−4	2.01e−1	112	1.85e−4	2.07e−1	112
	40	4.22e−4	5.21e−2	12	4.29e−4	5.28e−2	12	1.88e−4	2.68e−1	142	1.95e−4	2.81e−1	144
	80	4.32e−4	5.32e−2	12	4.36e−4	5.38e−2	12	1.97e−4	3.20e−1	162	2.10e−4	3.45e−1	173

		$\alpha = 0$						$\alpha = 2$					
		$\nu_0 = 1e^{-5}$			$\nu_0 = 1e^{-7}$			$\nu_0 = 1e^{-5}$			$\nu_0 = 1e^{-7}$		
		ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T
N	20	4.01e−4	4.83e−2	12	4.06e−4	4.85e−2	12	1.61e−4	1.95e−1	121	1.68e−4	2.01e−1	121
	40	4.21e−4	5.20e−2	12	4.27e−4	5.26e−2	12	1.64e−4	2.45e−1	149	1.66e−4	2.55e−1	152
	80	4.30e−4	5.30e−1	12	4.34e−4	5.33e−2	12	1.62e−4	2.79e−1	172	1.69e−4	2.98e−1	177

Remark 5. Korn's inequality is probably wrong in $W^{1,3}_{d^2}(\Omega)^3$ (Kalamajska's conjecture [5]). The well-posed character of the von Kármán model for the stationary Stokes problem is an open question.

Remark 6. The minimization technique of a functional used above to prove the existence of a velocity field cannot be applied to the Navier–Stokes equations. However, the numerical computations obtained on a discretization of stationary Navier–Stokes equations lead to the same conclusions as those mentioned above because the turbulent viscosity gives rise to a small local Reynolds number (see Table 1).

3. Numerical experiments

We finish this short note with some numerical results related to the problems from the previous section, with a particular attention to the case $\alpha = 2$. Indeed, from Remark 4, when using a finite element approximation on von Kármán model ($\alpha = 2$) and when the kinematic laminar viscosity ν_0 is "small" with respect to the numerical viscosity of the method, the obtained results can depend strongly on the mesh. In this case we take a very thin mesh close to the walls in order to overcome this situation.

In Table 1 we present numerical results of the non-linear stationary Stokes problem (4)–(6) discretized with the $\mathbb{P}_1$/bubble $- \mathbb{P}_1$ finite element method. The non-linearity is treated with Newton's method and each linear system is solved with the preconditioned GMRES algorithm. We display for different values of α and ν_0 the maximum normed velocity field u_{max}, the numerical viscosity ν_T (the numerical value of $l^{2-\alpha}\varkappa^\alpha d^\alpha_{\partial\Omega}|\boldsymbol{\epsilon}(\boldsymbol{u}_{max})|$) and the resulting Reynolds number $Re_T = \frac{u_{max} l}{\nu_T}$. The numerical results depend strongly on the mesh when $\alpha = 2$. The same behavior is observed for the stationary Navier–Stokes equations corresponding to (4)–(6).

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