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Complex analysis

Fekete–Szegö inequality for certain spiral-like functions

Inégalité de Fekete–Szegö pour certaines fonctions spiralées

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ABSTRACT

For $|\alpha| < \pi/2$, let $\mathcal{S}_\alpha$ denote the class of non-vanishing normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $\mathrm{Re}\, P_f(z) > 0$ in $\mathbb{D}$ where

$$P_f(z) = e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

The class $\mathcal{S}_\alpha$ consists of functions $f(z)$ for which $zf'(z)$ is spiral-like, which has been introduced and extensively studied by M.S. Robertson [24]. In the present paper, we obtain the sharp upper bound for the Fekete–Szegö functional $|a_3 - \lambda a_2^2|$ for the complex parameter λ when $f \in \mathcal{S}_\alpha$.

RÉSUMÉ

Pour $|\alpha| < \pi/2$, soit $\mathcal{S}_\alpha$ la classe des fonctions analytiques normalisées $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, non nulles dans le disque unité $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$ et satisfaisant $\mathrm{Re}\, P_f(z) > 0$ dans $\mathbb{D}$, où

$$P_f(z) = e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Pour $f(z) \in \mathcal{S}_\alpha$, la fonction $zf'(z)$ est spiralée, notion introduite et étudiée de façon approfondie par M.S. Robertson [24]. Dans la présente Note, nous obtenons une borne supérieure précise de la fonctionnelle de Fekete–Szegö $|a_3 - \lambda a_2^2|$, où λ est un paramètre complexe et $f \in \mathcal{S}_\alpha$.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Any function $f \in \mathcal{A}$ has the following series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function f is said to be univalent in $\mathbb{D}$ if it is one-to-one in $\mathbb{D}$. Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. A domain $z_0 \in \Omega \subseteq \mathbb{C}$ is said to be a star-like domain with respect to a point z_0 if the line segment joining z_0 to any point in Ω lies in Ω. A star-like domain with respect to the origin is simply said to be a star-like domain. A function $f \in \mathcal{A}$ is said to be a star-like function if f maps $\mathbb{D}$ onto a domain $f(\mathbb{D})$ that is star-like with respect to the origin. We denote the class of univalent star-like functions in $\mathcal{A}$ by $\mathcal{S}^*$. It is well known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

A domain $\Omega \subseteq \mathbb{C}$ is said to be convex if it is star-like with respect to every point in Ω. A function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. Let $\mathcal{C}$ denote the class of convex univalent functions in $\mathbb{D}$. A function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

It is well known that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$.

A domain $0 \in \Omega$ is said to be α-spiral-like if for each point $0 \neq w_0 \in \Omega$ the arc of the α-spiral from w_0 to the origin lies entirely in Ω. An analytic and univalent function f in the unit disk $\mathbb{D}$ with $f(0) = 0$ is said to be α-spiral-like if $f(\mathbb{D})$ is α-spiral-like. A function f is said to be spiral-like if it is α-spiral-like for some α. In 1932, Špaček [26] proved that a function $f \in \mathcal{A}$ is α-spiral-like for some real constant α ($|\alpha| < \pi/2$) if

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

The class of all α-spiral-like functions in $\mathbb{D}$ is denoted by $\mathcal{S}p(\alpha)$ and

$$\bigcup_{-\pi/2<\alpha<\pi/2} \mathcal{S}p(\alpha)$$

denotes the class of spiral-like functions in $\mathbb{D}$. In particular, $\mathcal{S}p(0)$ reduces to the class of star-like functions $\mathcal{S}^*$. For a general reference about many of these special classes, we refer the reader to [5].

In the present paper, we consider another family of functions $\mathcal{S}_\alpha$ that includes the class of convex functions as a proper subfamily. More precisely, for $-\pi/2 < \alpha < \pi/2$, we say that $f \in \mathcal{S}_\alpha$ if $f \in \mathcal{A}$ and is non-vanishing in $\mathbb{D}$ such that $\operatorname{Re} P_f(z) > 0$ in $\mathbb{D}$, where

$$P_f(z) = e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{1.2}$$

The class $\mathcal{S}_\alpha$ has been introduced by M.S. Robertson [24] in 1968. We note that $f \in \mathcal{S}_\alpha$ if and only if there exists a function $g \in \mathcal{S}^*$ such that

$$f'(z) = \left(\frac{g(z)}{z}\right)^{(\cos\alpha)\exp(-i\alpha)}. \tag{1.3}$$

The class $\mathcal{S}_\alpha$ consists of functions f for which $zf'(z)$ is spiral-like. In particular, the class $\mathcal{S}_0$ consists of the normalized convex functions in $\mathbb{D}$.

In 1968, Robertson [24] proved that $f \in \mathcal{S}_\alpha$ is univalent if $0 < \cos\alpha \le x_0$, where x_0 is the positive root $0.2315\cdots$ of the equation $16x^3 + 16x^2 + x - 1 = 0$. For general values of α ($|\alpha| < \pi/2$), a function in $\mathcal{S}_\alpha$ need not be univalent in $\mathbb{D}$. If $\mu + 1 = |\mu + 1|e^{-i\alpha}$ ($1/2 < \alpha < 1$), and if moreover $|\mu| \le 1$, $|\mu + 1| > 1$ and $|\mu - 1| > 1$, then the function

$$f^*_\alpha(z) = \frac{1}{\mu}\left(\frac{1}{(1-z)^\mu} - 1\right) = z + \frac{1}{2}(\mu+1)z^2 + \frac{1}{6}(\mu^2 + 3\mu + 2)z^3 + \cdots$$

belongs to $\mathcal{S}_\alpha \backslash \mathcal{S}$ in $\mathbb{D}$. For example, the function $f(z) = i(1-z)^i - i$ belongs to $\mathcal{S}_{\pi/4} \backslash \mathcal{S}$. In 1972, Libera and Zeigler [13] improved the range of α to $0 < \cos\alpha \le 0.2564\cdots$ so that $f \in \mathcal{S}_\alpha$ is univalent. In 1975, Chichra [3] has improved this range still further to $0 < \cos\alpha \le 0.2588\cdots$ to prove the univalency of $f \in \mathcal{S}_\alpha$ and indicated that this result is the best possible one obtainable exclusively from an application of Nehari's test for univalence [19]. It is interesting to note that in the same year Pfaltzgraff [20] has proved that functions in the class $\mathcal{S}_\alpha$ are univalent whenever $0 < \cos\alpha \le 1/2$. This settles the improvement of ranges of α for which functions in the class $\mathcal{S}_\alpha$ are univalent. On the other hand, Singh and Chichra [25] have proved that if $f \in \mathcal{S}_\alpha$ with $f''(0) = 0$, then f is univalent for all real values of α with $|\alpha| < \pi/2$.

Let $A(r)$ be defined by

$$\int_0^{2\pi}\int_0^{r} |f'(\rho e^{i\theta})|^2 \, \rho \, d\rho \, d\theta$$

and $L(r)$ be the length of the image of the circle $|z| = r$ under $f(z)$. In 1971, Liu [14] proved that if $f \in \mathcal{S}_\alpha$, then

$$\limsup_{r\to 1}\left(\sup_{f\in\mathcal{S}_\alpha} L(r)\right)\left(\pi A(r)\log\left(\frac{1+r}{1-r}\right)\right)^{-\frac{1}{2}} \le 2\cos\alpha .$$

In 2008, Ponnusamy, Vasudevarao and Yanagihara [23] obtained the region of variability for functions in the class $\mathcal{S}_\alpha$. Recently, the sharp arclength for functions in the class $\mathcal{S}_\alpha$ has been investigated by Vasudevarao [27].

2. Preliminaries

If f is a locally univalent function of the form (1.1), then the quantity $a_3 - a_2^2$ represents $\frac{1}{6}S_f(0)$, where $S_f(z) = \left(f''(z)/f'(z)\right)' - \frac{1}{2}\left(f''(z)/f'(z)\right)^2$ represents the Schwarzian derivative of f (see [10,19]). For a complex number λ, the coefficient functional $\phi_\lambda(f) = a_3 - \lambda a_2^2$ for functions f in the class $\mathcal{A}$ plays a significant role in the theory of univalent functions. For instance, maximizing $|\phi_\lambda(f)|$ over the class $\mathcal{S}$ or on its subclasses is known as the Fekete–Szegö problem.

For a function f in the class $\mathcal{S}$ given by (1.1), Fekete and Szegö [6] proved the following sharp inequality

$$\left|a_3 - \lambda a_2^2\right| \le 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{2.1}$$

in the case where λ is a real parameter, $0 \le \lambda < 1$ by using Loewner's method. Despite the fact that the Koebe function $z/(1-z)^2$ is an extremal for many problems in the class $\mathcal{S}$ and various of its subclasses, it is interesting to note that the Koebe function fails to be an extremal for the inequality (2.1) when $0 < \lambda < 1$. We note that the Koebe function is extremal for the inequality (2.1) only when $\lambda = 0$ and $\lambda = 1$. The inequality (2.1) is sharp in the following sense that for each λ in $[0, 1)$, there exists a function $f \in \mathcal{S}$ for which the equality holds.

In 1987, W. Koepf [11] solved the Fekete–Szegö problem for functions that are close to convex (see also [12]). Using a variational method, Pfluger [21] has given another treatment of the Fekete–Szegö inequality, including a description of the image domains under extremal functions. In 1986, Jenkins [8] proved the inequality (2.1) by means of his general coefficient theorem [7]. Using Jenkins' method, Pfluger [22] has proved that

$$\left|a_3 - \lambda a_2^2\right| \le 1 + 2\left|\exp\left(\frac{-2\lambda}{1-\lambda}\right)\right| \tag{2.2}$$

holds for the complex parameter λ in the unit disk $\mathbb{D}$ with $\mathrm{Re}\left(\frac{1}{1-\lambda}\right) \ge 1$. Ma and Minda [16–18] gave a complete answer to the Fekete–Szegö problem for the classes of strongly close-to-convex functions and strongly star-like functions. Subsequently, Choi, Kim and Sugawa [4] developed a new method for solving the Fekete–Szegö problem for classes of close-to-convex functions defined in terms of subordination. The Fekete–Szegö problem has a long and rich history in the literature (see, for instance, [1,2,9,15,28]).

In the present paper, we solve the Fekete–Szegö problem for functions in the class $\mathcal{S}_\alpha$ for complex parameter λ. As a particular case (by taking $\alpha = 0$), we obtain the sharp bound for the Fekete–Szegö functional for the class of convex functions $\mathcal{C}$.

3. Main results

Theorem 3.1. *Let $f \in \mathcal{S}_\alpha$ be given by (1.1) and $\lambda \in \mathbb{C}$. Then we have*

$$|a_3 - \lambda a_2{}^2| \le \begin{cases} \frac{1}{3}(\cos\alpha)\left|1 + (2-3\lambda)e^{-i\alpha}\cos\alpha\right| & \text{for } \left|\lambda - (1 + \frac{i}{3}\tan\alpha)\right| \ge \frac{1}{3\cos\alpha} \\ \frac{1}{3}\cos\alpha & \text{for } \left|\lambda - (1 + \frac{i}{3}\tan\alpha)\right| < \frac{1}{3\cos\alpha}. \end{cases} \tag{3.2}$$

The inequality (3.2) is sharp.

Proof. Let $f \in \mathcal{S}_\alpha$. Then there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ of the form

$$\omega(z) = \sum_{k=0}^{\infty} c_k z^k \tag{3.3}$$

such that

$$e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) = (\cos\alpha)\left(\frac{1+z\omega(z)}{1-z\omega(z)}\right) + i\sin\alpha. \tag{3.4}$$

A simple computation shows that

$$zf''(z)(1-z\omega(z)) = e^{-i\alpha}(\cos\alpha)(1+z\omega(z))f'(z) + (1-z\omega(z))(i\,e^{-i\alpha}\sin\alpha - 1)f'(z)$$

which is equivalent to

$$zf''(z) - z^2\omega(z)f''(z) = 2(\cos\alpha)e^{-i\alpha}z\omega(z)f'(z). \tag{3.5}$$

Using the series representations for functions f and ω given by (1.1) and (3.3), respectively, in (3.5), we obtain

$$z^2 f''(z)\,\omega(z) = 2a_2c_0z^2 + (2a_2c_1 + 6a_3c_0)z^3 + \cdots \tag{3.6}$$

$$\begin{aligned} zf''(z) - z^2 f''(z)\,\omega(z) = {} & 2a_2 z + (6a_3 - 2a_2c_0)z^2 \\ & + (12a_4 - (2a_2c_1 + 6a_3c_0))z^3 + \cdots. \end{aligned} \tag{3.7}$$

By substituting (3.6) and (3.7) in (3.5) and equating the coefficients of z and z^2, we obtain $a_2 = (\cos\alpha)\,e^{-i\alpha}c_0$ and

$$a_2 = (\cos\alpha)e^{-i\alpha}c_0, \tag{3.8}$$

$$a_3 = \frac{1}{6}\Big(2a_2c_0 + (2(\cos\alpha)e^{-i\alpha})(2a_2c_0 + c_1)\Big). \tag{3.9}$$

Further, a substitution of $a_2 = (\cos\alpha)e^{-i\alpha}c_0$ in (3.9) yields

$$a_3 = \frac{1}{3}(\cos\alpha)e^{-i\alpha}\left(c_0^2 + 2(\cos\alpha)e^{-i\alpha}c_0^2 + c_1\right). \tag{3.10}$$

Therefore, from (3.8) and (3.10) and $\lambda \in \mathbb{C}$, a simple computation gives

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{1}{3}(\cos\alpha)e^{-i\alpha}\left(c_0^2 + 2(\cos\alpha)e^{-i\alpha}c_0^2 + c_1\right) - \lambda(\cos^2\alpha)e^{-2i\alpha}c_0^2 \\ &= \frac{1}{3}(\cos\alpha)e^{-i\alpha}\left(c_1 + c_0^2 + c_0^2(2-3\lambda)(\cos\alpha)e^{-i\alpha}\right) \\ &= \frac{1}{3}(\cos\alpha)e^{-i\alpha}\left(c_1 + c_0^2\gamma(\alpha)\right) \end{aligned} \tag{3.11}$$

where $\gamma(\alpha) = 1 + (2-3\lambda)(\cos\alpha)e^{-i\alpha}$. In view of the Schwarz–Pick lemma, it is well known that

$$|c_0| \le 1 \quad \text{and} \quad |c_1| \le 1 - |c_0|^2. \tag{3.12}$$

Using (3.12) in (3.11) and then applying the triangle inequality yields

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &\le \frac{1}{3}(\cos\alpha)\left(|c_1| + |c_0|^2|\gamma(\alpha)|\right) \\ &\le \frac{1}{3}(\cos\alpha)\left(1 + (|\gamma(\alpha)| - 1)|c_0|^2\right). \end{aligned}$$

Therefore when $f \in \mathcal{S}_\alpha$ is given by (1.1) and $\lambda \in \mathbb{C}$, we obtain

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} \frac{1}{3}|\gamma(\alpha)|\cos\alpha & \text{for } |\gamma(\alpha)| \ge 1 \\ \frac{1}{3}\cos\alpha & \text{for } |\gamma(\alpha)| < 1. \end{cases} \tag{3.13}$$

A simple observation shows that

$$\begin{aligned} \big|\gamma(\alpha)\big| &= \left|1 + (2-3\lambda)(\cos\alpha)e^{-i\alpha}\right| \\ &= \left|e^{i\alpha} + (2-3\lambda)\cos\alpha\right| \\ &= |3(1-\lambda)\cos\alpha + i\sin\alpha| \\ &= 3(\cos\alpha)\left|1 - \lambda + \frac{i}{3}\tan\alpha\right|. \end{aligned} \tag{3.14}$$

Therefore $|\gamma(\alpha)| < 1$ if and only if $\left|\lambda - (1 + \frac{\mathrm{i}}{3}\tan\alpha)\right| < \frac{1}{3\cos\alpha}$. In view of (3.13) and (3.14), we obtain the following functional inequality for functions f in the class $\mathcal{S}_\alpha$:

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} \frac{1}{3}(\cos\alpha)\left|1 + (2 - 3\lambda)(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}\right| & \text{for } \left|\lambda - (1 + \frac{\mathrm{i}}{3}\tan\alpha)\right| \ge \frac{1}{3\cos\alpha} \\ \frac{1}{3}\cos\alpha & \text{for } \left|\lambda - (1 + \frac{\mathrm{i}}{3}\tan\alpha)\right| < \frac{1}{3\cos\alpha}. \end{cases}$$

To prove that the inequality (3.2) is sharp, we define f_1 and f_2 by

$$f_1'(z) = \frac{1}{(1-z)^{(2\cos\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha}}} \quad \text{and} \quad f_2'(z) = \frac{1}{(1-z^2)^{(\cos\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha}}} \quad \text{for } z \in \mathbb{D}.$$

Then a simple computation shows that

$$\mathrm{e}^{\mathrm{i}\alpha}\left(1 + \frac{zf_1''(z)}{f_1'(z)}\right) = (\cos\alpha)\left(\frac{1+z}{1-z}\right) + \mathrm{i}\sin\alpha$$

and

$$\mathrm{e}^{\mathrm{i}\alpha}\left(1 + \frac{zf_2''(z)}{f_2'(z)}\right) = (\cos\alpha)\left(\frac{1+z^2}{1-z^2}\right) + \mathrm{i}\sin\alpha$$

and hence $\mathrm{Re}\, P_{f_1}(z) > 0$ and $\mathrm{Re}\, P_{f_2}(z) > 0$ in the unit disk $\mathbb{D}$. Thus $f_1, f_2 \in \mathcal{S}_\alpha$.

Further, it is easy to see that

$$f_1'(z) = 1 + 2(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}z + (\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}(2(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha} + 1)z^2 + \cdots$$

and hence

$$a_2(f_1) = (\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha} \quad \text{and} \quad a_3(f_1) = \frac{1}{3}(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}(2(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha} + 1). \tag{3.15}$$

In view of (3.15) and $\lambda \in \mathbb{C}$, a simple computation gives

$$\begin{aligned} a_3(f_1) - \lambda a_2^2(f_1) &= \frac{1}{3}(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}(2(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha} + 1) - \lambda(\cos^2\alpha)e^{-2\mathrm{i}\alpha} \\ &= \frac{1}{3}(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}\left(1 + 2(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha} - 3\lambda(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}\right) \\ &= \frac{1}{3}(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}\left(1 + (2 - 3\lambda)(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}\right). \end{aligned}$$

This shows that the first inequality in (3.2) is sharp.

Next, it is not difficult to see that

$$f_2'(z) = 1 + (\cos\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha}z^2 + \cdots.$$

Therefor $a_2(f_2) = 0$ and $a_3(f_2) = \frac{1}{3}(\cos\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha}$, and hence for $\lambda \in \mathbb{C}$ we have

$$a_3(f_2) - \lambda a_2^2(f_2) = \frac{1}{3}(\cos\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha}.$$

Therefore, the second inequality in (3.2) is sharp. This completes the proof. □

In particular, for $\alpha = 0$, Theorem 3.1 reduces to the following interesting Fekete–Szegö inequality for the class of convex functions $\mathcal{C}$.

Corollary 3.16. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}$ and $\lambda \in \mathbb{C}$. Then we have*

$$|a_3 - \lambda {a_2}^2| \le \begin{cases} |1 - \lambda| & \text{for } |\lambda - 1| \ge \frac{1}{3} \\ \frac{1}{3} & \text{for } |\lambda - 1| < \frac{1}{3}. \end{cases} \tag{3.17}$$

The inequality (3.17) is sharp.

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