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Combinatorics

Note on some restricted Stirling numbers of the second kind

Note sur des restrictions des nombres de Stirling de deuxième espèce

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A B S T R A C T

The aim of this work is to establish some properties of the coefficients of the chromatic polynomials of special graphs. An application on (restricted) Stirling numbers of the second kind is considered.

R É S U M É

Le but de ce travail est d'établir quelques propriétés des coefficients des polynômes chromatiques de certains graphes. Nous donnons une application sur une restriction des nombres de Stirling de deuxième espèce.

1. Introduction

The object of our investigations is to establish connections between chromatic polynomials of some graphs and special (restricted) Stirling numbers of the second kind. We introduce variations of the Stirling numbers of the second kind counting the number of partitions with special conditions and we rely these numbers to the chromatic polynomials of special graphs and some of their properties. For a given graph $G=(V,E)$ of order $n\geq 1$, the expression of the chromatic polynomial in the factorial form is $P(G,\lambda)=\sum_{i=\chi(G)}^{n}\alpha_i(G)(\lambda)_i$, see [2, Thm. 1.4.1], recall that $(\lambda)_i=\lambda(\lambda-1)\cdots(\lambda-i+1)$ if $i\geq 1$ and $(\lambda)_0=1$, $\alpha_i(G)$ is the number of ways of partitioning V into i independent sets (i.e., no two vertices in the same set are adjacent in G). We denote by $\chi(G)$ the chromatic number of G.

This work is motivated by the work of Duncan et al. [3] on Stirling numbers of the second kind for graphs and by the work of Mihoubi et al. [9,8] on the $(r_1,\ldots,r_p)$-Stirling numbers of the second kind.

For any graph H, we give some recurrence relations for the coefficients $\alpha_k(G\cup H)$ for some graphs G and some results on log-concavity and Pólya frequency for sequences related to these coefficients. In the last section, we present an application on special (restricted) Stirling numbers of the second kind.

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2. Main results

Let H be any graph of h vertices, $h \geq 1$ and $(G_n; n \geq 0)$ be a sequence of graphs such that the order of G_n is n and G_0 is the graph with no vertices i.e., $P(G_0, \lambda) := 1$.

Proposition 2.1. *Let n, k be nonnegative integers. Assume that*

$$P(G_n, \lambda) = (\lambda - s_{n-1}) P(G_{n-1}, \lambda), \quad n > n_0,$$

for some non-positive integer n_0 and real numbers $s_0, \ldots, s_{n-1}$ such that $s_0 = 0$.

Assume that $n > n_0$. If $\chi(G_n) \leq k \leq n$ we obtain

$$\alpha_k(G_n) = (k - s_{n-1}) \alpha_k(G_{n-1}) + \alpha_{k-1}(G_{n-1}),$$

and $\alpha_k(G_n) = 0$ otherwise.

Proof. It is obvious that we have $\alpha_k(G_n) = 0$ if $k < \chi(G_n)$ or $k > n$.

Now, for $\chi(G_n) \leq k \leq n$, the result comes, trough a straightforward computation, from the identity $(\lambda - x)(\lambda)_k = (\lambda)_{k+1} + (k - x)(\lambda)_k$, which holds for any $k \in \mathbb{N}$. $\square$

To give some special cases of Proposition 2.1, let O_n be the empty graph (i.e. with no edge) of order n, K_n the complete graph of order n and T_n a tree of order n.

Corollary 2.1. *Let n, k be integers and*

$$H_n^{\square} := \square_n \cup H \quad \texttt{for} \quad \square = O,\ K,\ T.$$

Assume that $n > n_0^{\square}$, where $n_0^O = n_0^K = n_0^T - 1 = 0$.

If $\chi\left(H_n^{\square}\right) \leq k \leq n + h$, we obtain

$$\alpha_k\left(H_n^{\square}\right) = \left(k - s_{n-1}^{\square}\right) \alpha_k\left(H_{n-1}^{\square}\right) + \alpha_{k-1}\left(H_{n-1}^{\square}\right),$$

and $\alpha_k\left(H_n^{\square}\right) = 0$ otherwise, where $s_n^O = 0$, $s_n^K = n$, and $s_n^T = 1$.

Furthermore, $\chi\left(H_n^O\right) = \chi(H)$, $\chi\left(H_n^K\right) = \max(n, \chi(H))$ and $\chi\left(H_n^T\right) = \max(\chi(T_n), \chi(H))$.

Proof. It suffices to observe that $P\left(H_n^{\square}, \lambda\right) = \left(\lambda - s_{n-1}^{\square}\right) P\left(H_{n-1}^{\square}, \lambda\right)$, $n > n_0^{\square}$, and apply Proposition 2.1. Notice that the last equality and the values of $\chi\left(H_n^{\square}\right)$ are well-known results (see, e.g., [2, Sect. 2]). $\square$

For the following proposition, let us recall some definitions and results on log-concavity, Pólya frequency and q-log-convexity. A sequence $(u_n; n \geq 0)$ of nonnegative real numbers is called log-concave (LC) if $u_{i-1}u_{i+1} \leq u_i^2$ for all $i > 0$, and, it is called a Pólya frequency sequence (or a PF sequence) if all minors of the matrix $A = (u_{i-j})_{i,j \geq 0}$ have nonnegative determinants (where $u_k = 0$ if $k < 0$); for more information, see [5]. A sequence of real polynomials $(P_n(q), n \geq 0)$ is called q-log-convex if the polynomial $P_n(q)^2 - P_{n-1}(q)P_{n+1}(q)$ has nonnegative coefficients for all $n \geq 1$, see [10,11,13]. In particular, let $(T(n,k),\ n,k \geq 0)$ be sequence of nonnegative numbers satisfying for $n \geq k \geq 1$ the recurrence

$$T(n,k) = (a_1 n + a_2 k + a_3) T(n-1,k) + (b_1 n + b_2 k + b_3) T(n-1,k-1),$$

with $T(n,k) = 0$ unless $0 \leq k \leq n$, $T(0,0) > 0$, $a_1 \geq 0$, $a_1 + a_2 \geq 0$, $a_1 + a_3 \geq 0$ and $b_1 \geq 0$, $b_1 + b_2 \geq 0$, $b_1 + b_2 + b_3 \geq 0$. It is shown in [6, Thm. 2] that, for each fixed n, the sequence $(T(n,k),\ 0 \leq k \leq n)$ is log-concave. If we have $a_2 b_1 \geq a_1 b_2$ and $a_2(b_1 + b_2 + b_3) \geq (a_1 + a_3) b_2$, this sequence is Pólya frequency [12, Cor. 3] and further, by setting $T_n(q) = \sum_{k=0}^{n} T(n,k) q^k$, if $(a_2 b_1 - a_1 b_2) n + a_2 b_2 k + a_2 b_3 - a_3 b_2 \geq 0$ for $0 < k \leq n$, then, the sequence of polynomials $(T_n(q),\ n \geq 0)$ is q-log-convex [7, Thm. 4.1].

Proposition 2.2. *For $n, k \geq 0$, let*

$$U_{n,k}^{\square} = \alpha_{k+h+n_0^{\square}}\left(H_{n+n_0^{\square}}^{\square}\right) \quad \text{and} \quad U_n^{\square}(q) = \sum_{k=0}^{n} U_{n,k}^{\square} q^k.$$

Then, for $\square = O$, K, the sequence $\left(U_{n,k}^{\square},\ 0 \leq k \leq n\right)$ is log-concave and for $\square = O$, K, T, it is Pólya frequency sequence. For $\square = O$, K, the sequence of polynomials $\left(U_n^{\square}(q)\right)$ is q-log-convex.

Proof. We have $U^{O}_{0,0} = U^{K}_{0,0} = \alpha_h (H) = 1$ and $U^{T}_{0,0} = \alpha_{h+1} (T_1 \cup H) = 1$. Furthermore, for $n \geq k \geq 1$, Corollary 2.1 implies

$$U^{\square}_{n,k} = U^{\square}_{n-1,k-1} + \left(k + h + n^{\square}_0 - n^{\square}_{n-1}\right) U^{\square}_{n-1,k}.$$

So, the log-concavity follows from [6, Thm. 2], Pólya frequency follows from [12, Cor. 3] and q-log-convexity follows from [7, Thm. 4.1]. □

3. Application to the graph $K_{r_1,\ldots,r_p} \cup O_n$

For any integer $p \geq 2$, let $K_{r_1,\ldots,r_p}$ be the complete p-partite graph with a p-partition $(V_1, \ldots, V_p)$ such that $|V_i| = r_i \geq 1$, $i = 1, \ldots, p$, and $K^{O}_{r_1,\ldots,r_p} := K_{r_1,\ldots,r_p} \cup O_n$. We set $\mathbf{r}_p := (r_1, \ldots, r_p)$, $|\mathbf{r}_p| := \sum_{i=1}^{p} r_i$ and consider the graph $K_{n,\mathbf{r}_p} = K^{O}_{r_1,\ldots,r_p}$ with chromatic number $\chi\left(K_{n,\mathbf{r}_p}\right) = \max(p, 1) = p$.

Hence, the coefficient $\alpha_k\left(K_{n,\mathbf{r}_p}\right)$ represents the number of ways of partitioning the set of $n + |\mathbf{r}_p|$ vertices of $K_{n,\mathbf{r}_p}$ into k nonempty sets, and by the definition of the graph $K_{n,\mathbf{r}_p}$, any two elements x of the i-th block of the subgraph $K_{r_1,\ldots,r_p}$ and y of the j-th block of $K_{r_1,\ldots,r_p}$, with $i \neq j$, can't be in the same subset.

Now, we give connection between $\alpha_k\left(K_{n,\mathbf{r}_p}\right)$ and the Stirling numbers. Let us recall the main properties of these numbers.

As it is known, the Stirling number of the second kind $\left\{{n \atop k}\right\}$ and the r-Stirling number of the second kind $\left\{{n \atop k}\right\}_r$ count, respectively, the number of partitions of an n-set into k nonempty sets and the number of partitions of an n-set into k nonempty sets such that the r first elements are in different sets [1]. A generalization of these numbers is considered below. We start by giving the following definition.

Let $R_1, \ldots, R_p$ be subsets of the set $[n] := \{1, 2, \ldots, n\}$ with $|R_i| = r_i$ and $R_i \cap R_j = \emptyset$ for all $i, j = 1, \ldots, p$, $i \neq j$. The $K(r_1, \ldots, r_p)$-Stirling number of the second kind, denoted by $\left\{{n \atop k}\right\}_{K(\mathbf{r}_p)} := \left\{{n \atop k}\right\}_{K(r_1,\ldots,r_p)}$, counts the number of partitions of the set $[n]$ into k nonempty subsets such that if $x \in R_i$ and $y \in R_j$ with $i \neq j$, then x and y belong to different subsets of the partition.

From this definition, we may state the following:

$$\begin{aligned}
\left\{{n \atop k}\right\}_{K(r_1,\ldots,r_p)} &= 0, \ \text{ if } n < |\mathbf{r}_p| \text{ or } k < p, \\
\left\{{n \atop k}\right\}_{K(r_1,\ldots,r_p)} &= \left\{{n \atop k}\right\}_{|\mathbf{r}_p|} \ \text{if } r_1, \ldots, r_p \in \{0, 1\}, \\
\left\{{n \atop k}\right\}_{K(r_1,\ldots,r_p)} &= \left\{{n \atop k}\right\}_{K(r_{\sigma(1)},\ldots,r_{\sigma(p)})} \quad \text{for all permutations } \sigma \text{ on the set } [p].
\end{aligned}$$

So, by combining the above combinatorial interpretation of $\alpha_k\left(K_{n,\mathbf{r}_p}\right)$ and the definition of the $K(r_1, \ldots, r_p)$-Stirling numbers of the second kind, we conclude that $\alpha_k\left(K_{n,\mathbf{r}_p}\right) = \left\{{n+|\mathbf{r}_p| \atop k}\right\}_{K(\mathbf{r}_p)}$.

For the rest of the paper, assume that $r_1 \geq 1, \ldots, r_p \geq 1$.

So, for $H := K_{r_1,\ldots,r_p}$ in the first identity of Corollary 2.1, we obtain:

Corollary 3.1. *Let n, k be integers such that $p \leq k \leq n$ and $n \geq |\mathbf{r}_p|$. Then*

$$\left\{{n \atop k}\right\}_{K(\mathbf{r}_p)} = k\left\{{n-1 \atop k}\right\}_{K(\mathbf{r}_p)} + \left\{{n-1 \atop k-1}\right\}_{K(\mathbf{r}_p)}, \quad n \geq k \geq 1.$$

Proposition 3.1. *Let $B\left(\lambda; K_{n,\mathbf{r}_p}\right) := \sum_{k\geq 0} \left\{{n+|\mathbf{r}_p| \atop k}\right\}_{K(\mathbf{r}_p)} \lambda^k$. Then, we have*

$$\begin{aligned}
B\left(\lambda; K_{n,\mathbf{r}_p}\right) &= \lambda \exp(-\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\exp(\lambda) B\left(\lambda; K_{n-1,\mathbf{r}_p}\right)\right), \quad n \geq 1, \\
B\left(\lambda; K_{0,\mathbf{r}_p}\right) &= B_{r_1}(\lambda) \cdots B_{r_p}(\lambda),
\end{aligned}$$

where $B_n(\lambda) = \sum_{j=0}^{n} \left\{{n \atop j}\right\} \lambda^j$ is the classical Bell polynomial.

Proof. From Corollary 3.1 the polynomial $B\left(\lambda; K_{n,\mathbf{r}_p}\right)$ can be written as

$$B\left(\lambda; K_{n,\mathbf{r}_p}\right) = \sum_{k\geq 1} k\left\{{n+|\mathbf{r}_p|-1 \atop k}\right\}_{K(\mathbf{r}_p)} \lambda^k + \sum_{k\geq 1} \left\{{n+|\mathbf{r}_p|-1 \atop k-1}\right\}_{K(\mathbf{r}_p)} \lambda^k$$

which is exactly $\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(B\left(\lambda; K_{n-1,\mathbf{r}_p}\right)\right) + \lambda B\left(\lambda; K_{n-1,\mathbf{r}_p}\right) = \lambda \exp(-\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\exp(\lambda) B\left(\lambda; K_{n-1,\mathbf{r}_p}\right)\right)$.

The second identity follows from [2, Lemma 4.4.1]. □

Corollary 3.2. *For $n \geq |\mathbf{r}_p|$, the numbers $\left\{ {n \atop k} \right\}_{K(\mathbf{r}_p)}$, $k = p, p+1, \ldots, n$, are log-concave.*

Proof. We prove the result by induction on n. For $n = 0$, since the polynomial $B_r(\lambda)$, $r \geq 1$, has only real and non-positive roots (see for example [12]), so that the polynomial $B(\lambda; K_{0,\mathbf{r}_p}) = B_{r_1}(\lambda) \cdots B_{r_p}(\lambda)$. For $n \geq 1$, assume that the polynomial $B(\lambda; K_{n-1,\mathbf{r}_p})$ has only real and non-positive roots. From Proposition 3.1, the function $f_n(\lambda) := \exp(\lambda) B(\lambda; K_{n,\mathbf{r}_p})$ satisfies $f_n(\lambda) = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} f_{n-1}(\lambda)$. Since the polynomial $B(\lambda; K_{n-1,\mathbf{r}_p})$ is of degree $n - 1 + |\mathbf{r}_p|$, then, by the induction hypothesis, it has $n - 1 + |\mathbf{r}_p|$ real non-positive roots (the zero root is of multiplicity p). Apply Rolle's Theorem to $f_{n-1}(\lambda)$ to deduce that $f_n(\lambda)$ has $(n - 2 + |\mathbf{r}_p|) + 1 = n - 1 + |\mathbf{r}_p|$ real roots (the zero root is of multiplicity p) and the missing one must be real and negative. After that, apply Newton's inequality [4, p. 52] to complete the proof. □

Proposition 2.2 states that we have

Corollary 3.3. *For $0 \leq k \leq n$ let $W_{n,k} = \left\{ {n+|\mathbf{r}_p| \atop k+|\mathbf{r}_p|} \right\}_{K(\mathbf{r}_p)}$. Then, the sequence $(W_{n,k},\ 0 \leq k \leq n)$ is a Pólya frequency sequence.*

Other interesting applications can be obtained by investigating the graphs

$$G_{n,\mathbf{r}_p} := \left(K_{r_1} \cup \cdots \cup K_{r_p}\right)_n^O \quad \text{and} \quad T_{n,\mathbf{r}_p} := \left(T_{r_1} \cup \cdots \cup T_{r_p}\right)_n^O .$$

Applying our results on the graph $G_{n,\mathbf{r}_p}$ gives the $(r_1, \ldots, r_p)$-Stirling numbers of the second kind introduced by Mihoubi et al. in [9,8]. For more details, let $r_1, \ldots, r_p$ be positive integers and set

$$R_1 := \{1, \ldots, r_1\},\ R_2 := \{r_1 + 1, \ldots, r_1 + r_2\}, \ldots, R_p := \left\{r_1 + \cdots r_{p-1} + 1, \ldots, r_1 + \cdots + r_p\right\}.$$

We note that $\alpha_k(G_{n,\mathbf{r}_p})$ counts the number of partitions of the set $[n + |\mathbf{r}_p|]$ into k nonempty subsets such that the elements of each of the p sets $R_1, \ldots, R_p$ are in distinct subsets.

We also note that $\alpha_k(T_{n,\mathbf{r}_p})$ can be interpreted in terms of the Stirling numbers. Indeed, if for all $i = 1, \ldots, p$, the tree T_i is a path of r_i vertices, the coefficient $\alpha_k(T_{n,\mathbf{r}_p})$ represents the number of ways of partitioning the set of $n + |\mathbf{r}_p|$ vertices of $T_{n,\mathbf{r}_p}$ into k independent sets. In terms of Stirling numbers, the coefficient $\alpha_k(T_{n,\mathbf{r}_p})$ counts the number of partitions of the set $[n + |\mathbf{r}_p|]$ into k non-empty subsets such any two consecutive elements of R_i, $i = 1, \ldots, p$, cannot be in the same subset.

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