C. R. Acad. Sci. Paris, Ser. I 353 (2015) 653–658

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Numerical analysis

A mixed DG method for folded Naghdi's shell in Cartesian coordinates

Une méthode de Galerkin discontinue mixte pour une coque de Naghdi pliée en coordonnées cartésiennes

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ARTICLE INFO

Article history:
Received 16 December 2014
Accepted after revision 20 April 2015
Available online 11 May 2015

Presented by Olivier Pironneau

ABSTRACT

In this Note, a mixed formulation is proposed to solve Naghdi's equations for a thin linearly elastic shell. The unknowns of the problem are the displacement of the points of the middle surface, the rotation field of the normal vector to the middle surface of the shell and a Lagrange multiplier that is introduced in order to enforce the tangency requirement on the rotation. We prove the well posedness of the continuous and the discrete problems.

RÉSUMÉ

Dans cette Note, nous proposons une méthode mixte pour résoudre les équations du modèle de Naghdi de coques linéairement élastiques. Les inconnues du problème sont le déplacement des points de la surface moyenne, le vecteur de rotation de la normale à la surface moyenne et un multiplicateur de Lagrange introduit pour forcer le caractère tangentiel de la rotation. Nous démontrons le caractère bien posé du problème continu et du problème discret.

Version française abrégée

Dans cette Note, nous considérons le problème du modèle de coque de Naghdi pour une coque peu régulière (voir [2]). Ce problème est posé dans un domaine borné $\omega \subset \mathbb{R}^2$ à bord polygonal $\partial\omega$. Nous supposons que $\omega = \omega^+ \cup \Sigma \cup \omega^-$ et nous considérons une coque élastique homogène isotrope dont la surface moyenne $S = S^+ \cup \Gamma \cup S^-$ est donnée par (1). Les inconnues du problème sont respectivement le déplacement u des points de S et la rotation r de la normale unitaire à la surface S. Si $u, r \in H^1(\omega, \mathbb{R}^3)$; nous utiliserons le fait que la carte $\varphi^\pm \in W^{3,\infty}(\omega^\pm, \mathbb{R}^3)$, ce qui permet de définir les tenseurs linéarisés : de déformation (4), de changement de courbure (5) et de déformation de cisaillement transverse (6) comme des fonctions dans $L^2(\omega)$ (voir [2,3]). Ce dernier résultat et l'hypothèse de Naghdi, qui suppose que la rotation est

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http://dx.doi.org/10.1016/j.crma.2015.04.016
1631-073X/

tangentielle à la surface S, permettent de reformuler le problème de Naghdi sous la forme variationnelle (9) (voir [2,3]). Sur la base d'une nouvelle version du lemme du mouvement rigide valable pour des surfaces de classe $W^{2,\infty}(\omega, \mathbb{R}^3)$, A. Blouza [2] a pu démontrer l'existence et l'unicité de la solution du problème variationnel (9).

À cause de la contrainte imposée sur la rotation, le problème (9) ne peut être approché par des méthodes conformes pour une coque générale (voir [4]). Pour forcer le caractère tangentiel de la rotation, nous introduisons un multiplicateur de Lagrange. Nous considérons l'espace relaxé (introduit par Blouza et Le Dret [3]) $\mathbb{X}(\omega) = (H_0^1(\omega, \mathbb{R}^3))^2$ équipé de la norme (8) et l'espace $\mathbb{M}(\omega) = H^{-1}(\omega)$. Ensuite, nous considérons la formulation mixte (11) avec la forme $b(\cdot,\cdot)$ donnée par (12). Si le chargement $f \in L^2(\omega, \mathbb{R}^3)$, les hypothèses du théorème de Babuška–Brezzi sont vérifiées, si bien que le problème variationnel (11) mixte est bien posé. De plus, la solution (U, ψ) du problème mixte est telle que U est celle du problème de Naghdi (9).

Notre problème mixte (11) est discrétisé en utilisant une méthode des éléments finis discontinus mixtes. L'utilisation de la norme (17) et du problème auxiliaire (21) permet de démontrer le caractère bien posé du problème discret (16).

La formulation mixte discontinue (16) permet en particulier d'aborder les problèmes de jonction de coques de classe C^0 ou les coques présentant des arêtes. Cette formulation conduit à l'obtention d'une estimation d'erreur a priori quasi optimale et à une estimation d'erreur a posteriori efficace et fiable, ceci sans exiger aucune régularité additionnelle que celle de la solution du problème faible ; nous renvoyons à la Note compagnonne [8].

1. Introduction

The Koiter and Naghdi shell models are the most currently used in shell theory. The first one represents the Kirchhoff–Love theory, it neglects the shear deformation energy. But engineering experiments show that, for the case of shells with some geometrical singularities, the shear deformation energy cannot be neglected. In contrast, the Naghdi model [7], which enters in the framework of the Reissner–Mindlin theory, takes into consideration this energy and couples it with the same magnitude as the membrane energy. For the Naghdi model, the unknowns are the displacement of the points of the middle surface and the rotation field of the normal vector to the middle surface.

The classical formulation of shell theory uses the covariant and the contravariant components representation of the unknowns, which requires that the shell has a C^3-middle surface (see Ciarlet [5]). Therefore, this formulation is not appropriate for shell with low regularity. The formulation used here was introduced by Blouza [2], Blouza and Le Dret [3]; it is based on the idea of using a local basis-free formulation in which the displacement is described in Cartesian coordinates. This allows us to handle shells with a $W^{2,\infty}$-middle surface. Another advantage of using this formulation for folded shells is the fact that the transmission conditions are easily described and implemented using Cartesian coordinates.

The purpose of this work is to approximate the solution of Naghdi's shell model for a folded shell by using discontinuous mixed finite elements. The basic idea behind the use of our mixed finite element method consists in introducing a Lagrange multiplier in order to handle the tangency requirement on the rotation. On the other hand, the advantage of the use of discontinuous Galerkin methods stays on the fact that the trial functions are piecewise totally discontinuous polynomials. Hence, no continuity constraints are explicitly imposed on the trial functions across the element interfaces. In order to obtain a well-posed discrete problem, jump terms across interfaces or penalty terms are added to the weak formulation. For the case of folded shells, this technique is very useful to impose transmission conditions weakly.

This Note is organized as follows: in Section 2, we first briefly recall the geometry of the surface as well as Naghdi's shell model formulated in Cartesian coordinates and we point out the main difficulty for which the original problem cannot be implemented in a conforming way. In Section 3, we introduce a mixed formulation of Naghdi's model in which a Lagrange multiplier is introduced. We prove that the mixed problem is well posed and that it solves Naghdi's problem. We present in Section 4 the finite element discretization and we prove its well-posedness.

Later on, the notation $a \lesssim b$ is used for the estimate $a \leq c\, b$, where c is a generic constant that does not depend on any mesh size. The convention of summation of repeated indices, which run from 1 to 2 when they are Greek, is used.

2. Naghdi's model for elastic shells with weak regularity

2.1. The continuous problem

Let (e_1, e_2, e_3) be the canonical orthogonal basis of $\mathbb{R}^3$, $u \cdot v$ the inner product of $\mathbb{R}^3$, $u \times v$ the vector product of u and v. Let ω be a domain of $\mathbb{R}^2$. We suppose that $\omega = \omega^+ \cup \Sigma \cup \omega^-$ and we consider a shell whose middle surface $S = S^+ \cup \Gamma \cup S^-$ is given by

$$S^\pm = \varphi^\pm(\bar{\omega}^\pm) \text{ where } \varphi^\pm \in W^{3,\infty}(\omega^\pm, \mathbb{R}^3), \varphi \in W^{1,\infty}(\omega, \mathbb{R}^3), \text{ and } \Gamma = \varphi^+(\Sigma) = \varphi^-(\Sigma). \tag{1}$$

$\varphi^\pm$ is a one-to-one mapping such that the two vectors: $a_\alpha^\pm = \partial_\alpha \varphi^\pm$, $\alpha = 1, 2$ are linearly independent. The normal vector $a_3^\pm$ is given by $a_3^\pm = \dfrac{a_1^\pm \times a_2^\pm}{|a_1^\pm \times a_2^\pm|}$. The contravariant bases a^i are defined by the relation $a_i \cdot a^j = \delta_i^j$, δ_i^j being the Kronecker symbol. The covariant and contravariant components of the metric are given by: $(a_{\alpha\beta}) = (a_\alpha \cdot a_\beta)$ and $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$, $a = \det(a_{\alpha\beta})$, $\sqrt{a}$ is the area element of the midsurface in the chart φ and similarly $\ell = \sqrt{a^{\alpha\beta}\tau_\alpha\tau_\beta}$ is the length element on the boundary $\partial\omega$. We consider here the case of a homogeneous, isotropic material with Young modulus $E > 0$ and Poisson ratio $\nu, 0 \leqslant \nu < \frac{1}{2}$. We also denote by ε the thickness of the shell, which is assumed to be constant and positive.

Let $a^{\alpha\beta\rho\sigma}$ denote the contravariant components of the elasticity tensor, its components are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{2(1-\nu^2)}a^{\alpha\beta}a^{\rho\sigma}. \tag{2}$$

We note that the assumption (1) on the chart is made such that each component of the elasticity tensor is piecewise-$W^{1,\infty}$. Moreover, this tensor satisfies the usual symmetry properties and is uniformly strictly positive i.e., there exists a positive constant c_0 such that

$$a^{\alpha\beta\rho\sigma}(x)\tau_{\alpha\beta}\tau_{\rho\sigma} \geq c_0|\tau|^2 \text{ for a.e. } x \in \omega, \forall\tau, \text{ symmetric tensor of order 2}. \tag{3}$$

Let $u \in H^1(\omega, \mathbb{R}^3)$ be the middle surface displacement and $r \in H^1(\omega, \mathbb{R}^3)$ the rotation of the normal vector. Following [2], the covariant components of the change of metric tensor, $\gamma_{\alpha\beta}(u)$, the covariant components of the change of curvature tensor $\chi_{\alpha\beta}(u, r)$ and the covariant components of the change of transverse shear tensor $\delta_{\alpha 3}(u, r)$ read

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}\left(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha\right), \tag{4}$$

$$\chi_{\alpha\beta}(u, r) = \frac{1}{2}\left(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3\right) + \frac{1}{2}\left(\partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha\right), \tag{5}$$

$$\delta_{\alpha 3}(u, r) = \frac{1}{2}\left(\partial_\alpha u \cdot a_3 + r \cdot a_\alpha\right). \tag{6}$$

The stress resultant $n^{\alpha\beta}(u)$, the stress couple $m^{\alpha\beta}(u, r)$ and the transverse shear force $q^\beta(u, r)$ read $n^{\alpha\beta}(u) = \varepsilon a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u)$, $m^{\alpha\beta}(u, r) = \frac{\varepsilon^3}{12}a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(u, r)$ and $q^\beta(u, r) = \frac{\varepsilon E}{1+\nu}a^{\alpha\beta}\delta_{\alpha 3}(u, r)$.

2.2. The variational formulation of the linear Naghdi model for shells with little regularity

Let us now consider the functional space $\mathbb{V}(\omega)$, introduced by Blouza and Le Dret [3], which is appropriate in the context of shells with little regularity,

$$\mathbb{V}(\omega) = \left\{(v, s) \in \left(H^1(\omega, \mathbb{R}^3)\right)^2; s \cdot a_3 = 0 \text{ in } \omega, v = s = 0 \text{ on } \partial\omega\right\}, \tag{7}$$

equipped with the norm:

$$\|(v, s)\|_{\mathbb{V}(\omega)} = \left(\|v\|^2_{H^1(\omega, \mathbb{R}^3)} + \|s\|^2_{H^1(\omega, \mathbb{R}^3)}\right)^{\frac{1}{2}}. \tag{8}$$

The variational formulation of the problem reads:

$$\text{Find } U = (u, r) \in \mathbb{V}(\omega) \text{ such that } \quad \mathbf{a}((u, r), (v, s)) = \mathcal{L}((v, s)), \forall V = (v, s) \in \mathbb{V}(\omega), \tag{9}$$

where

$$\mathbf{a}((u, r), (v, s)) = \int_\omega \left(n^{\rho\sigma}(u)\gamma_{\rho\sigma}(v) + m^{\rho\sigma}(U)\chi_{\rho\sigma}(V) + q^\beta(U)\delta_{\beta 3}(V)\right)\sqrt{a}\,\mathrm{d}x, \qquad \mathcal{L}((v, s)) = \int_\omega f \cdot v\sqrt{a}\,\mathrm{d}x.$$

Theorem 2.1. *(See [2].) Let $f \in L^2(\omega, \mathbb{R}^3)$ be a given force resultant density. Then the variational problem (9) has a unique solution in $\mathbb{V}(\omega)$.*

Remark 1. For the general shell situation, the constraint $s \cdot a_3$ cannot be implemented in a standard conforming way (see [1,4]).

3. A mixed formulation

Now we propose a mixed formulation for problem (9). Let us first consider the relaxed[1] functional space:

[1] I.e., without the constraint $s \cdot a_3 = 0$.

$$\mathbb{X}(\omega) = H^1_0\big(\omega, \mathbb{R}^3\big) \times H^1_0(\omega, \mathbb{R}^3) \tag{10}$$

equipped with the norm (8) and set $\mathbb{M}(\omega) = H^{-1}(\omega) = \big(H^1_0(\omega)\big)'$.

We consider the following variational mixed problem:

$$\begin{cases} \text{Find } (U, \psi) = (u, r, \psi) \in \mathbb{X}(\omega) \times \mathbb{M}(\omega) \text{ such that} \\ \mathbf{a}(U, V) + b(V, \psi) = \mathcal{L}(V), \forall V \in \mathbb{X}(\omega), \\ b(U, \phi) = 0, \forall \phi \in \mathbb{M}(\omega). \end{cases} \tag{11}$$

For $V = (v, s) \in \mathbb{X}(\omega), \phi \in \mathbb{M}(\omega)$, the bilinear form $b(\cdot, \cdot)$, reads

$$b(V, \phi) =: < \phi, s \cdot a_3 >_{H^{-1}, H^1_0}, \tag{12}$$

where $< \cdot, \cdot >_{H^{-1}, H^1_0}$, denotes the duality pair between $H^{-1}(\omega)$ and $H^1_0(\omega)$.

Observe that the space $\mathbb{V}(\omega)$ is characterized as $\mathbb{V}(\omega) = \{V \in \mathbb{X}(\omega), b(V, \phi) = 0, \forall \phi \in \mathbb{M}(\omega)\}$.

Proposition 3.1. *The bilinear form $b(\cdot, \cdot)$ is continuous on $\mathbb{X}(\omega) \times \mathbb{M}(\omega)$ and it satisfies the following inf–sup condition:*

$$\inf_{\phi \in \mathbb{M}(\omega) \setminus \{0\}} \sup_{(v,s) \in \mathbb{X}(\omega) \setminus \{0\}} \frac{b((v, s), \phi)}{\|(v, s)\|_{\mathbb{X}(\omega)} \|\phi\|_{\mathbb{M}(\omega)}} \gtrsim 1. \tag{13}$$

Proof. Obviously, the bilinear form $b(\cdot, \cdot)$ is continuous on $\mathbb{X}(\omega) \times \mathbb{M}(\omega)$.

Let us prove the inf–sup condition (13). Indeed, for an arbitrary $\phi \in \mathbb{M}(\omega) \setminus \{0\}$, since the mapping $v \mapsto \Delta v$ is an isomorphism from $H^1_0(\omega)$ onto $H^{-1}(\omega)$ (see [6] for instance), then there exists $\eta \in H^1_0(\omega)$ such that $-\Delta\eta = \phi$. Take $V = (0, s)$ with $s = \eta a_3$ then, $b(V, \phi) = \int_\omega |\nabla\eta|^2 \, dx$. Hence, (13) holds. □

We easily deduce the following theorem.

Theorem 3.1. *Let $f \in L^2(\omega, \mathbb{R}^3)$ be a given force resultant density. Then the mixed problem (11) has a unique solution in $\mathbb{X}(\omega) \times \mathbb{M}(\omega)$, which is such that U is the solution of Naghdi's problem (9). Moreover this solution satisfies*

$$\|U\|_{\mathbb{X}(\omega)} + \|\psi\|_{\mathbb{M}(\omega)} \lesssim \|f\|_{L^2(\omega)^3}.$$

4. A discrete discontinuous approximation for the mixed formulation (11)

Let $(\mathcal{T}_h)_{h>0}$ be a regular affine family of triangulations which covers the domain ω. Let $\mathcal{E}_h$ be the set of (open) edges in $\mathcal{T}_h$. Let us introduce the finite dimensional spaces

$$\mathbb{V}_h = \{V_h = (v_h, s_h) \in L^2(\omega)^3 \times L^2(\omega)^3 / (v_h, s_h)_{|T} \in (\mathbb{P}_k(T))^6, \ \forall T \in \mathcal{T}_h\}, \tag{14}$$

$$\mathbb{M}_h = \{\phi_h \in L^2(\omega) / \phi_{h|T} \in \mathbb{P}_k(T), \ \forall T \in \mathcal{T}_h\}. \tag{15}$$

Then we consider the following discrete problem:

$$\begin{cases} \text{Find } (U_h, \psi_h) \in \mathbb{V}_h \times \mathbb{M}_h \text{ such that} \\ a_h(U_h, V_h) + b(V_h, \psi_h) = \mathcal{L}(V_h), \forall V_h = (v_h, s_h) \in \mathbb{V}_h, \\ b(U_h, \phi_h) = 0, \forall \phi_h \in \mathbb{M}_h. \end{cases} \tag{16}$$

Here we take

$$a_h(U_h, V_h) = \tilde{a}(U_h, V_h) + c(U_h, V_h) + d(U_h, V_h),$$

$$\begin{aligned} \tilde{a}(U_h, V_h) = &\sum_{T \in \mathcal{T}_h} \int_T \big(n^{\rho\sigma}(u_h)\, \gamma_{\rho\sigma}(v_h) + m^{\rho\sigma}(U_h)\, \chi_{\rho\sigma}(V_h) + q^\beta(U_h)\, \delta_{\beta 3}(V_h)\big) \sqrt{a}\, dx \\ &+ \sum_{T \in \mathcal{T}_h} h_T^{-2} \int_T (r_h \cdot a_3)(s_h \cdot a_3)\, dx, \end{aligned}$$

$$\begin{aligned} c(U_h, V_h) = &- \sum_{e \in \mathcal{E}_h} \int_e (\{\!\{\nu_\rho(T^\rho(U_h))\}\!\} \cdot [\![v_h]\!] + \{\!\{\nu_\rho(T^\rho(V_h))\}\!\} \cdot [\![u_h]\!])\, de \\ &- \sum_{e \in \mathcal{E}_h} \int_e (\{\!\{\nu_\rho(m^{\rho\sigma}(U_h) a_\sigma)\}\!\} \cdot [\![s_h]\!] + \{\!\{\nu_\rho(m^{\rho\sigma}(V) a_\sigma)\}\!\} \cdot [\![r_h]\!]) \sqrt{a}\, de, \end{aligned}$$

$$d(U_h, V_h) = \sum_{e\in\mathcal{E}_h} \frac{\kappa}{|e|} \int_e ([\![u_h]\!]\cdot[\![v_h]\!] + [\![r_h]\!]\cdot[\![s_h]\!])\sqrt{a}\,\mathrm{d}e,$$

where $T^\rho(V_h) = (n^{\rho\sigma}(v_h)a_\sigma + m^{\rho\sigma}(V_h)\partial_\sigma a_3 + q^\rho(V_h)a_3)\sqrt{a}$, $\forall V_h \in \mathbb{V}_h$, $[\![v_h]\!] = v_h^+ - v_h^-$ is the jump of v_h and $\{\!\{v_h\}\!\} = \frac{1}{2}(v_h^+ + v_h^-)$ is its average across the edge $e \in \mathcal{E}_h$ and $\kappa > 0$ is a penalty parameter.

4.1. Mesh-dependent norms

Let us define the following quantities:

$$||(v,s)||_h^2 = \sum_{T\in\mathcal{T}_h}\sum_{\alpha\beta}(\|\gamma_{\alpha\beta}(v)\|_{0,T}^2 + \|\chi_{\alpha\beta}(v,s)\|_{0,T}^2 + \|\delta_{\alpha 3}(v,s)\|_{0,T}^2) + h_T^{-2}\|s\cdot a_3\|_{0,T}^2)$$
$$+ \sum_{e\in\mathcal{E}_h}\frac{\kappa}{|e|}(\|[\![v]\!]\|_{L^2(e)^3}^2 + \|[\![s]\!]\|_{L^2(e)^3}^2), \tag{17}$$

$$\|\phi_h\|_{-1,h} = (\sum_{T\in\mathcal{T}_h} h_T^2\|\phi_h\|_{0,T}^2)^{1/2}. \tag{18}$$

Proposition 4.1. *$\|\cdot\|_h$ (resp. $\|\cdot\|_{-1,h}$) defines a norm on the space $\mathbb{V}_h$ (resp. $\mathbb{M}_h$).*

Proof. Let

$$\sum_{\alpha\beta}\sum_{T\in\mathcal{T}_h}(\|\gamma_{\alpha\beta}(v)\|_{0,T}^2 + \|\chi_{\alpha\beta}(v,s)\|_{0,T}^2 + \|\delta_{\alpha 3}(v,s)\|_{0,T}^2) = 0, \tag{19}$$

$$\sum_{e\in\mathcal{E}_h}\frac{\kappa}{|e|}(\|[\![v]\!]\|_{L^2(e)^3}^2 + \|[\![s]\!]\|_{L^2(e)^3}^2) + \sum_{T\in\mathcal{T}_h} h_T^{-2}\|s\cdot a_3\|_{0,T}^2 = 0. \tag{20}$$

Eq. (20) implies that $(v,s) \in \mathbb{V}(\omega)$. Then Eq. (19) means that $\mathbf{a}((v,s);(v,s)) = 0$. Hence (v,s) must be identically zero since $\mathbf{a}(,)$ is $\mathbb{V}(\omega)$-elliptic. □

Proposition 4.2. *The bilinear form $a_h(.,.)$ is continuous on $\mathbb{V}_h$ and for κ large enough, we have:*

$$a_h((v_h,s_h),(v_h,s_h)) \geq \frac{1}{2}\|(v_h,s_h)\|_h^2,\ \forall (v_h,s_h) \in \mathbb{V}_h.$$

Proof. Based on some discrete trace inequalities. □

Now we need to prove the inf–sup condition for the bilinear form $b(\cdot,\cdot)$. Let us first define on $\mathbb{M}_h$ the following symmetric bilinear form: For $\sigma > 0$, we set

$$k_h(\eta_h,\theta_h) = \sum_{T\in\mathcal{T}_h}\int_T (\nabla\eta_h\nabla\theta_h + h_T^{-2}\eta_h\theta_h)\,\mathrm{d}x + \sum_{e\in\mathcal{E}_h}\Big(\frac{\sigma}{|e|}\int_e [\![\eta_h]\!][\![\theta_h]\!]\,\mathrm{d}s - \int_e (\{\!\{\nabla\eta_h\}\!\}[\![\theta_h]\!] + [\![\eta_h]\!]\{\!\{\nabla\theta_h\}\!\})\,\mathrm{d}s\Big).$$

Lemma 4.1. *Let us define a_3^h by $a^h_{3|T} = \frac{1}{|T|}\int_T a_3(x)\,\mathrm{d}x$. Then for h small enough, one has $a_3^h\cdot a_3 \geq \frac{1}{2}$.*

Lemma 4.2. *Let $\phi_h \in \mathbb{M}_h$. Then for σ sufficiently large, there exists a unique solution $\eta_h \in \mathbb{M}_h$ of*

$$k_h(\eta_h,\theta_h) = \int_\omega (a_3^h\cdot a_3)\phi_h\theta_h\,\mathrm{d}x, \quad \forall\theta_h \in \mathbb{M}_h. \tag{21}$$

Proof. This is a direct consequence of Lax–Milgram's lemma, since for σ sufficiently large, the bilinear form k_h is coercive on $\mathbb{M}_h$. □

Proposition 4.3. *For h small enough, the bilinear form $b(\cdot,\cdot)$ satisfies the uniform inf–sup condition*

$$\inf_{\phi_h\in\mathbb{M}_h\setminus\{0\}}\ \sup_{(v_h,s_h)\in\mathbb{V}_h\setminus\{0\}} \frac{b((v_h,s_h),\phi_h)}{\|(v_h,s_h)\|_h\|\phi_h\|_{-1,h}} \gtrsim 1. \tag{22}$$

Proof. Fix σ large enough and h small enough in order to apply Lemmas 4.1 and 4.2. Take $V_h = (\mathbf{0}, s_h) = (\mathbf{0}, \eta_h a_3^h)$, then we have:

$$b(V_h, \phi_h) = \int_\omega (a_3^h \cdot a_3)\eta_h \phi_h \, dx = k_h(\eta_h, \eta_h). \tag{23}$$

Taking in (21) $\theta_{h|T} = h_T^2 \phi_{h|T}$, for all $T \in \mathcal{T}_h$, we have:

$$k_h(\eta_h, \theta_h) = \int_\omega (a_3^h \cdot a_3)\phi_h \theta_h \, dx = \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (a_3^h \cdot a_3)\phi_h^2 \gtrsim \|\phi_h\|_{-1,h}^2. \tag{24}$$

Then by Cauchy–Schwarz's inequality, we obtain:

$$\|\phi_h\|_{-1,h}^2 \lesssim (k_h(\eta_h, \eta_h))^{1/2} (k_h(\theta_h, \theta_h))^{1/2}.$$

On the other hand, we have:

$$k_h(\theta_h, \theta_h) \lesssim \sum_{T\in\mathcal{T}_h} h_T^4 |\phi_h|_{1,T}^2 + \sum_{T\in\mathcal{T}_h} h_T^2 \|\phi_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} \frac{h_T^4}{|e|} \|[\![\phi_h]\!]\|_{0,e}^2 \lesssim \|\phi_h\|_{-1,h}^2.$$

Hence

$$\|\phi_h\|_{-1,h} \lesssim k_h(\eta_h, \eta_h)^{1/2}. \tag{25}$$

In addition, we have:

$$\begin{aligned} \|V_h\|_h^2 &= \sum_{T\in\mathcal{T}_h} \sum_{\alpha\beta} (\|\gamma_{\alpha\beta}(s_h)\|_{0,T}^2 + \|\delta_{\alpha 3}(\mathbf{0}, s_h)\|_{0,T}^2) + \sum_{e\in\mathcal{E}_h} \frac{\kappa}{|e|} (\|[\![s_h]\!]\|_{(L^2(e))^3}^2) + \sum_{T\in\mathcal{T}_h} h_T^{-2} \|s_h \cdot a_3\|_{0,T}^2 \\ &\lesssim \sum_{T\in\mathcal{T}_h} \|\nabla \eta_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} h_T^{-2} \|\eta_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \|[\![\eta_h]\!]\|_{0,e}^2 \lesssim k_h(\eta_h, \eta_h). \end{aligned} \tag{26}$$

The estimates (24), (25) and (26) imply that

$$\frac{b(V_h, \phi_h)}{\|V_h\|_h \|\phi_h\|_{-1,h}} \gtrsim \frac{k_h(\eta_h, \eta_h)}{k_h(\eta_h, \eta_h)} \gtrsim 1.$$

Hence (22) is satisfied. □

Theorem 4.3. *For any $f \in L^2(\omega)^3$, the discrete problem* (16) has a unique solution in $\mathbb{V}_h \times \mathbb{M}_h$. Moreover this solution satisfies

$$\|U_h\|_{\mathbb{X}(\omega)} + \|\psi_h\|_{\mathbb{M}(\omega)} \lesssim \|f\|_{L^2(\omega)^3}.$$

Proof. The proof is a direct consequence of Propositions 4.2 and 4.3 (see [6] §4.1). □

Remark 2. Using a combination between the a priori and the a posteriori techniques, the discrete formulation (16) allows the derivation of a quasi-optimal a priori error estimate and an efficient and reliable a posteriori error estimate. Thereby the analysis does not require any additional regularity other than that of the weak solution; we refer to the companion Note [8] for the details.

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