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Algebraic geometry

On a question of Mehta and Pauly

Sur une question de Mehta et Pauly

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ABSTRACT

In this short note, we provide explicit examples in characteristic p on certain smooth projective curves where for a given semistable vector bundle $\mathcal{E}$ the length of the Harder–Narasimhan filtration of $F^*\mathcal{E}$ is longer than p. This negatively answers a question of Mehta and Pauly raised in [2].

RÉSUMÉ

Dans cette courte note, nous donnons des exemples explicites en caracteristique p sur certaines courbes projectives lisses où, pour un fibré vectoriel semi-stable donné $\mathcal{E}$, la longeur de la filtration d'Harder–Narasimhan de $F^*\mathcal{E}$ est plus grande que p. Cela répond negativement à une question posée par Mehta et Pauly dans [2].

0. Introduction

In [2, page 2], Mehta and Pauly asked whether for a smooth projective curve over a field of characteristic $p > 0$ and $\mathcal{E}$ a semistable bundle on X the length of the Harder–Narasimhan filtration of $F^*\mathcal{E}$ is at most p. In [4, Construction 2.13], this is answered negatively. Examples are constructed based on a result of Sun [3]. The bundles for which examples are obtained in [4] have rank $\geq 2p$ (in fact, examples are constructed for any np with $n \geq 2$) and are over curves of large genus, since restriction theorems and Bertini's Theorem are used. The purpose of this short note is to provide surprisingly simple down-to-earth examples in characteristic p for certain smooth plane curves and bundles of rank $p+1 \leq r \leq \lfloor \frac{3p+1}{2} \rfloor$. In characteristic 2, negative examples exist on any smooth projective curve of genus ≥ 2. We note that our examples are only polystable, while one should be able to obtain stable bundles using the methods outlined in [4].

1. The example

Proposition 1.1. *Let X be a smooth projective curve over an algebraically closed field k of positive characteristic. Let $\mathcal{E}_i$, $i = 1, \dots, n$ be semistable rank-two bundles of slope μ on X such that the $F^*\mathcal{E}_i$ split as $F^*\mathcal{E}_i = \mathcal{L}_i \oplus \mathcal{G}_i$ with $\mu(\mathcal{L}_i) > \mu(\mathcal{G}_i)$. Assume, moreover, that*

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$\mu(\mathcal{L}_i) > \mu(\mathcal{L}_{i+1})$ for all $i = 1, \ldots, n-1$. Then $\mathcal{S} = \bigoplus_{i=1}^n \mathcal{E}_i$ is semistable and $F^*\mathcal{S}$ is unstable and its Harder–Narasimhan filtration is:

$$0 \subset \mathcal{L}_1 \subset \mathcal{L}_1 \oplus \mathcal{L}_2 \subset \ldots \subset \bigoplus_{i=1}^n \mathcal{L}_i \subset \bigoplus_{i=1}^n \mathcal{L}_i \oplus \mathcal{G}_n \subset \bigoplus_{i=1}^n \mathcal{L}_i \oplus \mathcal{G}_n \oplus \mathcal{G}_{n-1} \subset \ldots \subset F^*\mathcal{S}.$$

In particular, the Harder–Narasimhan filtration of $F^*\mathcal{S}$ has length $2n$.

Proof. Clearly $\mathcal{S}$ is semistable. We have $\mu(\mathcal{G}_i) = 2\mu - \mu(\mathcal{L}_i)$, which implies $\mu(\mathcal{G}_i) < \mu(\mathcal{G}_{i+1})$ for all i. We also have $\mu(\mathcal{L}_i) > \mu(\mathcal{G}_j)$ for all i, j. Indeed, we may assume that $i > j$ then $\mu(\mathcal{L}_i) - \mu(\mathcal{G}_j) = \mu(\mathcal{L}_j) - \mu(\mathcal{G}_i)$ and by assumption $\mu(\mathcal{L}_i) > \mu(\mathcal{L}_j) > \mu(\mathcal{G}_j)$. Hence, $\mu(\mathcal{L}_j) > \mu(\mathcal{G}_i)$.

It follows that the slopes of the quotients $\mathcal{Q}_i$ of the filtration form a strictly decreasing sequence. As all $\mathcal{Q}_i$ are semistable as line bundles, this is the Harder–Narasimhan filtration of $F^*\mathcal{S}$. □

Example 1.2. By [1, Theorem 1] any smooth projective curve X of genus ≥ 2 admits a semistable rank two bundle $\mathcal{E}$ with trivial determinant such that $F^*\mathcal{E}$ is not semistable. Then $\mathcal{S} = \mathcal{E} \oplus \mathcal{O}_X$ is a semistable vector bundle and the Harder–Narasimhan filtration of $F^*\mathcal{S}$ has length $3 > 2$. Indeed, if $0 \subset \mathcal{L} \subset F^*\mathcal{E}$ is a Harder–Narasimhan filtration of $F^*\mathcal{E}$ then $0 \subset \mathcal{L} \subset \mathcal{L} \oplus \mathcal{O}_X \subset F^*\mathcal{S}$ is one for $F^*\mathcal{S}$.

Lemma 1.3. *Let X be a smooth projective curve and $\mathcal{E}$ a rank 2 vector bundle on X. If $\mathcal{E}$ is given by an extension $0 \neq c \in \operatorname{Ext}^1(\mathcal{M}, \mathcal{L})$ with $\deg \mathcal{L} < \deg \mathcal{M}$ and $F^*(c) = 0$ then $\mathcal{E}$ is semistable.*

Proof. Assume, on the contrary, that $\mathcal{E}$ is unstable and let $\mathcal{N}$ denote the maximal destabilizing subbundle $\mathcal{E}$. The maximal destabilizing subbundle of $F^*\mathcal{E} = F^*\mathcal{M} \oplus F^*\mathcal{L}$ is $F^*\mathcal{M}$. Since the Harder–Narasimhan filtration is unique and in the rank 2 case automatically strong, we must have $F^*\mathcal{M} = F^*\mathcal{N}$. Hence, $\mathcal{N} = \mathcal{M} \otimes \mathcal{T}$ for some p-torsion bundle $\mathcal{T}$.

Consider now the natural inclusion $i : \mathcal{M} \otimes \mathcal{T} \to \mathcal{E}$ and the projection $p : \mathcal{E} \to \mathcal{M}$. The Frobenius pull-back of the composition $p \circ i$ is the identity. In particular $p \circ i : \mathcal{M} \otimes \mathcal{T} \to \mathcal{M}$ is non-zero. Since both line bundles are of the same degree, this map is an isomorphism. Hence, if $\mathcal{E}$ is not semistable, then the sequence has to split, which contradicts the assumption $c \neq 0$. □

Example 1.4. Let now p be any prime and k an algebraically closed field of characteristic p. We consider the plane curve:

$$X = V_+(x^{3p} + xy^{3p-1} + yz^{3p-1}) \subseteq \mathbb{P}^2_k.$$

By the Jacobian criterion, this is a smooth curve. We will construct $\lfloor \frac{3p+1}{2} \rfloor$ rank-two bundles of slopes $-\frac{3p}{2}$ as in Proposition 1.1. The direct sum over at least $\frac{p+1}{2}$ of these bundles then constitutes the desired example.

Consider the cohomology class

$$c = \frac{x^3}{y^2 z^2} \in H^1(X, \mathcal{O}_X(-1)),$$

which is non-zero. Also note that its Frobenius pull-back

$$F^*(c) = \frac{x^{3p}}{y^{2p} z^{2p}} = \frac{-xy^{3p-1} - yz^{3p-1}}{y^{2p} z^{2p}} = -(\frac{xy^{p-1}}{z^{2p}} + \frac{z^{p-1}}{y^{2p-1}})$$

is zero. Moreover, multiplication by z yields a map $\mathcal{O}_X(-1) \to \mathcal{O}_X$ and the induced map on cohomology maps c to $\frac{x^4}{y^2 z^2}$, which is still non-zero. Let $P_1, \ldots, P_{3p}$ be the (distinct) points on X where z vanishes.[2] In particular, the cokernel of multiplication by z is just $\bigoplus_{i=1}^{3p} k(P_i)$, where $k(P_i)$ is the skyscraper sheaf at P_i.

Multiplication by z factors as

$$\mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X(-1 + \textstyle\sum_{i=1}^l P_i) \longrightarrow \mathcal{O}_X$$

for any $l \leq 3p$. Indeed, the image of the line bundle in the middle is just the sum of the image of $\mathcal{O}_X(-1)$ in $\mathcal{O}_X$ and the preimage of $\sum_{i=1}^l k(P_i)$. In particular, we get an induced factorization on cohomology and we denote the image of c in $H^1(X, \mathcal{O}_X(-1 + \sum_{i=1}^l P_i))$ by c_l. Note that c_l is non-zero, while $F^*(c_l)$ is zero.

Assume now that l is even. These cohomology classes then define extensions $\mathcal{E}_l$ as follows. Let I be the odd numbers from 1 to l and let J be the even numbers from 1 to l. Then

[2] We could also work with multiplication by x which yields one reduced point and one with multiplicity $3p - 1$.

$$c_l \in H^1(X, \mathcal{O}_X(-1 + \sum_{i=1}^{l} k(P_i))) = \mathrm{Ext}^1(\mathcal{O}_X(-\sum_{j\in J} P_j), \mathcal{O}_X(-1 + \sum_{i\in I} P_i))$$

yield extensions

$$0 \longrightarrow \mathcal{O}_X(-1 + \textstyle\sum_{i\in I} P_i) \longrightarrow \mathcal{E}_l \longrightarrow \mathcal{O}_X(-\textstyle\sum_{j\in J} P_j) \longrightarrow 0\,.$$

The $\mathcal{E}_l$ all have slope $-\frac{3p}{2}$ and pulling back along Frobenius splits the above sequence. By Lemma 1.3 the $\mathcal{E}_l$ are semistable. Hence, the $\mathcal{E}_l$ satisfy the hypothesis of Proposition 1.1, and we obtain the desired examples.

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