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Complex analysis

Some properties related to a certain class of starlike functions

Quelques propriétés liées à une classe de fonctions étoilées

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ABSTRACT

This paper considers a class Δ^* of normalized starlike functions f analytic in the open unit disk $|z|<1$ satisfying the inequality that

$$\left|\left\{\frac{zf'(z)}{f(z)}\right\}^2-1\right|<2\left|\frac{zf'(z)}{f(z)}\right|$$

in $|z|<1$. We first show that the class $\mathcal{S}^*(q)$ (defined below) is a subclass of Δ^* and then obtain some useful properties of these classes of functions.

RÉSUMÉ

Nous considérons dans cette Note une classe Δ^* de fonctions étoilées normalisées f, analytiques dans le disque unité ouvert $|z|<1$ et y satisfaisant l'inégalité

$$\left|\left\{\frac{zf'(z)}{f(z)}\right\}^2-1\right|<2\left|\frac{zf'(z)}{f(z)}\right|.$$

Nous montrons d'abord que la classe $S^*(q)$ (définie ci-dessous) est une sous-classe de Δ^*, puis nous obtenons quelques propriétés utiles de ces classes de fonctions.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U}=\{z: \ |z|<1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprised of functions f normalized by $f(0)=0$, $f'(0)=1$, and let $\mathcal{S}\subset\mathcal{A}$ denote the class of functions that are univalent in $\mathbb{U}$. Let a function f be analytic univalent in the unit disc $\mathbb{U}=\{z: \ |z|<1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0)=0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0=0$ if and only if

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$$\Re\mathfrak{e}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \quad (z\in\mathbb{U}). \tag{1.1}$$

It is well known that if an analytic function f satisfies (1.1) and $f(0)=0$, $f'(0)\neq 0$, then f is univalent and starlike in $\mathbb{U}$. The set of all functions $f\in\mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$. We say that an analytic function f is subordinate to an analytic function g, and write $f(z)\prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0)=0$, $|\omega(z)|<1$ for $|z|<1$ and $f(z)=g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z)\prec g(z) \iff f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}). \tag{1.2}$$

We will need the following basic lemma in the theory of differential subordinations.

Lemma 1.1. *(See [9], see also [10, p. 24].) Assume that $\mathcal{Q}$ is the set of analytic functions that are injective on $\overline{\mathbb{U}}\setminus E(f)$, where $E(f):=\{\zeta:\zeta\in\partial\mathbb{U} \text{ and } \lim_{z\to\zeta}f(z)=\infty\}$, and are such that $f'(\zeta)\neq 0$ $(\zeta\in\partial(\mathbb{U})\setminus E(f))$. Let $\psi\in\mathcal{Q}$ with $q(0)=a$ and let $\varphi(z)=a+a_mz^m+\cdots$ be analytic in $\mathbb{U}$ with $\varphi(z)\not\equiv a$ and $m\in\mathbb{N}$. If $\varphi\not\prec\psi$ in $\mathbb{U}$, then there exist points $z_0=r_0e^{i\theta}\in\mathbb{U}$ and $\zeta_0\in\partial\mathbb{U}\setminus E(\psi)$, for which $\varphi(|z|<r_0)\subset\psi(\mathbb{U})$, $\varphi(z_0)=\psi(\zeta_0)$ and $z_0\varphi'(z_0)=s\zeta_0\psi'(\zeta_0)$, for some $s\geq m$.*

2. Main result

Let Δ^* be defined by

$$\Delta^*=\left\{f\in\mathcal{S}^*:\left|\left\{\frac{zf'(z)}{f(z)}\right\}^2-1\right|<2\left|\frac{zf'(z)}{f(z)}\right|, z\in\mathbb{U}\right\}. \tag{2.1}$$

We prove the following theorem, which would be used to obtain an equivalent class of Δ^*.

Theorem 2.1. *If $p(z)\in\mathcal{H}$ with $p(0)=1$, then*

$$p(z)\prec q(z):=z+\sqrt{1+z^2},\quad q(0)=1, \tag{2.2}$$

implies that $\Re\mathfrak{e}\{p(z)\}>0$ and

$$\left|p^2(z)-1\right|<2|p(z)|,\quad z\in\mathbb{U}. \tag{2.3}$$

Proof. We first show that $q(z)$ is univalent in $\mathbb{U}$. Assume that $q(z_1)=q(z_2)$, for some $z_1,z_2\in\mathbb{U}$, then

$$z_1-z_2=\sqrt{1+z_2^2}-\sqrt{1+z_1^2}. \tag{2.4}$$

Upon squaring (2.4), we get

$$1+z_1z_2=\sqrt{1+z_2^2}\sqrt{1+z_1^2}. \tag{2.5}$$

Again squaring (2.5), we are easily lead to $z_1=z_2$, and hence, $q(z)$ is univalent in $\mathbb{U}$. Evidently, then by virtue of (1.2), the subordination (2.2) is equivalent to

$$p(\mathbb{U})\subset q(\mathbb{U}). \tag{2.6}$$

In order to prove that $\Re\mathfrak{e}\{p(z)\}>0$, $z\in\mathbb{U}$, it suffices to show that $\Re\mathfrak{e}\left\{q(e^{it})\right\}\geq 0$, $t\in[0,2\pi)$. Let $z=e^{it}$, $t\in[0,2\pi)$, then

$$e^{it}+\sqrt{e^{2it}+1}=\begin{cases}\cos t+i\sin t+\sqrt{2\cos t}(\cos t/2+i\sin t/2) & \text{for } t\in[0,\pi/2),\\ i & \text{for } t=\pi/2,\\ \cos t+i\sin t+\sqrt{|2\cos t|}(\sin t/2-i\cos t/2) & \text{for } t\in(\pi/2,3\pi/2),\\ -i & \text{for } t=3\pi/2,\\ \cos t+i\sin t+\sqrt{2\cos t}(-\cos t/2-i\sin t/2) & \text{for } t\in(3\pi/2,2\pi).\end{cases} \tag{2.7}$$

Now, some simple calculations show that $\Re\mathfrak{e}\left\{e^{it}+\sqrt{e^{2it}+1}\right\}=0$, if and only if $t=\pi/2$, or if $t=3\pi/2$, which implies that $\Re\mathfrak{e}\{q(z)\}>0$ in $\mathbb{U}$ (see Fig. 1). From (2.7), we can also find that $q(e^{it})$ is a union of two circular arcs: smaller arc $|w+1|=\sqrt{2}$ and greater arc $|w-1|=\sqrt{2}$.

Finally, we will prove (2.3). It follows from (2.2) that

$$(p(z)-w(z))^2=1+w^2(z) \tag{2.8}$$

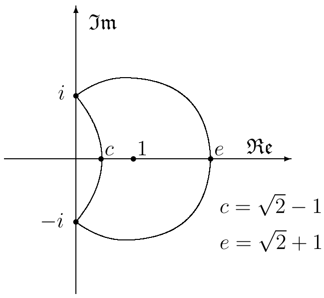

Fig. 1. $q(e^{it})$.

for some analytic function $w(z)$ such that

$$|w(z)| < 1, \ \ z \in \mathbb{U}, \text{ and } \ w(0) = 0.$$

From (2.8), we readily get

$$p(z)^2 - 1 = 2p(z)w(z), \ \ |w(z)| < 1,$$

which establishes (2.3) and this completes the proof of Theorem 2.1. □

Corollary 2.2. *Let $f \in \mathcal{A}$, then*

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z, \ \ z \in \mathbb{U} \tag{2.9}$$

implies that $f \in \mathcal{S}^*$ and

$$\left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right|, \ \ z \in \mathbb{U}. \tag{2.10}$$

Proof. Corollary 2.2 follows at once if we put

$$p(z) = \frac{zf'(z)}{f(z)}, \ \ z \in \mathbb{U}$$

in Theorem 2.1. □

We recall here the class $\mathcal{S}^*(q)$, which was recently introduced by the authors in [11] as follows:

Definition 2.3. Let $\mathcal{S}^*(q)$ denote the class of functions f analytic in $\mathbb{U}3$ normalized by $f(0) = f'(0) - 1 = 0$, and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z = q(z), \ \ z \in \mathbb{U}, \tag{2.11}$$

where the branch of the square root is chosen to be $q(0) = 1$.

In view of Corollary 2.2, $\mathcal{S}^*(q)$ is a subclass of Δ^* defined by (2.1). If we interpret the condition (2.10) geometrically, then we observe that the product of the distances of $zf'(z)/f(z)$ from the foci -1 and 1 is less than twice the distance of $zf'(z)/f(z)$ from the origin. The shape of the domain for $zf'(z)/f(z)$ will be described in Theorem 2.4 below and the shape of $q(\mathbb{U})$ is already depicted above in Fig. 1. Note here that the function $w(z) = \sqrt{1+z}$ maps $\mathbb{U}$ onto a set bounded by Bernoulli lemniscate, and the class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \sqrt{1+z}$ was considered in [13], while $zf'(z)/f(z) \prec \sqrt{1+cz}$ was considered in [1]. In [12], Rønning defined the class

$$\mathcal{S}_p = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \mathfrak{Re} \frac{zf'(z)}{f(z)}, \ z \in \mathbb{U} \right\}.$$

Interpreting this condition geometrically, we see that $zf'(z)/f(z)$ lies inside the parabola $(\mathfrak{Im} w)^2 < 2\mathfrak{Re} w - 1$, and this way the known class of k-starlike functions was seen to be connected with certain conic domains. For some recent results for k-starlike functions, we refer to [14]. In recent papers [2–5,8], certain function classes were considered, which were defined by $zf'(z)/f(z) \prec \widehat{q}(z)$, where $\widehat{q}(z)$ was not univalent, which made the consideration of geometric properties for such classes much more difficult.

Theorem 2.4. *If $f(z) \in \Delta^*$, then*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{U} \tag{2.12}$$

and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \sqrt{2} \quad \text{and} \quad \left|\frac{zf'(z)}{f(z)} + 1\right| > \sqrt{2}, \quad z \in \mathbb{U}. \tag{2.13}$$

Proof. In view of (1.1), the condition (2.12) follows at once, because $\Delta^* \subset \mathcal{S}^*$. Let $f(z) \in \Delta^*$, then $f \in \mathcal{S}^*$ and

$$\left|\left\{\frac{zf'(z)}{f(z)}\right\}^2 - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right|, z \in \mathbb{U}. \tag{2.14}$$

If we put

$$x + \mathrm{i}y = \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{U},$$

then we obtain $x > 0$, because $f(z) \in \mathcal{S}^*$. It easy to show that

$$\left|(x + \mathrm{i}y)^2 - 1\right| = 2|x + \mathrm{i}y|$$

yields

$$\left((x^2 - 1) + y^2\right)^2 = 4x^2. \tag{2.15}$$

Thus, we obtain

$$|(x^2 - 1) + y^2| = 2x, \tag{2.16}$$

and from (2.16), we at once have

$$(x - 1)^2 + y^2 = 2 \text{ or } (x + 1)^2 + y^2 = 2.$$

By noting that $x > 0$, (2.13) follows from (2.14). □

Interpreting the conditions (2.12) and (2.13) geometrically, we note that $zf'(z)/f(z)$ lies in the right half plane, inside the disc $|w - 1| < \sqrt{2}$, but is outside the disc $|w + 1| < \sqrt{2}$. We now prove the following result related to the subordination.

Theorem 2.5. *If $p(z) \in \mathcal{H}$ with $p(0) = 1$ satisfies*

$$\mathfrak{Re}\left\{\frac{zp'(z)}{p(z)}\right\}^2 < \frac{1}{2}, \quad z \in \mathbb{U}, \tag{2.17}$$

then

$$p(z) \prec q(z) = z + \sqrt{1 + z^2}, \quad z \in \mathbb{U}. \tag{2.18}$$

Proof. We want to prove that $p(z) \prec q(z)$. If $p(z) \not\prec q(z)$, then there exist points z_0, $|z_0| < 1$ and ζ_0, $|\zeta_0| = 1$, $\zeta_0 \neq 1$ for which

$$p(z_0) = q(\zeta_0), \quad p(|z| < |z_0|) \subset q(\mathbb{U}), \quad |\zeta_0| = 1.$$

In view of Lemma 1.1, it follows that there exists $k \geq 1$ such that

$$\left\{\frac{z_0 p'(z_0)}{p(z_0)}\right\}^2 = \left\{\frac{k\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right\}^2 = \left\{\frac{k\zeta_0}{\sqrt{1 + \zeta_0^2}}\right\}^2 = \frac{k^2\zeta_0^2}{1 + \zeta_0^2}.$$

For $|\zeta_0| = 1$, we have

$$\frac{\zeta_0^2}{1 + \zeta_0^2} = \frac{1}{2} + \mathrm{i}\,\frac{\tan(\arg \zeta_0)}{2}, \quad \text{which gives} \quad \mathfrak{Re}\left\{\frac{z_0 p'(z_0)}{p(z_0)}\right\}^2 = \frac{k^2}{2} \geq \frac{1}{2}.$$

But this contradicts our assumption (2.17), and therefore, $p(z) \prec q(z)$ in $\mathbb{U}$. □

Corollary 2.6. *If $f \in \mathcal{A}$ and*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\}^2 < \frac{1}{2}, \quad z \in \mathbb{U}, \tag{2.19}$$

then

$$f(z) \in \mathcal{S}^*(q). \tag{2.20}$$

Proof. If we put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{U},$$

then the condition (2.17) becomes (2.19). By Theorem 2.5, we have

$$p(z) = \frac{zf'(z)}{f(z)} \prec q(z) = z + \sqrt{1+z^2}, \quad z \in \mathbb{U},$$

which proves the assertion (2.20). □

Theorem 2.7. *If $f \in \mathcal{A}$ and*

$$\left|\frac{zf'(z)}{f(z)} - \sqrt{2}\right| < 1, \quad z \in \mathbb{U}, \tag{2.21}$$

then

$$f(z) \in \mathcal{S}^*(q). \tag{2.22}$$

Proof. The condition (2.21) implies that $zf'(z)/f(z)$ lies in a disc with center $\sqrt{2}$ and radius 1. To prove (2.22), it suffices to show that

$$\left|q(\mathrm{e}^{\mathrm{i}t}) - \sqrt{2}\right| \geq 1 \tag{2.23}$$

for $t \in [0, \pi]$, because $q(\mathbb{U})$ is symmetric with respect to real axis, see Fig. 1. For the case $t \in [0, \pi/2)$, we have by applying (2.7) (after some elementary calculations):

$$\begin{aligned}
&\left|q(\mathrm{e}^{\mathrm{i}t}) - \sqrt{2}\right|^2 \\
&\quad = 3 - 2(\sqrt{2}-1)\cos t - 2(\sqrt{2}-1)\sqrt{2}\sqrt{\cos t}\cos(t/2) \\
&\quad \geq 3 + 2 - 2\sqrt{2} - 2(\sqrt{2}-1)\sqrt{2} \\
&\quad = 1.
\end{aligned}$$

For $t = \pi/2$, (2.23) becomes $\left|\mathrm{i} - \sqrt{2}\right| \geq 1$. When $t \in (\pi/2, \pi]$, then by applying (2.7), we have

$$\begin{aligned}
&\left|q(\mathrm{e}^{\mathrm{i}t}) - \sqrt{2}\right|^2 \\
&\quad = 3 + 2(\sqrt{2}+1)|\cos t| - 2(\sqrt{2}+1)\sqrt{2|\cos t|}\sin(t/2) \\
&\quad = 3 + 2(\sqrt{2}+1)\left\{|\cos t| - \sqrt{2|\cos t|}\sin(t/2)\right\}.
\end{aligned} \tag{2.24}$$

We shall now find the smallest value of the last expression within the braces. If $t \in (\pi/2, \pi]$ and $\sin(t/2) = x$, then

$$x \in (\sqrt{2}/2, 1] \ \text{and} \ |\cos t| = 2x^2 - 1,$$

hence

$$|\cos t| - \sqrt{2|\cos t|}\sin(t/2) = 2x^2 - 1 - x\sqrt{2(2x^2-1)} := h(x), \quad x \in (\sqrt{2}/2, 1].$$

We have

$$h'(x) = \frac{\sqrt{2}(\sqrt{(4x^2-1)^2-1} - (4x^2-1))}{\sqrt{2x^2-1}} < 0, \quad x \in (\sqrt{2}/2, 1],$$

therefore, applying

$$\min_{x\in(\sqrt{2}/2,1]} h(x)=h(1)=1-\sqrt{2},$$

in (2.24), we obtain

$$\begin{aligned}&\left|q(\mathrm{e}^{\mathrm{i}t})-\sqrt{2}\right|^2\\&\quad\geq 3+2(\sqrt{2}+1)(1-\sqrt{2})\\&\quad=3-2=1.\end{aligned}$$

This completes the proof of (2.22). □

Let $\mathcal{S}^*(A,B)$ denote the class of functions $f\in\mathcal{A}$, $-1\leq B<A\leq 1$, satisfying the condition that

$$\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{U},$$

which was introduced and studied in [6] (see also [7]). By setting $A=\sqrt{2}-1$ and $B=1-\sqrt{2}$, then Theorem 2.7 becomes the following corollary.

Corollary 2.8. *If $f\in\mathcal{S}^*(\sqrt{2}-1,1-\sqrt{2})$, then $f(z)\in\mathcal{S}^*(q)$.*

Corollary 2.9. *Let $g(z)=z+az^n$, $n\geq 2$, $z\in\mathbb{U}$. Then $g\in\mathcal{S}^*(q)$ if and only if*

$$\begin{cases}|a|\leq\frac{2-\sqrt{2}}{n+1-\sqrt{2}} & \text{for } n\geq 4,\\ |a|\leq\frac{2-\sqrt{2}}{4-\sqrt{2}} & \text{or } 1<|a|\leq\sqrt{2}/(2-\sqrt{2}) \text{ for } n=3,\\ |a|\leq\frac{2-\sqrt{2}}{3-\sqrt{2}} & \text{or } 1<|a| \text{ for } n=2.\end{cases}\tag{2.25}$$

Proof. Since $g(z)=z+az^n$, it readily follows that

$$G(z):=\frac{zg'(z)}{g(z)}=\frac{1+n\,a\,z^{n-1}}{1+a\,z^{n-1}},n\geq 2,\quad z\in\mathbb{U}.$$

The function $G(z)$ maps the unit disc onto a disc symmetric with respect to the real axis. Therefore, on using (1.2), we conclude that the function $g\in\mathcal{S}^*(q)$ if and only if

$$\sqrt{2}-1\leq\frac{1-n|a|}{1-|a|}\ \text{and}\ \frac{1+n|a|}{1+|a|}\leq\sqrt{2}+1.$$

This establishes (2.25) and the proof is complete. □

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