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Partial differential equations

Invariance of the support of solutions for a sixth-order thin film equation ☆

Invariance du support des solutions d'une équation du sixième ordre modélisant un film mince

Changchun Liu, Xiaoli Zhang

Department of Mathematics, Jilin University, Changchun 130012, China

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ABSTRACT

In this paper, we study the invariance of the support of solutions for a sixth-order nonlinear parabolic equation, which arises in the industrial application of the isolation oxidation of silicium. Based on the suitable integral inequalities, we establish the invariance of the support of solutions.

RÉSUMÉ

Dans cet article, on étudie l'invariance des solutions d'une équation parabolique du sixième ordre issue d'une application industrielle, l'isolement de l'oxydation du silicium. À partir d'inégalités intégrales, on établit l'invariance du support des solutions.

1. Introduction

In this paper, we consider the sixth-order thin film equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[m(u)(k\nabla\Delta^2 u + \nabla(|u|^{p-1}u))\right], \quad \text{in } Q_T,\ p > 2, \tag{1.1}$$

where $Q_T = \Omega \times (0, T)$, Ω is a bounded domain in R^2 with smooth boundary and $m(u) = |u|^n$, $n > 0$, $k > 0$ are constants.

On the basis of physical consideration, as usual the equation (1.1) is supplemented with the natural boundary-value conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta^2 u}{\partial n}\right|_{\partial\Omega} = 0,\ t > 0, \tag{1.2}$$

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E-mail address: liucc@jlu.edu.cn (C. Liu).

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and the initial value condition

$$u(x,0)=u_0(x). \tag{1.3}$$

The equation (1.1) arises in the industrial application of the isolation oxidation of silicon [4–6]. The pure sixth-order thin film equation (without the lower-order term) also arises when considering the motion of a thin film of viscous fluid driven by an overlying elastic plate [7,9]. The values of n can be motivated by, inter alia, applications to power-law fluids spreading over horizontal substrates, with $n>3$ corresponding to shear-thickening fluids and $n<3$ to shear-thinning ones. Liu [11] considered the problem (1.1)–(1.3). He proved the existence, the nonnegativity and the expansion of the support of weak solutions for one dimension. By the combination of the energy techniques with some methods based on the framework of Campanato spaces, Liu [12] proved the existence and the nonnegativity for two space dimensions.

We also refer to the following relevant equation

$$\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}\left(u^n\frac{\partial^3 u}{\partial x^3}\right), \tag{1.4}$$

which has been extensively studied. F. Bernis and A. Friedman [3] proved that if $n\ge 2$ the support of the solutions $u(\cdot,t)$ is nondecreasing with respect to t for the initial boundary value problems (see also [1]). Yin and Gao [14] proved that the $u(\cdot,t)$ has compact support for $0<n<1$. F. Bernis [2] proved the similar result for $0<n<2$. Hulshof and Shishkov [8] established an estimate for the finite speed of propagation of the support of compactly supported nonnegative solutions with $2\le n<3$. Liu [10] studied the finite speed of propagation of perturbations of solutions for the convective Cahn–Hilliard equation with $0<n<1$. Liu and Qu [13] considered the equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(u^n\left(\frac{\partial^3 u}{\partial x^3}-\alpha\frac{\partial u}{\partial x}+\beta\right)\right)=0.$$

They proved that if $\frac{4}{3}\le n<2$, this equation has the finite speed of propagation property for the nonnegative strong solutions. The upper bound for the speed of the support of this solution is obtained.

Because of the degeneracy, the problem (1.1)–(1.3) does not admit classical solutions in general. So, we introduce the weak solutions in the following sense

Definition. A function u is said to be a weak solution to (1.1)–(1.3) if the following conditions are satisfied:

(1) $u\in C^{1/2}(\overline{Q}_T)$, $u\in L^\infty(0,T;H^2(\Omega))$, $|u|^{n/2}\nabla\Delta^2 u\in L^2(P)$.
(2) For $\varphi\in C^1(\overline{Q}_T)$ and $Q_T=\Omega\times(0,T)$,

$$-\int_\Omega u(x,T)\varphi(x,T)\mathrm{d}x+\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x+\iint_{Q_T} u\frac{\partial\varphi}{\partial t}\mathrm{d}x\,\mathrm{d}t$$
$$=\iint_P |u|^n(k\nabla\Delta^2 u+\nabla(|u|^{p-1}u))\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

where $P=\overline{Q}_T\backslash(\{u(x,t)=0\}\cup\{t=0\})$.

2. Invariance of the support of solutions

We consider the weak solution u constructed in Theorem 4.3 of [12], then $u=\lim\limits_{\delta\to 0}u_\delta$, where u_δ is the classical positive solution to (1.1), (1.2) with initial data $u_\delta(x,0)=u_0(x)+\delta$, $\delta>0$. We have

Theorem 2.1. *Suppose that $0\le u_0(x)\in H^2(\Omega)$ and $n\ge 4$. Then the support of the weak solution u is non-decreasing with respect to t.*

Proof. To prove the theorem, it suffices to verify that for any $x_0\in\Omega$ with $u_0(x_0)>0$, and we have $u(x_0,t)>0$ for all $t>0$. Let $\varepsilon>0$ be fixed, such that $u_0(x)>0$ holds in $\overline{\Omega}\cap B_\varepsilon(x_0)$, where $B_\varepsilon(x_0)$ is the ball center at x_0 and radius ε. Choose a nonnegative smooth function $\xi(x)$, such that

$$\left.\frac{\partial\xi}{\partial n}\right|_{\partial\Omega}=0,\ \xi(x)=1,\ \text{in } B_\varepsilon(x_0), \tag{2.1}$$

$$\int_\Omega \xi(x)u_0^{2-n}(x)\,\mathrm{d}x\le C<\infty. \tag{2.2}$$

Let

$$G_0(s) = \frac{s^{2-n}}{(n-2)(n-1)} + \frac{sA^{1-n}}{n-1} - \frac{A^{2-n}}{n-2}.$$

Multiplying both sides of the equation (1.1), by $\xi G_0'(u_\delta)$, and then integrating over Q_t, we obtain

$$\begin{aligned}
&\int_\Omega \xi G_0(u_\delta)\,\mathrm{d}x - \int_\Omega \xi G_0(u_{0\delta})\,\mathrm{d}x \\
&\quad = -\iint_{Q_t} (k\nabla\Delta^2 u_\delta + \nabla(|u_\delta|^{p-1}u_\delta))\nabla u_\delta \xi\,\mathrm{d}x\,\mathrm{d}s - \iint_{Q_t} (k\nabla\Delta^2 u_\delta + \nabla(|u_\delta|^{p-1}u_\delta))\, j(u_\delta)\nabla\xi\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \equiv I_1 + I_2,
\end{aligned} \tag{2.3}$$

where

$$j(u_\delta) = -u_\delta^n \int_{u_\delta}^{A} \frac{\mathrm{d}s}{s^n} = \frac{1}{n-1}A^{1-n}u_\delta^n - \frac{1}{n-1}u_\delta.$$

For I_1, we have

$$\begin{aligned}
I_1 &= \iint_{Q_t} (k\Delta^2 u_\delta + |u_\delta|^{p-1}u_\delta))(\Delta u_\delta \xi + \nabla\xi\nabla u_\delta)\,\mathrm{d}x\,\mathrm{d}s \\
&= -k\iint_{Q_t} |\nabla\Delta u_\delta|^2\xi\,\mathrm{d}x\,\mathrm{d}s + \iint_{Q_t} |u_\delta|^{p-1}u_\delta\Delta u_\delta\xi\,\mathrm{d}x\,\mathrm{d}s - \iint_{Q_t} |u_\delta|^{p-1}u_\delta\nabla\xi\nabla u_\delta\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

For I_2, we obtain

$$\begin{aligned}
I_2 &= \iint_{Q_t} (k\Delta^2 u_\delta + |u_\delta|^{p-1}u_\delta))(j'(u_\delta)\nabla u_\delta\nabla\xi + j\Delta\xi)\,\mathrm{d}x\,\mathrm{d}s \\
&= -k\iint_{Q_t} \nabla\Delta u_\delta(j''|\nabla u_\delta|^2\nabla\xi + j'\Delta u_\delta\nabla\xi + 2j'\nabla u_\delta\Delta\xi + j(u_\delta)\nabla\Delta\xi)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \iint_{Q_t} (|u_\delta|^{p-1}u_\delta)(j'(u_\delta)\nabla u_\delta\nabla\xi + j\Delta\xi)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

By [12], we know that

$$\int_\Omega (\Delta u_\delta)^2\,\mathrm{d}x \le C, \quad \sup_{Q_T}|u_\delta| \le C.$$

Hence, we have

$$\left|\iint_{Q_t} |u_\delta|^{p-1}u_\delta\Delta u_\delta\xi\,\mathrm{d}x\,\mathrm{d}s\right| \le C,$$

and

$$\left|-\iint_{Q_t} |u_\delta|^{p-1}u_\delta\nabla\xi\,\nabla u_\delta\,\mathrm{d}x\,\mathrm{d}s\right| \le C.$$

We now choose ξ to have the form $\xi = \zeta^r$, where ζ is a smooth nonnegative function and $r \ge 6$, then

$$\begin{aligned}
&\left|-k\iint_{Q_t} \nabla\Delta u_\delta(j''|\nabla u_\delta|^2\nabla\xi + j'\Delta u_\delta\nabla\xi + 2j'\nabla u_\delta\Delta\xi + j(u_\delta)\nabla\Delta\xi)\,\mathrm{d}x\,\mathrm{d}s\right| \\
&\quad \le C\iint_{Q_t} |\nabla\Delta u_\delta||\nabla u_\delta|\zeta^{r-1}\,\mathrm{d}x\,\mathrm{d}t + C\iint_{Q_t} |\nabla\Delta u_\delta||\Delta u_\delta|\zeta^{r-1}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$+ C \iint_{Q_t} |\nabla \Delta u_\delta||\nabla u_\delta|\zeta^{r-2}\, dx\, dt + C \iint_{Q_t} |\nabla \Delta u_\delta|\zeta^{r-3}\, dx\, dt$$
$$\le \frac{k}{2} \iint_{Q_t} |\nabla \Delta u_\delta|^2 \xi\, dx\, ds + C \iint_{Q_t} \zeta^{r-6}\, dx\, dt + C$$
$$\le \frac{k}{2} \iint_{Q_t} |\nabla \Delta u_\delta|^2 \xi\, dx\, ds + C,$$

and

$$\left| \iint_{Q_t} (|u_\delta|^{p-1} u_\delta)(j'(u_\delta)\nabla u_\delta \nabla \xi + j\Delta \xi)\, dx\, ds \right| \le C.$$

Hence, it follows from (2.3), that

$$\int_\Omega \xi u_\delta^{2-n}(x,t)\, dx \le \int_\Omega \xi u_0^{2-n}(x)\, dx + C \le C.$$

Let $E_\varepsilon = \Omega \cap \overline{B_\varepsilon(x_0)}$, then we have

$$\int_{E_\varepsilon} u_\delta^{2-n}(x,t)\, dx \le C.$$

Letting $\delta \to 0$, we get

$$\int_{E_\varepsilon} u^{2-n}(x,t)\, dx \le C.$$

Since $u(x,t) \in C^{1/2,1/12}, n \ge 4$, the similar argument as in Theorem 4.2 of [12] shows that $u(x,t) > 0$ for any $x \in E_\varepsilon, t > 0$. Therefore, $\operatorname{supp} u_0 \subset \operatorname{supp} u(\cdot,t)$. The proof is complete. □

Theorem 2.2. *If $n \ge 4$, then any nonnegative solution u of the problem (1.1)–(1.3) satisfies*

$$\operatorname{supp} u(\cdot,t) \subset \operatorname{supp} u_0, \ \text{for } t > 0. \tag{2.4}$$

Proof. Arguing by contradiction we may suppose that there exist a time $t > 0$, a constant $\delta > 0$ and a smooth positive function φ with support in Ω such that

$$u(x,t) > \delta > 0, \ \text{for } x \in \operatorname{supp} \varphi,$$
$$\operatorname{supp} \varphi \cap \operatorname{supp} u_0 = \emptyset.$$

Let $\sigma > 0$ be constant. It follows from a standard approximation procedure that we may take $\psi = \frac{\varphi}{u+\sigma}$ as a test function, and hence

$$\int_\Omega \varphi(x) \ln(u(x,t)+\sigma)\, dx - \int_\Omega \varphi(x) \ln(u_0(x)+\sigma)\, dx$$
$$= \iint_P \nabla \Delta^2 u \frac{\nabla \varphi u^n}{u+\sigma}\, dx\, dt - \iint_P \nabla \Delta^2 u \frac{\varphi \nabla u u^n}{(u+\sigma)^2}\, dx\, dt$$
$$+ \iint_P \nabla(|u|^{p-1}u) \frac{\nabla \varphi u^n}{u+\sigma}\, dx\, dt - \iint_P \nabla(|u|^{p-1}u) \frac{\varphi \nabla u u^n}{(u+\sigma)^2}\, dx\, dt. \tag{2.5}$$

By the choice of φ,

$$\int_\Omega \varphi(x) \ln(u(x,t)+\sigma)\, dx - \int_\Omega \varphi(x) \ln(u_0(x)+\sigma)\, dx \to +\infty, \ \text{as } \sigma \to 0. \tag{2.6}$$

On the other hand, since $n \ge 4$, $u^{\frac{n}{2}} \nabla \Delta^2 u \in L^2(P \cap Q_t)$, $\nabla u \in L^2(Q_t)$ and u being bounded in Q_t, Höder's inequality implies that the last two terms in (2.5) are uniformly bounded: we have that

$$\left|\iint\limits_P \nabla\Delta^2 u \frac{\nabla\varphi u^n}{u+\sigma}\,dx\,dt\right| \le \left(\iint\limits_P u^n|\nabla\Delta^2 u|^2\,dx\,dt\right)^{1/2}\left(\iint\limits_P u^{n-2}\left(\frac{|\nabla\varphi|u^n}{u+\sigma}\right)^2\,dx\,dt\right)^{1/2} \le K_1,$$

and

$$\left|\iint\limits_P \nabla\Delta^2 u \frac{\varphi u_x u^n}{(u+\sigma)^2}\,dx\,dt\right| \le \left(\iint\limits_P u^n|\nabla\Delta^2 u|^2\,dx\,dt\right)^{1/2}\left(\iint\limits_P u^{n-4}\frac{\varphi^2|\nabla u|^2 u^4}{(u+\sigma)^4}\,dx\,dt\right)^{1/2} \le K_2.$$

Similarly, we have

$$\left|\iint\limits_P \nabla(|u|^{p-1}u)\frac{\nabla\varphi u^n}{u+\sigma}\,dx\,dt\right| \le C\iint\limits_P u^{p-1}|\nabla u|\frac{|\nabla\varphi|u^n}{u+\sigma}\,dx\,dt \le K_3,$$

and

$$\left|\iint\limits_P \nabla(|u|^{p-1}u)\frac{\varphi\nabla u u^n}{(u+\sigma)^2}\,dx\,dt\right| \le C\iint\limits_P u^{p-1}|\nabla u|^2 u^{n-2}\frac{\varphi u^2}{(u+\sigma)^2}\,dx\,dt \le K_4,$$

where $K_i, i=1,2,3,4$ are constants independent of σ. Combined with (2.5) and (2.6) this leads to a contradiction. □

Theorem 2.3. *If $n \ge 4$, then any nonnegative solution u of the problem (1.1)–(1.3) satisfies*

$$\text{supp}\, u(\cdot,t) = \text{supp}\, u_0, \quad \text{for } t>0. \tag{2.7}$$

Proof. It follows at once from Theorem 2.1 and Theorem 2.2 that the support of u does not depend on t. □

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