Dynamical systems

Generalised Lüroth expansions and a family of Minkowski's question-mark functions

Développement de Lüroth généralisé et une famille de fonctions point d'interrogation de Minkowski

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A R T I C L E   I N F O

A B S T R A C T

Minkowski's question-mark function is a singular homeomorphism of the unit interval that maps the set of quadratic surds into the rationals. This function has deserved the attention of several authors since the beginning of the twentieth century. Using different representations of real numbers by infinite sequences of integers, called α-Lüroth expansions, we obtain different instances of the standard shift map on infinite symbols, all of them topologically conjugated to the Gauss Map. In this note we prove that each of these conjugations share properties with Minkowski's question-mark function: all of them are singular homeomorphisms of the interval, and in the "rational" cases, they map the set of quadratic surds into the set of rational numbers. In this sense, this family is a natural generalisation of Minkowski's question-mark function.

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R É S U M É

La fonction point d'interrogation de Minkowski est un homéomorphisme singulier de l'intervalle unité tel que l'image des irrationnels quadratiques est contenue dans les rationnels. Cette fonction a attiré l'attention de plusieurs auteurs depuis le début du XXe siècle. En utilisant différents développements des nombres réels avec des séquences infinies de nombres entiers, appelées développements α-Lüroth, on obtient différentes instances de l'opérateur de décalage sur l'espace des mots sur une infinité de symboles, tous topologiquement conjugués à l'application de Gauss. Dans cette note, nous prouvons que toutes ces conjugaisons ont des propriétés en commun avec la fonction point d'interrogation : elles sont toutes des homéomorphismes singuliers de l'intervalle et, dans les cas « rationnels », l'image des irrationnels quadratiques est contenue dans les rationnels. En ce sens, cette famille est une généralisation naturelle de la fonction point d'interrogation de Minkowski.

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A *quadratic surd* is an irrational solution of a quadratic polynomial with integer coefficients. Denote by $\mathbb{A}_2$ the set of quadratic surds in the unit interval $\mathcal{I} := [0, 1] \subset \mathbb{R}$, and denote by $\mathbb{Q}$ the set of rational numbers contained in $\mathcal{I}$. Minkowski, in [8], defined a function which is a one-to-one correspondence between the set $\mathbb{A}_2 \cup \mathbb{Q}$ and the set $\mathbb{Q}$. This function is called *Minkowski's question-mark function*, and is denoted by $? : \mathcal{I} \to \mathcal{I}$.

On the other hand, the function ? is a singular function, that is, its derivative exists and $?'(x) = 0$ for Lebesgue almost every $x \in \mathcal{I}$. Functions like Cantor maps, sometimes called *devil staircases*, are singular maps if they are defined using a Lebesgue measure zero Cantor set on the interval; however, these functions are constant on each connected component of the complement of such a Cantor set. The function ? is strictly increasing and singular, so it is often called a *slippery devil staircase*.

Minkowski's question-mark function is defined using Farey fractions and mediants: set $?(0/1) = 0$, $?(1/1) = 1$ and define it in the following way: if p/q and p'/q' are consecutive Farey rationals of order $n > 1$, set:

$$? \left(\frac{p + p'}{q + q'} \right) = \frac{?(p/q) + ?(p'/q')}{2}.$$

After Minkowski, several generalisations have been constructed, beginning with Denjoy and Salem; see [3,4,10]. Much more recently, Kessebohmer, Munday and Stratmann in [5], and Munday in [9], study this map from the point of view of dynamical systems, and of the relation with the generalised Lüroth expansion of numbers. In this note, we exploit further this relation to obtain a family of functions which are generalisations of Minkowski's question-mark function.

Let us recall the continued fraction expression of a number. Given $x \in \mathcal{I}$, the continued fraction expansion of x is a sequence of positive integers cf $(x) := [a_1, a_2, \ldots] \in \mathbb{N}^{\mathbb{N}}$ such that:

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

For rational numbers, this sequence is finite and is not unique. In fact, $[a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1]$. Conversely, if $x \in \mathcal{I} \setminus \mathbb{Q}$, then cf (x) is infinite and unique. Given $x \in \mathcal{I}$ such that cf $(x) = [a_1, a_2, \ldots]$, the formula given in [10] is:

$$?(x) = \frac{1}{2^{a_1 - 1}} - \frac{1}{2^{a_1 + a_2 - 1}} + \frac{1}{2^{a_1 + a_2 + a_3 - 1}} - \cdots. \tag{1}$$

Notice that for rational points $x \in \mathbb{Q}$, the function $?(x)$ is well defined and its value is rational, since the formula involves only a finite number of terms.

Generalised α-Lüroth expansions are representations of real numbers by a sequence of integers, depending on a partition of the interval. They where studied in [1] and they are a generalisation of the classical *Lüroth expansion* of [7]. In this note, we will restrict our attention to the class of α-Lüroth expansions that were introduced in [5] (see also [9]). Each of these representations depends on a countable partition of $\mathcal{I}$ defined by a sequence with the following properties: let $\alpha = \{t_j\}_{j \geq 1} \subset \mathcal{I}$ be a strictly decreasing sequence such that $t_1 = 1$, and $t_j \to 0$ when $j \to \infty$. This sequence defines a partition of $\mathcal{I}$ by non-trivial intervals:

$$\left\{ (t_{j+1}, t_j] \mid j \geq 1 \right\} \tag{2}$$

of length $t_j - t_{j+1} > 0$. In this way, for any $x \in \mathcal{I}$, there is a sequence of positive integers $\mathrm{L\ddot{u}r}_\alpha(x) := (a_n)_{n \geq 1}$ (finite or infinite) such that:

$$x = t_{a_1} + \sum_{n=2}^{\infty} (-1)^{n+1} \prod_{j=1}^{n-1} (t_{a_j} - t_{a_j+1}) t_{a_n}. \tag{3}$$

In this way, $\mathrm{L\ddot{u}r}_\alpha(x)$ is called the α-Lüroth expansion of x with respect to the given partition α. This expression can be found in [5] and [9]. Moreover, except for a countable set in $\mathcal{I}$, this expression is unique. It is not difficult to see that the formula in (1), can be derived from (3), setting $t_n = 2^{-(n-1)}$, for $n \geq 1$. On the other hand, if we set $t_n = 1/n$, we obtain the standard alternating Lüroth expansion of x.

We will show that, for any sequence α, there is a singular homeomorphism of the interval $?_\alpha : \mathcal{I} \to \mathcal{I}$. Moreover, if $\alpha \subset \mathbb{Q}$, then $?_\alpha$ maps quadratic surds into the set of rational numbers. In other words, to define $?_\alpha(x)$ for $x \in \mathcal{I}$, one needs to compute the continued fraction expression of x. The real number that has this sequence of integers as its generalised α-Lüroth expansion is $?_\alpha(x)$. The purpose of this note is to prove the following theorem, which states that these maps are generalisations of Minkowski's question-mark function.

Main Theorem. *Let α be a sequence as in* (2). *The function:*

$$?_\alpha(x) := t_{a_1} + \sum_{n=2}^{\infty} (-1)^{n+1} \prod_{j=1}^{n-1} (t_{a_j} - t_{a_j+1}) t_{a_n},$$

where cf $(x) = [a_1, a_2, \ldots]$, is a singular homeomorphism of $\mathcal{I}$. Moreover, if $\alpha \subset \mathbb{Q}$, then $?_\alpha(\mathbb{A}_2) \subset \mathbb{Q}$.

The proof of the fact that these functions are singular is a consequence of the fact that ergodic invariant measures for dynamical system either coincide or are mutually singular.

1. Proof of results

Let $\Sigma := \mathbb{N}^{\mathbb{N}}$ be the set of infinite sequences of positive integers with the product topology. Let $\sigma : \Sigma \to \Sigma$ be the shift map, that is $\sigma((a_1, a_2, \ldots)) = (a_2, a_3, \ldots)$. On the other hand, consider the Gauss map $G : \mathcal{I} \to \mathcal{I}$ defined by:

$$G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,$$

for $x \neq 0$, and $G(0) = 0$. It is a well-known fact that the set $\mathbb{A}_2$ corresponds to the set of pre-periodic points of G, $\text{Per}(G) := \{y \in \mathcal{I} \mid \exists n, m \in \mathbb{N}, \text{ such that } G^{n+m}(x) = G^n(x)\}$. Furthermore, the Gauss map is a dynamical factor of σ. In fact, the inverse map $\text{cf}^{-1} : \Sigma \to \mathcal{I}$ is a continuous and surjective map that satisfies $\sigma \circ \text{cf}^{-1} = \text{cf}^{-1} \circ G$. If we set the partition γ of $\mathcal{I}$, given by $I_n = ((n+1)^{-1}, n^{-1}]$, for $n \geq 1$, we have that if $\text{cf}(x) = (a_1, a_2, \ldots)$, then $G^k(x) \in I_{a_k}$, for any $k \geq 1$. That is, the continued fraction expression of a point x describes the itinerary x under the iterations of G, with respect to the partition γ.

The following class of maps, introduced in [5], are also factors of the shift map σ.

Definition 1.1. Given a sequence α as in (2), an α-Lüroth map is a piecewise linear map $L_\alpha : \mathcal{I} \to \mathcal{I}$ defined by $L_\alpha(0) = 0$ and:

$$L_\alpha(x) = \begin{cases} \dfrac{t_j - x}{t_j - t_{j+1}}, & \text{if } x \in [t_{j+1}, t_j], \ j \in \mathbb{N}. \end{cases} \tag{4}$$

For these maps, the α-Lüroth expansion of x codifies the itinerary of the orbit of x, under the iterations of L_α, with respect to the partition α. In fact, we have that $\text{Lür}_\alpha \circ \sigma = L_\alpha \circ \text{Lür}_\alpha$. Precisely, if $\text{Lür}_\alpha(x) := (a_1, a_2, \ldots)$ then $L_\alpha^k(x) \in (t_{a_k+1}, t_{a_k}]$, for any $k \geq 1$. Moreover, if $\alpha \subset \mathbb{Q}$, then the set of pre-periodic points is $\mathbb{Q}$.

1.1. Invariant measures

Denote by λ the Lebesgue measure on $\mathcal{I}$. The Gauss measure, given by $\mu_G(A) := \frac{1}{\log 2} \int_A \frac{dx}{1+x}$, for any Borel set $A \subset \mathcal{I}$, is the unique invariant measure for G that is absolutely continuous with respect to λ. On the other hand, the Lebesgue measure is the unique ergodic invariant measure that is absolutely continuous with respect to λ, for any α-Lüroth map, for any given sequence α. The following lemma, whose proof we omit, is a consequence of a stronger statement that can be found in [5]. The uniqueness of this measure can also be obtained from the Folklore Theorem of existence and uniqueness of absolutely continuous invariant measures for Markov maps; see [2].

Lemma 1.2. *If T is a generalised linear α-Lüroth map, then λ, the Lebesgue measure, is the unique T-invariant and ergodic measure that is absolutely continuous with respect to the Lebesgue measure.*

Now we are in a position to state and prove the following theorem, which proves the Main Theorem.

Theorem 1.3. *Given a partition α as above, the map $?_\alpha : \mathcal{I} \to \mathcal{I}$ is a increasing homeomorphism such that $?_\alpha$ is singular and that $L_\alpha \circ ?_\alpha = ?_\alpha \circ G$. Moreover, if $\alpha \subset \mathbb{Q}$ then $?_\alpha(\mathbb{A}_2) \subset \mathbb{Q}$.*

Proof. Let α be a partition as above. If we define $?_\alpha := \text{Lür}_\alpha^{-1} \circ \text{cf} : \mathcal{I} \to \mathcal{I}$, then clearly $L_\alpha \circ ?_\alpha = ?_\alpha \circ G$, and hence is a topological conjugation between G and L_α. Notice that $?_\alpha'(x)$ exists for λ-a.e. $x \in \mathcal{I}$, since the function $?_\alpha$ is increasing. Moreover, if $\alpha \subset \mathbb{Q}$ then $?_\alpha(\mathbb{A}_2) = \text{Per}(L_\alpha) \subset \mathbb{Q}$, since $?_\alpha$ is a conjugation.

To prove that $?_\alpha$ is singular, we need to show that there is a Borel set $\tilde{B}$ such that $\lambda(\tilde{B}) = 0$ and $\lambda(?_\alpha(\tilde{B})) = 1$; see Theorem 3.72 of [6], for instance. For that, consider the pull-back measure $\mu_\alpha(B) := \lambda(?_\alpha(B))$, for any Borel set B on $\mathcal{I}$. The measure μ_α is invariant and ergodic for G. This follows from the fact that λ is an invariant ergodic measure for L_α.

So, we have two ergodic measures for G. Therefore, either $\mu_\alpha = \mu_G$ or they are singular to each other. If $\mu_\alpha = \mu_G$, we have that for any Borel set B:

$$\int\limits_{?_\alpha(B)} dx = \int\limits_B ?_\alpha^{-1}(x) dx = \frac{1}{\log 2} \int\limits_B \frac{dx}{1+x}.$$

Therefore, $?_\alpha^{-1}(x) = 1/(\log 2(1+x))$, for λ-a.e. $x \in \mathcal{I}$. Observe that the right-hand side is a decreasing function on x, and that $?_\alpha^{-1}(x)$ is increasing. This is a contradiction. Then, μ_α and μ_G are mutually singular. Hence, there exists a measurable set $\tilde{B} \subset \mathcal{I}$ such that $\lambda(\tilde{B}) = 0$ and $\lambda(?_\alpha(\tilde{B})) = 1$, and we are done. $\square$

Acknowledgements

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