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Error analysis for a mixed DG method for folded Naghdi's shell

Analyse d'erreur pour une méthode de Galerkin discontinue mixte pour une coque de Naghdi pliée

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ABSTRACT

In this Note, we perform some error analyses of a mixed discontinuous formulation for Naghdi's equations for thin linearly elastic shells. Using a combination between the a priori and the a posteriori techniques, we derive a quasi-optimal a priori error estimate and an efficient and reliable a posteriori error estimate. Thereby the analysis does not require any additional regularity other than of the weak solution.

RÉSUMÉ

Dans cette Note, nous fournissons des estimations d'erreur d'une méthode d'éléments finis discontinus mixte proposée pour résoudre les équations du modèle de Naghdi de coques linéairement élastiques. En utilisant une combinaison entre les techniques de l'analyse a priori et celles de l'analyse a posteriori, nous donnons une estimation d'erreur a priori quasi-optimale et une estimation d'erreur a posteriori fiable et efficace. L'analyse utilisée n'exige aucune regularité additionnelle autre que celle de la solution du problème faible.

Version française abrégée

Dans cette Note, nous nous intéressons à l'estimation d'erreur d'une méthode d'éléments finis discontinus mixte proposée dans la Note compagnonne [6] pour résoudre les équations du modèle de Naghdi de coques linéairement élastiques ([2]). Le problème est posé dans un domaine borné $\omega \subset \mathbb{R}^2$ à bord polygonal $\partial\omega$. Nous supposons que $\omega = \omega^+ \cup \Sigma \cup \omega^-$ et nous considérons une coque élastique homogène isotrope dont la surface moyenne S est donnée par :

$$S^\pm = \varphi^\pm\left(\bar{\omega}^\pm\right) \text{ où } \varphi^\pm \in W^{3,\infty}\left(\omega^\pm, \mathbb{R}^3\right), \varphi \in W^{1,\infty}\left(\omega, \mathbb{R}^3\right).$$

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Les inconnues du problème sont respectivement le déplacement u des points de S et la rotation r de la normale unitaire à la surface S.

Nous considérons la formulation mixte (14) avec une nouvelle forme $b(\cdot,\cdot)$ donnée par (16), où $\mathbb{X}(\omega) = (H_0^1(\omega, \mathbb{R}^3) \times (H_0^1(\omega))^2$ est l'espace relaxé (introduit par Blouza et Le Dret [3]) et est équipé de la norme (12) et $\mathbb{M}(\omega) = H^{-1}(\omega)$. Si le chargement $f \in L^2(\omega, \mathbb{R}^3)$, la formulation variationnelle (14) admet une solution unique, puisque la forme bilinéaire $\mathbf{a}(.,.)$ est continue et coercive sur $\mathbb{V}(\omega) := \ker b$ et la forme $b(.,.)$, est continue sur $\mathbb{X}(\omega)$ et satisfait la condition inf–sup [6]. La solution du problème mixte est celle de Naghdi.

Notre problème est ensuite discrétisé en utilisant une méthode d'éléments finis discontinus mixte avec une attention particulière portée au cas où la solution du problème faible possède seulement la régularité H^1 et au cas où la base covariante $\{a_i\}_{1\le i\le 3}$ est de classe C^0, ce qui correspond, par exemple, à une jonction de coque de classe C^0 ou une coque présentant des arêtes. Si (U,ψ) est la solution du problème (14) et (U_h,ψ_h) est celle du problème (20), nous montrons que :

$$\|U - U_h\|_h + \|\psi - \psi_h\|_{-1,h} \lesssim \inf_{(V_h,\phi_h)\in \mathbb{V}_h\times\mathbb{M}_h} (\|U - V_h\|_h + \|\psi - \phi_h\|_{-1,h}) + Osc, \tag{1}$$

les espaces $\mathbb{V}_h$ et $\mathbb{M}_h$ étant donnés par (18) et (19) et les normes $\|\cdot\|_h$ et $\|\cdot\|_{-1,h}$ sont définies par (21) et (22). Osc représente des termes d'oscillation des données et des coefficients. Enfin, la notation $a \lesssim b$ signifie que $a \le c\, b$, où c est une constante générique indépendante de h.

Concernant l'analyse d'erreur a posteriori, l'estimateur η_h étant défini par (28), nous montrons qu'il est efficace et fiable modulo les oscillations des données. Autrement dit, nous démontrons que :

$$\|U - U_h\|_h + \|\psi - \psi_h\|_{\mathbb{M}(\omega)} \lesssim \eta_h, \text{ et que } \eta_h \lesssim \|U - U_h\|_h + \|\psi - \psi_h\|_{-1,h} + Osc.$$

Notons finalement que la différence entre $\|\psi - \psi_h\|_{\mathbb{M}(\omega)}$ et $\|\psi - \psi_h\|_{-1,h}$ est superconvergente par rapport à l'erreur globale. Ainsi, notre estimateur est efficace et fiable.

1. Introduction

The classical shell theory uses the covariant and contravariant components representation of the unknowns. This formulation requires that the shell has a C^3-middle surface (see [3]). For the classical formulation, i.e. the covariant based formulation, there is a considerable literature devoted to the development of finite-element methods for linear shell models (see [1]).

The formulation used here was introduced in [2,3]; it is based on the idea of using a local basis-free formulation in which the unknowns are described in Cartesian coordinates. This new formulation allows us to handle shells with little regularity, i.e. shells with a $W^{2,\infty}$-middle surface. The construction and the implementation of conforming finite-element methods for this formulation is computationally quite complicated, for a general shell due to the constraint imposed on the rotation field, which must be tangential to the middle surface. For the case of shells with little regularity, there seems to be few mathematical studies. Let us just mention the following approaches. In [4], two finite-element discretizations of Naghdi's equations are considered, a penalized version, which intends to approximate the above-mentioned tangency, the second approach consisting in handling this constraint via the introduction of a Lagrange multiplier, which involves the Laplace–Beltrami operator on the shell midsurface. The corresponding system is a second-order elliptic system of PDEs. Using some additional regularity assumptions, standard and optimal a priori error estimates were obtained for both approaches. Still in the context of shells with little regularity and for the Cartesian based formulation, an a posteriori analysis for the mixed formulation was done in [1]. Provided that the solution of the continuous problem has the H^2-regularity, the discrete problem leads to the construction of error indicators that satisfy optimal estimates.

This Note is organized as follow: in Section 2, we first briefly recall Naghdi's shell model formulated in Cartesian coordinates. We recall a mixed formulation of Naghdi's model in which a Lagrange multiplier is introduced; this mixed problem is well posed and it solves Naghdi's problem (this was proved in [6]). We present in Section 3 our discontinuous mixed finite element discretization. Section 4 is devoted to error analyses: first we present an a priori analysis and second we propose an a posteriori estimator that yields an upper bound and a lower bound of the error.

In what follows, we adopt the notation $a \lesssim b$ for $a \le c\, b$, where c is a generic constant that must not depend on any mesh-size. The convention of summation of repeated indices, which run from 1 to 2 when they are Greek, is used.

2. The continuous problem

Let ω be a domain of $\mathbb{R}^2$. We suppose that $\omega = \omega^+ \cup \Sigma \cup \omega^-$ and we consider a shell whose middle surface $S = S^+ \cup \Gamma \cup S^-$, is given by

$$S^\pm = \varphi^\pm(\bar\omega^\pm) \text{ where } \varphi^\pm \in W^{3,\infty}\left(\omega^\pm, \mathbb{R}^3\right), \varphi \in W^{1,\infty}\left(\omega, \mathbb{R}^3\right), \text{ and } \Gamma = \varphi^+(\Sigma) = \varphi^-(\Sigma), \tag{2}$$

$\varphi^\pm$ is a one-to-one mapping such that the two vectors: $a_\alpha^\pm = \partial_\alpha \varphi^\pm$, $\alpha = 1, 2$ are linearly independent. The normal vector $a_3^\pm$ is given by $a_3^\pm = \frac{a_1^\pm \times a_2^\pm}{|a_1^\pm \times a_2^\pm|}$. The contravariant bases a^i are defined by the relation $a_i \cdot a^j = \delta_i^j$, δ_i^j being the Kronecker symbol. The covariant and contravariant components of the metric are given by: $(a_{\alpha\beta}) = a_\alpha \cdot a_\beta$ and $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}, a = \det(a_{\alpha\beta})$, $\sqrt{a}$ is the area element of the midsurface in the chart φ. We consider here the case of a homogeneous, isotropic material with Young's modulus $E > 0$ and Poisson's ratio $\nu, 0 \leqslant \nu < \frac{1}{2}$. We also denote by ε the thickness of the shell, which is assumed to be constant and positive.

Let $a^{\alpha\beta\rho\sigma}$ denote the contravariant components of the elasticity tensor, whose components are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{2(1-\nu^2)}a^{\alpha\beta}a^{\rho\sigma}. \tag{3}$$

Moreover, this tensor satisfies the usual symmetry properties and is uniformly strictly positive, i.e. there exists a positive constant c_0 such that

$$a^{\alpha\beta\rho\sigma}(x)\tau_{\alpha\beta}\tau_{\rho\sigma} \geq c_0|\tau|^2 \text{ for a.e. } x \in \omega, \forall \tau, \text{ symmetric tensor of order 2.} \tag{4}$$

Let $u \in H^1(\omega, \mathbb{R}^3)$ be the middle surface displacement and $r \in H^1(\omega, \mathbb{R}^3)$ the rotation of the normal vector. Following [2], the covariant components of the change of metric tensor $\gamma_{\alpha\beta}(u)$, the covariant components of the change of curvature tensor $\chi_{\alpha\beta}(u, r)$ and the covariant components of the change of transverse shear tensor $\delta_{\alpha 3}(u, r)$ read

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}\left(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha\right), \tag{5}$$

$$\chi_{\alpha\beta}(u, r) = \frac{1}{2}\left(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3\right) + \frac{1}{2}\left(\partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha\right), \tag{6}$$

$$\delta_{\alpha 3}(u, r) = \frac{1}{2}\left(\partial_\alpha u \cdot a_3 + r \cdot a_\alpha\right). \tag{7}$$

The stress resultant $n^{\alpha\beta}(u)$, the stress couple $m^{\alpha\beta}(u, r)$ and the transverse shear force $q^\beta(u, r)$ read

$$n^{\alpha\beta}(u) = \varepsilon a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u), \tag{8}$$

$$m^{\alpha\beta}(u, r) = \frac{\varepsilon^3}{12}a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(u, r), \tag{9}$$

$$q^\beta(u, r) = \frac{\varepsilon E}{1+\nu}a^{\alpha\beta}\delta_{\alpha 3}(u, r). \tag{10}$$

Of course, the expressions (3)–(10) are understood piecewise for the case of a folded shell.

2.1. A mixed formulation

In this Note, we are interested in a mixed formulation for solving Naghdi's equations introduced in the companion Note [6]. Let us first consider the functional spaces:

$$\mathbb{X}(\omega) = H_0^1\left(\omega, \mathbb{R}^3\right) \times H_0^1(\omega, \mathbb{R}^3), \tag{11}$$

equipped with the norm

$$\|(v, s)\|_{\mathbb{X}(\omega)} = \left(\|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \|s\|^2_{H^1(\omega,\mathbb{R}^3)}\right)^{\frac{1}{2}}, \tag{12}$$

and

$$\mathbb{M}(\omega) = H^{-1}(\omega) = \left(H_0^1(\omega)\right)'. \tag{13}$$

We consider the following variational mixed problem:

$$\begin{cases} \text{Find } (U, \psi) = (u, r, \psi) \in \mathbb{X}(\omega) \times \mathbb{M}(\omega) \text{ such that} \\ \mathbf{a}(U, V) + b(V, \psi) = \mathcal{L}(V), \forall V \in \mathbb{X}(\omega), \\ b(U, \phi) = 0, \forall \phi \in \mathbb{M}(\omega). \end{cases} \tag{14}$$

For $U = (u, r), V = (v, s) \in \mathbb{X}(\omega), \phi \in \mathbb{M}(\omega)$, the bilinear forms $\mathbf{a}(\cdot,\cdot)$, $b(\cdot,\cdot)$, and the linear form $\mathcal{L}$ read

$$\mathbf{a}(U, V) = \int_\omega \left(n^{\rho\sigma}(u)\gamma_{\rho\sigma}(v) + m^{\rho\sigma}(U)\chi_{\rho\sigma}(V) + q^\beta(U)\delta_{\beta 3}(V)\right)\sqrt{a}\,\mathrm{d}x, \tag{15}$$

$$b(V, \phi) = \langle \phi, s \cdot a_3 \rangle_{H^{-1}, H_0^1}, \tag{16}$$

$$\mathcal{L}(V) = \int_\omega f \cdot v\sqrt{a}\,\mathrm{d}x, \tag{17}$$

where $\langle \cdot, \cdot \rangle_{H^{-1},H_0^1}$ denotes the duality product between $H^{-1}(\omega)$ and $H_0^1(\omega)$.
The well-posedness of problem (14) was proved in [6].

3. A discrete discontinuous approximation for the mixed formulation (14)

Let $(\mathcal{T}_h)_{h>0}$ be a regular affine family of triangulations that cover the domain ω. Let $\mathcal{E}_h$ be the set of (open) edges in $\mathcal{T}_h$. Let us introduce the finite dimensional spaces

$$\mathbb{V}_h = \{V_h = (v_h, s_h) \in L^2(\omega)^3 \times L^2(\omega)^3 / (v_h, s_h)_{|T} \in (\mathbb{P}_k)^6,\ \forall T \in \mathcal{T}_h\}, \tag{18}$$

$$\mathbb{M}_h = \{\phi_h \in L^2(\omega)/\phi_{h|T} \in \mathbb{P}_k,\ \forall T \in \mathcal{T}_h\}. \tag{19}$$

Then we consider the following discrete problem:

$$\begin{cases} \text{find } (U_h, \psi_h) \in \mathbb{V}_h \times \mathbb{M}_h \text{ such that} \\ a_h(U_h, V_h) + b(V_h, \psi_h) = \mathcal{L}(V_h), \forall V_h = (v_h, s_h) \in \mathbb{V}_h, \\ b(U_h, \phi_h) = 0, \forall \phi_h \in \mathbb{M}_h. \end{cases} \tag{20}$$

Here we take $b(V_h, \phi_h) = \int_\omega (s_h \cdot a_3)\phi_h \,\mathrm{d}x$ and $a_h(U_h, V_h) = \tilde{a}(U_h, V_h) + c(U_h, V_h) + d(U_h, V_h)$. The bilinear forms $\tilde{a}(\cdot,\cdot)$, $c(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are given by

$$\begin{aligned} \tilde{a}(U_h, V_h) &= \sum_{T\in\mathcal{T}_h} \int_T \left(n^{\rho\sigma}(u_h)\,\gamma_{\rho\sigma}(v_h) + m^{\rho\sigma}(U_h)\,\chi_{\rho\sigma}(V_h) + q^\beta(U_h)\,\delta_{\beta 3}(V_h)\right)\sqrt{a}\,\mathrm{d}x \\ &\quad + \sum_{T\in\mathcal{T}_h} h_T^{-2} \int_T (r_h \cdot a_3)(s_h \cdot a_3)\,\mathrm{d}x, \\ c(U_h, V_h) &= -\sum_{e\in\mathcal{E}_h} \int_e \left(\left\{\!\!\left\{ \nu_\rho(T^\rho(U_h))\right\}\!\!\right\} \cdot [\![v_h]\!] + \left\{\!\!\left\{\nu_\rho(T^\rho(V_h)a_\sigma)\right\}\!\!\right\} \cdot [\![u_h]\!]\right) \mathrm{d}e \\ &\quad - \sum_{e\in\mathcal{E}_h} \int_e (\left\{\!\!\left\{\nu_\rho(m^{\rho\sigma}(U_h)a_\sigma)\right\}\!\!\right\} \cdot [\![s_h]\!] + \left\{\!\!\left\{\nu_\rho(m^{\rho\sigma}(V)a_\sigma)\right\}\!\!\right\} \cdot [\![r_h]\!])\sqrt{a}\,\mathrm{d}e, \\ d(U_h, V_h) &= \sum_{e\in\mathcal{E}_h} \frac{\kappa}{|e|} \int_e ([\![u_h]\!] \cdot [\![v_h]\!] + [\![r_h]\!] \cdot [\![s_h]\!])\sqrt{a}\,\mathrm{d}e. \end{aligned}$$

Here and below $T^\rho(V_h) = (n^{\rho\sigma}(v_h)a_\sigma + m^{\rho\sigma}(V_h)\partial_\sigma a_3 + q^\rho(V_h)a_3)\sqrt{a}$, $[\![s]\!] = s^+ - s^-$ denotes the jump of s and $\{s\} = \frac{1}{2}(s^+ + s^-)$ is its average across the edge $e \in \mathcal{E}_h$. Again problem (20) is well posed for $f \in L^2(\omega)$, see [6].

3.0.1. Mesh-dependent norms

Let us define the following quantities:

$$\begin{aligned} \|(v,s)\|_h^2 &= \sum_{T\in\mathcal{T}_h} \Big(\sum_{\alpha\beta} (\|\gamma_{\alpha\beta}(v)\|_{0,T}^2 + \|\chi_{\alpha\beta}(v,s)\|_{0,T}^2 + \|\delta_{\alpha 3}(v,s)\|_{0,T}^2) + h_T^{-2}\|s\cdot a_3\|_{0,T}^2\Big) \\ &\quad + \sum_{e\in\mathcal{E}_h} \frac{\kappa}{|e|} (\|[\![v]\!]\|_{(L^2(e))^3}^2 + \|[\![s]\!]\|_{(L^2(e))^3}^2), \quad \kappa > 1. \end{aligned} \tag{21}$$

$$\|\phi_h\|_{-1,h}^2 = \sum_{T\in\mathcal{T}_h} h_T^2 \|\phi_h\|_{0,T}^2. \tag{22}$$

4. Error analysis for the mixed formulation (20)

4.1. Preliminaires

Let us introduce the space

$$\mathbb{X}_h = \{V_h \in C^0(\bar{\omega})^3 \times C^0(\bar{\omega})^3 | V_{h_{|T}} \in \mathbb{P}^1(T)^3 \times \mathbb{P}^1(T)^3, \forall T \in \mathcal{T}_h; V_{h_{|\partial\omega}} = 0\}. \tag{23}$$

Then we introduce the averaging operator (also called enriching operator, see [5])

$$\mathbb{E}_h(v_h, s_h)(x) = \begin{cases} 0 & \text{if } x \text{ is a vertex on } \partial\omega, \\ \frac{1}{N_x}\sum_{T'\subset\omega_x}(v_h, s_h)_{|T'}(x) & \text{if } x \text{ is a vertex in } \omega, \end{cases} \tag{24}$$

where ω_x is the set of the triangles in $\mathcal{T}_h$ that share x as a common vertex and N_x the number of triangles in ω_x. Following [5], we have the following lemma.

Lemma 4.1. *The map* $\mathbb{E}_h$ *(enriching operator) from* $\mathbb{V}_h$ *onto* $\mathbb{X}_h$ *satisfies:*

$$\|\mathbb{E}_h(v_h, s_h)\|_h \lesssim \|(v_h, s_h)\|_h, \ \forall (v_h, s_h) \in \mathbb{V}_h, \tag{25}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{-2}\|(v_h, s_h) - \mathbb{E}_h(v_h, s_h)\|^2_{L^2(T)^6} \lesssim \sum_{e\in\mathcal{E}_h} \frac{1}{|e|}(\|[\![v_h]\!]\|^2_{L^2(e)^3} + \|[\![s_h]\!]\|^2_{L^2(e)^3}), \forall (v_h, s_h) \in \mathbb{V}_h. \tag{26}$$

Using the variational formulation (14), inverse estimates and Green's formula, one can prove the next Lemmas.

Lemma 4.2. *Let* (U, ψ) *be the solution of problem (14). Then for any* $(V_h, \phi_h) \in \mathbb{V}_h \times \mathbb{M}_h$ *we have*

$$h_T^2\|f\sqrt{a} + \partial_\rho(T^\rho(V_h))\|^2_{L^2(T)^3} + h_T^2\|M(V_h, \phi_h)\|^2_{L^2(T)^3} \lesssim |||U - V_h|||^2_{\mathbb{V}(T)} + \|\phi_h - \psi\|^2_{-1,h,T} + Osc_T^2,$$

where we have set

$$M(V_h, \phi_h) = \partial_\rho(m^{\rho\sigma}(V_h)a_\sigma\sqrt{a}) - q^\beta(V_h)a_\beta\sqrt{a} - \phi_h a_3,$$
$$|||V|||^2_{\mathbb{V}(T)} = \sum_{\alpha\beta}(\|\gamma_{\alpha\beta}(v)\|^2_{0,T} + \|\chi_{\alpha\beta}(v, s)\|^2_{0,T} + \|\delta_{\alpha 3}(v, s)\|^2_{0,T}),$$
$$\|\psi\|_{-1,h,T} = h_T\|\psi\|_{L^2(T)}, \text{ and } Osc_T^2 = osc_T(f)^2 + osc_T(\partial_\rho(T^\rho(V_h))^2 + osc_T(M(V_h, \phi_h))^2,$$
$$osc_T(f) = h_T\|f - \tilde{f}_T\|_{L^2(T)^3}, \ \tilde{f}_T \text{ being the } L^2(T)^3\text{-projection of } f \text{ on } \mathbb{P}_{k-1}(T)^3,$$
$$osc_T(\partial_\rho(T^\rho(V_h)) = h_T\|\partial_\rho(T^\rho(V_h)) - (\partial_\rho(T^\rho(V_h)))^h\|_{0,T},$$
$$osc_T(M(V_h, \phi_h)) = h_T\|M(V_h, \phi_h) - (M(V_h, \phi_h))^h\|_{0,T},$$
$$v^h_{|T} \text{ is the } L^2(T)^3\text{-projection of } v \text{ on } \mathbb{P}_k(T)^3.$$

Lemma 4.3. *Let* (U, ψ) *be the solution of problem (14). Then for any* $(V_h, \phi_h) \in \mathbb{V}_h \times \mathbb{M}_h$ *and any* $e \in \mathcal{E}_h^i$ *we have*

$$|e|\|[\![\nu_\rho(T^\rho(V_h))]\!]\|^2_{L^2(e)^3} + |e|\|[\![\nu_\rho m^{\rho\sigma}(V_h)a_\sigma]\!]\|^2_{L^2(e)^3} \lesssim \sum_{T\subset\omega_e}(|||U - V_h|||^2_{\mathbb{V}(T)} + \|\phi_h - \psi\|^2_{-1,h,T} + Osc_T^2) + Osc_e^2,$$

where

$$Osc_e^2 = |e|(\|T^\rho(V_h) - (T^\rho(V_h))^h)\|^2_{0,e} + \|m^{\rho\sigma}(V_h)a_\sigma - (m^{\rho\sigma}(V_h)a_\sigma)^h\|^2_{0,e}).$$

4.2. Medius analysis

Theorem 4.4. *Let* (U, ψ) *be the solution of problem* (14) and (U_h, ψ_h) the solution of problem (20). Then

$$\|U - U_h\|_h + \|\psi - \psi_h\|_{-1,h} \lesssim \inf_{(V_h,\phi_h)\in\mathbb{V}_h\times\mathbb{M}_h}(\|U - V_h\|_h + \|\psi - \phi_h\|_{-1,h}) + (\sum_{T\in\mathcal{T}_h} Osc_T^2 + \sum_{e\in\mathcal{E}_h} Osc_e^2)^{1/2}, \tag{27}$$

Proof. For any $(V_h, \phi_h) \in \mathbb{V}_h \times \mathbb{M}_h$, by Propositions 4.2 and 4.3 of [6], we have

$$\|V_h - U_h\|_h + \|\phi_h - \psi_h\|_{-1,h} \lesssim \sup_{(W_h,\lambda_h)\in\mathbb{V}_h\times\mathbb{M}_h\setminus\{0\}} \frac{a_h(V_h - U_h, W_h) + b(W_h, \phi_h - \psi_h) + b(V_h - U_h, \lambda_h)}{\|W_h\|_h + \|\lambda_h\|_{-1,h}}.$$

But owing to problem (20), and the fact that $b(U, \lambda_h) = 0$, we have

$$a_h(V_h - U_h, W_h) + b(W_h, \phi_h - \psi_h) + b(V_h - U_h, \lambda_h)$$
$$= a_h(V_h, W_h) + b(W_h, \phi_h) - \int_\omega f \cdot w_h\sqrt{a}\,dx + b(V_h - U, \lambda_h).$$

For the last term of this right-hand side, the Cauchy–Schwarz inequality directly yields $b(V_h - U, \lambda_h) \le \|U - V_h\|_h\|\lambda_h\|_{-1,h}$.

So in order to prove (27), it remains to prove that

$$\sup_{W_h\in\mathbb{V}_h\setminus\{0\}} \frac{a_h(V_h, W_h) - \int_\omega f \cdot w_h\sqrt{a}\,dx + b(W_h, \phi_h)}{\|W_h\|_h} \lesssim \|U - V_h\|_h + \|\psi - \phi_h\|_{-1,h} + Osc.$$

Let $(\tilde{w}_h, \tilde{t}_h) = \mathbb{E}_h(w_h, t_h)$; then by integration by part, through the Cauchy–Schwarz inequality and Lemmas 4.1, 4.2 and 4.3, one can get the estimate (27). □

4.3. A posteriori analysis

In this subsection, we will derive a posteriori error estimates. Thereby the analysis does not require the H^2-regularity or any regularity other than that of the weak solution. Let us define the local error indicator η_T and the global error indicator η_h by

$$\eta_T^2 = (\eta_T^{(1)})^2 + (\eta_T^{(2)})^2 + (\eta_T^{(3)})^2 + (\eta_T^{(4)})^2 + (\eta_T^{(5)})^2 + (\eta_h^{(6)})^2, \quad \eta_h^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2. \tag{28}$$

$$\eta_T^{(1)} = h_T \| f\sqrt{a} + \partial_\rho((n^{\rho\sigma}(u_h)a_\sigma + m^{\rho\sigma}(U_h)\partial_\sigma a_3 + q^\rho(U_h)a_3\sqrt{a}))\|_{L^2(T)^3},$$

$$\eta_T^{(2)} = h_T \|\partial_\rho(T^\rho(U_h))\|_{L^2(T)^3},$$

$$\eta_T^{(3)} = \sum_{e\subset\partial T} |e|^{\frac{1}{2}} \|[\![\nu_\rho(T^\rho(U_h))]\!]\|_{L^2(e)^3},$$

$$\eta_T^{(4)} = \sum_{e\subset\partial T} |e|^{\frac{1}{2}} \|[\![\nu_\rho m^{\rho\sigma}(U_h)a_\sigma\sqrt{a}]\!]\|_{L^2(e)^3}, \quad (\eta_T^{(5)})^2 = \sum_{e\subset\partial T} \frac{\kappa}{|e|}(\|[\![u_h]\!]\|^2_{L^2(e)^3} + \|[\![r_h]\!]\|^2_{L^2(e)^3}),$$

$$(\eta_T^{(6)})^2 = h_T^{-2}\|r_h\cdot a_3\|^2_{L^2(T)} + |r_h\cdot a_3|^2_{H^1(T)}.$$

Theorem 4.5. *For $f \in L^2(\omega, \mathbb{R}^3)$ a given force resultant density, the following a posteriori error estimate holds between the solution (U, ψ) of problem (14) and the solution (U_h, ψ_h) of (20)*

$$\|U - U_h\|_h + \|\psi - \psi_h\|_{\mathbb{M}(\omega)} \lesssim \eta_h. \tag{29}$$

Proof. Let $V \in \mathbb{X}(\omega)$, $V_h \in \mathbb{V}_h$, $\tilde{V}_h = \mathbb{E}_h(V_h)$ and $\tilde{U}_h = \mathbb{E}_h(U_h)$. By definition of the norm $\|\cdot\|_h$, we have:

$$\|U - U_h\|_h^2 = \sum_{T\in\mathcal{T}_h}(|||U - U_h|||^2_{\mathbb{V}(T)} + h_T^{-2}\|(r - r_h)\cdot a_3\|^2_{0,T}) + \sum_{e\in\mathcal{E}_h}\frac{\kappa}{|e|}(\|[\![u - u_h]\!]\|^2_{L^2(e)^3} + \|[\![r - r_h]\!]\|^2_{L^2(e)^3}).$$

As $[\![u]\!] = [\![s]\!] = 0$, the estimates can be directly obtained for the last two terms. For the first term, both U and $\tilde{U}_h$ being in $\mathbb{X}(\omega)$, we have $\sum_{T\in\mathcal{T}_h} |||U - \tilde{U}_h|||^2_{\mathbb{V}(T)} \lesssim \|U - \tilde{U}_h\|^2_{\mathbb{X}(\omega)}$.

The well-posedness of problem (14) amounts to say:

$$\|U - \tilde{U}_h\|_{\mathbb{X}(\omega)} + \|\psi - \psi_h\|_{\mathbb{M}(\omega)} \lesssim \sup_{(V,\phi)\in\mathbb{X}(\omega)\times\mathbb{M}(\omega)\setminus\{0\}} \frac{|\mathbf{a}(U - \tilde{U}_h, V) + b(U - \tilde{U}_h, \phi) + b(V, \psi - \psi_h)|}{\|V\|_{\mathbb{X}(\omega)} + \|\phi\|_{\mathbb{M}(\omega)}}.$$

By using the Clément interpolant, Lemmas 4.1, 4.2 and 4.3, we prove that

$$\sup_{(V,\phi)\in\mathbb{X}(\omega)\times\mathbb{M}(\omega)/\{0\}} \frac{|\mathbf{a}(U - \tilde{U}_h, V) + b(U - \tilde{U}_h, \phi) + b(\tilde{U}_h, \psi - \psi_h)|}{\|V\|_{\mathbb{X}(\omega)} + \|\phi\|_{\mathbb{M}(\omega)}} \lesssim \eta_h. \quad \square$$

By using Lemmas 4.2 and 4.3 and inverse inequalities, we can prove that the error estimator also provides a lower bound for the true error up to data oscillation if the normal vector a_3 is piecewise polynomial, i.e,

$$\eta_h \lesssim \|U - U_h\|_h + \|\psi - \psi_h\|_{-1,h} + Osc. \tag{30}$$

Note finally that the difference between $\|\psi - \psi_h\|_{\mathbb{M}(\omega)}$ and $\|\psi - \psi_h\|_{-1,h}$ is superconvergent with respect to the global error. Hence our estimator is reliable and efficient.

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