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On cosymplectic groupoids

Sur les groupoïdes cosymplectiques

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ABSTRACT

This Note is concerned with cosymplectic groupoids and their infinitesimal counterparts. Examples of cosymplectic groupoids include those obtained by integrating the 1-jet bundle of some Poisson manifolds endowed with an infinitesimal automorphism.

RÉSUMÉ

Cette Note est consacrée aux groupoïdes cosymplectiques et à leurs objets infinitésimaux associés. Des exemples de groupoïdes cosymplectiques sont donnés, en particulier ceux provenant de l'integration du fibré des 1-jets de certaines variétés de Poisson munies d'un automorphisme infinitésimal.

Version française abrégée

La notion de structure cosymplectique sur une variété lisse de dimension impaire a été introduite par Libermann en 1958 [5]. Rappelons qu'une structure cosymplectique sur une variété lisse M de dimension $2n+1$ est la donnée d'un couple (ω, η) formé d'une 2-forme fermée ω et d'une 1-forme fermée η, tel que $\omega^n \wedge \eta \neq 0$ partout. Nous nous intéressons ici aux structures cosymplectiques sur les groupoïdes de Lie. En somme, un groupoïde de Lie au-dessus de M est une variété lisse G munie de deux submersions surjectives $s, t : G \to M$ (appelées source et but, respectivement), d'une application unité $\epsilon : M \to G$, d'une application inverse $i : G \to G$ et d'une multiplication $m : G_2 \to G$ vérifiant certaines relations de compatibilité (voir [8] pour plus de détails), où $G_2 = \{(g, h) \in G \times G \mid \mathrm{t}(g) = \mathrm{s}(h)\}$ est l'ensemble des couples composables.

- Une fonction $\sigma \in C^\infty(G, \mathbb{R})$ est dite multiplicative si $\sigma(m(g_1, g_2)) = \sigma(g_1) + \sigma(g_2)$, $\forall (g_1, g_2) \in G_2$, où m est la multiplication de G.
- Une k-form α sur G est dite *multiplicative* si $m^*\alpha = \mathrm{pr}_1^*\alpha + \mathrm{pr}_2^*\alpha$, où les $\mathrm{pr}_i : G_2 \to G$ sont les projections canoniques de G_2 sur G.

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Définition 0.1. Un groupoïde cosymplectique est un groupoïde de Lie $G \underset{t}{\overset{s}{\rightrightarrows}} M$ muni d'une structure cosymplectique (ω, η) telle que ω et η soient toutes les deux multiplicatives.

Nous obtenons les résultats suivants.

Proposition 0.2. *Soient $(G \underset{t}{\overset{s}{\rightrightarrows}} M, \omega, \eta)$ un groupoïde cosymplectique de dimension $2n+1$ avec sa structure de Poisson associée π et son champ de Reeb $\mathbf{R}$. Alors, la variété de base M est de dimension n et elle admet un unique couple (π_0, E_0) formé d'un tenseur de Poisson π_0 et d'un automorphisme infinitésimal $E_0 = s_*\mathbf{R}$ tels que s soit une application de Poisson et t, une application anti-Poisson.*

La preuve de cette proposition repose, en partie, sur l'idée de poissonification, qui peut se formuler de la manière suivante :

Proposition 0.3. *Soit $G \underset{t}{\overset{s}{\rightrightarrows}} M$ un groupoïde de Lie muni d'une fonction multiplicative σ. Alors, il y a une correspondance bi-univoque entre les structures cosymplectiques multiplicatives (ω, η) sur G et les 2-formes symplectiques multiplicatives sur $G \times_\sigma \mathbb{R}$ de la forme $\Omega = p^*\omega + d\tau \wedge p^*\eta$, où $p : G \times \mathbb{R} \to G$ est la projection canonique.*

Nous observons que, sur la variété base M d'un groupoïde cosymplectique G, le couple induit (π_0, E_0) obtenu dans la proposition 0.2 détermine une structure d'algébroïde de Lie sur le fibré J^1M des 1-jets qui dépend uniquement de la classe de cohomologie de E_0 [3]. Ceci découle d'un phénomène plus général concernant des actions des algèbres de Lie sur les algébroïdes de Lie. On a :

Proposition 0.4. *Toute action infinitésimale d'une algèbre de Lie $\mathfrak{g}$ sur un algébroïde de Lie A, par automorphismes d'algébroïde de Lie, engendre un couple d'algébroïdes de Lie croisés $(A, M \rtimes \mathfrak{g})$.*

Une conséquence de ce résultat est :

Corollaire 0.5. *Soit (A, A^*) un bi-algébroïde de Lie au-dessus de M. Tout 1-cocycle $\alpha \in \Gamma(A^*)$ donne lieu à un couple d'algébroïdes de Lie croisés $(M \rtimes \mathbb{R}, A)$. En outre, le produit croisé associé $(M \rtimes \mathbb{R}) \bowtie A$ dépend uniquement de la classe de cohomologie de $[\alpha] \in H^1(A)$.*

Le cas particulier où $(A, A^*) = (T^*M, TM)$ est le bi-algébroïde d'une variété de Poisson sur M donne :

Corollaire 0.6. *Soit (M, π) une variété de Poisson munie d'un automorphisme infinitésimal E. Alors, son fibré vectoriel des 1-jets, noté par J^1M, est muni d'une structure d'algébroïde de Lie au-dessus de M qui dépend uniquement de la classe de cohomologie de E et qui est telle que T^*M et le fibré trivial $M \times \mathbb{R}$ soient toutes les deux des sous-algébroïdes de Lie de J^1M.*

Pour revenir aux groupoïdes cosymplectiques, nous donnons :

Lemme 0.7. *Le champ de Reeb $\mathbf{R}$ associé à un groupoïde cosymplectique (G, ω, η) est invariant à droite. Plus précisément, il existe une section X_0 de l'algébroïde de Lie AG de G dont le champ de vecteurs invariant à droite correspondant $\overrightarrow{X_0}$ coïncide avec $\mathbf{R}$.*

Définition 0.8. Soit $(G \underset{t}{\overset{s}{\rightrightarrows}} M, \omega, \eta)$ un groupoïde cosymplectique avec son champ de Reeb $\mathbf{R} = \overrightarrow{X_0}$. Une fonction multiplicative σ sur G est appelée fonction de Reeb si son 1-cocycle associé $\alpha_0 \in \Gamma(A^*G)$ vérifie les relations : $\alpha_0(X_0) = 0$ et $a_*(\alpha_0) = -a(X_0) = -s_*\mathbf{R}$, où a et a_* sont applications ancres de l'algébroïde de Lie AG de G et son dual A^*G, respectivement.

Notons que, si une fonction de Reeb existe, alors elle est unique et complètement déterminée par le champ de Reeb $\mathbf{R}$. Des exemples de groupoïdes cosymplectiques peuvent être construits par intégration de l'algébroïde de Lie sur le fibré des 1-jets de certaines variétés de Poisson munies d'un automorphisme infinitésimal [3].

1. Basic definitions and facts

1.1. Cosymplectic manifolds

Recall that a cosymplectic structure on a $(2n+1)$-dimensional smooth manifold [5] M is given by a pair (ω, η) consisting of a closed 2-form ω and a closed 1-form η on M such that $\eta \wedge \omega^n$ is a volume form. Cosymplectic manifolds were called *canonical manifolds* by A. Lichnerowicz [6]. On a cosymplectic manifold (M, ω, η), there exists a unique vector field $\mathbf{R}$, called the Reeb vector field, which is completely determined by the relations $\iota_{\mathbf{R}}\eta = 1$ and $\iota_{\mathbf{R}}\omega = 0$. It is known that any cosymplectic structure (ω, η) on M determines a Poisson structure π on M given by:

$$\pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\} = \omega(X_f, X_g),$$

for all $f, g \in C^\infty(M)$, where the Hamiltonian vector X_f is defined by: $\eta(X_f) = 0$ and $\iota_{X_f}\omega = \mathrm{d}f - (\mathbf{R}\cdot f))\eta$. The Reeb vector field is a Poisson vector field of (M, π). On $M \times \mathbb{R}$, the bivector field $\Pi = \pi + \frac{\partial}{\partial\tau} \wedge \mathbf{R}$, (where τ is the standard coordinate on $\mathbb{R}$) has a maximal rank and defines a symplectic structure that is called the symplectization of the cosymplectic structure. In fact, $(M \times \mathbb{R}, \Pi)$ is symplectic if and only if (M, ω, η) is cosymplectic.

1.2. Lie groupoids and Lie algebroids

We assume that the reader is familiar with symplectic groupoids, but we will briefly review a few facts about Lie algebroids and Lie groupoids (see Reference [8] for their definitions and more details).

Let A be a Lie algebroid over M together with a 1-cocycle $\phi \in \Gamma(A^*)$. The pull-back of A to $M \times \mathbb{R}$ admits a Lie algebroid structure over $M \times \mathbb{R}$, denoted by $(A \times_\phi \mathbb{R}, [\![\cdot,\cdot]\!]^\phi, \rho^\phi)$, where the smooth sections of $A \times_\phi \mathbb{R}$ are of the form $\bar{X}(x, \tau) = X_\tau(x)$, with $X_\tau \in \Gamma(A)$ for all $\tau \in \mathbb{R}$, and

$$\begin{aligned} [\![\bar{X}, \bar{Y}]\!]^\phi(x, \tau) &= [\![X_\tau, Y_\tau]\!](x) + \phi(X_\tau)(x)\frac{\partial \bar{Y}}{\partial \tau} - \phi(Y_\tau)(x)\frac{\partial \bar{X}}{\partial \tau}, \\ \rho^\phi(\bar{X})(x, \tau) &= \rho(X_\tau)(x) + \phi(X_\tau)(x)\frac{\partial}{\partial \tau}, \end{aligned} \tag{1}$$

where $\dfrac{\partial \bar{X}}{\partial \tau} \in \Gamma(A \times_\phi \mathbb{R})$ denotes the derivative of $\bar{X}$ with respect to τ.

Given a Lie groupoid $G \overset{\mathrm{s}}{\underset{\mathrm{t}}{\rightrightarrows}} M$ together with a multiplicative function σ. One can define a right action of G on the canonical projection $p_1 : M \times \mathbb{R} \to M$ as follows:

$$(x, \tau)\cdot g = (\mathrm{s}(g), \sigma(g) + \tau), \text{ for } (x, \tau, g) \in M \times \mathbb{R} \times G, \text{ with } \mathrm{t}(g) = x.$$

We get the corresponding action groupoid $G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}$, denoted by $G \times_\sigma \mathbb{R}$, with structural functions:

$$\begin{aligned} &\mathrm{s}_\sigma(g, \tau) = (\mathrm{s}(g), \sigma(g) + \tau), \qquad \mathrm{t}_\sigma(g', \tau') = (\mathrm{t}(g'), \tau'), \\ &m_\sigma((g, \tau), (g', \tau')) = (gg', \tau), \text{ if } \mathrm{s}_\sigma(g, \tau) = \mathrm{t}_\sigma(g', \tau'). \end{aligned} \tag{2}$$

Let AG be the Lie algebroid of G. The multiplicative function σ induces a 1-cocycle α on AG given by

$$\langle \alpha_x,\ X_x\rangle = (X\cdot\sigma)_x, \quad \text{for } x \in M \text{ and } X \in \Gamma(AG). \tag{3}$$

The Lie algebroid of $G \times_\sigma \mathbb{R}$ can be identified with $AG \times_\alpha \mathbb{R}$. Conversely, one has the following:

Proposition 1. *(See [1].) Let A be a Lie algebroid over M, $\alpha \in \Gamma(A^*)$ a 1-cocycle and $A \times_\alpha \mathbb{R}$ the Lie algebroid given by Equation (1). If $G(A)$ (resp., $G(A \times_\alpha \mathbb{R})$) is the Weinstein groupoid of A (resp., $A \times_\alpha \mathbb{R}$) and σ is the multiplicative function associated with α, then $G(A \times_\alpha \mathbb{R}) \cong G(A) \times_\sigma \mathbb{R}$. Moreover, A is integrable if and only $A \times_\alpha \mathbb{R}$ is integrable.*

Definition 1.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let A be a Lie algebroid over a smooth manifold M. An infinitesimal action of $\mathfrak{g}$ on A, by Lie algebroid automorphisms, is defined by a map $\phi : \mathfrak{g} \times \Gamma(A) \to \Gamma(A)$ having the properties:

(i) $\phi(u)$ is a derivative endomorphism, that is, $\phi(u)(fX) = f\phi(u)X + \Big(\varsigma(\phi(u))\cdot f\Big)X$;
(ii) $\phi(u)$ is a derivative for the Lie bracket, that is, $\phi(u)[X, Y] = [\phi(u)X, Y] + [X, \phi(u)Y]$,

for all $f \in C^\infty(M, \mathbb{R})$, $u \in \mathfrak{g}$, $X, Y \in \Gamma(A)$. Here $\varsigma(\phi(u)) \in \mathfrak{X}(M)$ denotes the symbol of $\phi(u)$.

2. Main results and examples

Definition 2.1. A *cosymplectic groupoid* is a Lie groupoid G endowed with a cosymplectic structure (ω, η) such that both ω and η are multiplicative.

Proposition 2.2. *Let (G, ω, η) be a cosymplectic groupoid of dimension $2n + 1$ together with its associated Poisson tensor π and its Reeb vector field $\mathbf{R}$. Then its base manifold M is n-dimensional and is equipped with a unique pair (π_0, E_0) consisting of a Poisson structure π_0 and a Poisson vector field $E_0 = s_*\mathbf{R}$ such s is Poisson and t is anti-Poisson.*

Proof. It is clear that $\pi_0 = s_*\pi = -t_*\pi$ is a Poisson tensor on M and the vector field $E_0 = s_*\mathbf{R}$ satisfies $[\pi_0, E_0] = 0$. Now, let $p : G \times \mathbb{R} \to G$ be the canonical projection onto the first factor and let $\Omega = p^*\omega + d\tau \wedge p^*\eta$, where τ is the standard coordinate on $\mathbb{R}$. We wish to prove that $\Omega = p^*\omega + \mathrm{d}\tau \wedge p^*\eta$ is multiplicative. We can identify the set $(G \times \mathbb{R})_2$ of composable pairs of arrows of $G \times \mathbb{R}$ with $G_2 \times \mathbb{R}$ as follows: $((g,\tau),(g',\tau)) \mapsto ((g,g'),\tau)$, where G_2 is the set of composable pairs of arrows of G. The projections maps of $(G \times \mathbb{R})_2$ onto $G \times \mathbb{R}$ become $pr_i(g_1, g_2, \tau) = (g_i, \tau)$, $i = 1, 2$ and $m((g,g'),\tau) = (gg',\tau)$. By a simple calculation, one gets that $m^*\Omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega$ using the fact that ω and η are multiplicative. Thus $(G \times \mathbb{R}, \Omega)$ is a symplectic groupoid over $M \times \mathbb{R}$. Obviously, $\dim M = n$ since $M \times \mathbb{R}$ is a Lagrangian submanifold of $G \times \mathbb{R}$. The inverse of Ω is the Poisson tensor $\Pi = \pi + \partial_\tau \wedge \mathbf{R}$. □

The Poissonization idea in the proof of the previous proposition can be generalized as follows:

Proposition 2.3. *Let G be a Lie groupoid equipped with a multiplicative function σ. Then there is a one-to-one correspondence between multiplicative cosymplectic structures (ω, η) on G and multiplicative symplectic structures on $G \times_\sigma \mathbb{R}$ of the form $\Omega = p^*\omega + \mathrm{d}\tau \wedge p^*\eta$.*

Proof. In Proposition 2.2, we considered the symplectic structure on $G \times \mathbb{R}$ (i.e. $\sigma = 0$). But the same proof works when $\sigma \neq 0$. Indeed, in the general case, $(G \times \mathbb{R})_2$ can be identified with $G_2 \times \mathbb{R}$ via the map: $((g, \tau - \sigma(g)), (g', \tau) \mapsto ((g, g'), \tau)$. Thus any multiplicative cosymplectic structure on G determines a multiplicative symplectic structure on $G \times_\sigma \mathbb{R}$. Conversely, if $(G \times_\sigma \mathbb{R}, \Omega = p^*\omega + \mathrm{d}\tau \wedge p^*\eta)$ is a symplectic groupoid, then considering the terms that contain $\mathrm{d}\tau$ in $m^*\Omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega$, one sees that η must be multiplicative. The remaining terms show that ω is multiplicative. Thus, (G, ω, η) is cosymplectic. □

The 1-jet bundle J^1M of the base manifold M of any cosymplectic groupoid has a Lie algebroid structure that depends only on the cohomology class of $E_0 = s_*\mathbf{R}$. This Lie algebroid structure on J^1M can be considered as a special case of Lu's construction of matched pairs [7]. Recall that a matched pair of Lie algebroids is a pair (A, B) of Lie algebroids over the same base manifold M whose direct sum $A \oplus B$ is equipped with a Lie algebroid structure for which both A and B are Lie subalgebroids. It also turns out that Lu's Lie algebroid construction, in the special case of J^1M, can also be viewed as a consequence of a more general phenomenon about infinitesimal actions of Lie algebras on Lie algebroids. We have:

Proposition 2.4. *Let $\mathfrak{g}$ be a Lie algebra and let A be a Lie algebroid over a smooth manifold M. Any infinitesimal action of $\mathfrak{g}$ on A, by Lie algebroid automorphisms, determines a matched pair of Lie algebroids $(M \rtimes \mathfrak{g}, A)$.*

Here, all Lie algebras $\mathfrak{g}$ considered are finite-dimensional. The proof of Proposition 2.4 uses the following:

Lemma 2.5. *Let $\phi : \mathfrak{g} \times \Gamma(A) \to \Gamma(A)$ be an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a Lie algebroid $A \to M$ by Lie algebroid automorphisms. Then one has: $[\varsigma(\phi(u)), a(X)] = a(\phi(u)X)$, $\forall\, u \in \mathfrak{g}$, $X \in \Gamma(A)$, where a and ς denote the anchor map of A and the symbol map of ϕ, respectively.*

Proof. By Proprieties (i) and (ii) of the infinitesimal action ϕ (see Definition 1.1), one gets:

$$\phi(u)[X, fY] = [\phi(u)X,\ fY] + [X,\ (\varsigma(\phi(u)) \cdot f)Y + f\phi(u)Y],\ \forall f \in C^\infty(M, \mathbb{R}),\ u \in \mathfrak{g} \text{ and } X, Y \in \Gamma(A).$$

Expand both sides of this last equation using the Leibniz property for the Lie algebroid bracket and Property (ii) to get: $a(\phi(u)X) \cdot f + a(X) \cdot (\varsigma(\phi(u)) \cdot f) = \varsigma(\phi(u)) \cdot (a(X) \cdot f)$. There follows the relation: $[\varsigma(\phi(u)), a(X)] = a(\phi(u)X)$. □

Proof of Proposition 2.4. We choose a basis for $\mathfrak{g}$ in order to extend ϕ to sections of the trivial vector bundle $M \times \mathfrak{g}$ (which are viewed as functions in $C^\infty(M, \mathfrak{g})$). We take this extension of ϕ to define the representations:

$$\begin{aligned} \rho:\ \Gamma(A) \otimes C^\infty(M,\mathfrak{g}) &\longrightarrow C^\infty(M,\mathfrak{g}) \\ (X, U) &\longmapsto a(X)\cdot U \end{aligned} \qquad \text{and} \qquad \begin{aligned} \sigma:\ C^\infty(M,\mathfrak{g}) \otimes \Gamma(A) &\longrightarrow \Gamma(A) \\ (U, X) &\longmapsto \phi(U)X \end{aligned}.$$

Using these maps and Lemma 2.5, we show that $(A, M \rtimes \mathfrak{g})$ is a matched pair of Lie algebroids. The Lie bracket on the bicrossed product $A \bowtie (M \rtimes \mathbb{R})$ is such that $[X \oplus 0, 0 \oplus U] = -\sigma_U X \oplus \rho_X U$, for all $X \in \Gamma(A)$ and for all $U \in C^\infty(M, \mathfrak{g})$. □

As a consequence of Proposition 2.4, we get the following:

Corollary 2.6. *Let (A, A^*) be a Lie bialgebroid over M. Any 1-cocycle $\alpha \in \Gamma(A^*)$ induces a matched pair of Lie algebroids $(A, M \rtimes_\alpha \mathbb{R})$ over M. Moreover, the associated bicrossed product $A \bowtie (M \rtimes_\alpha \mathbb{R})$ depends only on the cohomology class of α. Here, $M \rtimes_\alpha \mathbb{R}$ is viewed as the Lie algebroid induced by α.*

Proof. Let (A, A^*) be a Lie bialgebroid over M. Given a 1-cocycle $\alpha \in \Gamma(A^*)$, we consider the infinitesimal action $\phi : \mathbb{R} \times \Gamma(A) \to \Gamma(A)$ such that $\phi(1, X) = \mathcal{L}_\alpha X$ and $\phi(u, X) = u\phi(1, X)$. Then ϕ satisfies **(i)** as well as **(ii)** since

$$L_\alpha[X_1, X_2] = [L_\alpha X_1, X_2] + [X_1, L_\alpha X_2], \quad \forall X_1, X_2 \in \Gamma(A).$$

Then apply Proposition 2.4 to get Corollary 2.6. □

The special case of Corollary 2.6 where $(A, A^*) = (T^*M, TM)$ is the Lie bialgebroid of a Poisson manifold can be re-stated as follows:

Corollary 2.7. *(See [3].) Let (M, π) be a Poisson manifold. Any Poisson vector field E_0 on (M, π) determines a matched pair of Lie algebroids $(T^*M, M \rtimes_{E_0} \mathbb{R})$. Moreover, up to isomorphism, the bicrossed product Lie algebroid structure on J^1M depends only on the cohomology class of E_0.*

Coming back to cosymplectic groupoids, we have the statement:

Lemma 2.8. *Let (G, ω, η) be a cosymplectic groupoid, then its Reeb vector field* $\mathbf{R}$ *is right invariant.*

The property of right invariance for $\mathbf{R}$ is similar to that of the contact groupoid case [2,4].

Recall that a multiplicative function σ on a Lie groupoid G over M determines a Lie algebroid 1-cocycle $\alpha_0 \in \Gamma(A^*G)$ defined by: $\alpha_0(X_x) = (X \cdot \sigma)_x$ for all $X \in \Gamma(AG), x \in M$. We introduce the following:

Definition 2.9. Let (G, ω, η) be a cosymplectic groupoid over M together with its Reeb vector field $\mathbf{R} = \overrightarrow{X_0}$. A multiplicative function σ on G is called a Reeb function if its associated Lie algebroid 1-cocycle α_0 satisfies the conditions: $\alpha_0(X_0) = 0$ and $a_*(\alpha_0) = -a(X_0) = -s_*\mathbf{R}$, where a and a_* are the anchor maps of the Lie algebroid AG of G and its dual A^*G, respectively.

Remark. Let (G, ω, η) be a cosymplectic groupoid over M together with its Reeb vector field $\mathbf{R} = \overrightarrow{X_0}$. If a Reeb function σ exists, then it is unique and completely determined by the Reeb vector field $\mathbf{R}$ since there a unique α_0 satisfying $a_*(\alpha_0) = -s_*\mathbf{R}$. The uniqueness of α_0 can be easily seen by considering the associated symplectic groupoid $G \times \mathbb{R} \simeq G \times_{X_0} \mathbb{R}$ with its Lie algebroid $A^*G \times_{X_0} \mathbb{R}$. Then the anchor $\bar{a}_* : A^*G \times_{X_0} \mathbb{R} \to T(M \times \mathbb{R})$ is bijective (see Theorem 5.3 in [9]) and it must send $(\alpha_0, 0)$ to $-s_*\mathbf{R}$. □

Examples

- Obviously, any symplectic groupoid (G, Ω) determines a cosymplectic groupoid $(G \times \mathbb{R}, \Omega, \eta = dt)$.
- Let (M, ω, η) be a cosymplectic manifold and let $G = M \times M \times \mathbb{R}$ with the canonical projections pr_i on the i-th factor. Set $\overline{\omega} = \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$, $\bar{\eta} = \mathrm{pr}_1^*\eta - \mathrm{pr}_2^*\eta$. Then $(G, \overline{\omega}, \bar{\eta}, \sigma = 0)$ is a cosymplectic groupoid.
- Non-trivial cosymplectic groupoids can be obtained by integration of some Lie algebroids coming from matched pairs of Lie algebroids. One has:

Theorem 2.10. *(See [3].) Let E_0 be a Poisson vector field of a Poisson manifold (M, π_0). If the corresponding Lie algebroid structure on J^1M is integrable, then its associated source-simply connected Lie groupoid $G_c(M)$ admits a cosymplectic structure.*

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