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Complex analysis

Complementability of exponential systems ☆

Complémentabilité des systèmes d'exponentielles

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A R T I C L E   I N F O

A B S T R A C T

We prove that any incomplete systems of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^2(-\pi, \pi)$ is a subset of some complete and minimal system of exponentials. In addition, we prove an analogous statement for systems of reproducing kernels in de Branges spaces.

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R É S U M É

Nous démontrons que tout système incomplet d'exponentielles complexes $\{e^{i\lambda_n t}\}$ dans $L^2(-\pi, \pi)$ est un sous-ensemble d'un système complet et minimal d'exponentielles. De plus, nous montrons un résultat analogue pour des systèmes de noyaux reproduisants dans les espaces de de Branges.

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1. Introduction

Let a sequence $\Lambda \subset \mathbb{C}$ be such that the system of exponential functions $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is complete and minimal in $L^2(-\pi, \pi)$. Numerous papers are devoted to the different aspects of the theory of exponential systems such as completeness, Riesz basis property, Riesz sequence property, linear summation methods etc. We refer to [9,11,6,5] for details. The geometry of such sequences is not well understood. Nevertheless, there exist a non-geometric necessary and sufficient conditions for the sequence $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ to be complete and minimal; see Theorem 4 in Lecture 18 of [7].

The systems of exponentials are much more regular than an arbitrary system of vectors in $L^2(-\pi, \pi)$. For example, it is easy to verify that if $e^{iat} \in \overline{\text{Span}}\{e^{i\lambda t}\}_{\lambda \in \Lambda}$, $a \notin \Lambda$, then $\overline{\text{Span}}\{e^{i\lambda t}\}_{\lambda \in \Lambda} = L^2(-\pi, \pi)$.

The system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is said to be a Riesz basis if there exists an isomorphism $T : L^2(-\pi, \pi) \to L^2(-\pi, \pi)$ such that $T(e^{int}) = e^{i\lambda_n t} \cdot \|e^{i\lambda_n t}\|^{-1}$, $n \in \mathbb{Z}$ (or, which is the same, $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is an image of an orthogonal basis under a bounded invertible operator). Moreover, $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is said to be a Riesz sequence if it is a Riesz basis of the closure of the space spanned by elements from $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$.

The next question was raised in [8, p. 196]:

Question. *Can every Riesz sequence of complex exponentials be complemented up to a complete and minimal system of complex exponentials?*

K. Seip has shown in [10, Theorem 2.8] that system $\{e^{\pm i(n+\sqrt{n})t}\}_{n>1}$ is a Riesz sequence of complex exponentials in $L^2(-\pi, \pi)$ that cannot be complemented up to a Riesz basis. We will give a positive answer to the question. Moreover, we do not need the Riesz sequence assumption.

Theorem 1.1. *If $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is an incomplete system in $L^2(-\pi, \pi)$, then there exists a sequence $S \subset \mathbb{R}$, $\Lambda \cap S = \emptyset$ such that the system $\{e^{i\lambda t}\}_{\lambda \in \Lambda \cup S}$ is complete and minimal in $L^2(-\pi, \pi)$.*

In [8], this was proved only for very specific real sequences Λ ($\Lambda = \{n + \delta_n\}_{n \in \mathbb{Z}}$, δ_n is an increasing sequence). It is interesting to note that there exist a lot of complete exponential systems with no complete and minimal subsequences. The simplest example is any sequence $\{\lambda_n\}$ such that $|\lambda_n| \to 0$ (but it may happen even if $|\lambda_n| \to \infty$; see Section 4).

The proof of Theorem 1.1 is nonconstructive and based on the deep and significant de Branges theory of entire functions. In Section 3 we will prove the analogous theorem for the systems of reproducing kernels in *an arbitrary de Branges space*. It would be interesting to find a direct and constructive proof of Theorem 1.1.

In the next section, we transfer our problem to the Paley–Wiener space of entire functions. Section 3.1 is devoted to the de Branges theory. Section 3.2 is devoted to the proof of our result.

2. Paley–Wiener space

We reformulate our problem in the Paley–Wiener space

$$\mathcal{PW}_\pi := \left\{ F : F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{itz} dt, \ g \in L^2(-\pi, \pi) \right\}.$$

The Fourier transform $\mathcal{F}$ acts unitarily from $L^2(-\pi, \pi)$ onto $\mathcal{PW}_\pi$; under this transform, the exponential functions become the reproducing kernels in $\mathcal{PW}_\pi$,

$$\mathcal{F}\left(e^{-i\bar{\lambda}t}\right) = \frac{\sin \pi(z - \bar{\lambda})}{\pi(z - \bar{\lambda})} =: k_\lambda^{\mathcal{PW}_\pi}(z),$$

$$\left(F, k_\lambda^{\mathcal{PW}_\pi}\right)_{\mathcal{PW}_\pi} = F(\lambda).$$

This gives us the following description (see, e.g., Theorem 4 in Lecture 18 of [7]).

Proposition 2.1. *The system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is complete and minimal in $L^2(-\pi, \pi)$ if and only if the following three conditions hold:*

(i) *the product*

$$G(z) := \lim_{R \to \infty} \prod_{|\lambda| < R} \left(1 - \frac{z}{\lambda}\right) \tag{2.1}$$

converges to an entire function G;
(ii) *For some (any) $\lambda \in \Lambda$, we have $\frac{G(z)}{z - \lambda} \in \mathcal{PW}_\pi$;*
(iii) *There is no non-zero entire T such that $GT \in \mathcal{PW}_\pi$.*

If the system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is incomplete, then the product (2.1) may diverge. But since Λ satisfies Blaschke's condition $\sum_{\lambda \in \Lambda} \frac{|\operatorname{Im} \lambda|}{|\lambda|^2} < \infty$, the product

$$G_\Lambda(z) := \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) e^{\Re(\lambda^{-1})z} \tag{2.2}$$

converges. Function G_Λ is such that $\frac{G(z)}{G(\bar{z})}$ is *a ratio of two Blaschke products* in $\mathbb{C}^+$.

Let $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ be an incomplete system in $L^2(-\pi, \pi)$ and G_Λ be a canonical product from (2.2). Put

$$\mathcal{H}_{\Lambda, \pi} := \{T : T - \text{ entire, and } G_\Lambda T \in \mathcal{PW}_\pi\}.$$

It is easy to verify that $\mathcal{H}_{\Lambda, \pi}$ is a nontrivial Hilbert space of entire functions (with respect to the norm inherited from the Paley–Wiener space).

Proposition 2.2. *Let $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ be an incomplete system. Then the system $\{e^{i\lambda t}\}_{\lambda \in \Lambda \cup S}$ is complete and minimal in $L^2(-\pi, \pi)$ if and only if the set S is a minimal uniqueness set in $\mathcal{H}_{\Lambda, \pi}$ (i.e. S is a uniqueness set, but $S \setminus \{s_0\}$ is not a uniqueness set for some (any) $s_0 \in S$).*

So, it remains to prove that, in $\mathcal{H}_{\Lambda, \pi}$, there exists a minimal uniqueness set.

3. De Branges spaces of entire functions

3.1. Preliminary steps

It is well known that $\mathcal{PW}_\pi = \mathcal{E}_\pi \cap L^2(\mathbb{R})$, where $\mathcal{E}_\pi$ is a class of entire functions of exponential type at most π. In other words, we have *global growth condition* and *inclusion onto some weighted Hilbert space on the real axis*. The key idea is that the space $\mathcal{H}_{\Lambda,\pi}$ is of the same kind.

Informally speaking, the de Branges spaces form a general class of spaces of such type. Now we remind two possible ways to define de Branges spaces.

1. An entire function E is said to be in the Hermite–Biehler class if $|E(z)| > |E^*(z)|$, $z \in \mathbb{C}_+$, where $E^*(z) = \overline{E(\bar{z})}$. With any such function we associate the *de Branges space* $\mathcal{H}(E)$, which consists of all entire functions F such that F/E and F^*/E, restricted to $\mathbb{C}_+$, belong to the Hardy space $H^2 = H^2(\mathbb{C}_+)$. The inner product in $\mathcal{H}(E)$ is given by

$$(F, G)_{\mathcal{H}(E)} = \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2} \, dt.$$

2. We will say that $\mathcal{H}$ is a de Branges space if $\mathcal{H}$ is a nontrivial Hilbert space, whose elements are entire functions that satisfy the following axioms:
   - (A1) whenever F is in the space and has a nonreal zero w, the function $F(z)\frac{z-\bar{w}}{z-w}$ is in the space and has the same norm as F;
   - (A2) for every nonreal number w, the evaluation functional $F \mapsto F(w)$ is continuous;
   - (A3) the function F^* belongs to the space whenever F belongs to the space and it always has the same norm as F.

These two definitions are equivalent (see [4, Theorem 23]). Sometimes it is natural to require the additional assumption that for any $w \in \mathbb{C}$, there exists $F \in \mathcal{H}$ such that $F(w) \neq 0$. This corresponds to the situation when E has no real zeros. In what follows, we will consider only such spaces.

Any de Branges space has a reproducing kernel which is given by

$$k_w(z) = \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{2\pi i(\bar{w} - z)} = \frac{\overline{A(w)}B(z) - \overline{B(w)}A(z)}{\pi(z - \bar{w})}, \tag{3.1}$$

where the entire functions A and B are defined by $A = \frac{E+E^*}{2}$, $B = \frac{E^*-E}{2i}$. Functions A and B have only simple real zeros, and $E = A - iB$.

If $E(z) = e^{-i\pi z} = \cos(\pi z) - i\sin(\pi z)$, then $\mathcal{H}(E) = \mathcal{PW}_\pi$.

Any de Branges space has an orthogonal basis that consists of reproducing kernels. More precisely,

[4, Theorem 22]: *Let $\mathcal{H}(E)$ be a given space and let $\varphi(t) := -\arg E(t)$ be a phase function associated with $E(z)$. If α is a given real number, the system $\{k_{t_n}(z)\}_{\varphi(t_n)-\alpha \in \pi\mathbb{Z}}$ is an orthonormal basis for all $\alpha \in [0, \pi)$, except maybe one.*

For simplicity, we will assume that system $\{k_{t_n}(z)\}$ is an orthogonal basis for $\alpha = 0$. So, $\{t_n\} = \{z : A(z) = 0\}$.

It is interesting to note that if a Hilbert space of entire functions contains *two orthogonal bases of reproducing kernels*, then it is equal to some de Branges space of entire functions (see [3]).

In particular, in any de Branges space, there exists a minimal uniqueness set $\{x : A(x) = 0\}$.

Further we will use the following simple fact about de Branges spaces, which follows from the definition of $\mathcal{H}(E)$.

Proposition 3.1. *Let F belong to $\mathcal{H}(E)$ and B be a Blaschke product such that FB is an entire function. Then $FB \in \mathcal{H}(E)$.*

3.2. Proof of Theorem 1.1

We claim that $\mathcal{H}_{\Lambda,\pi}$ is a de Branges space (there exists E such that $\mathcal{H}(E) = \mathcal{H}_{\Lambda,\pi}$). It is easy to see that for any $w \in \mathbb{C}$, there exists $F \in \mathcal{H}_{\Lambda,\pi}$ such that $F(w) \neq 0$. Without loss of generality we can assume that $\{z : A(z) = 0\} \cap \Lambda = \emptyset$ (we can always choose another α in [4, Theorem 22]). Put $S = \{z : A(z) = 0\}$. From Proposition 2.2 we get that system $\{e^{i\lambda t}\}_{\lambda \in \Lambda \cup S}$ is a complete and minimal system in $L^2(-\pi, \pi)$.

To prove the claim, we will use the axiomatic description of de Branges spaces. We need only to verify that $\mathcal{H}_{\Lambda,\pi}$ satisfies axioms (A1)–(A3). Let G_Λ be the canonical product from (2.2).

(A2) For any $w \in \mathbb{C}$ the point evaluation functional $F \mapsto F(w)$ is a bounded linear functional on $\mathcal{H}_{\Lambda,\pi}$. Indeed, if $w \notin \Lambda$, then

$$\left|F(w)\right| = \left|G_\Lambda(w)F(w)/G_\Lambda(w)\right| \leq \left|G_\Lambda(w)\right|^{-1}\|G_\Lambda F\|_{\mathcal{PW}_\pi} \cdot \left\|k_w^{\mathcal{PW}_\pi}\right\|$$
$$= \left|G_\Lambda(w)\right|^{-1}\|F\|_{\mathcal{H}_{\Lambda,\pi}} \cdot \left\|k_w^{\mathcal{PW}_\pi}\right\|.$$

If $w \in \Lambda$, then we can write $F(z) = \frac{1}{2\pi} \int_0^{2\pi} F(w + \varepsilon e^{i\theta}) d\theta$, and use the same estimate.

(A1) If $F \in \mathcal{H}_{\Lambda,\pi}$ and $F(w) = 0$, then $F(z)\frac{z-\bar{w}}{z-w} \in \mathcal{H}_{\Lambda,\pi}$ and $\|F(z)\frac{z-\bar{w}}{z-w}\| = \|F\|$.

(A3) If $F \in \mathcal{H}_{\Lambda,\pi}$, then $F^* \in \mathcal{H}_{\Lambda,\pi}$. Indeed, if $G_\Lambda F \in \mathcal{P}W_\pi$, then $G_\Lambda^* F^* \in \mathcal{P}W_\pi$. Using Proposition 3.1 and the fact that $\frac{G_\Lambda}{G_\Lambda^*}$ is a ratio of two Blaschke products we get that $G_\Lambda F^* \in \mathcal{P}W_\pi$. □

3.3. Reproducing kernels in de Branges spaces

Theorem 1.1 can be generalized to an arbitrary de Branges space.

Theorem 3.2. *Let $\mathcal{H}(E)$ be a de Branges space. If $\{k_\lambda\}_{\lambda \in \Lambda_1}$ is an incomplete system of reproducing kernels in $\mathcal{H}(E)$, then there exists a system $\{k_\lambda\}_{\lambda \in \Lambda_2}$ such that the system $\{k_\lambda\}_{\lambda \in \Lambda_1 \cup \Lambda_2}$ is complete and minimal in $\mathcal{H}(E)$.*

Proof. Let

$$\mathcal{H}_{\Lambda_1, \mathcal{H}(E)} := \left\{ T : T - \text{ entire and } G_{\Lambda_1} T \in \mathcal{H}(E) \right\}.$$

Now we can repeat the arguments from the previous subsection. □

Theorem 3.2 can be useful for the research of strong M-basis property in de Branges spaces (see [2, Section 8]).

4. Concluding remarks and open questions

1. It is well known that there is no $F \in \mathcal{P}W_\pi \setminus \{0\}$ that vanishes only on the imaginary axis. On the other hand, the set $\{z : F(z) = 0\}$ has to satisfy the Blaschke condition. Hence, the system $\{e^{nt}\}_{n \in \mathbb{Z}}$ is a complete set in $L^2(-\pi, \pi)$ that contains no complete and minimal subset.
2. It is interesting to know whether Theorem 3.2 holds true for the Fock space of entire functions.
3. It is not clear whether it is possible to keep some regularity of the system of reproducing kernels when we complement our system.

Question 1. *Is it true that any incomplete system of reproducing kernels with infinite defect $\dim(L^2(-\pi, \pi) \ominus \overline{\text{Span}} \{e^{i\lambda t}\}_{\lambda \in \Lambda}) = \infty$ can be complemented up to a strong M-basis?*

For a detailed review about the strong M-basis property for a system of complex exponentials in $L^2(-\pi, \pi)$, see [1] and references therein.

References

[1] A. Baranov, Yu. Belov, A. Borichev, Hereditary completeness for systems of exponentials and reproducing kernels, Adv. Math. 235 (2013) 525–554.
[2] A. Baranov, Y. Belov, A. Borichev, Strong M-basis property for systems of reproducing kernels in de Branges spaces, http://arxiv.org/abs/1309.6915.
[3] Yu. Belov, T. Mengestie, K. Seip, Unitary discrete Hilbert transforms, J. Anal. Math. 112 (2010) 383–395.
[4] L. de Branges, Hilbert Spaces of Entire Functions, Prentice–Hall, Englewood Cliffs, 1968.
[5] S.V. Hruscev, N.K. Nikolskii, B.S. Pavlov, Unconditional bases of exponentials and of reproducing kernels, in: Complex Analysis and Spectral Theory, Leningrad, 1979/1980, in: Lecture Notes in Mathematics, vol. 864, Springer, Berlin–New York, 1981, pp. 214–335.
[6] P. Koosis, The Logarithmic Integral, I, Cambridge University Press, Cambridge, UK, 1988.
[7] B.Ya. Levin, Lectures on Entire Functions, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, USA, 1996.
[8] A. Nakamura, Basis properties and complements of complex exponential systems, Hokkaido Math. J. 36 (1) (2007) 193–206.
[9] R. Redheffer, Completeness of sets of complex exponentials, Adv. Math. 24 (1977) 1–62.
[10] K. Seip, On the connection between exponential bases and certain related sequence in $L^2[-\pi, \pi]$, J. Funct. Anal. 130 (1995) 131–160.
[11] R. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, San Diego–London, 2001.