Combinatorics/Topology

Connectivity of pseudomanifold graphs from an algebraic point of view

Connexité des graphes de pseudo-variétés d'un point de vue algébrique

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A R T I C L E I N F O

A B S T R A C T

The connectivity of graphs of simplicial and polytopal complexes is a classical subject going back at least to Steinitz, and the topic has since been studied by many authors, including Balinski, Barnette, Athanasiadis, and Björner. In this note, we provide a unifying approach that allows us to obtain more general results. Moreover, we provide a relation to commutative algebra by relating connectivity problems to graded Betti numbers of the associated Stanley–Reisner rings.

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R É S U M É

La connexité des graphes des complexes simpliciaux et polytopaux est un sujet classique remontant au moins à Steinitz. Il a été étudié depuis par de nombreux auteurs, dont Balinski, Barnette, Athanasiadis et Björner. Dans cette note, nous présentons une approche unifiée nous permettant d'obtenir des résultats plus généraux. De plus, nous faisons un lien avec l'algèbre commutative en rapprochant les problèmes de connexité des nombres de Betti gradués des anneaux de Stanley–Reisner associés.

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1. Introduction

A graph G is said to be *k-connected* if it has at least k vertices and removing any subset of vertices of cardinality less than k results in a connected graph. The *connectivity number* κ_G of G is the maximum number k such that G is k-connected.

The classical Steinitz's theorem [8] (see also [9, Lecture 4]) asserts that a graph G is the underlying graph (1-skeleton) of a 3-polytope if and only if G is 3-connected and planar. In 1961, Balinski extended the "only if" direction of Steinitz's theorem

by showing that the underlying graph of a d-polytope is d-connected, cf. [9, p. 95]. David Barnette showed that the same bound is also valid for the connectivity number of underlying graphs of $(d-1)$-dimensional simplicial pseudomanifolds [2].

Athanasiadis [1] showed that if the pseudomanifold is a flag simplicial complex (i.e. the clique complex of its 1-skeleton), then this lower bound can be improved to $2d-2$. Björner and Vorwerk quantified this connection using the notion of banner simplicial complexes [4].

The purpose of this note is to provide a unifying approach that allows us to obtain more general results in the simplicial case.

2. Basics in commutative algebra

Our aim in this section is to relate global properties of a simplicial complex to the connectivity number of its underlying graph using the Hochster formula from commutative algebra. We start by recalling some notions, and refer to [7,6] for exact definitions and more details.

Let Δ be a simplicial complex on the vertex set $[n] := \{1, \ldots, n\}$. Let $\Bbbk$ be a field and $S = \Bbbk[x_1, \ldots, x_n]$ the polynomial ring in n variables over $\Bbbk$. The *Stanley–Reisner ideal* $I_\Delta \subset S$ of Δ is the ideal generated by monomials $\mathbf{x}_F := \prod_{i \in F} x_i$ for all F not in Δ. The quotient ring $\Bbbk[\Delta] = S/I_\Delta$ is called the *face ring* of Δ. Let

$$\mathbf{F}_{\Bbbk[\Delta]} := 0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to \Bbbk[\Delta] \to 0,$$

be the minimal graded free resolution of $\Bbbk[\Delta]$, with $F_i = \bigoplus_j S(-j)^{b_{i,j}}$ in homological degree i. The number $b_{i,j} = b_{i,j}(\Bbbk[\Delta])$ is the *graded Betti number* of S/I in *homological degree* i and *internal degree* j. The length of the j-th row in the Betti table will be denoted by $lp_j(\Bbbk[\Delta])$, that is

$$lp_j(\Bbbk[\Delta]) := \max\{i \mid b_{i,i+j-1}(\Bbbk[\Delta]) \neq 0\}.$$

We also denote by $t_i(\Bbbk[\Delta])$ the *maximum internal degree* of a minimal generator in the homological degree i that is $\max\{j \mid b_{i,j} \neq 0\}$. The *projective dimension* of $\Bbbk[\Delta]$ is the maximum i such that $b_{i,j} \neq 0$, for some j. The *regularity* of $\Bbbk[\Delta]$ is defined to be $\max_i\{t_i(\Bbbk[\Delta]) - i\}$.

Definition 1. Let Δ be a simplicial complex such that $\Bbbk[\Delta]$ has regularity r. Set $m = lp_{r+1}(\Bbbk[\Delta])$. We say S/I satisfies the property $\mathfrak{A}$ if

(1) $lp_2(\Bbbk[\Delta]) \leq m$,
(2) $b_{m-i,m-i+1}(\Bbbk[\Delta]) \leq b_{i,i+r-1}(\Bbbk[\Delta])$, for all $0 \leq i \leq m$.

We also say that $\Bbbk[\Delta]$ satisfies the *property* $\mathfrak{B}_s$ if for all $i < s$ one has $t_i(\Bbbk[\Delta]) < r + i - 1$.

Remark 2 (*Relations to Poincaré duality and the flag property*).

(1) If Δ is Gorenstein, then it satisfies the property $\mathfrak{A}$. However, the property only requires a much simpler property than Poincaré–Lefschetz duality; a simple inequality shall be enough, see Lemma 7.
(2) If Δ is flag, then it is easy to see that it $t_s(\Bbbk[\Delta]) \leq 2s$ for all s and therefore $\Bbbk[\Delta]$ satisfies $\mathfrak{B}_{r-1}$.

Proposition 3. *Let Δ be a simplicial complex such that $\Bbbk[\Delta]$ has regularity r. Moreover, assume that $\Bbbk[\Delta]$ satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_s$. Then one has*

$$s \leq lp_{r+1}(\Bbbk[\Delta]) - lp_2(\Bbbk[\Delta]).$$

Proof. We have

$$
\begin{aligned}
lp_{r+1}(\Bbbk[\Delta]) - lp_2(\Bbbk[\Delta]) &= m - \max\{j \mid b_{j,j+1}(\Bbbk[\Delta]) \neq 0\} \\
&= \min\{m - j \mid b_{j,j+1}(\Bbbk[\Delta]) \neq 0\} \\
&\geq \min\{m - j \mid b_{m-j,m-j+r-1}(\Bbbk[\Delta]) \neq 0\} \qquad \text{(Property } \mathfrak{A}) \\
&= \min\{k \mid b_{k,k+r-1}(\Bbbk[\Delta]) \neq 0\} \\
&\geq \min\{k \mid t_k(\Bbbk[\Delta]) \geq k + r - 1\}
\end{aligned}
$$

where the last term is at least s by Property $\mathfrak{B}_s$. $\square$

The following observation relates our study to the connectivity number.

Observation 4. Let Δ be a simplicial complex on the vertex set $[n]$ and κ be the connectivity number of its underlying graph. Then one has

$$\kappa + lp_2(\Bbbk[\Delta]) = n - 1.$$

Proof. We consider (throughout) homology to be computed with $\Bbbk$-coefficients. By Hochster's formula [6, Theorem 8.1.1], it suffices to observe that

$$\kappa = n - \max\left\{\#W : \widetilde{H}_0(X_W) \neq 0\right\} = n - 1 - lp_2(\Bbbk[\Delta]).$$

Alternatively (and algebraically) it suffices to observe that Δ and $\mathrm{Cl}(G)$, the clique complex of the underlying graph G of Δ, both have the same connectivity. Hence, it suffices to verify the result in the case of flag complexes and we may assume that $\Delta = \mathrm{Cl}(G)$. The result follows from [5, Theorem 3.1]. □

Theorem 5. *Let Δ be a $(d-1)$-dimensional simplicial complex with nontrivial top-homology. Also, assume that $\Bbbk[\Delta]$ satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_s$. Then the underlying graph is $(d+s-1)$-connected.*

Proof. Note that the regularity of $\Bbbk[\Delta]$ is equal to d since Δ has nontrivial top-homology. So, it follows from Proposition 3 that

$$s \leq lp_{d+1}(\Bbbk[\Delta]) - lp_2(\Bbbk[\Delta]).$$

Note that $lp_{d+1}(\Bbbk[\Delta]) = n - d$. So, by Observation 4 we get

$$s \leq n - d - (n - \kappa_\Delta - 1),$$

where κ_Δ stands for the connectivity number of the underlying graph of Δ. Therefore

$$\kappa_\Delta \geq d + s - 1. \qquad \square$$

Remark 6. As a special case of Theorem 5, we can consider Δ to be Gorenstein*. Then $\Bbbk[\Delta]$ satisfies the property $\mathfrak{B}_1$. Moreover, if Δ is also flag, then it satisfies the property $\mathfrak{B}_{d-1}$, since $t_i(\Bbbk[\Delta]) \leq 2i$ for any i and $\Bbbk[\Delta]$ has regularity d.

3. A Poincaré–Lefschetz-type inequality for minimal cycles

Recall that a *minimal d-cycle* Σ (w.r.t. a coefficient ring R) is a pure d-dimensional complex that supports precisely one homology d-class ζ whose support is the complex itself. For instance, every pseudomanifold is a minimal cycle (over $\mathbb{Z}/2\mathbb{Z}$); and so is every triangulation of a closed, connected manifold.

Lemma 7. *Let Σ denote any minimal d-cycle and W a subset of the vertex-set $\mathrm{V}(\Sigma)$. Then*

$$\dim \widetilde{H}_0(\Sigma_W) \leq \dim \widetilde{H}_{d-1}(\Sigma_{V(\Sigma)\setminus W}).$$

Proof. Since Σ supports a global d-cycle (by minimality), we have an injection

$$H_0(\Sigma_W) \hookrightarrow H_d(\Sigma, \Sigma \setminus \Sigma_W).$$

To see this, notice that the restriction $\widetilde{\zeta}$ of the global d-cycle ζ to any component of $(\Sigma, \Sigma \setminus \Sigma_W)$ is a relative cycle for $(\Sigma, \Sigma \setminus \Sigma_W)$. Since Σ is a minimal d-cycle, this relative cycle is not a boundary.

Now, notice that $\Sigma \setminus \Sigma_W$ admits an ambient deformation retraction in Σ to $\Sigma_{V(\Sigma)\setminus W}$ (cf. [3, Lemma 4.27]). In particular, $H_i(\Sigma, \Sigma \setminus \Sigma_W) \cong H_i(\Sigma, \Sigma_{V(\Sigma)\setminus W})$ for all $i \in \mathbb{N}$. Finally, the exact sequence

$$0 \longrightarrow H_d(\Sigma) \longrightarrow H_d(\Sigma, \Sigma_{V(\Sigma)\setminus W}) \longrightarrow H_{d-1}(\Sigma_{V(\Sigma)\setminus W}) \longrightarrow \cdots$$

implies

$$\dim H_{d-1}(\Sigma_{V(\Sigma)\setminus W}) + \dim \widetilde{H}_d(\Sigma)$$
$$\geq \dim H_d(\Sigma, \Sigma_{V(\Sigma)\setminus W})$$
$$\geq \dim H_0(\Sigma_W).$$

Recalling that $\dim \widetilde{H}_d(\Sigma) = 1$ finishes the proof. □

4. Applications to connectivity of pseudomanifold graphs

Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set $V(\Delta)$. Recall the notion of banner complexes of [4]:

○ a subset W of $V(\Delta)$ is called *complete* if every two vertices of W form an edge of Δ;
○ a complete set $W \subseteq V(\Delta)$ is *critical* if $W \setminus \{v\}$ is a face of Δ for some $v \in W$;
○ we say that Δ is *banner* if every critical complete set W of size at least d is a face of Δ;
○ we define the *banner number* of Δ to be

$$b(\Delta) = \min \left\{ b \quad : \quad \begin{array}{l} lk_\sigma \Delta \text{ is banner or the boundary of the 2-simplex} \\ \text{for all faces } \sigma \in \Delta \text{ of cardinality } b \text{ and degree } d \end{array} \right\},$$

where the *degree* of a face is the maximal cardinality of a facet containing it.

Note that our notions of banner complexes and banner numbers are slightly more general then the ones introduced in [4]. However, if the complex is pure, the definitions coincide.

Lemma 8. *Let Δ be a $(d-1)$-dimensional simplicial complex.*

(a) *If σ is a face of degree d in Δ, then $b(lk_\sigma \Delta) \leq \max\{0, b(\Delta) - \#\sigma\}$.*
(b) *If Δ has nontrivial top-homology and $b(\Delta) < d - 2$, then every induced subcomplex of Δ having nontrivial $(d-2)$-homology has at least $2d - 2 - b(\Delta)$ vertices.*

Proof. The part (a) is clear from the definition. For claim (b), let us first show that, if Δ is banner, then every induced subcomplex Γ of Δ such that $\widetilde{H}_{d-2}(\Gamma) \neq 0$ has at least $2d - 2$ vertices by induction on d. If $d = 3$, this is clear because in this situation Δ must be flag.

Let $d > 3$. We may assume that no induced subcomplex of Δ has a nontrivial $(d-1)$-dimensional cycle: indeed, such a subcomplex is forced to have dimension $d - 1$, so it would be banner and we could replace Δ with it. Furthermore, we may assume that Γ is a minimal induced subcomplex with the property that $\widetilde{H}_{d-2}(\Gamma) \neq 0$. Under such a minimality assumption, for any vertex u of Γ, it follows from the exact sequence

$$\widetilde{H}_{d-2}(\Gamma_{V(\Gamma)\setminus u}) \to \widetilde{H}_{d-2}(\Gamma) \to \widetilde{H}_{d-3}(lk_u \Gamma)$$

that the link of u in Γ admits a nontrivial homology cycle in dimension $d - 3$. Also, it can be easily seen (from the banner property of Δ) that the 1-skeleton of Γ is not a complete graph. So, we may take a vertex v of Γ such that the vertex set of $lk_v \Gamma$ is a proper subset of $V(\Gamma) \setminus u$. Since $lk_v \Gamma$ is an induced subcomplex of $lk_v \Delta$, which is banner and admits a nontrivial $(d-3)$-cycle, by induction $lk_v \Gamma$ has at least $2d - 4$ vertices. And the conclusion follows in the banner case.

The claim (b) now follows by induction on the banner number and claim (a). $\square$

Remark 9. While a flag simplicial complex (not necessarily of dimension $d - 1$) supporting a nontrivial $(d-1)$-cycle has at least $2d$ vertices, this is false for banner complexes. Take the boundary of a d-simplex, and join one facet with an external edge: the resulting complex is a $(d+1)$-dimensional banner complex supporting a nontrivial $(d-1)$-cycle, but with only $d + 3$ vertices.

Lemma 10. *Let Δ be a pure $(d-1)$-dimensional complex with nontrivial top-homology. Then $\Bbbk[\Delta]$ satisfies the property $\mathfrak{B}_{d-b(\Delta)-1}$.*

Proof. By Hochster's formula [6, Theorem 8.1.1] we have that $\mathrm{reg}(\Bbbk[\Delta]) = d$, since Δ has a nontrivial top-homology. If $b_{i,i+d-1}(\Bbbk[\Delta]) \neq 0$, then there must exist a subset $W \subseteq V(\Delta)$ of cardinality $i + d - 1$ such that Δ_W supports a nontrivial $(d-2)$-cycle. By part (b) of Lemma 8, thus:

$$i \geq d - b(\Delta) - 1. \qquad \square$$

Theorem 11. *Let Δ be a $(d-1)$-dimensional minimal cycle. Then the underlying graph of Δ is $(2d - b(\Delta) - 2)$-connected.*

Proof. It follows from Hochster's formula that $b_{n-d,n}(\Bbbk[\Delta]) = \dim \widetilde{H}_{d-1}(\Delta) \neq 0$. Thus, one has $lp_{d+1}(\Bbbk[\Delta]) = n - d$. On the other hand, for a subset W of the vertices of Δ from Lemma 7 we have

$$\dim \widetilde{H}^0(\Delta_{V(\Delta)\setminus W}) \leq \dim \widetilde{H}_{d-2}(\Delta_W).$$

Now, summing over all subsets W of cardinality $i + d - 1$, again by Hochster's formula we get

$$b_{n-i-d,n-i-d+1}(\Bbbk[\Delta]) \leq b_{i,i+d-1}(\Bbbk[\Delta]).$$

Hence, $\Bbbk[\Delta]$ satisfies the property $\mathfrak{A}$. Now, since $\Bbbk[\Delta]$ satisfies the property $\mathfrak{B}_{d-b(\Delta)-1}$, Lemma 10 above, the result follows from Theorem 5. $\square$

Corollary 12. *Let Δ be a flag (or more generally banner) $(d-1)$-dimensional minimal cycle. Then the underlying graph of Δ is $(2d-2)$-connected.*

Proof. If Δ is a banner complex, then $b(\Delta) = 0$. □

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