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On metric Diophantine approximation in matrices and Lie groups

Approximation diophantienne métrique dans les matrices et les groupes de Lie

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A R T I C L E I N F O

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A B S T R A C T

We study the Diophantine exponent of analytic submanifolds of $m \times n$ real matrices, answering questions of Beresnevich, Kleinbock, and Margulis. We identify a family of algebraic obstructions to the extremality of such a submanifold, and give a formula for the exponent when the submanifold is algebraic and defined over $\mathbb{Q}$. We then apply these results to the determination of the Diophantine exponent of rational nilpotent Lie groups.

R É S U M É

Nous étudions l'exposant diophantien des sous-variétés analytiques de matrices réelles $m \times n$ et répondons à certaines questions posées par Beresnevich, Kleinbock et Margulis. Nous identifions une famille d'obstructions algébriques à l'extrémalité d'une telle sous-variété, et donnons une formule pour l'exposant lorsque celle-ci est définie sur $\mathbb{Q}$. Enfin, nous appliquons ces résultats à la détermination de l'exposant diophantien des groupes de Lie nilpotents rationnels.

1. Introduction

In their breakthrough paper [11], Kleinbock and Margulis have solved a long-standing conjecture of Sprindzuk regarding metric Diophantine approximation on submanifolds of $\mathbb{R}^n$, stating roughly speaking that non-degenerate submanifolds are extremal in the sense that almost every point on them has similar Diophantine properties to those of a random vector in $\mathbb{R}^n$ (i.e. it is not very well approximable, see below). Doing so, they used new methods coming from dynamics and based on quantitative non-divergence estimates (going back to early work of Margulis [13] and Dani [5]) for certain flows on the

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non-compact homogeneous space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. They suggested at the end of their paper to extend their results to the case of submanifolds of matrices $M_{m,n}(\mathbb{R})$, a natural set-up for such questions. This was studied further in [12,4] and the problem appears in Gorodnik's list of open problems [7].

In this note, we announce a set of results [2] that give a fairly complete picture of what happens in the matrix case as far as extremality is concerned. We identify a natural family of obstructions to extremality (Theorem 4.1) and show that they are in some sense the only obstructions to be considered (Theorem 4.3). Our results also extend to the matrix case the previous works of Kleinbock [8,9] regarding degenerate submanifolds of $\mathbb{R}^n$. When the submanifold is algebraic and defined over $\mathbb{Q}$, we obtain a formula for the exponent (Theorem 5.1).

In a second part of this note, we state new results regarding Diophantine approximation on Lie groups, in the spirit of our earlier work [1]. These results, which are applications of the theorems described in the first part of this note, concern the Diophantine exponent of nilpotent Lie groups and were our initial motivation for studying Diophantine approximation on submanifolds of matrices. The submanifolds to be considered here are images of certain word maps. Depending on the structure of the Lie algebra and its ideal of laws, these submanifolds can be degenerate. The relevant obstructions can nevertheless be identified and this leads to a formula for the Diophantine exponent of an arbitrary rational nilpotent Lie group (Theorem 7.2). A number of examples are also worked out explicitly.

2. Diophantine approximation on submanifolds of $\mathbb{R}^n$

A vector $x \in \mathbb{R}^n$ is called *extremal* (or *not very well approximable*), if for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$|q \cdot x + p| > \frac{c_\varepsilon}{\|q\|^{n+\varepsilon}}$$

for all $p \in \mathbb{Z}$ and all $q \in \mathbb{Z}^n \setminus \{0\}$. Here $q \cdot x$ denotes the standard scalar product in $\mathbb{R}^n$ and $\|q\| := \sqrt{q \cdot q}$ the standard Euclidean norm.

As is well known, almost every $x \in \mathbb{R}^n$ is extremal. An important question in metric Diophantine approximation is that of understanding the Diophantine properties of points x that are allowed to vary inside a fixed submanifold $\mathcal{M}$ of $\mathbb{R}^n$. The submanifold $\mathcal{M}$ is called *extremal* if almost every point on $\mathcal{M}$ is extremal. A key result here is Theorem 2.1.

Theorem 2.1. *(See Kleinbock–Margulis, [11].) Let U be an open connected subset of $\mathbb{R}^k$ and $\mathcal{M} := \{\mathbf{f}(x); x \in U\}$, where $\mathbf{f} : U \to \mathbb{R}^n$ is a real analytic map. Assume that $\mathcal{M}$ is not contained in a proper affine subspace of $\mathbb{R}^n$, then $\mathcal{M}$ is extremal.*

This answered a conjecture of Sprindzuk. The proof made use of homogeneous dynamics via the so-called *Dani correspondence* between Diophantine exponents and the rate of escape to infinity of a diagonal flow in the space of lattices. We will also utilize these tools.

3. Diophantine approximation on submanifolds of matrices

It is natural to generalize this setting to that of submanifolds of matrices, namely submanifolds $\mathcal{M} \subset M_{m,n}(\mathbb{R})$. The Diophantine problem now becomes that of finding good integer approximations (by a vector $p \in \mathbb{Z}^m$) of the image $M \cdot q$ of an integer vector $q \in \mathbb{Z}^n$ under the linear endomorphism $M \in M_{m,n}(\mathbb{R})$. The case $m = 1$ corresponds to the above classical case (that of linear forms), while the dual case $n = 1$ corresponds to a simultaneous approximation.

It turns out that it is more natural to study the slightly more general problem of approximating 0 by the image $M \cdot q$ of an integer vector q. One can pass from the old problem to the new one by embedding $\mathcal{M}$ inside $M_{m,m+n}(\mathbb{R})$, via the embedding (I_m denotes the $m \times m$ identity matrix):

$$M_{m,n}(\mathbb{R}) \to M_{m,m+n}(\mathbb{R})$$
$$M \mapsto (I_m | M)$$

From now on, we will consider an arbitrary connected analytic submanifold $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$, given as $\mathcal{M} := \{\mathbf{f}(x); x \in U\}$, where $\mathbf{f} : U \to M_{m,m+n}(\mathbb{R})$ is a real analytic map from a connected open subset U in some $\mathbb{R}^k$.

Definition 3.1 *(Diophantine exponent).* We say that a matrix $M \in M_{m+n,n}(\mathbb{R})$ has Diophantine exponent $\beta(M) \geq 0$, if $\beta(M)$ is the supremum of all numbers $\beta \geq 0$ for which there are infinitely many $q \in \mathbb{Z}^{m+n}$ such that

$$\|M \cdot q\| < \frac{1}{\|q\|^\beta}.$$

4. The pigeonhole argument and the obstructions to extremality

By the pigeonhole principle (Dirichlet's theorem), the lower bound $\beta(M) \geq \frac{m}{n}$ holds for all M. Indeed one compares the number of integer points in a box of side length T in $\mathbb{Z}^{m+n}$ with the volume occupied by the image of this box under M in $\mathbb{R}^m$. Furthermore, instead of considering the full box of side length T in $\mathbb{Z}^{m+n}$, we could have restricted attention to the intersection of this box with a *rational* subspace $W \leq \mathbb{R}^{m+n}$. The same argument would have then given the lower bound:

$$\beta(M) \geq \frac{\dim W}{\dim MW} - 1.$$

Of course it may happen, given M, that for some exceptional subspace W, $\frac{\dim W}{\dim MW} - 1 > \frac{n}{m} = \frac{n+m}{m} - 1$. And this may well also happen for all $M \in \mathcal{M}$, provided $\mathcal{M}$ lies in the following algebraic subvariety $\mathcal{P}_{W,r}$ of $M_{m,m+n}(\mathbb{R})$

$$\mathcal{P}_{W,r} := \bigl\{M \in M_{m,m+n}(\mathbb{R});\, \dim MW \leq r\bigr\}, \tag{1}$$

where W is a fixed rational subspace of $\mathbb{R}^{m+n}$ and r a non-negative integer such that

$$\frac{\dim W}{r} - 1 > \frac{n}{m}. \tag{2}$$

By convention, we agree that (2) is satisfied if $r = 0$. We will call the subvariety $\mathcal{P}_{W,r}$ of $M_{m,m+n}(\mathbb{R})$ a *pencil of endomorphisms* with parameters W and r (defined also for arbitrary, non-rational, subspaces W). Note that when $m = 1$, and $r = 0$, this notion reduces to the notion of linear subspace (the orthogonal of W) of $\mathbb{R}^{n+1}$ (or affine subspace of $\mathbb{R}^n$). Hence asking that the submanifold $\mathcal{M}$ be not contained in any of those pencils $\mathcal{P}_{W,r}$ satisfying (2) is analogous in the matrix context to the condition of Theorem 2.1 that $\mathcal{M}$ be not contained in an affine subspace. The following result is close in spirit to that of [4], which gave a sufficient geometric condition for strong extremality. Our condition is strictly weaker, but only implies extremality:

Theorem 4.1 *(Extremal submanifolds). Let $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$ be a connected real analytic submanifold. Assume that $\mathcal{M}$ is not contained in any of the pencils $\mathcal{P}_{W,r}$, where W, r range over all non-zero linear subspaces $W \leq \mathbb{R}^{m+n}$ and non-negative integers r such that (2) holds. Then $\mathcal{M}$ is extremal, i.e. $\beta(M) = \frac{n}{m}$ for Lebesgue almost every $M \in \mathcal{M}$.*

4.1. Non-extremal submanifolds

A general result of Kleinbock [10] implies that the Diophantine exponent of a random point of $\mathcal{M}$ is always well defined. Namely there is $\beta = \beta(\mathcal{M}) \in [0, +\infty]$ such that for Lebesgue almost every $x \in U$,

$$\beta\bigl(\mathbf{f}(x)\bigr) = \beta(\mathcal{M}).$$

Our first result is a general upper bound:

Theorem 4.2 *(Upper bound on the exponent). Let $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$ be an analytic submanifold as defined above. Then*

$$\beta(\mathcal{M}) \leq \max\left\{\frac{\dim W}{r} - 1;\, \mathcal{P}_{W,r} \supset \mathcal{M}\right\}.$$

Of course Theorem 4.1 is an immediate consequence of this bound.

In [8,9] Kleinbock showed that the Diophantine exponent of an analytic submanifold of $\mathbb{R}^n$ depends only on its linear span. Our next result is a matrix analogue of this fact. Note that the Diophantine exponent of a matrix M depends only on its kernel $\ker M$. As M varies in the submanifold $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$, consider the set of these kernels as a subset of the Grassmannian and take its linear span in the Plücker embedding. Denote by $\mathcal{H}(\mathcal{M})$ the matrices M whose kernel lies in this linear span. The set $\mathcal{H}(\mathcal{M})$ is an algebraic subvariety containing $\mathcal{M}$ and contained in every pencil containing $\mathcal{M}$.

Theorem 4.3 *(Optimality of the exponent). We have:*

$$\beta(\mathcal{M}) = \beta\bigl(\mathcal{H}(\mathcal{M})\bigr).$$

In particular $\beta(\mathcal{M}) = \beta(\mathrm{Zar}(\mathcal{M}))$, where $\mathrm{Zar}(\mathcal{M})$ denotes the Zariski closure of $\mathcal{M}$, and $\beta(\mathcal{M}) = \beta(\Omega)$ for any open subset $\Omega \subset \mathcal{M}$.

In particular $\mathcal{M}$ is extremal if and only if $\mathcal{H}(\mathcal{M})$ is extremal.

5. Lower bounds on the exponent and rationality

Theorem 4.2 gives a general upper bound on the exponent. The pigeonhole argument described at the beginning of Section 4 yields a lower bound on $\beta(\mathcal{M})$ in terms of the exponents of the *rational obstructions* in which $\mathcal{M}$ is contained, i.e. the pencils $\mathcal{P}_{W,r}$ with W a rational subspace of $\mathbb{R}^{m+n}$. Hence, for a general analytic submanifold $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$, we only have the following general upper and lower bound:

$$\max_{\mathcal{P}_{W,r}\supset\mathcal{M},W \text{ rational}} \frac{\dim W - r}{r} \le \beta(\mathcal{M}) \le \max_{\mathcal{P}_{W,r}\supset\mathcal{M}} \frac{\dim W - r}{r}. \tag{3}$$

For a submanifold $\mathcal{M}$ in general position, the upper and lower bounds are typically distinct. However, we will prove Theorem 5.1.

Theorem 5.1 *(Subvarieties defined over $\mathbb{Q}$). Assume that the Zariski-closure of the connected real analytic submanifold $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$ is defined over $\mathbb{Q}$. Then the upper and lower bounds in (3) coincide, and hence are equal to $\beta(\mathcal{M})$. In particular, $\beta(\mathcal{M}) \in \mathbb{Q}$.*

The proof of Theorem 5.1 is based on the following combinatorial lemma, which is used here with $G = \mathrm{Gal}(\mathbb{C}|\mathbb{Q})$ and will be used once again later on in the applications to nilpotent groups with $G = \mathrm{GL}_k$.

Let V be a finite dimensional vector space over a field and $\phi : \mathrm{Grass}(V) \to \mathbb{N} \cup \{0\}$ a function on the Grassmannian, which is non-decreasing (for set inclusion) and *submodular* in the sense that for every two subspaces W_1 and W_2, we have

$$\phi(W_1 + W_2) + \phi(W_1 \cap W_2) \le \phi(W_1) + \phi(W_2).$$

Lemma 5.2 *(Submodularity lemma). Let G be a group acting by linear automorphisms on V. If ϕ is invariant under G, then the following minimum is attained on a G-invariant subspace*

$$\min_{W\in\mathrm{Grass}(V)\setminus\{0\}} \frac{\phi(W)}{\dim W}.$$

6. Diophantine approximation on Lie groups

Inspired by works of Gamburd–Jakobson–Sarnak [6] and Bourgain–Gamburd [3] on the spectral gap problem for finitely generated subgroups of compact Lie groups, we defined in a previous article [1] the notion of Diophantine subgroup of an arbitrary Lie group G. The definition is as follows. Any finite symmetric subset $S := \{1, s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ in G generates a subgroup $\Gamma \le G$. If for all $n \in \mathbb{N}$

$$\inf\{d(1,\gamma);\, \gamma \in S^n \setminus \{1\}\} > \frac{1}{|S^n|^\beta},$$

then we say that (Γ, S) is *β-Diophantine*. And we say that Γ is *Diophantine* if it is β-Diophantine for some finite β. Here $d(\cdot,\cdot)$ denotes a fixed Riemannian metric on G and $|S^n|$ is the cardinality of the n-th product set $S^n := S \cdot \ldots \cdot S$. It is easily seen that being Diophantine does not depend on the choice of S or $d(\cdot,\cdot)$. And if G is nilpotent, this is also true of being β-Diophantine.

The connected Lie group G is said to be *Diophantine on k letters* if for almost every choice of k group elements $s_1, \dots, s_k$ chosen independently with respect to the Haar measure, the subgroup they generate is Diophantine. Finally, one says that G is *Diophantine* if it is Diophantine on k letters for every integer k.

While it is conjectured that all semisimple Lie groups are Diophantine, there are examples of non-Diophantine Lie groups. Indeed a construction was given in [1] for each integer $k \in \mathbb{N}$ of a connected Lie group that is Diophantine on k letters, but not on $k+1$ letters. Our examples are certain nilpotent Lie groups without a rational structure. We showed in that paper that the first examples arise in nilpotency class 6 and higher. In fact, every nilpotent Lie group G with nilpotency class at most 5, or derived length at most 2 (i.e. metabelian), is Diophantine.

7. Diophantine exponent of nilpotent Lie groups

If G is nilpotent, $|S^n|$ grows like n^{α_S}, where α_S is an integer given by the Bass–Guivarc'h formula. If the k elements s_i's forming S are chosen at random with respect to Haar measure, then α_S is almost surely a fixed integer, which is a polynomial in k (see [1]).

Proposition 7.1 *(Zero-one law). Let G be a simply connected nilpotent Lie group, and pick an integer $k \ge \dim G/[G,G]$. There is a number $\beta_k \in [0, +\infty]$, such that if $\beta > \beta_k$ (resp. $\beta < \beta_k$), then with respect to Haar measure almost every (resp. almost no) k-tuple in G generates a β-Diophantine subgroup.*

The proof of this is based on the ergodicity of the group of rational automorphisms of the free Lie algebra on k letters acting on $(\mathrm{Lie}(G))^k$. When the nilpotent Lie group G is rational (i.e. admits a $\mathbb{Q}$-structure) the exponent β_k can be computed explicitly using Theorem 5.1. We have the following.

Theorem 7.2 *(A formula for the exponent). Assume that G is a rational simply connected nilpotent Lie group. There is a rational function $F \in \mathbb{Q}(X)$ with coefficients in $\mathbb{Q}$ such that for all large enough k,*

$$\beta_k = F(k).$$

In particular $\beta_k \in \mathbb{Q}$. When $k \to \infty$, β_k converges to a limit β_∞ with $0 < \beta_\infty \leq 1$.

For example, if G is the $(2m+1)$-dimensional Heisenberg group and $k \geq 2m$, then $\beta_k = 1 - \frac{1}{k} - \frac{2}{k^2}$. More generally, if G is any 2-step nilpotent group not necessarily rational, then $\beta_k = (1 - \frac{1}{k})\frac{1}{\dim[G,G]} - \frac{2}{k^2}$ for $k \geq \dim G/[G,G]$.

We also obtain closed formulas for β_k in the case when G is the group of $n \times n$ unipotent upper-triangular matrices, e.g. if $n = 4$, and $k \geq 3$, then $\beta_k = \frac{k^3-k-3}{k^3+k^2-k}$. And in the case when G is an s-step free nilpotent group on m generators, e.g., if $m = 2$ and $s = 3$, then $\beta_k = \frac{k^3-k-6}{2(k^3+k^2-k)}$. These formulas involve the dimensions of the maximal (for the natural partial order on Young diagrams) irreducible GL_k-submodule of the free Lie algebra on k generators modulo the ideals of laws of G.

The reduction to Theorem 5.1 proceeds as follows. Since k is large, one can restrict attention to the last term $G^{(s)}$ in the central descending series. Given a $\mathbb{Z}$-basis $e_1, \ldots, e_{m+n}$ of the s-homogeneous part of the relatively free Lie algebra of G on k generators $\mathcal{F}_{k,G}$ (see [1]), the submanifold $\mathcal{M}_{k,G}$ of matrices to be considered is the image of $(\mathrm{Lie}(G))^k$ under the (polynomial) map sending $\mathsf{x} \in (\mathrm{Lie}(G))^k$ to the $(n+m) \times m$ matrix whose columns are the $e_i(\mathsf{x})$. Here $m = \dim G^{(s)}$. Computing the exponent amounts to first identify the pencils $\mathcal{P}_{W,r}$ in which $\mathcal{M}_{k,G}$ sits and then compute the maximum of the ratios $\frac{\dim W}{r}$. Using the submodularity lemma (Lemma 5.2) applied for the GL_k action of linear substitutions, we may restrict attention to those pencils corresponding to subspaces W of $\mathcal{F}_{k,G}$ that are fully invariant ideals. Determining those ideals is usually possible, depending on G, thanks to the known representation theory of the free Lie algebra viewed as a GL_k-module.

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