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Homogeneous Hermitian holomorphic vector bundles and the Cowen–Douglas class over bounded symmetric domains ☆

Fibrés vectoriels homogènes holomorphes hermitiens et classe de Cowen–Douglas sur des domaines bornés symétriques

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ABSTRACT

It is known that all the vector bundles of the title can be obtained by holomorphic induction from representations of a certain parabolic group on finite-dimensional inner product spaces. The representations, and the induced bundles, have composition series with irreducible factors. We write down an equivariant constant coefficient differential operator that intertwines the bundle with the direct sum of its irreducible factors. As an application, we show that in the case of the closed unit ball in $\mathbb{C}^n$ all homogeneous n-tuples of Cowen–Douglas operators are similar to direct sums of certain basic n-tuples.

RÉSUMÉ

Il est bien connu que les fibrés vectoriels homogènes holomorphes hermitiens peuvent être obtenus par induction holomorphe à partir des representations de dimension finie d'un certain groupe parabolique. Les représentations, ainsi que les fibrés induits, ont des séries de composition à quotients irréductibles. On montre qu'il existe un opérateur différentiel invariant à coefficients constants qui entrelace le fibré et la somme directe de ses quotients irréductibles. Comme application, on montre que tous les n-tuples d'opérateurs homogènes de la classe de Cowen–Douglas associés à la boule dans $\mathbb{C}^n$ sont similaires à des sommes directes de certains n-tuples fondamentaux.

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1. Holomorphic vector bundles

Let $\mathfrak{g}$ be a simple non-compact Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that $\mathfrak{k}$ is not semi-simple. Then $\mathfrak{k}$ is the direct sum of its center and of its semisimple part, $\mathfrak{k} = \mathfrak{z} + \mathfrak{k}_{ss}$, and there is an element $\hat{z}$ that generates $\mathfrak{z}$ and $\mathrm{ad}(\hat{z})$ is a complex structure on $\mathfrak{p}$.

The complexification $\mathfrak{g}^{\mathbb{C}}$ is then the direct sum $\mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ of the $i, 0, -i$ eigenspaces of $\mathrm{ad}(\hat{z})$. We let $G^{\mathbb{C}}$ denote the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and we let $G, K^{\mathbb{C}}, K, P^{\pm}, \ldots$ be the analytic subgroups corresponding to $\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{k}, \mathfrak{p}^{\pm}, \ldots$ We denote by $\tilde{G}$ the universal covering group of the group G and by $\tilde{K}, \tilde{K}_{ss}, \ldots$ its analytic subgroups corresponding to $\mathfrak{k}, \mathfrak{k}_{ss}, \ldots$.

$K^{\mathbb{C}}P^-$ is a parabolic subgroup of $G^{\mathbb{C}}$. $P^+K^{\mathbb{C}}P^-$ is open dense in $G^{\mathbb{C}}$. The corresponding decomposition $g^+g^0g^-$ of any g in $P^+K^{\mathbb{C}}P^-$ is unique. The natural map $G/K \to G^{\mathbb{C}}/K^{\mathbb{C}}P^-$ is a holomorphic embedding, its image is in the orbit of P^+. Applying now $\exp_{\mathfrak{p}^+}^{-1}$, we get the Harish–Chandra realization of G/K as a bounded symmetric domain $\mathcal{D} \subset \mathfrak{p}^+$. The action of $g \in G$ on $z \in \mathcal{D}$, written $g \cdot z$, is then defined by $\exp(g \cdot z) = (g \exp z)^+$. We will use the notation $k(g, z) = (g \exp z)^0$ and $\exp Y(g, z) = (g \exp z)^-$, so we have

$$g \exp z = (\exp(g \cdot z))k(g, z) \exp(Y(g, z)).$$

The $\tilde{G}$-homogeneous Hermitian holomorphic vector bundles (hHhvb) over $\mathcal{D}$ are obtained by holomorphic induction from representations (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ on finite-dimensional inner product spaces V such that $\rho(\mathfrak{k})$ is skew Hermitian. We write ρ^0, ρ^- for the restrictions of ρ to $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^-$, respectively. The representation space V is the orthogonal direct sum of its subspaces V_λ $(\lambda \in \mathbb{R})$ on which $\rho^0(\hat{z}) = \mathrm{i}\lambda$. It is easy to see that $\rho^-(Y)V_\lambda \subset V_{\lambda-1}$ for $Y \in \mathfrak{p}^-$. We also have

$$\rho^-([Z, Y]) = [\rho^0(Z), \rho^-(Y)], \;\; Z \in \mathfrak{k}^{\mathbb{C}}, \; Y \in \mathfrak{p}^-. \tag{1}$$

We note that if representations ρ^0 and ρ^- of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^-$, respectively, are given, then they will together give a representation of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ if and only if equation (1) holds. We call (ρ, V) and the induced bundle, indecomposable if it is not the orthogonal sum of sub-representations, respectively, sub-bundles. We restrict ourselves to describing these.

Proposition 1.1. *Every indecomposable holomorphic homogeneous Hermitian vector bundle E can be written as a tensor product $L_{\lambda_0} \otimes E'$, where L_{λ_0} is the line bundle induced by a character χ_{λ_0} and E' is the lift to $\tilde{G}$ of a G-homogeneous holomorphic Hermitian vector bundle, which is the restriction to G and $\mathcal{D}$ of a $G^{\mathbb{C}}$-homogeneous vector bundle induced in the holomorphic category by a representation of $K^{\mathbb{C}}P^-$.*

The proof involves some structural properties of $G^{\mathbb{C}}$, which we omit in this short Announcement.

As shown in [1], $P^+ \times \tilde{K}^{\mathbb{C}} \times P^-$ can be given a structure of complex analytic local group such that (writing $\pi : \tilde{K}^{\mathbb{C}} \to K^{\mathbb{C}}$) $\mathrm{id} \times \pi \times \mathrm{id}$ is the universal local group covering of $P^+K^{\mathbb{C}}P^-$. We write $\tilde{G}_{\mathrm{loc}}$ for this local group and abbreviate $\mathrm{id} \times \pi \times \mathrm{id}$ to π. By [1], $\tilde{G}$, $\tilde{K}^{\mathbb{C}}P^-$, $P^+\tilde{K}^{\mathbb{C}}$ are closed subgroups of $\tilde{G}^{\mathbb{C}}_{\mathrm{loc}}$ and $\tilde{G} \exp \mathcal{D} \subset \tilde{G}^{\mathbb{C}}_{\mathrm{loc}}$. Defining $g \cdot z = \pi(g) \cdot z$ and $Y(g, z) = Y(\pi(g), z)$ we have the decomposition

$$g \exp z = (\exp g \cdot z)\tilde{k}(g, z) \exp Y(g, z), \;\; (g \in \tilde{G}, z \in \mathcal{D})$$

in $\tilde{G}_{\mathrm{loc}}$. We write $\tilde{b}(g, z) = \tilde{k}(g, z) \exp Y(g, z)$; then $\tilde{b}(g, z)$ satisfies the multiplier identity and $\tilde{b}(kp^-, 0) = kp^-$ for $kp^- \in \tilde{K}^{\mathbb{C}}P^-$.

Hence given a representation (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ as above, the holomorphically induced bundle has a canonical trivialization such that the sections are the elements of $\mathrm{Hol}(\mathcal{D}, V)$, and $\tilde{G}$ acts via the multiplier

$$\rho(\tilde{b}(g, z)) = \rho^0(\tilde{k}(g, z))\rho^-(\exp Y(g, z)).$$

If $f \in \mathrm{Hol}(\mathcal{D}, V)$, then we write Df for the derivative: $Df(z)X = (D_X f)(z)$ for $X \in \mathfrak{p}^+$. Thus $Df(z)$ is a $\mathbb{C}$-linear map from $\mathfrak{p}^+$ to V.

Lemma 1.2. *For any holomorphic representation τ of $\tilde{K}^{\mathbb{C}}$ and any $g \in \tilde{G}$, $z \in \mathcal{D}$, $X \in \mathfrak{p}^+$,*

$$D_X \tau\big(\tilde{k}(g, z)^{-1}\big) = -\tau\big([Y(g, z), X]\big)\tau\big(\tilde{k}(g, z)^{-1}\big).$$

Furthermore,

$$D_X Y(g, z) = \frac{1}{2}\big[Y(g, z), [Y(g, z), X]\big].$$

This is proved by refining the arguments of [4, p. 65].

2. The main results about vector bundles

If in the set-up of Section 1, each subspace V_λ is irreducible under $\mathfrak{k}^{\mathbb{C}}$; we call the corresponding representations and the vector bundles *filiform*. We consider this case first.

We have seen that every indecomposable filiform representation is a direct sum of subspaces $V_{\lambda-j}$, which we denote by V_j, carrying an irreducible representation ρ_j^0 of $\mathfrak{k}^{\mathbb{C}}$ $(0 \le j \le m)$; furthermore, we have non-zero $\mathfrak{k}^{\mathbb{C}}$-equivariant maps $\rho_j^- : \mathfrak{p}^- \to \mathrm{Hom}(V_{j-1}, V_j)$. The space of such maps is 1-dimensional: this is an equivalent restatement of the known fact that $\mathfrak{p}^- \otimes V_{j-1}$ as a representation of $\mathfrak{k}^{\mathbb{C}}$ is multiplicity free [2, Corollary 4.4]. We denote the orthogonal projection from $\mathfrak{p}^- \otimes V_{j-1}$ to V_j by P_j. We define for $Y \in \mathfrak{p}^-$, $v \in V_{j-1}$,

$$\tilde{\rho}_j(Y)v = P_j(Y \otimes v). \tag{2}$$

Then $\tilde{\rho}_j$ has the $\mathfrak{k}^{\mathbb{C}}$-equivariant property, and it follows that $\rho_j^- = y_j \tilde{\rho}_j$ with some $y_j \neq 0$. We write $y = (y_1, \dots, y_m)$ and denote by E^y the induced vector bundle. We observe here that the vector bundle E^y is uniquely determined by $\rho_0^0, P_1, \dots, P_m$ and y, but these data cannot be arbitrarily chosen: the $\tilde{\rho}_j$ $(1 \le j \le m)$ together must give a representation of the Abelian Lie algebra $\mathfrak{p}^-$. In terms of P_j, this condition amounts to

$$P_{j+1}\big(Y' \otimes P_j(Y \otimes v)\big) = P_{j+1}\big(Y \otimes P_j(Y' \otimes v)\big) \tag{3}$$

for all $Y, Y' \in \mathfrak{p}^-$ and $v \in V_{j-1}$.

We denote by ι the identification of $(\mathfrak{p}^+)^*$ with $\mathfrak{p}^-$ under the Killing form, and, for any vector space W, extend it to a map from $\mathrm{Hom}(\mathfrak{p}^+, W)$ to $\mathfrak{p}^- \otimes W$, that is, for $Y \in \mathfrak{p}^-$, $w \in W$,

$$\iota(B(\cdot, Y)w) = Y \otimes w.$$

In what follows, n denotes the complex dimension of $\mathcal{D}$.

Lemma 2.1. *Given ρ_{j-1}^0, ρ_j^0, as above, there exists a constant c_j, independent of λ, such that for all $Y \in \mathfrak{p}^-$, we have*

$$P_j \iota \rho_{j-1}^0([Y, \cdot]) = (c_j - \tfrac{\lambda - j + 1}{2n})\tilde{\rho}_j(Y).$$

Furthermore, there exist constants u, v such that for all $1 \le j \le m$,

$$c_j = u + (j-1)v.$$

This follows from $\mathfrak{k}^{\mathbb{C}}$-equivariance and some computation. The following two lemmas can be proved by computations based on Lemmas 1.2 and 2.1.

Lemma 2.2. *For all $1 \le j \le m-1$, and holomorphic $F : \mathcal{D} \to V_j$,*

$$\begin{aligned} &P_{j+1}\iota D^{(z)}\big(\rho_j^0(\tilde{k}(g,z)^{-1})F(gz)\big) \\ &\quad = -(c_{j+1} - \tfrac{\lambda - j}{2n})\tilde{\rho}_{j+1}\big(Y(g,z)\big)\big(\rho_j^0(\tilde{k}(g,z)^{-1})F(gz)\big) + \rho_{j+1}^0(\tilde{k}(g,z)^{-1})\big((P_{j+1}\iota D F)(gz)\big), \end{aligned}$$

where $D^{(z)}$ denotes the differentiation with respect to z.

Lemma 2.3. *For all $1 \le j \le m-1$, with the constants c_j of Lemma 2.1,*

$$P_{j+1}\iota D^{(z)}\tilde{\rho}_j(Y(g,z)) = \frac{1}{2}(c_j - c_{j+1} - \tfrac{1}{2n})\tilde{\rho}_{j+1}(Y(g,z))\tilde{\rho}_j(Y(g,z)).$$

Now let E^y be an indecomposable filiform hHhvb as described above. Writing $0 = (0, \dots, 0)$, E^0 makes sense, it is the direct sum of irreducible vector bundles in the composition series of E^y.

If $f \in \mathrm{Hol}(\mathcal{D}, V)$, we write f_j for the component of f in V_j, that is, the projection of f onto V_j.

Theorem 2.4. *Assume that λ is regular in the sense that*

$$c_{jk} = \tfrac{1}{(j-k)!}\prod_{i=1}^{j-k}\big\{u - \tfrac{\lambda}{2n} + \tfrac{2k+i-1}{2}\big(v + \tfrac{1}{2n}\big)\big\}^{-1}$$

is meaningful for $0 \le k \le j \le m$. Then the operator $\Gamma : \mathrm{Hol}(\mathcal{D}, V) \to \mathrm{Hol}(\mathcal{D}, V)$ given by

$$(\Gamma f_j)_\ell = \begin{cases} c_{\ell j}\, y_\ell \cdots y_{j+1}(P_\ell \iota D) \cdots (P_{j+1}\iota D) f_j & \text{if } \ell > j, \\ f_j & \text{if } \ell = j, \\ 0 & \text{if } \ell < j \end{cases}$$

intertwines the actions of $\tilde{G}$ on the trivialized sections of E^0 and E^y.

The proof is by induction based on the preceding lemmas.

Next we pass from the filiform case to the general case. Now (ρ_0, V_0) is a direct sum of representations $(\rho_j^{0\alpha}, V_j^{\alpha})$ with inequivalent irreducible representations α of $\mathfrak{k}_{ss}^{\mathbb{C}}$, and $\rho_j^{0\alpha} = \chi_{\lambda-j}(I_{m_{j\alpha}} \otimes \alpha)$. For pairs of (α, β) that are admissible in the sense that $\beta \subset \mathrm{Ad}_{\mathfrak{p}^-} \otimes \alpha$, we write $P_{\alpha\beta}$ for the corresponding projection and define maps $\tilde{\rho}_{\alpha\beta}$ for $Y \in \mathfrak{p}^-$. Then

$$\rho_j^-(Y) = \oplus_{\alpha,\beta} y_j^{\alpha\beta} \otimes \tilde{\rho}_{\alpha\beta}(Y)$$

with $y_j^{\alpha\beta} \in \mathrm{Hom}(\mathbb{C}^{m_\alpha}, \mathbb{C}^{m_\beta})$ such that $y_{j+1}^{\beta\gamma} y_j^{\alpha\beta} = 0$ unless $(\alpha\beta)$ and $(\beta\gamma)$ are admissible and the analogue of (3) holds. We let $\mathbb{E}^y$ denote the bundle holomorphically induced by ρ, and let $\mathbb{E}^0$ be the (direct sum) bundle gotten by changing all the $y^{\alpha\beta}$ to 0. The general version of Γ is now going to be (for $j < \ell$)

$$(\Gamma f_j)_\ell = \oplus_{\alpha_j,\dots,\alpha_\ell} c_{\ell j}^{\alpha_j,\dots,\alpha_\ell} \left(y_\ell^{\alpha_{\ell-1}\alpha_\ell} \cdots y_{j+1}^{\alpha_j \alpha_{j+1}}\right) \otimes \left(P_{\alpha_{\ell-1}\alpha_\ell} \iota D\right) \cdots \left(P_{\alpha_j j+1} \iota D\right) f_j^{\alpha}.$$

For $j \geq \ell$, it is unchanged.

Theorem 2.5. *Suppose that $\mathbb{E}^y$ is induced by an indecomposable ρ. Then there exist constants $c_{\ell j}^{\alpha_j,\dots,\alpha_\ell}$ such that Γ intertwines the actions of $\tilde{G}$ on the trivialized sections of $\mathbb{E}^0$ and $\mathbb{E}^y$.*

We note that if $\mathcal{D}$ is the disc in one variable then Theorem 2.5 specializes to Theorem 3.1 of [3].

3. Hilbert spaces of sections

With notations preserved, for general $\mathcal{D}$, we consider first the case where ρ is irreducible. Then automatically ρ^0 is also irreducible and $\rho^- = 0$. We write $\rho^0 = \chi_\lambda \otimes \sigma$, where σ is an irreducible representation of $\mathfrak{k}_{ss}$. For every σ, there is an (explicitly known) set of λ-s such that the sections of the corresponding holomorphically induced vector bundle have a $\tilde{G}$-invariant inner product. This is Harish–Chandra's holomorphic discrete series and its analytic continuation. In the canonical trivialization it gives Hilbert spaces $\mathcal{H}_{\rho^0} = \mathcal{H}_{\sigma,\lambda}$, which are known to have reproducing kernels $K_{\sigma,\lambda}(z, w)$. If we set

$$\tilde{\mathcal{K}}(z, w) = \tilde{k}(\exp -\bar{w}, z),$$

(where $\bar{w}$ denotes conjugation with respect to $\mathfrak{g}$) we have, slightly extending [4, Chap. II, §5]

$$K_{\sigma,\lambda}(z, w) = (\chi_\lambda \otimes \sigma)(\tilde{\mathcal{K}}(z, w)).$$

In particular, it is known that the inner product is *regular* in the sense that all K-types (i.e polynomials) have non-zero norm in $\mathcal{H}_{\sigma,\lambda}$ if and only if $\lambda < \lambda_\sigma$ for a certain known constant λ_σ.

In the following theorem, we consider a bundle $\mathbb{E}^y$ as in Section 2. The corresponding $\mathbb{E}^0$ is then a direct sum of irreducible bundles as above. Its sections have a $\tilde{G}$-invariant inner product if and only if this is true for each summand. In this case, we have a Hilbert space $\mathcal{H}^0 = \oplus \mathcal{H}_{\rho_j^0}$.

Theorem 3.1. *The sections of $\mathbb{E}^y$ have a $\tilde{G}$-invariant regular inner-product if and only if the same is true for $\mathbb{E}^0$. In this case, the map Γ is a unitary isomorphism of $\mathcal{H}^0$ onto the Hilbert space $\mathcal{H}^y$ of sections of $\mathbb{E}^y$. The space $\mathcal{H}^y$ (as well as $\mathcal{H}^0$) has a reproducing kernel.*

For the proof one observes that Γ has an inverse of the same form (only the constants c_{jk} change). Γ being a holomorphic differential operator, the image of $\mathcal{H}^0$ is also a Hilbert space of holomorphic functions with a reproducing kernel. One can verify that this is the sought after $\mathcal{H}^y$.

Theorem 3.2. *Suppose $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$. Let σ_0, σ_1 be irreducible representations of $\mathfrak{k}_{ss}^{\mathbb{C}}$ such that $\sigma_1 \subset \mathrm{Ad}_{\mathfrak{p}^-} \otimes \sigma_0$ and let P be the corresponding projection. Then if $\lambda < \lambda_{\sigma_0}$, we have $\lambda - 1 < \lambda_{\sigma_1}$ and $P\iota D$ is a bounded linear transformation from $\mathcal{H}_{\sigma_0,\lambda}$ to $\mathcal{H}_{\sigma_1,\lambda-1}$.*

By the theory of reproducing kernels, for this it is enough to prove that ($D^{(z)}$ and $D^{(w)}$ denote the differentiation with respect to the variable z and w respectively)

$$CK_{\sigma_1,\lambda-1}(z, w) - (P\iota D^{(z)})K_{\sigma_0,\lambda}(z, w)(P\iota D^{(w)})^*$$

is positive definite for some $C > 0$. (In general, we say that a kernel K taking values in $\mathrm{Hom}(V, V)$ is positive definite if, for any $z_1, \dots, z_n$ in $\mathcal{D}$ and $v_1, \dots, v_n$ in V,

$$\sum_{j,k=1}^{n} \langle K(z_j, z_k)v_k, v_j\rangle \geq 0$$

holds.)

Remark 3.3. When $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$, the spaces $\mathcal{H}^0$ and $\mathcal{H}^y$ are equal as sets. This follows from Theorem 3.2 and the closed graph theorem.

4. Homogeneous Cowen–Douglas tuples

For any bounded domain $\mathcal{D} \subseteq \mathbb{C}^n$, the n-tuple $\boldsymbol{T} = (T_1, \dots, T_n)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is said to be homogeneous (relative to the holomorphic automorphism group $\mathrm{Aut}(\mathcal{D})$) if the joint (Taylor) spectrum of $\boldsymbol{T}$ is in $\overline{\mathcal{D}}$ and for every g in $\mathrm{Aut}(\mathcal{D})$, the n-tuple $g(\boldsymbol{T}) = g(T_1, \dots, T_n)$ is unitarily equivalent to $\boldsymbol{T}$.

Another important class of n-tuples of commuting operators associated with the domain $\mathcal{D} \subseteq \mathbb{C}^n$ is the extended Cowen–Douglas class $\mathrm{B}'_k(\mathcal{D})$. Its elements are n-tuples of bounded operators that can be realized as adjoints of the multiplications M_j by the coordinate functions on some Hilbert space of holomorphic $\mathbb{C}^k$- valued functions on $\mathcal{D}$ possessing a reproducing kernel K and containing all $\mathbb{C}^k$- valued polynomials as a dense set. (The strict Cowen–Douglas class $\mathrm{B}_k(\mathcal{D})$ as originally defined consists of the n-tuples of bounded operators $(T_1, \dots, T_n)$ that can be realized like this and in addition satisfy the condition that $\oplus(T_j - z_j)$ mapping the Hilbert space into the n-fold direct sum with itself has closed range.)

We wish to investigate, for bounded symmetric domains $\mathcal{D}$, the homogeneous n-tuples in $\mathrm{B}'_k(\mathcal{D})$. For the case of the unit disc, there is a complete description and classification of these in [3]. (In that case, it turns out that the homogeneous operators in $\mathrm{B}'_k(\mathbb{D})$ are the same as in $\mathrm{B}_k(\mathbb{D})$.)

Theorem 4.1. *For any bounded symmetric $\mathcal{D}$, an n-tuple $\boldsymbol{T}$ in $\mathrm{B}'_k(\mathcal{D})$ is homogeneous if and only if the corresponding holomorphic Hermitian vector bundle is homogeneous under $\tilde{G}$.*

The proof (not entirely trivial) is the same as in [3, Theorem 2.1].

For a bounded symmetric $\mathcal{D}$, we call a n-tuple $\boldsymbol{T}$ in $\mathrm{B}'_k(\mathcal{D})$ and its corresponding bundle E basic if E is induced by an irreducible ρ. From the results of Section 3, E is basic if and only if it is induced by some $\chi_\lambda \otimes \sigma$ with $\lambda < \sigma_\lambda$.

Theorem 4.2. *If $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$, all homogeneous n-tuples in $\mathrm{B}'_k(\mathcal{D})$ are similar to direct sums of basic homogeneous n-tuples.*

The proof is based on Remark 3.3. The similarity arises as the identity map between $\mathcal{H}^0$ to $\mathcal{H}^y$, which clearly intertwines the operators M_j on the respective Hilbert spaces.

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