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Differential geometry

On the rank of a product of manifolds

Sur le rang d'un produit de variétés

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ABSTRACT

This note gives an example of closed smooth manifolds M and N for which the rank of $M \times N$ is strictly greater than $\operatorname{rank} M + \operatorname{rank} N$.

RÉSUMÉ

Cette note donne un exemple de deux variétés compactes M et N pour lesquelles le rang de $M \times N$ est strictement plus grand que $\operatorname{rang} M + \operatorname{rang} N$.

1. Introduction

Milnor defined the *rank* of a smooth manifold M as the maximal number of commuting vector fields on M that are linearly independent at each point.

One of the questions raised by Milnor at the Seattle Topology Conference of 1963, and echoed by Novikov [2], was

is $\operatorname{rank}(M \times N) = \operatorname{rank}(M) + \operatorname{rank}(N)$

whenever M and N are smooth closed manifolds?

In this note we give a negative answer to this question.

2. The main result

We need a simple result about mapping tori.

Let $f : X \to X$ be a diffeomorphism of a manifold X and let

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$$M(f) = \frac{I \times X}{(0,x) \sim (1, f(x))}$$

be the mapping torus of f where $I = [0,1]$.

Equivalently, $M(f) = \dfrac{\mathbb{R} \times X}{\mathbb{Z}}$ where the action of $\mathbb{Z}$ on $\mathbb{R} \times X$ is given by $\alpha(k)(t,x) = (t+k, f^k(x))$. $M(f)$ is a fiber bundle over S^1 with fiber X. We note that $\pi_1(M(f)) = \pi_1(X) *_f \mathbb{Z}$ where $*$ denotes the semi-direct product and $f_*\colon \pi_1(X) \to \pi_1(X)$.

Proposition 2.1. *Consider two periodic diffeomorphisms $f\colon X \to X$ and $g\colon Y \to Y$ with periods m and n respectively. Assume m and n are relatively prime, i.e., there are integers c, d such that $mc + nd = 1$.*

Then $M(f) \times M(g)$ is diffeomorphic to $M(h)$ where $h\colon S^1 \times X \times Y \to S^1 \times X \times Y$ is defined by $h(\theta, x, y) = (\theta, f^{-d}(x), g^c(y))$. Moreover $h^{m-n} = (id, f, g)$.

Proof. $M(f) \times M(g)$ can be identified with the quotient of $\mathbb{R}^2 \times X \times Y$ under the action of $\mathbb{Z}^2$ given by $\beta(z)(u,x,y) = (u+z, f^{z_1}(x), g^{z_2}(y))$, where $z = (z_1, z_2) \in \mathbb{Z}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ and $(x,y) \in X \times Y$.

Set $\lambda = (m,n)$ and $\mu = (-d, c)$. Since $mc + nd = 1$, $\mathcal{B} = \{\lambda, \mu\}$ is at the same time a basis of $\mathbb{Z}^2$ as a $\mathbb{Z}$-module and a basis of $\mathbb{R}^2$ as a vector space. On the other hand

$$\beta(\lambda)(u,x,y) = (u+\lambda, x, y) \quad \text{and} \quad \beta(\mu)(u,x,y) = (u+\mu, f^{-d}(x), g^c(y)).$$

Therefore the action β referred to the new basis $\mathcal{B}$ of $\mathbb{Z}^2$ and $\mathbb{R}^2$ is written now:

$$\beta(k,r)(a,b,x,y) = (a+k, b+r, \varphi^r(x), \gamma^r(y))$$

where $\varphi = f^{-d}$ and $\gamma = g^c$.

As the action of the first factor of $\mathbb{Z}^2$ on $X \times Y$ is trivial, identifying S^1 with $\dfrac{\mathbb{R}}{\mathbb{Z}}$ shows that $M(f) \times M(g)$ is diffeomorphic to $M(h)$.

Finally from $(-n)(-d) = 1 - cm$ and $cm = 1 - dn$ follows that $h^{m-n} = (id, f, g)$. □

On the other hand:

Lemma 2.1. *Let $f\colon N \to N$ be a diffeomorphism and let $X_1, \dots, X_k$ be a family of commuting vector fields on N that are linearly independent everywhere. Assume $f_* X_i = \sum_{j=1}^k a_{ij} X_j$, $i = 1, \dots, k$, where the matrix $(a_{ij}) \in GL(k, \mathbb{R})$. Then $\operatorname{rank}(M(f)) \geq k$.*

Proof. It suffices to construct k commuting vector fields $\widetilde{X}_1, \dots, \widetilde{X}_k$ on $I \times N$ that are linearly independent at each point and such that every $\widetilde{X}_i(t,x)$ equals $X_i(x)$ if t is close to zero and $f_* X_i(x)$ when t is close to 1 ($X_1, \dots, X_k$ are considered vector fields on $I \times N$ in the obvious way).

If $|\, a_{ij} \,| > 0$ consider an interval $[a,b] \subset (0,1)$ and a (differentiable) map $(\varphi_{ij})\colon I \to GL(k, \mathbb{R})$ such that $\varphi_{ij}([0,a]) = \delta_{ij}$ and $\varphi_{ij}([b,1]) = a_{ij}$, and set $\widetilde{X}_i(t,x) = \sum_{j=1}^k \varphi_{ij}(t) X_j(x)$.

When $|\, a_{ij} \,| < 0$ first take an interval $[c,d] \subset (0, 1/2)$ and a function $\rho\colon [0, 1/2] \to \mathbb{R}$ such that $\rho([0,c]) = 1$, $\rho([d, 1/2]) = -1$, and on $[0, 1/2] \times N$ set $\widetilde{X}_1(t,x) = \rho(t) X_1(x) + (1 - \rho^2(t)) \frac{\partial}{\partial t}$ and $\widetilde{X}_i(t,x) = X_i(x)$, $i = 2, \dots, k$.

The matrix of coordinates of $f_* X_1, \dots, f_* X_k$ with respect to the basis $\{-X_1, X_2, \dots, X_k\}$ has positive determinant, so by doing as before we can extend $\widetilde{X}_1, \dots, \widetilde{X}_k$ to $[1/2, 1] \times N$ by means of an interval $[a,b] \subset (1/2, 1)$ and a suitable map $(\varphi_{ij})\colon [1/2, 1] \to GL(k, \mathbb{R})$. □

Proposition 2.1 and Lemma 2.1 quickly yield a counterexample.

Assume X is a torus $\mathbb{T}^k = \dfrac{\mathbb{R}^k}{\mathbb{Z}^k}$ and f is the map induced by a nontrivial element of $GL(k, \mathbb{Z})$. Then by the above lemma applied to $\frac{\partial}{\partial \theta_j}$, $j = 1, \dots, k$, $\operatorname{rank}(M(f)) \geq k$. But $M(f)$ has non-Abelian fundamental group, so it is not a torus and $\operatorname{rank}(M(f)) = k$. (If M is a closed connected n-manifold of rank n, then M is diffeomorphic to the n-torus.)

For the same reason, if $Y = \mathbb{T}^r$ and g is induced by a nontrivial element of $GL(r, \mathbb{Z})$, then $\operatorname{rank}(M(g)) = r$.

If f and g are periodic with relatively prime periods m and n, respectively, then by Proposition 2.1, $M(f) \times M(g) = M(h)$ where $h\colon \mathbb{T}^{k+r+1} \to \mathbb{T}^{k+r+1}$ is induced by a nontrivial element of $GL(k+r+1, \mathbb{Z})$. Moreover $\operatorname{rank}(M(h)) = k+r+1$. Therefore:

$$\operatorname{rank}(M(f) \times M(g)) > \operatorname{rank}(M(f)) + \operatorname{rank}(M(g)).$$

For instance, set $k = r = 2$ and consider f, g induced by the elements in $SL(2, \mathbb{Z}) \subset GL(2, \mathbb{Z})$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

respectively, so $M(f)$ and $M(g)$ are orientable. Then the period of f is 2 and that of g equals 3.

An even simpler but non-orientable counterexample can be constructed as follows. Take r and g as before, $k = 1$ and f induced by (-1). Then $M(f)$ is the Klein bottle which has rank 1 and M(g) has rank 2; however, $M(f) \times M(g)$ is diffeomorphic to $M(h)$ and hence has rank 4.

Remark 1. The *file* of a manifold M was defined by Rosenberg [3] to be the largest integer k such that $\mathbb{R}^k$ acts locally free on M. When M is closed file(M) equals rank(M) but file($\mathbb{R} \times S^2$) = 1, [3], while rank($\mathbb{R} \times S^2$) = 3.

The analog of Milnor's question for the file of a product of noncompact manifolds also fails. Indeed, let $\mathbb{R}^4_e$ be any exotic $\mathbb{R}^4$. Then file($\mathbb{R}^4_e$) ≤ 3 otherwise $\mathbb{R}^4_e = \mathbb{R}^4$. But $\mathbb{R}^4_e \times \mathbb{R} = \mathbb{R}^5$, because there in no exotic $\mathbb{R}^5$, so file($\mathbb{R}^4_e \times \mathbb{R}$) $= 5 >$ file($\mathbb{R}^4_e$) + file($\mathbb{R}$).

Orientable closed connected n-manifolds of rank $n-1$ are completely described in [4,1,5].

References

[1] G. Chatelet, H. Rosenberg, Manifolds which admit $\mathbb{R}^n$ actions, Publ. Math. Inst. Hautes Études Sci. 43 (1974) 245–260.

[2] S.P. Novikov, The topology summer institute, in: Seattle, USA, 1963, Russ. Math. Surv. 20 (1965) 145–167, http://www.mi.ras.ru/~snovikov/16.pdf.

[3] H. Rosenberg, Singularities of $\mathbb{R}^2$ actions, Topology 7 (1968) 143–145.

[4] H. Rosenberg, R. Roussarie, D. Weil, A classification of closed oriented 3-manifold of rank two, Ann. of Math. (2) 91 (1970) 449–464.

[5] D. Tischler, Manifolds M^n of rank $n-1$, Proc. Amer. Math. Soc. 94 (1985) 158–160.