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New identity and Korn's inequalities on a surface

Nouvelle identité et nouvelles inégalités de Korn sur une surface

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ABSTRACT

We establish an identity satisfied by smooth-enough vector fields defined on a surface $S \subset \mathbb{R}^3$ with values in $\mathbb{R}^3$. As consequences of this identity, we establish several new Korn inequalities for vector fields that vanish on the entire boundary of this surface.

RÉSUMÉ

Nous établissons une identité satisfaite par des champs de vecteurs suffisamment réguliers définis sur une surface $S \subset \mathbb{R}^3$ à valeurs dans $\mathbb{R}^3$. Comme conséquences de cette identité, nous établissons plusieurs nouvelles inégalités de Korn pour des champs de vecteurs qui s'annulent sur tout le bord de cette surface.

1. Preliminaries

This section briefly reviews the notions of differential geometry and the function spaces used throughout this Note. The main results are then stated in the next sections, viz., an identity satisfied by every displacement field of a surface (Theorem 1); several new Korn inequalities on a surface (Theorems 2 to 4); and a new "infinitesimal rigid displacement lemma on a surface" (Theorem 5).

In all that follows, Greek indices (except ε and ν) range in the set $\{1, 2\}$, while Latin indices range in the set $\{1, 2, 3\}$, and the repeated index summation convention is used in conjunction with these rules. A generic point in a two-dimensional open subset $\omega \subset \mathbb{R}^2$ is denoted $y = (y_\alpha)$. Partial derivatives, in the classical or distributional sense, of functions or vector fields defined on ω are denoted $\partial_\alpha := \partial/\partial y_\alpha$, $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$, etc.

The inner product and vector product of vectors $\boldsymbol{u} \in \mathbb{R}^3$ and $\boldsymbol{v} \in \mathbb{R}^3$ are respectively denoted $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \wedge \boldsymbol{v}$.

A given surface $S := \boldsymbol{\theta}(\overline{\omega}) \subset \mathbb{R}^3$ in the three-dimensional Euclidean space, defined by means of an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$, is the domain of definition of all vector fields appearing in this Note. Such displacement fields of the surface S are identified with vector fields $\boldsymbol{\eta} : \overline{\omega} \to \mathbb{R}^3$.

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Since $\boldsymbol{\theta}$ is an immersion, the three vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y) \quad \text{and} \quad \boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$$

are linearly independent and thus form a basis in $\mathbb{R}^3$ at each point $y \in \overline{\omega}$; its dual basis at the same point $y \in \overline{\omega}$, denoted $(\boldsymbol{a}^i(y))$, is defined by

$$\boldsymbol{a}^i(y) \cdot \boldsymbol{a}_j(y) = \delta_i^j \quad \big(\delta_j^i \text{ denotes Kronecker's delta}\big).$$

Thus, any vector field $\boldsymbol{\eta} : \overline{\omega} \to \mathbb{R}^3$ can be decomposed as

$$\boldsymbol{\eta}(y) = \eta_i(y) \boldsymbol{a}^i(y) \quad \text{for all } y \in \omega.$$

Note that, at each $y \in \overline{\omega}$, the two vectors $\boldsymbol{a}_\alpha(y)$, resp. $\boldsymbol{a}^\beta(y)$, form the covariant, resp. the contravariant, basis in the tangent plane at $\boldsymbol{\theta}(y)$ to the surface S, while the vector $\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y)$ is unit and normal to this plane.

The *first fundamental form* of $S = \boldsymbol{\theta}(\overline{\omega})$ is defined by means of its covariant, resp. contravariant, components

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad \text{resp. } a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta.$$

Note that, at each $y \in \overline{\omega}$, the matrix $(a^{\alpha\beta}(y))$ is the inverse of the matrix $(a_{\alpha\beta}(y))$, that

$$a := \det(a_{\alpha\beta}) = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|^2,$$

and that $\sqrt{a(y)}\,\mathrm{d}y$ is the area element along the surface S.

The *second fundamental form* of $S = \boldsymbol{\theta}(\overline{\omega})$ is defined by means of its covariants, mixed, and contravariant, components, respectively given by

$$b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3 = -\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3, \qquad b_\alpha^\beta := b_{\alpha\sigma} a^{\sigma\beta}, \quad \text{and} \quad b^{\alpha\beta} := a^{\alpha\sigma} b_\sigma^\beta.$$

The *mean curvature* and *total curvature* of $S = \boldsymbol{\theta}(\overline{\omega})$ are denoted and defined by

$$H(y) := \frac{1}{2}\operatorname{tr}\big(b_\alpha^\beta(y)\big) \quad \text{and} \quad K(y) := \det\big(b_\alpha^\beta(y)\big) \quad \text{at each } y \in \overline{\omega}.$$

Finally, the *Christoffel symbols of the second kind* associated with the immersion $\boldsymbol{\theta}$ are denoted and defined by

$$\Gamma_{\alpha\beta}^\tau := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}^\tau = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\tau,$$

and the *covariant derivatives of a tangent vector field* $\xi_\alpha \boldsymbol{a}^\alpha \in \mathcal{C}^1(\omega; \mathbb{R}^3)$ to the surface $S = \boldsymbol{\theta}(\overline{\omega})$ are denoted and defined by

$$\xi_\alpha|_\beta := \partial_\beta \xi_\alpha - \Gamma_{\alpha\beta}^\sigma \xi_\sigma.$$

The space $H_0^1(\omega)$ is equipped with the norms

$$\|\varphi\|_{1,\omega} := \left\{\sum_\omega \left[\varphi^2 + \sum_\alpha (\partial_\alpha \varphi)^2\right] \mathrm{d}y\right\}^{1/2} \quad \text{and} \quad |\varphi|_{1,\omega,\boldsymbol{\theta}} := \left\{\int_\omega a^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \sqrt{a}\, \mathrm{d}y\right\}^{1/2},$$

the space $H^{-1}(\omega)$ is equipped with the norms

$$\|f\|_{-1,\omega} := \sup\big\{f(\varphi);\ \varphi \in H_0^1(\omega), \|\varphi\|_{1,\omega} \le 1\big\} \quad \text{and} \quad |f|_{-1,\omega,\boldsymbol{\theta}} := \sup\big\{f(\varphi);\ \varphi \in H_0^1(\omega), |\varphi|_{1,\omega,\boldsymbol{\theta}} \le 1\big\},$$

and the space $H_0^1(\omega; \mathbb{R}^3) := \{\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i;\ \eta_i \in H_0^1(\omega)\}$ is equipped with the norms

$$\|\boldsymbol{\eta}\|_{1,\omega} := \left\{\int_\omega \sum_{i=1}^3 \left[(\eta_i)^2 + \sum_{\alpha=1}^2 (\partial_\alpha \eta_i)^2\right] \mathrm{d}y\right\}^{1/2} \quad \text{and} \quad \|\boldsymbol{\eta}\|_{1,\omega,\boldsymbol{\theta}} := \left\{\int_\omega a^{\alpha\beta} \partial_\alpha \boldsymbol{\eta} \cdot \partial_\beta \boldsymbol{\eta} \sqrt{a}\, \mathrm{d}y\right\}^{1/2}.$$

Note that, on each space, the norms defined above are equivalent (as a consequence of Banach open mapping theorem).

Complete proofs of the results announced in this Note, as well as new proofs of several other Korn inequalities on a surface, will be found in [4].

2. An identity satisfied by the displacement fields of a surface

Given any smooth enough vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$, the *linearized change of metric*, and *change of curvature*, *tensor fields* associated with the deformed surface $(\boldsymbol{\theta} + \boldsymbol{\eta})(\omega)$ are respectively denoted and defined by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}_\beta + \boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}) \quad \text{and} \quad \rho_{\alpha\beta}(\boldsymbol{\eta}) := \left(\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \boldsymbol{\eta}\right) \cdot \boldsymbol{a}_3.$$

Theorem 1. *Let ω be any open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an immersion. Given any vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i \in \mathcal{C}^3(\omega; \mathbb{R}^3)$, let*

$$w_\alpha(\boldsymbol{\eta}) := \partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{\eta} - \left(\partial_\beta \boldsymbol{\eta} \cdot \boldsymbol{a}^\beta\right)\eta_\alpha - 2(\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}_3)\eta_3 \in \mathcal{C}^2\left(\omega; \mathbb{R}^3\right).$$

Then the following identity holds in ω:

$$\begin{aligned} &a^{\alpha\beta}\partial_\alpha \boldsymbol{\eta} \cdot \partial_\beta \boldsymbol{\eta} + \left(\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}^\alpha\right)^2 + a^{\alpha\beta} w_\alpha(\boldsymbol{\eta})|_\beta \\ &\quad = 2a^{\alpha\beta}a^{\sigma\tau}\gamma_{\alpha\sigma}(\boldsymbol{\eta})\gamma_{\beta\tau}(\boldsymbol{\eta}) + 2Ha^{\alpha\beta}(\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}_3)\eta_\beta + \left[2\left(b^{\alpha\beta} - Ha^{\alpha\beta}\right)\gamma_{\alpha\beta}(\boldsymbol{\eta}) - a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\right]\eta_3. \end{aligned}$$

Sketch of proof. The announced identity for vector fields $\boldsymbol{\eta} : \omega \to \mathbb{R}^3$ defined over the two-dimensional open set ω will be obtained as the limit when $\varepsilon \to 0^+$ of another identity (see (1) below), when specific vector fields $\boldsymbol{v} : \Omega^\varepsilon \to \mathbb{R}^3$ defined over the three-dimensional open set $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ are used in this identity. A generic point in Ω^ε is denoted $x = (y, x_3)$, with $y \in \omega$ and $x_3 \in (-\varepsilon, \varepsilon)$.

To this end, the Riemannian metric $(a_{\alpha\beta})$ induced on ω by the immersion $\boldsymbol{\theta}$ is extended to a Riemannian metric (g_{ij}) defined on Ω^ε (with $\varepsilon > 0$ sufficiently small, so that the matrix $(g_{ij}(x))$ is positive–definite at each $y \in \overline{\Omega}^\varepsilon$) by letting

$$g_{\alpha\beta}(\cdot, x_3) := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + (x_3)^2 b^\sigma_\alpha b_{\sigma\beta} \quad \text{in } \omega, \qquad g_{\alpha 3} = g_{3\alpha} := 0 \quad \text{in } \Omega^\varepsilon, \quad \text{and} \quad g_{33} := 1 \quad \text{in } \Omega^\varepsilon.$$

At each $x \in \overline{\Omega}^\varepsilon$, the inverse of the matrix $(g_{ij}(x))$ is denoted $(g^{ij}(x))$. The *Christoffel symbols of the second kind* associated with the metric (g_{ij}) are denoted and defined by

$$\Gamma^k_{ij} := \frac{1}{2}g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}),$$

and the *covariant derivatives of a one-form* (u_i), $u_i \in \mathcal{C}^1(\Omega^\varepsilon)$, are denoted and defined by

$$u_i\|_j := \partial_j u_i - \Gamma^k_{ij}u_k.$$

Then one proves first that any one-form (u_i) with $u_i \in \mathcal{C}^1(\Omega^\varepsilon)$ satisfies the identity

$$g^{ik}g^{j\ell}u_i\|_j u_k\|_\ell + \left(g^{ij}u_i\|_j\right)^2 - 2g^{ik}g^{j\ell}e_{ij}e_{k\ell} + g^{ij}\left[g^{k\ell}(u_\ell\, u_i\|_k - u_i\, u_\ell\|_k)\right]\Big\|_j = 0 \quad \text{in } \Omega^\varepsilon. \tag{1}$$

Given any vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i \in \mathcal{C}^3(\omega; \mathbb{R}^3)$, define the functions $\zeta_\alpha \in \mathcal{C}^2(\omega)$, $v_i \in \mathcal{C}^2(\Omega^\varepsilon)$, $e_{ij} \in \mathcal{C}^1(\Omega^\varepsilon)$, and $w_i \in \mathcal{C}^1(\Omega^\varepsilon)$, by letting

$$\begin{aligned} &\zeta_\alpha := \partial_\alpha \eta_3 + b^\sigma_\alpha \eta_\sigma \quad \text{and} \quad v_\alpha(\cdot, x_3) := (\eta_\alpha - x_3\zeta_\alpha) - x_3 b^\sigma_\alpha(\eta_\sigma - x_3\zeta_\sigma) \quad \text{and} \quad v_3(\cdot, x_3) := \eta_3 \quad \text{in } \omega, \\ &e_{ij} := \frac{1}{2}(v_i\|_j + v_j\|_i) \quad \text{and} \quad w_i := g^{jk}(v_k\, v_i\|_j - v_i\, v_k\|_j) \quad \text{in } \Omega^\varepsilon. \end{aligned}$$

Then taking the average over the transverse variable $x_3 \in (-\varepsilon, \varepsilon)$ of the identity (1) with u_i replaced by v_i yields the relation

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left[g^{ik}g^{j\ell}v_i\|_j v_k\|_\ell + \left(g^{ij}v_i\|_j\right)^2 - 2g^{ik}g^{j\ell}e_{ij}e_{k\ell} + g^{ij}w_i\|_j\right](y, x_3)\,\mathrm{d}x_3 = 0 \quad \text{in } \omega,$$

from which one obtains, by letting $\varepsilon \to 0$, the following relation:

$$\left[g^{ik}g^{j\ell}v_i\|_j v_k\|_\ell + \left(g^{ij}v_i\|_j\right)^2 - 2g^{ik}g^{j\ell}e_{ij}e_{k\ell} + g^{\alpha\beta}w_\alpha\|_\beta\right](\cdot, 0) + \lim_{\varepsilon \to 0}\frac{1}{2\varepsilon}\left[w_3(y, \varepsilon) - w_3(y, -\varepsilon)\right] = 0 \quad \text{in } \omega.$$

The identity announced in Theorem 1 is then obtained by expressing each term of the above identity in terms of the vector field $\boldsymbol{\eta}$. More specifically, a series of computations (some of which are somewhat delicate) shows that the following relations hold in ω:

$$\begin{aligned}
&\big[g^{ik}g^{j\ell}v_i\|_j v_k\|_\ell\big](\cdot,0)=a^{\alpha\beta}\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}+a^{\alpha\beta}\zeta_\alpha\zeta_\beta \quad\text{and}\quad \zeta_\alpha=\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}_3,\\
&\big[g^{ij}v_i\|_j\big](\cdot,0)=\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}^\alpha \quad\text{and}\quad \big[g^{ik}g^{j\ell}e_{ij}e_{k\ell}\big](\cdot,0)=a^{\alpha\beta}a^{\sigma\tau}\gamma_{\alpha\sigma}(\boldsymbol{\eta})\gamma_{\beta\tau}(\boldsymbol{\eta}),\\
&\big[g^{\alpha\beta}w_\alpha\|_\beta\big](\cdot,0)=a^{\alpha\beta}\omega_\alpha(\boldsymbol{\eta})|_\beta-2Ha^{\alpha\beta}\big[\eta_\alpha\zeta_\beta-\eta_3\gamma_{\alpha\beta}(\boldsymbol{\eta})\big],\\
&\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\big[w^3(y,\varepsilon)-w^3(y,-\varepsilon)\big]=-a^{\alpha\beta}\zeta_\alpha\zeta_\beta+\eta_3\big[a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})-2b^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\big]. \qquad\square
\end{aligned}$$

3. New Korn inequalities on a surface

In this section, we prove that the identity of Theorem 1 is the basis for establishing several new Korn inequalities (see Theorems 2 to 4), all of which substantially improve some classical Korn inequalities, by replacing the "full" linearized change of curvature tensor field that appears in their right-hand sides by only its *trace*. Note, that, by contrast with the "classical" derivations of Korn's inequalities, ours do *not* use *J.L. Lions lemma* (as in, e.g., [3]).

Theorem 2. *Let ω be a bounded open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be an immersion. Then the following identity holds for each $0<\varepsilon<1$ and for each vector field $\boldsymbol{\eta}=\eta_i\boldsymbol{a}^i$ with components $\eta_i\in H_0^1(\omega)$:*

$$\begin{aligned}
&\int_\omega\big[a^{\alpha\beta}\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}+(\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}^\alpha)^2\big]\sqrt{a}\,\mathrm{d}y\\
&\quad\le\int_\omega\big(8H^2+\varepsilon(2H^2-K)\big)a^{\alpha\beta}\eta_\alpha\eta_\beta\sqrt{a}\,\mathrm{d}y+\frac{2\varepsilon}{1-\varepsilon}\int_\omega(H^2-K)(\eta_3)^2\sqrt{a}\,\mathrm{d}y\\
&\qquad+\frac{4}{\varepsilon}\Big(\int_\omega a^{\alpha\beta}a^{\sigma\tau}\gamma_{\alpha\sigma}(\boldsymbol{\eta})\gamma_{\beta\tau}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y+\big|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\big|^2_{-1,\omega,\boldsymbol{\theta}}\Big).
\end{aligned}$$

Sketch of proof. It suffices to establish the above inequality for smooth vector fields $\boldsymbol{\eta}=\eta_i\boldsymbol{a}^i$ that vanish on the boundary of ω. The identity of Theorem 1 together with Stokes' integral formula imply that such vector fields satisfy the inequality

$$\begin{aligned}
&\int_\omega\big[a^{\alpha\beta}\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}+(\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}^\alpha)^2\big]\sqrt{a}\,\mathrm{d}y\\
&\quad=2\int_\omega a^{\alpha\beta}a^{\sigma\tau}\gamma_{\alpha\sigma}(\boldsymbol{\eta})\gamma_{\beta\tau}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y+\int_\omega 2Ha^{\alpha\beta}(\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\eta_\beta\sqrt{a}\,\mathrm{d}y\\
&\qquad+\int_\omega 2\big(b^{\alpha\beta}-Ha^{\alpha\beta}\big)\gamma_{\alpha\beta}(\boldsymbol{\eta})\eta_3\sqrt{a}\,\mathrm{d}y-\int_\omega a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\eta_3\sqrt{a}\,\mathrm{d}y.
\end{aligned}$$

Then the inequality of Theorem 2 is a consequence of the above inequality combined with the following inequalities (ε denotes any number in the interval $(0,1)$)

$$\begin{aligned}
&\int_\omega 2Ha^{\alpha\beta}(\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\eta_\beta\sqrt{a}\,\mathrm{d}y\le\int_\omega\frac14a^{\alpha\beta}\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}+4H^2a^{\alpha\beta}\eta_\alpha\eta_\beta\sqrt{a}\,\mathrm{d}y,\\
&\int_\omega 2\big(b^{\alpha\beta}-Ha^{\alpha\beta}\big)\gamma_{\alpha\beta}(\boldsymbol{\eta})\eta_3\sqrt{a}\,\mathrm{d}y\le\int_\omega\Big[\frac{\varepsilon}{1-\varepsilon}\big(H^2-K\big)(\eta_3)^2+\frac{2(1-\varepsilon)}{\varepsilon}a^{\alpha\sigma}a^{\beta\tau}\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\sigma\tau}(\boldsymbol{\eta})\Big]\sqrt{a}\,\mathrm{d}y\\
&-\int_\omega a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\eta_3\sqrt{a}\,\mathrm{d}y\le\frac{2}{\varepsilon}\big|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\big|^2_{-1,\omega,\boldsymbol{\theta}}+\frac14\int_\omega\big[a^{\alpha\beta}\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}+\varepsilon\big(4H^2-2K\big)a^{\sigma\tau}\eta_\sigma\eta_\tau\big]\sqrt{a}\,\mathrm{d}y
\end{aligned}$$

(these relations are established by a series of long, but otherwise straightforward, computations involving in particular the Cayley–Hamilton theorem applied to the matrix field (b_α^β)). $\square$

Theorem 3. (a) *Let ω be a bounded open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be an immersion. Then there exists a constant $C_1=C_1(\omega,\boldsymbol{\theta})$ such that, for each vector field $\boldsymbol{\eta}=\eta_i\boldsymbol{a}^i$ with components $\eta_i\in H_0^1(\omega)$,*

$$\sum_i\|\eta_i\|_{1,\omega}\le C_1\Big(\sum_i\|\eta_i\|_{0,\omega}+\sum_{\alpha,\beta}\big\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega}+\big\|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\big\|_{-1,\omega}\Big).$$

(b) *Assume in addition that the boundary of ω is of class $\mathcal{C}^2$. Then there exists a constant $C_2 = C_2(\omega, \boldsymbol{\theta})$ such that, for each vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ with components $\eta_\alpha \in H_0^1(\omega)$ and $\eta_3 \in H_0^1(\omega) \cap H^2(\omega)$,*

$$\sum_\alpha \|\eta_\alpha\|_{1,\omega} + \|\eta_3\|_{2,\omega} \le C_2 \Big(\sum_\alpha \|\eta_\alpha\|_{0,\omega} + \|\eta_3\|_{1,\omega} + \sum_{\alpha,\beta} \big\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega} + \big\|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega} \Big).$$

Sketch of proof. The inequality (a) is obtained by combining the inequality of Theorem 2 with $\varepsilon = 1/2$ with the inequalities

$$\big|f\sqrt{a}\big|_{-1,\omega,\boldsymbol{\theta}} \le D_1 \|f\|_{-1,\omega} \quad \text{for all } f \in H^{-1}(\omega),$$
$$\|\boldsymbol{\eta}\|_{1,\omega} \le D_2 \|\boldsymbol{\eta}\|_{1,\omega,\boldsymbol{\theta}} \quad \text{for all } \boldsymbol{\eta} \in H^1\big(\omega; \mathbb{R}^3\big),$$

which hold for some constants D_1 and D_2 independent of f and $\boldsymbol{\eta}$.

The inequality (b) is obtained from inequality (a) by noting that, since the function $\eta_3 \in H_0^1(\omega) \cap H^2(\omega)$ satisfies the second-order elliptic partial differential equation (see the definition of $\rho_{\alpha\beta}(\boldsymbol{\eta})$):

$$a^{\alpha\beta}\partial_{\alpha\beta}\eta_3 = a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta}) + a^{\alpha\beta}\Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 + b^\beta_\alpha b^\alpha_\beta \eta_3 - 2b^{\alpha\beta}\partial_\alpha\eta_\beta + \big(2b^{\alpha\beta}\Gamma^\sigma_{\alpha\beta} - b^{\alpha\sigma}|_\alpha\big)\eta_\sigma \quad \text{in } \omega,$$

there exists a constant $D_3 = D_3(\omega, \boldsymbol{\theta})$ independent of $\boldsymbol{\eta}$ such that (see, e.g., Gilbarg & Trudinger [5])

$$\|\eta_3\|_{2,\omega} \le D_3 \Big\{ \sum_\sigma \|\eta_i\|_{1,\omega} + \big\|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega} \Big\}. \qquad \square$$

Theorem 4. *Let ω be a bounded open subset of $\mathbb{R}^2$ with a boundary $\gamma := \partial\omega$ of class $\mathcal{C}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an immersion.*

(a) *Let (τ^α) denote a unit tangent vector field to the boundary of ω. Assume that*

$$\sup_{y \in \partial\omega} \big|\big(b_{\alpha\beta}\tau^\alpha\tau^\beta\big)(y)\big| > 0.$$

Then there exists a constant $C_3 = C_3(\omega, \boldsymbol{\theta})$ such that, for each vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ with components $\eta_i \in H_0^1(\omega)$,

$$\sum_i \|\eta_i\|_{1,\omega} \le C_3 \Big(\sum_{\alpha,\beta} \big\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega} + \big\|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\big\|_{-1,\omega} \Big).$$

(b) *Let $\gamma_0 \subset \partial\omega$ denote a non-empty relatively open subset of the boundary of ω. Then there exists a constant $C_4 = C_4(\omega, \boldsymbol{\theta}, \gamma_0)$ such that, for each vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ with components $\eta_\alpha \in H_0^1(\omega)$ and $\eta_3 \in H_0^1(\omega) \cap H^2(\omega)$ with $\partial_\nu\eta_3 = 0$ on γ_0,*

$$\sum_\alpha \|\eta_\alpha\|_{1,\omega} + \|\eta_3\|_{2,\omega} \le C_4 \Big(\sum_{\alpha,\beta} \big\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega} + \big\|a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\big\|_{0,\omega} \Big).$$

The inequality (a), resp. (b), in Theorem 4 is deduced by means of a contradiction argument from inequality (a), resp. (b), in Theorem 3, combined with the following improved version of the classical “infinitesimal rigid displacement lemma”.

Theorem 5. *Let ω be a bounded open subset of $\mathbb{R}^2$ with a boundary $\gamma := \partial\omega$ of class $\mathcal{C}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an immersion.*

(a) *Let (τ^α) denote a unit tangent vector field to the boundary of ω. Assume that*

$$\sup_{y \in \partial\omega} \big|\big(b_{\alpha\beta}\tau^\alpha\tau^\beta\big)(y)\big| > 0.$$

Then any vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ with components $\eta_i \in H^1(\omega)$ that satisfy the relations

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \textit{in } \omega \quad \textit{and} \quad \eta_i = 0 \quad \textit{on } \gamma$$

vanishes in ω.

(b) *Let γ_0 be a non-empty relatively open subset of γ. Then any vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ with components $\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$ that satisfy the relations*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \textit{in } \omega \quad \textit{and} \quad \eta_i = \partial_\nu\eta_3 = 0 \quad \textit{on } \gamma_0$$

vanishes in ω.

Sketch of proof. (a) The assumptions

$$\eta_i \in H_0^1(\omega) \quad \text{and} \quad a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \text{in } \omega$$

imply that $a^{\alpha\beta}\partial_{\alpha\beta}\eta_3 \in L^2(\omega)$ and $\eta_3 = 0$ on γ. Hence $\eta_3 \in H^2(\omega)$; see, e.g., [5]. The assumption $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω implies that $(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) \in H^1(\omega)$, which, combined with the identities

$$2\partial_{\alpha\beta}\eta_\sigma = \partial_\alpha(\partial_\beta\eta_\sigma + \partial_\sigma\eta_\beta) + \partial_\beta(\partial_\alpha\eta_\sigma + \partial_\sigma\eta_\beta) - \partial_\sigma(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) \quad \text{in } \omega,$$

imply that $\eta_\alpha \in H^2(\omega)$. Then the assumptions $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω and $\eta_i = 0$ on γ together imply that

$$(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) = 0 \quad \text{and} \quad \tau^\alpha\partial_\alpha\eta_\beta = 0 \quad \text{on } \gamma.$$

Hence $\nu^\alpha\partial_\nu\eta_\beta + \nu^\beta\partial_\nu\eta_\alpha = 0$ on γ, which in turn implies that $\sum_{\alpha,\beta}[(\nu^\alpha)^2\nu^\beta\partial_\nu\eta_\beta + (\nu^\beta)^2\nu^\alpha\partial_\nu\eta_\alpha] = 2\nu^\alpha\partial_\nu\eta_\alpha = 0$ on γ; then that $\sum_\alpha[(\nu^\alpha)^2\partial_\nu\eta_\beta + \nu^\beta(\nu^\alpha\partial_\nu\eta_\alpha)] = \partial_\nu\eta_\beta = 0$ on γ.

The assumption $\sup_{y\in\partial\omega}|(b_{\alpha\beta}\tau^\alpha\tau^\beta)(y)| > 0$ implies that there exists a non-empty relatively open subset γ_0 of $\partial\omega$ such that $b_{\alpha\beta}\tau^\alpha\tau^\beta \neq 0$ on γ_0, and the assumption $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω implies that

$$\big(b_{\alpha\beta}\tau^\alpha\tau^\beta\big)\eta_3 = \frac{1}{2}\big(\tau^\beta\partial_\tau\eta_\beta + \tau^\alpha\partial_\tau\eta_\alpha\big) - \Gamma^\sigma_{\alpha\beta}\tau^\alpha\tau^\beta\eta_\sigma \quad \text{in } \omega.$$

Combined with the observation that

$$\partial_\nu(\partial_\tau\eta_\beta) = \nu^\alpha\partial_\alpha\big(\tau^\lambda\partial_\lambda\eta_\beta\big) = \partial_\tau(\partial_\nu\eta_\beta) + \nu^\alpha\big(\partial_\alpha\tau^\lambda\big)\partial_\lambda\eta_\beta - \tau^\lambda\big(\partial_\lambda\nu^\alpha\big)\partial_\alpha\eta_\beta = 0 \quad \text{on } \gamma_0,$$

this implies that

$$\partial_\nu\eta_3 = \frac{1}{2}\partial_\nu\left[\big(b_{\alpha\beta}\tau^\alpha\tau^\beta\big)^{-1}\left(\tau^\beta\partial_\tau\eta_\beta + \frac{1}{2}\tau^\alpha\partial_\tau\eta_\alpha - \Gamma^\sigma_{\alpha\beta}\tau^\alpha\tau^\beta\eta_\sigma\right)\right] = 0 \quad \text{on } \gamma_0.$$

This means that the vector field $\boldsymbol{\eta}$ satisfies all the assumptions of part (b) of Theorem 5. Consequently, it must vanish in ω, as we now prove.

(b) Consider a vector field $\boldsymbol{\eta} = \eta_i\boldsymbol{a}^i : \omega \to \mathbb{R}^3$ with $\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$ that satisfy the relations

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \text{in } \omega \quad \text{and} \quad \eta_i = \partial_\nu\eta_3 = 0 \quad \text{on } \gamma_0.$$

Then there exists an open ball B centered at a point $y_0 \in \gamma_0$ such that $B \cap \partial\omega \subset \gamma_0$, and there exists an immersion $\tilde{\boldsymbol{\theta}} \in \mathcal{C}^3(\overline{\omega \cup B}; \mathbb{R}^3)$ such that $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}}|_\omega$. Let $\tilde{\omega} := \omega \cup B$ and let $\tilde{\eta}_i : \tilde{\omega} \to \mathbb{R}$ denote the extensions of the functions $\eta_i : \omega \to \mathbb{R}$ defined by $\tilde{\eta}_i = 0$ in $\tilde{\omega} - \omega$. Then, on the one hand, $\tilde{\eta}_\alpha \in H^1(\tilde{\omega})$, $\tilde{\eta}_3 \in H^2(\tilde{\omega})$, and $\tilde{\eta}_i = 0$ in the open subset $\tilde{\omega} - \overline{\omega}$ of $\tilde{\omega}$.

On the other hand, the relations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω imply that, with self-explanatory notations (similar to those introduced in Section 1),

$$\frac{1}{2}(\partial_\alpha\tilde{\eta}_\beta + \partial_\beta\tilde{\eta}_\alpha) - \tilde{\Gamma}^\sigma_{\alpha\beta}\tilde{\eta}_\sigma = \tilde{b}_{\alpha\beta}\tilde{\eta}_3 \quad \text{in } \tilde{\omega},$$
$$\tilde{a}^{\alpha\beta}\partial_{\alpha\beta}\tilde{\eta}_3 = \tilde{a}^{\alpha\beta}\tilde{\Gamma}^\sigma_{\alpha\beta}\partial_\sigma\tilde{\eta}_3 + \tilde{b}^\beta_\alpha\tilde{b}^\alpha_\beta\tilde{\eta}_3 - 2\tilde{b}^{\alpha\beta}\theta_\alpha\tilde{\eta}_\beta + \big(2\tilde{b}^{\alpha\beta}\tilde{\Gamma}^\sigma_{\alpha\beta} - \tilde{b}^{\alpha\sigma}|_\alpha\big)\tilde{\eta}_\sigma \quad \text{in } \tilde{\omega}.$$

Therefore, the unique continuation property of solutions to elliptic systems (see, e.g., [1,2], or [6]) implies that $\tilde{\eta}_i = 0$ in $\tilde{\omega}$. □

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