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Algebraic geometry

Remarks on minimal rational curves on moduli spaces of stable bundles ☆

Remarques sur les courbes rationnelles minimales sur les espaces des modules de faisceaux stables

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ABSTRACT

Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic zero, and M be the moduli space of stable bundles of rank 2 and with fixed determinant $\mathcal{L}$ of degree d on the curve C. When $g = 3$ and d is even, we prove that, for any point $[W] \in M$, there is a minimal rational curve passing through $[W]$, which is not a Hecke curve. This complements a theorem of Xiaotao Sun.

RÉSUMÉ

Soient C une courbe projective lisse de genre $g \geq 2$ et M l'espace des modules de faisceaux stables de rang 2 et de déterminant fixe $\mathcal{L}$ de degré d sur C. Nous prouvons que, lorsque $g = 3$ et d est pair, il existe, pour tout point $[W] \in M$, une courbe rationnelle minimale passant par $[W]$, qui n'est pas une courbe de Hecke. Cela complète un théorème de Xiaotao Sun.

1. Introduction

Throughout this paper, we assume that C is a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic zero. Let $M := SU_C(r, \mathcal{L})$ be the moduli space of stable vector bundles of rank $r \geq 2$ and with the fixed determinant $\mathcal{L}$ of degree d, which is a smooth quasi-projective Fano variety with $Pic(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2(r,d)\Theta$, where Θ is an ample divisor ([9,1]). By a rational curve of M, we mean a nontrivial proper morphism $\phi : \mathbb{P}^1 \to M$ and its degree is defined to be $\deg \phi^*(-K_M)$ (with respect to the ample anti-canonical line bundle $-K_M$).

In [10], Xiaotao Sun has determined all rational curves of minimal degree passing through generic points of M except in the case where $g = 3$, $r = 2$, and d is even.

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Theorem 1.1. *(Theorem 1 of [10]) If $g \geq 3$, then any rational curve $\phi : \mathbb{P}^1 \to M$ passing through the generic point has degree at least $2r$. It has degree $2r$ if and only if it is a Hecke curve* ***unless*** $g = 3$, $r = 2$, ***and*** d ***is even.***

This implies that all the rational curves of $(-K_M)$-degree smaller than $2r$, called *small rational curves*, must lie in a proper closed subset [3,4]. In this note, we remark that the condition in Sun's Theorem is necessary:

Theorem 1.2. *If $g = 2$, $r = 2$ and d is odd, then, for any $[W] \in M$, there exists a rational curve passing through it, which has degree 2.*

If $g = 3$, $r = 2$ and d is even, then, for any point $[W] \in M$, there exists a rational curve of degree 4 passing through it, which is not a Hecke curve.

Recall that, by Lemma 2.1 of [10], any rational curve $\phi : \mathbb{P}^1 \to M$ is defined by a vector bundle E on $f : X = C \times \mathbb{P}^1 \to C$. **If E is semi-stable on generic fiber** $X_\xi = f^{-1}(\xi)$ (tensoring a pullback of line bundle on $\mathbb{P}^1$, we can assume the restriction of E to a generic fiber is of the form $O_{X_\xi}^{\oplus r}$), according to the arguments of section 2 in [10], there is a finite set $S \subset C$ of points and a vector bundle V on C such that E just suits in the exact sequence

$$0 \to f^*V \to E \to \bigoplus_{p \in S} \mathcal{Q}_p \to 0$$

where $\mathcal{Q}_p$ is a vector bundle on $X_p = \{p\} \times \mathbb{P}^1$. The curves defined by such E were said to be of **Hecke type** in [8,11] (since a Hecke curve by definition is defined by a vector bundle E suited in $0 \to f^*V \to E \to \mathcal{O}_{X_p}(-1) \to 0$). **If E is not semi-stable on the generic fiber** X_ξ (curves defined by such E were said of **split type** in [11]) and the curve has minimal degree $2(r,d)$, then E must suit in

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0$$

where $\pi : X \to \mathbb{P}^1$ is the projection and V_1, V_2 are stable vector bundles on C of rank r_1, r_2, and degrees d_1, d_2 satisfy $r_1 d - r d_1 = (r,d)$. Note that rational curves of degree $2(r,d)$ have degree 1 with respect to Θ because of $-K_M = 2(r,d)\Theta$, which will be called lines in M.

The rational curves we constructed in Theorem 1.2 are of split type (thus they are not Hecke curves). We have in fact a more general result. Let $M = \mathcal{SU}_C(2, \mathcal{L})$ be the moduli space of rank-two stable bundles with fixed determinant $\mathcal{L}$ on a smooth projective curve C of genus $g \geq 3$. Let $M_s \subset M$ be the locus of stable bundles $[W] \in M$ with the Segre invariant $s(W) = s$ (refer to Section 3 for the definition of Segre invariant). Then we have the following theorem.

Theorem 1.3. *When d is even, for any $[W] \in M_2$, there is a rational curve of split type passing through it, which has degree 4. If d is odd, for any $[W] \in M_1$, there is a rational curve of split type passing through it, which has degree 2.*

When $g = 3$ and d is even, we have $M_2 = M$ (see Lemma 3.1). Thus Theorem 1.2 is a corollary of Theorem 1.3.

2. Rational curves of split type

Let C be a smooth projective curve with genus $g \geq 2$ over an algebraically closed field of characteristic zero, W be a stable bundle of rank r and of degree d with determinant $\mathcal{L}$ over C. Assume that there is a stable subbundle V_1 of W such that

$$r_1 d - d_1 r = (r,d), \tag{1}$$

where $r_1 = \operatorname{rank} V_1$, $d_1 = \deg V_1$ and $d = \deg W$. Let $V_2 := V/V_1$ be the quotient bundle, then W fits a non-trivial extension

$$0 \to V_1 \to W \to V_2 \to 0. \tag{2}$$

It is known that there is a family of vector bundles $\{\mathcal{E}_p\}_{p \in P}$ on C parametrized by $P = \mathbb{P}Ext^1(V_2, V_1)$ so that for each $p \in P$, $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of V_2 by V_1 given by p (see Lemma 2.3 of [9]). Let l be a line in $P = \mathbb{P}Ext^1(V_2, V_1)$ passing through the point p_0, where p_0 is the point in P given by (2). If it happens that $\mathcal{E}_p$ is stable for each $p \in l$, then

$$\{\mathcal{E}_p\}_{p \in l}$$

will define a rational curve of degree $2(r,d)$ (with respect to $-K_M$) passing through $[W] \in SU_C(r, \mathcal{L})$ ([10,4]). Such a rational curve in $SU_C(r, \mathcal{L})$ will be called a **rational curve of split type**.

It is known that an extension $0 \to E \to W \to F \to 0$, where E, W, F are vector bundles on C, gives rise to an element $\delta(W) \in H^1(C, Hom(F, E))$, which is the image of the identity homomorphism in $H^0(C, Hom(F, F))$ by the connecting homomorphism $H^0(C, Hom(F, F)) \to H^1(C, Hom(F, E))$. This gives a one:one correspondence between the set of equivalent classes of extensions of F by E and $H^1(C, Hom(F, E))$ (refer to section 2 in [9]).

Lemma 2.1. *Let d be an even number, and $0 \to L_1 \to W \to L_2 \to 0$ be any non-trivial extension of L_2 by L_1, where L_1 (resp. L_2) is a line bundle of degree $\frac{d}{2} - 1$ (resp. $\frac{d}{2} + 1$). Then*

(i) W is semi-stable;

(ii) W is non-stable if and only if the element $\delta(W) \in H^1(C, L_2^{-1} \otimes L_1)$ corresponding to W is in the kernel of the map

$$H^1(C, L_2^{-1} \otimes L_1) \longrightarrow H^1(C, L_2^{-1} \otimes L_1 \otimes L_x),$$

for some $x \in C$, where $L_x = \mathcal{O}_C(x)$ is the line bundle defined by x. In this case, W is S-equivalent to $L_2 \otimes L_x^{-1} \oplus L_1 \otimes L_x$ (refer to section 2 of [7] for the definition of S-equivalent).

Proof. (i) See Lemma 2.2 in [4] and [5].

(ii) Let L' be a line bundle of degree $\frac{d}{2}$. Then, since $H^0(C, \mathrm{Hom}(L', L_1)) = 0$, it is easy to see that $H^0(C, \mathrm{Hom}(L', W)) \neq 0$ if and only if L' is of the form $L_2 \otimes L_x^{-1}$ for some $x \in C$ and the natural map $L_2 \otimes L_x^{-1} \to L_2$ can be lifted into a map $L_2 \otimes L_x^{-1} \to W$.

Consider the commutative diagram of vector bundles

$$\begin{array}{ccccc} 0 \to Hom(L_2, L_1) & \longrightarrow & Hom(L_2, W) & \longrightarrow & Hom(L_2, L_2) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to Hom(L_2 \otimes L_x^{-1}, L_1) & \longrightarrow & Hom(L_2 \otimes L_x^{-1}, W) & \longrightarrow & Hom(L_2 \otimes L_x^{-1}, L_2) \to 0, \end{array}$$

where the horizontal sequences are exact and the vertical maps are induced by the natural map $L_2 \otimes L_x^{-1} \to L_2$. From this, we deduce the commutative diagram

$$\begin{array}{ccccc} 0 \to H^0(C, Hom(L_2, W)) & \longrightarrow & H^0(C, Hom(L_2, L_2)) & \longrightarrow & H^1(C, Hom(L_2, L_1)) \to \cdots \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to H^0(C, Hom(L_2 \otimes L_x^{-1}, W)) & \longrightarrow & H^0(C, Hom(L_2 \otimes L_x^{-1}, L_2)) & \longrightarrow & H^1(C, Hom(L_2 \otimes L_x^{-1}, L_1)) \to \cdots \end{array}$$

which implies the lemma. □

Remark 2.2. Lemma 2.1 (ii) asserts that the non-stable bundles in $\mathbb{P}H^1(L_2^{-1} \otimes L_1)$ correspond precisely to the image of C in $\mathbb{P}H^1(L_2^{-1} \otimes L_1)$ under the map given by the linear system $K_C \otimes L_1^{-1} \otimes L_2$. Which implies that the dimension of the subset of non-stable bundles in $\mathbb{P}H^1(L_2^{-1} \otimes L_1)$ is at most 1.

3. Proof of Theorem 1.3

Let C be a smooth irreducible curve over an algebraically closed field of characteristic zero, W a vector bundle of rank 2 over C, set

$$m(W) := \max\{\deg(L) | L \subset W \text{ is a sub line bundle of } W\}. \tag{3}$$

A sub line bundle L of W of maximal degree $m(W)$ is called a **maximal sub line bundle**. The **Segre invariant** is defined by

$$s(W) := \deg(W) - 2m(W). \tag{4}$$

Note that $s(W) \equiv \deg(W) \pmod 2$ and that W is stable (resp. semi-stable) if and only if $s(W) \geq 1$ (resp. $s(W) \geq 0$). Nagata proved in [6] that

$$s(W) \leq g.$$

It is easy to see that

Lemma 3.1. *If $g = 3$, then, for any stable bundle W over C of rank 2 and with even degree d, we have $s(W) = 2$.*

In general, the function $s : M \longrightarrow \mathbb{Z}$ defined by $[W] \longmapsto s(W)$ is lower semicontinuous and gives a stratification of M into locally closed subsets M_s according to the value of s. Then, by Proposition 3.1 in [2], we have

Proposition 3.2. *([2]) Suppose that $1 \leq s \leq g - 2$ and $s \equiv d \pmod 2$. Then M_s is an irreducible algebraic variety of dimension $2g + s - 2$.*

The proof of Theorem 1.3 follows the following two propositions.

Proposition 3.3. *Suppose that $g \geq 3$, $r = 2$, d is even and M_2 is non-empty. Then, for any $[W] \in M_2$, there is a rational curve of split type passing through it, which has degree 4.*

Proof. For any $[W] \in M_2$, there is a sub line bundle L_1 of W with $\deg L_1 = \frac{d}{2} - 1$, where $d = \deg \mathcal{L}$. Let $L_2 := W/L_1$ be the quotient bundle, which has degree $\frac{d}{2} + 1$. It is easy to see that

$$1 \times d - (\frac{d}{2} - 1) \times 2 = 2 = (2, d).$$

Let $i : L_1 \to W$ be the natural injection, then

$$0 \longrightarrow L_1 \xrightarrow{\ i\ } W \longrightarrow L_2 \longrightarrow 0$$

is a non-trivial extension (otherwise, we have $W \cong L_1 \oplus L_2$, which contradicts the stability of W).

It is known that there is a family of vector bundles $\mathcal{E}$ on C parametrized by $P_{(L_1,L_2)} = \mathbb{P}Ext^1(L_2, L_1)$ so that for each $p \in P_{(L_1,L_2)}$, the $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of L_2 by L_1 given by p (see Lemma 2.3 of [9]). More precisely, there is a universal extension

$$0 \to f^*L_1 \otimes \pi^*\mathcal{O}_{P_{(L_1,L_2)}}(1) \to \mathcal{E} \to f^*L_2 \to 0 \tag{5}$$

on $C \times P_{(L_1,L_2)}$, where $f : C \times P_{(L_1,L_2)} \to C$ and $\pi : C \times P_{(L_1,L_2)} \to P_{(L_1,L_2)}$ are projections. Then $\mathcal{E}$ is a family of semi-stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$ (Lemma 2.1). Thus, the universal extension (5) defines a morphism

$$\Phi_{(L_1,L_2)} : \ P_{(L_1,L_2)} \longrightarrow U_C(2, \mathcal{L}), \tag{6}$$

where $U_C(2, \mathcal{L})$ denotes the moduli space of semi-stable bundles of rank 2 and with fixed determinant $\mathcal{L}$, which is a projective compactification of M.

It is easy to see that $P_{(L_1,L_2)}$ is a projective space of dimension $g \geq 3$. By Lemma 2.1 and Remark 2.2, there is a line l in $P_{(L_1,L_2)}$ passing through

$$q = [0 \longrightarrow L_1 \xrightarrow{\ i\ } W \longrightarrow L_2 \longrightarrow 0]$$

such that $\mathcal{E}_p$ is stable for each $p \in l$. Thus, $\Phi_{(L_1,L_2)}(l) \subset M = SU_C(2, \mathcal{L})$ and

$$\Phi_{(L_1,L_2)}|_l : l \to M = SU_C(2, \mathcal{L}) \tag{7}$$

is a rational curve of split type passing through the point $[W] \in M$. □

Proposition 3.4. *Suppose $g \geq 2$, $r = 2$, d is odd and M_1 is non-empty. Then, for any $[W] \in M_1$, there is a rational curve of split type passing through it, which has degree 2.*

Proof. Let $[W]$ be a point in M_1, then we have $s(W) = 1$ and there is a sub line bundle L_1 of W with $\deg L_1 = \frac{d-1}{2}$, where $d = \deg \mathcal{L}$. Let $L_2 := W/L_1$, which is a line bundle of degree $\frac{d+1}{2}$. It is easy to see that

$$1 \times d - \frac{d-1}{2} \times 2 = 1 = (2, d).$$

Let $\iota : L_1 \to W$ be the natural injection, then

$$0 \longrightarrow L_1 \xrightarrow{\ \iota\ } W \longrightarrow L_2 \longrightarrow 0$$

is a non-trivial extension because W is a stable bundle.

It is known that there is a family of vector bundles $\{\mathcal{E}_p\}$ on C parametrized by $P_{(L_1,L_2)} = \mathbb{P}Ext^1(L_2, L_1)$ such that for each $p \in P_{(L_1,L_2)}$, $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of L_2 by L_1 given by p (Lemma 2.3 of [9]). By Lemma 3.1 of [10], $\{\mathcal{E}_p\}$ is a family of stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$, which defines a morphism

$$\Psi_{(L_1,L_2)} : \ P_{(L_1,L_2)} \longrightarrow SU_C(2, \mathcal{L}) = M. \tag{8}$$

Let l be a line in $P_{(L_1,L_2)}$ passing through

$$q = [0 \longrightarrow L_1 \xrightarrow{\ \iota\ } W \longrightarrow L_2 \longrightarrow 0],$$

then

$$\Psi_{(L_1,L_2)}|_l : l \longrightarrow M = SU_C(2, \mathcal{L}) \tag{9}$$

is a rational curve of split type passing through the point $[W] \in M$, which has degree 2. □

When $g = 2$, the same as Lemma 3.1, we have:

Lemma 3.5. *If $g = 2$, $r = 2$ and d is odd, for any $[W] \in M$, $s(W) = 1$.*

By Lemma 3.5 and Proposition 3.4, we have:

Proposition 3.6. *If $g = 2$, $r = 2$ and d is odd, then, for any $[W] \in M$, there exists a rational curve of split type passing through it, which has degree 2.*

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